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Variants of the Littlewood conjecture, their connection to uniformly distributed sequences, and the exact order of the discrepancy of van der Corput–Kronecker-type sequences

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Résumé. Les objectifs de cet article sont doubles. Premièrement, il traite de la conjecture de Littlewood et de ses variantes en relation avec les suites équidistribuées. Le second objectif est de déterminer l'ordre exact de la discrépance des suites de type van der Corput–Kronecker qui sont basées sur des contre-exemples récents à la conjecture de Littlewood X-adique sur les corps finis. Notre résultat sur l'ordre exact de la discrépance étaye la conjecture bien établie dans la théorie de l'équidistribution, selon laquelle $D_N \leq c\frac{\log^s N}{N}$, avec $c > 0$ pour tout $N > 1$, constitue la meilleure borne supérieure possible pour la discrépance D_N d'une suite dans $[0,1)^s$.

Abstract. The aims of this paper are twofold. First, it discusses the Littlewood conjecture and its variants with respect to uniformly distributed sequences. The second aim is to determine the exact order of the discrepancy of the van der Corput–Kronecker-type sequences which are based on recent counterexamples to the X-adic Littlewood conjecture over finite fields. Our result on the exact order of the discrepancy supports the well-established conjecture in the theory of uniform distribution, which states that $D_N \leq c\frac{\log^s N}{N}$, with $c > 0$ for all $N > 1$ is the best possible upper bound for the discrepancy D_N of a sequence in $[0,1)^s$.

1. Uniform distribution, discrepancy and the Littlewood conjecture

The Littlewood Conjecture and its variants have been the subject of frequent investigation in recent decades. Similar problems to these variants have arisen independently in the theory of uniform distribution, where the discrepancy of so-called Kronecker, Kronecker-type, Halton–Kronecker and Halton–Kronecker-type sequences has been investigated. This manuscript aims to provide an overview of variants of the Littlewood conjecture and

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their relevance to specific uniformly distributed sequences. This overview is intentionally brief and does not provide a complete list of all references, but rather aims to offer a concise starting point.

We start with the definition of the uniform distribution and the discrepancy of a sequence, before giving two well-established examples of one-dimensional uniformly distributed, low-discrepancy sequences.

Definition 1.1. A sequence $(\boldsymbol{x}_n)_{n\geq 0}$ in $[0,1)^s$ is *uniformly distributed* if

$$\lim_{N\to\infty} \frac{\#\{0 \leq n < N : \boldsymbol{x}_n \in J\}}{N} = \mathrm{vol}(J)$$

for every subinterval $J = \prod_{i=1}^{s}[a_i, b_i)$ of $[0,1)^s$ with $0 \leq a_i < b_i \leq 1$ for $i = 1, \ldots, s$.

The *discrepancy* D_N of the first $N \in \mathbb{N}$ points of a sequence $(\boldsymbol{x}_n)_{n\geq 0}$ in $[0,1)^s$ is defined as

$$D_N(\boldsymbol{x}_n) := \sup_J \left| \frac{\#\{0 \leq n < N : \boldsymbol{x}_n \in J\}}{N} - \mathrm{vol}(J) \right|$$

where the supremum is extended over all such subintervals $J = \prod_{i=1}^{s}[a_i, b_i)$ of $[0,1)^s$. The star-discrepancy D_N^* is obtained when taking the supremum over subintervals of the form $\prod_{i=1}^{s}[0, b_i)$.

Note that for a sequence $(\boldsymbol{x}_n)_{n\geq 0}$ in $[0,1)^s$ we always have

$$D_N^* \leq D_N \leq 2^s D_N^* \tag{1.1}$$

for every N (see e.g. [41, Proposition 2.4]).

Example 1.2.

(1) Let $b \geq 2$, $b \in \mathbb{N}$. Additionally, for $n \in \mathbb{N}_0$, let $n_i \in \{0, 1, \ldots, b-1\}$ and let $n = \sum_{i=0}^{\infty} n_i b^i$ be the unique base b expansion of n. Then, define the *b-adic radical inverse function* $\varphi_b(n)$ of $n \in \mathbb{N}_0$ as

$$\varphi_b(n) = \frac{n_0}{b} + \frac{n_1}{b^2} + \frac{n_2}{b^3} + \cdots .$$

The so-called *b-adic van der Corput sequence* $(\varphi_b(n))_{n\geq 0}$ is well known as uniformly distributed sequence and its discrepancy satisfies

$$N D_N(\varphi_b(n)) = O(\log N)$$

(see e.g. [41, Chapter 3]).

(2) Let $\alpha \in \mathbb{R}$. Let the symbol $\{\,\cdot\,\}$ denote the fractional part function in $\mathbb{R}$. Then the *$(n\alpha)$-sequence*, $(\{n\alpha\})_{n\geq 0}$, is uniformly distributed if and only if α is irrational (see e.g. [17, Theorem 1.59]). Represent $\alpha \in [0,1)$ in its simple continued fraction $\alpha = [a_1, a_2, a_3, \ldots]$

with convergents p_j/q_j then the discrepancy of the $(n\alpha)$-*sequence* satisfies

$$\Omega\left(\sum_{i=1}^{m(N)} a_i\right) = ND_N(\{n\alpha\}) = O\left(\sum_{i=1}^{m(N)} a_i\right)$$

where $q_{m(N)-1} < N \leq q_{m(N)}$ (cf. e.g. [17, Corollary 1.64]). In particular, for an irrational α, $ND_N(\{n\alpha\}) = O(\log N)$ if and only if $\frac{1}{m}\sum_{i=1}^{m} a_i$ is bounded (cf. e.g. [17, Corollary 1.65]). Consequently, if the coefficients in the simple continued fraction of α are bounded, then

$$\Omega(\log N) = ND_N(\{n\alpha\}) = O(\log N)\,. \tag{1.2}$$

Later on the symbol $\{\cdot\}$ is also used for vectors in $\mathbb{R}^n$, which is applied then componentwise. Furthermore "$= \Omega(f(N))$" means "$\geq cf(N)$ for infinitely many $N \in \mathbb{N}$ with an implied constant $c > 0$, which might depend on various parameters but which is independent of N". Similarly, "$O(f(N))$" denotes "$\leq Cf(N)$ for all $N \in \mathbb{N}$ (or at least for all sufficiently large N) with an implied constant $C > 0$ which may depend on various parameters, but which is independent of N."

The bounds in (1.2) are best possible in N, since a well-known general result of Schmidt [48] states that, for every one-dimensional sequence $(x_n)_{n\geq 0}$ in $[0,1)$ its discrepancy satisfies $ND_N(x_n) = \Omega(\log N)$.

In the case of an s-dimensional sequence $(\boldsymbol{x}_n)_{n\geq 0}$ in $[0,1)^s$, the best-known lower bound for its discrepancy is due to Bilyk, Lacey and Vagharshakyan [8], and is of the form

$$ND_N(\boldsymbol{x}_n) = \Omega(\log^{s/2+\eta_s} N)$$

where both the implied positive constant and $0 < \eta_s < 1/2$ depend on s, but are independent of N. This result improves upon and generalises earlier results by Roth [46], Beck [5], and Bilyk and Lacey [7].

In the theory of uniform distribution it is frequently conjectured,

$$ND_N(\boldsymbol{x}_n) = O(\log^s N) \tag{1.3}$$

is the best possible upper bound in N that can be achieved for the discrepancy of an s-dimensional sequence $(\boldsymbol{x}_n)_{n\geq 0}$ in $[0,1)^s$. This is why sequences that satisfy a bound of the form given in equation (1.3) are called *low-discrepancy sequences*. It is therefore of great interest to determine whether there is a lower bound for the discrepancy of a concrete example of an s-dimensional low-discrepancy sequence with $s > 1$ of the form $ND_N = \Omega(\log^s N)$, which then supports the conjecture that (1.3) is best possible. It should be emphasised that a counterexample to $ND_N = \Omega(\log^s N)$ in any dimension $s > 1$ would raise many new questions and create numerous open problems in the theory of uniform distribution, as well as in the field of numerical integration algorithms known as quasi-Monte Carlo methods. (The interested reader is refered to e.g. [15] and [41] for more details on quasi-Monte Carlo algorithms). One of the most famous multi-dimensional low-discrepancy sequence is the so-called *Halton sequence*, $(\varphi_{b_1}(n), \dots, \varphi_{b_s}(n))_{n\geq 0}$, in pairwise coprime bases $b_1, \dots, b_s \geq 2, \in \mathbb{N}$. This sequence is uniformly distributed and its discrepancy satisfies

$$ND_N(\varphi_{b_1}(n), \dots, \varphi_{b_s}(n)) = O(\log^s N)$$

(see e.g. [41, Theorem 3.6]). In [34] Levin ensured the conjectured lower bound

$$ND_N(\varphi_{b_1}(n), \dots, \varphi_{b_s}(n)) = \Omega(\log^s N)$$

for the discrepancy of the Halton sequence. This work was supplemented by further work of Levin [35, 36, 37], which ensures the conjectured lower discrepancy bound for various classes of low-discrepancy sequences. (See also [21] for a previous work and [27] for a further lower-discrepancy bound of the form $ND_N(\boldsymbol{x}_n) = \Omega(\log^s N)$ for a class of low-discrepancy sequences.)

Now we return to the $(n\alpha)$-sequences. It is well known that the coefficients in the simple continued fraction of a real number α are bounded if and only if

$$\alpha \in \mathbf{Bad} := \Big\{\beta \in \mathbb{R} \, : \, \inf_{h\in\mathbb{Z}\setminus\{0\}} |h| \cdot \|h\beta\| \geq c_\beta > 0\Big\}.$$

Here $\|\cdot\|$ denotes the distance-to-the-nearest-integer function in $\mathbb{R}$. Hence, if $\alpha \in \mathbf{Bad}$ then (1.2) holds for the discrepancy of the $(n\alpha)$-sequence.

For a vector $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_s) \in \mathbb{R}^s$ the so-called *Kronecker sequence* is defined as $(\{n\boldsymbol{\alpha}\})_{n\geq 0}$, and serves as a multidimensional version of the $(n\alpha)$-sequence. It is known that a Kronecker sequence, $(\{n\boldsymbol{\alpha}\})_{n\geq 0}$, is uniformly distributed if and only if $1, \alpha_1, \alpha_2, \dots, \alpha_s$ are rationally independent.

Although the Kronecker sequence has been studied extensively and intensively, the following two issues remain unresolved:

- Is it true that for $s \geq 2$ we always have $ND_N(\{n(\alpha_1, \dots, \alpha_s)\}) = \Omega(\log^s N)$.
- Is it possible to identify for $s \geq 2$ an example $(\alpha_1, \dots, \alpha_s) \in \mathbb{R}^s$ such that

$$ND_N(\{n(\alpha_1, \dots, \alpha_s)\}) = O(\log^s N). \tag{1.4}$$

For the discrepancy of a Kronecker sequence, $(\{n\boldsymbol{\alpha}\})_{n\geq 0}$, many proof techniques rely on lower bounds of

$$|h_1 \cdots h_s| \cdot \|h_1\alpha_1 + \dots + h_s\alpha_s\| \tag{1.5}$$

depending on $(h_1, \dots, h_s) \in (\mathbb{Z} \setminus \{0\})^s$. In short, one can say, the larger the value of (1.5), the smaller the obtained upper discrepancy bound.

We define a generalization of **Bad** namely the set $\mathbf{Bad^s}$ as

$$(1.6)\quad \left\{ (\alpha_1,\ldots,\alpha_s)\in\mathbb{R}^s : \begin{array}{r} \inf\limits_{(h_1,\ldots,h_s)\in\mathbb{Z}^s\setminus\{\mathbf{0}\}} \max\{1,|h_1|\}\cdots\max\{1,|h_s|\} \\ \cdot\, \|h_1\alpha_1+\cdots+h_s\alpha_s\| \\ \geq c_{\alpha_1,\ldots,\alpha_s}>0 \end{array} \right\}.$$

The famous Littlewood Conjecture, which is unresolved for almost 100 years, states that for $s>1$ the set $\mathbf{Bad^s}$ is empty. In other words, for every $(\alpha_1,\ldots,\alpha_s)\in\mathbb{R}^s$ with $s>1$ we have

$$\inf_{(h_1,\ldots,h_s)\in\mathbb{Z}^s\setminus\{\mathbf{0}\}} \max\{1,|h_1|\}\cdots\max\{1,|h_s|\}\cdot\|h_1\alpha_1+\cdots+h_s\alpha_s\|=0.$$

Although Littlewood's conjecture is widely accepted, the question of a multi-dimensional, low-discrepancy Kronecker sequence remains meaningful. Note that the existence of a vector $(\alpha_1,\ldots,\alpha_s)\in\mathbb{R}^s$ such that (1.4) holds, is not equivalent to the existence of a vector $(\alpha_1,\ldots,\alpha_s)\in\mathbf{Bad^s}$. Example 1.2 (2) shows that this is even not the case for $s=1$. For example choose $\alpha=[0;a_1,a_2,\ldots]$ with $a_{k!}=k$ and $a_j=1$ else, then $\alpha\notin\mathbf{Bad}$ but $\frac{1}{m}\sum_{j=1}^m a_j$ is bounded. Using [17, Corollary 1.65] together with $m=O(\log N)$ we obtain for this α, $ND_N(\{n\alpha\})=O(\log N)$. Note that $m=O(\log N)$ follows immediately from the fact that $q_{m(N)-1}$ can be bounded from below by the Fibonacci numbers.

The Littlewood conjecture is usually found in literature in its most common form: for every $(\alpha_1,\alpha_2)\in\mathbb{R}^2$ we have

$$\liminf_{h\to\infty} h\cdot\|h\alpha_1\|\cdot\|h\alpha_2\|=0.$$

Cassels and Swinnerton-Dyer [13] proved the equivalence with its dual form: for every $(\alpha_1,\alpha_2)\in\mathbb{R}^2$ we have

$$\inf_{(h_1,h_2)\in\mathbb{Z}\times\mathbb{Z}\setminus\{(0,0)\}} \max\{|h_1|,1\}\max\{|h_2|,1\}\|h_1\alpha_1+h_2\alpha_2\|=0$$

For weaker versions of (1.6), as

$$\inf_{(h_1,\ldots,h_s)\in(\mathbb{Z}\setminus\{\mathbf{0}\})^s} |h_1\cdots h_s|\cdot\varphi(|h_1\cdots h_s|)\cdot\|h_1\alpha_1+\cdots+h_s\alpha_s\|=c_{\alpha_1,\ldots,\alpha_s}>0$$

with a suitable function φ, there are several results (see e.g. Schmidt [47], Beck [6], Bugeaud & Moshchevitin [12]) with consequences for the discrepancy of the related Kronecker sequences. A result of Beck [6, Theorem 1] states for example:

Theorem 1.3. *For almost all $(\alpha_1,\ldots,\alpha_s)\in\mathbb{R}^s$, in the sense of Lebesgue measure, we have for every $\varepsilon>0$,*

$$ND_N(\{n(\alpha_1,\ldots,\alpha_s)\})=O\left((\log N)^s(\log\log N)^{1+\varepsilon}\right)$$

with an implied constant depending on $(\alpha_1,\ldots,\alpha_s)$ and ε but independent of N.

Finally the work of Einsiedler, Katok, and Lindenstrauss [18] should be definitely mentioned when discussing the Littlewood conjecture. They proved (see [18, Theorem 1.5]) that the Hausdorff dimension of the set $\mathbf{Bad^s}$, i.e. the set of all counterexamples to the Littlewood conjecture, is zero.

The Littlewood conjecture was stated in the 1930s, after which many variants of it arose and were studied. The subsequent Subsections 1.1 and 1.2 introduce variants that are particularly important for sequences in the theory of uniform distribution.

1.1. The *p*-adic Littlewood conjecture. Let $\mathbb{P}$ be the set of prime numbers. Let $p \in \mathbb{P}$. The so-called *p-adic Littlewood conjecture* (see Mathan and Teulié [39]) predicts for every $\alpha \in \mathbb{R}$ that

$$\inf_{h\in\mathbb{Z}\setminus\{0\},k\geq 0} |h| \cdot \|hp^k\alpha\| = 0$$

or equivalently that the following set $\mathbf{Bad}_p$ is empty.

$$\mathbf{Bad}_p := \Big\{\alpha \in \mathbb{R} : \inf_{h\in\mathbb{Z}\setminus\{0\},k\geq 0} |h| \cdot \|hp^k\alpha\| \geq c_{\alpha,p} > 0\Big\}.$$

Note that for a counterexample α to the p-adic Littlewood conjecture, i.e. $\alpha \in \mathbf{Bad}_p$, the continued fraction coefficients of $\{p^k\alpha\}$ must be uniformly bounded for all $k \in \mathbb{N}_0$. From [28, Proposition 2] for such $\alpha \in \mathbf{Bad}_p$ it follows that the two-dimensional sequence $((\varphi_p(n), \{n\alpha\}))_{n\geq 0}$ would be a low-discrepancy sequence, i.e.

$$ND_N((\varphi_p(n), \{n\alpha\})) = O(\log^2 N).$$

The p-adic Littlewood conjecture is still open.

For weaker forms of the p-adic Littlewood conjecture such as

$$\inf_{h\in\mathbb{Z}\setminus\{0\},k\geq 0} |h| \cdot \varphi(h, p^k) \cdot \|hp^k\alpha\| \geq c_{\alpha,p} > 0$$

with a suitable function φ, there are metrical results as well as results for algebraic α (see [28, 32, 43, 10]) with consequences for the discrepancy of the related sequence, $((\varphi_p(n), \{n\alpha\}))_{n\geq 0}$. See e.g. the following two theorems:

Theorem 1.4 ([28, 32])**.** *Let $p \in \mathbb{P}$. For almost every α in $[0, 1]$, in the sense of Lebesgue measure, we have*

$$ND_N((\varphi_p(n), \{n\alpha\})) = O(\log^{2+\varepsilon} N) \qquad \textit{for every } \varepsilon > 0,$$

with an implied constant, which may depend on ε and α but which is independent of N.

Theorem 1.5 ([16])**.** *Let $p \in \mathbb{P}$. For a real number α that is irrational as well as algebraic we have*

$$ND_N((\varphi_p(n), \{n\alpha\})) = O(N^{\varepsilon}) \qquad \textit{for every } \varepsilon > 0,$$

with an implied constant, which may depend on ε and α but which is independent of N.

See Einsiedler and Kleinbock [19] for a recent work on the p-adic Littlewood conjecture, which proves that the Hausdorff dimension of $\mathbf{Bad}_p$ is zero. For more recent work on putative counterexamples to the p-adic Littlewood conjecture, see [9].

We would like to note that, according to [19, Theorem 1.3], the following variant of the Littlewood conjecture is true: Let p_1, p_2 be distinct prime numbers. Then for every $\alpha \in \mathbb{R}$ we have

$$\inf_{h\in\mathbb{Z}\setminus\{0\}, r_1,r_2\geq 0} |h| \cdot \|hp_1^{r_1}p_2^{r_2}\alpha\| = 0. \tag{1.7}$$

This excludes existence of an $\alpha \in \mathbb{R}$ such that the continued fraction coefficients of $\{p_1^{r_1}p_2^{r_2}\alpha\}$ can be uniformly bounded for all $r_1, r_2 \in \mathbb{N}_0$. Consequently, there is no obvious candidate $\alpha \in \mathbb{R}$ such that from [28, Proposition 2] we know $ND_N((\varphi_{p_1}(n), \varphi_{p_2}(n), \{n\alpha\})) = O(\log^3 N)$. Nevertheless, [19, Theorem 1.3] does not contradict existence of such an α.

1.2. The Littlewood conjecture in the field of formal Laurent series and its X-adic version. Replacing the set of real numbers with the set of formal Laurent series introduces further variants of the Littlewood conjecture. Before we state these variants, we will clarify the set of formal Laurent series and some relevant notions.

We write $\mathbb{F}[X]$ for the ring of polynomials, $\mathbb{F}(X)$ for the field of rational functions, and

$$\mathbb{F}((X^{-1})) = \left\{ \sum_{i=j}^{\infty} a_i X^{-i} : j \in \mathbb{Z}, a_i \in \mathbb{F} \text{ for all } i \geq j, a_j \neq 0 \right\}$$

for the field of formal Laurent series over a field $\mathbb{F}$. For

$$\theta = \sum_{i=j}^{\infty} a_i X^{-i} \in \mathbb{F}((X^{-1}))$$

satisfying $a_j \neq 0$ we define as an extension of the degree of a polynomial the *degree evaluation* $\deg(\theta)$ of θ as

$$\deg(\theta) := -j,$$

the absolute value $|\theta|$ of θ as

$$|\theta| := 2^{-j} = 2^{\deg(\theta)},$$

the fractional part of θ as

$$\langle\theta\rangle := \sum_{i=\max\{1,j\}}^{\infty} a_i X^{-i},$$

and finally

$$\|\theta\| := |\langle\theta\rangle|$$

denotes the distance-to-the-nearest-polynomial.

Davenport & Lewis [14] asked whether or not

$$\inf_{(Q_1,\ldots,Q_s)\in\mathbb{F}[X]^s\setminus\{\mathbf{0}\}} \max\{1,|Q_1|\}\cdots\max\{1,|Q_s|\}\cdot\|Q_1\theta_1+\cdots+Q_s\theta_s\| = 0 \tag{1.8}$$

for every $(\theta_1,\ldots,\theta_s) \in \mathbb{F}((X^{-1}))^s$ with $s > 1$ in analogy to the Littlewood conjecture. In other words, one can say an element in $(\theta_1,\ldots,\theta_s) \in \mathbb{F}((X^{-1}))^s$ with $s > 1$ and

$$\inf_{(Q_1,\ldots,Q_s)\in\mathbb{F}[X]^s\setminus\{\mathbf{0}\}} \max\{1,|Q_1|\}\cdots\max\{1,|Q_s|\}\cdot\|Q_1\theta_1+\cdots+Q_s\theta_s\| \\ = c_{\theta_1,\ldots,\theta_s} > 0$$

would be a counterexample to this variant of the Littlewood conjecture (1.8). Davenport and Lewis [14] ensured the existence of a counterexample in the case where the field $\mathbb{F}$ is infinite. Their work was complemented by Baker [4] who provided an explicit counterexample in this case. An attempt by Armitage 1969 to construct a counter example over the finite field $\mathbb{F}_p$ with $p > 3$ failed as the proof was erroneous (see [3]). Consequently, the function field analog of the Littlewood conjecture over a finite field remains an open problem.

In particular, a counterexample over a finite field $\mathbb{F} = \mathbb{F}_p$ with $p \in \mathbb{P}$, might be used to obtain low-discrepancy *Kronecker-type sequences* in the sense of Larcher and Niederreiter [33]. Note that Kronecker-type sequences are also known as digital Kronecker sequences in the literature (see e.g. [38, 44]). For the construction of a Kronecker-type sequence first choose instead of $\alpha_1,\ldots,\alpha_s \in \mathbb{R}$, formal Laurent series $\theta_1,\ldots,\theta_s \in \mathbb{F}_p((X^{-1}))$. Then for an index $n \in \mathbb{N}_0$ define the polynomial $n(X) = n_0 + n_1X + n_2X^2 + \cdots$ via the unique base p expansion of n, i.e. $n = n_0 + n_1p + n_2p^2 + \cdots$. Then compute

$$\langle\theta_i n(X)\rangle \qquad \text{and finally set } X = p \text{, which is denoted by “}|_p\text{”,}$$

in order to obtain a real number in $[0,1)$. The so-constructed sequence $(\boldsymbol{x}_n)_{n\geq 0}$ in $[0,1)^s$ with

$$\boldsymbol{x}_n = (\langle\theta_1 n(X)\rangle|_p,\ldots,\langle\theta_s n(X)\rangle|_p)$$

is then called a Kronecker-type sequence. Larcher and Niederreiter [33, Theorem 2] identified counterexamples to the function field analog of the Littlewood conjecture over a finite field as potential candidates for constructing low-discrepancy Kronecker-type sequences.

Theorem 1.6. *If* $(\theta_1, \ldots, \theta_s) \in \mathbb{F}_p((X^{-1}))^s$ *is a counterexample to the Littlewood conjecture* (1.8) *then the Kronecker-type sequence satisfies*

$$ND_N((\langle\theta_1 n(X)\rangle|_p, \ldots, \langle\theta_s n(X)\rangle|_p)) = O(\log^s N)$$

with an implied constant only depending on $\theta_1, \ldots, \theta_s$, p, *and* s.

Similar as for the original Littlewood conjecture weaker versions of (1.8) were studied and e.g. as a consequence the following result is known.

Theorem 1.7 ([31])**.** *For almost all* $(\theta_1, \ldots, \theta_s) \in \mathbb{F}_p((X^{-1}))^s$, *in the sense of Haar-measure, we have for every* $\varepsilon > 0$*:*

$$ND_N((\langle\theta_1 n(X)\rangle|_p, \ldots, \langle\theta_s n(X)\rangle|_p)) = O\big((\log N)^s(\log\log N)^{2+\varepsilon}\big).$$

Note that there are unsolved problems in $\mathbb{R}$ that have been solved over finite fields. One prominent example is the function field analog of the famous Riemann hypothesis (see e.g. [22]). Nevertheless, the following two items are still open for Kronecker-type sequences:

- Is it true that for $s \geq 2$ we always have
$$ND_N((\langle\theta_1 n(X)\rangle_p, \ldots, \langle\theta_s n(X)\rangle_p)) = \Omega(\log^s N)?$$
- Is it possible to identify for $s \geq 2$ an example $(\theta_1, \ldots, \theta_s) \in \mathbb{R}^s$ such that
$$ND_N(\langle\theta_1 n(X)\rangle|_p, \ldots, \langle\theta_s n(X)\rangle|_p) = O(\log^s N)?$$

The last variant, which we would like to discuss in this manuscript, is the so-called $P(X)$*-adic Littlewood conjecture* [39] with $P(X)$ an irreducible polynomial in $\mathbb{F}[X]$, the corresponding analog of the unsolved p-adic Littlewood conjecture. We state here (see (1.9)) the $P(X)$-adic Littlewood conjecture for the probably most simplest choice of an irreducible polynomial $P(X) = X$. For a general irreducible polynomial $P(X)$ replace X by $P(X)$ in (1.9). We refer the interested reader to e.g. [23, 39, 44]. The X-adic Littlewood conjecture claims for every $\theta \in \mathbb{F}((X^{-1}))$,

$$\inf_{r\in\mathbb{N}_0,\, Q\in\mathbb{F}[X]\setminus\{\mathbf{0}\}} |Q| \cdot \|QX^r\theta\| = 0. \tag{1.9}$$

Note that from a counterexample to the X-adic Littlewood conjecture (1.9) one can derive a counterexample to the $P(X)$-adic Littlewood conjecture by substituting X by $P(X)$ (cf. [45, Theorem 1.0.3]).

If the ground field $\mathbb{F}$ is infinite counterexamples are already known, see [39, Théorème 4.3]. However, the method in [39] fails for finite fields $\mathbb{F}$. Note that a counterexample over a finite field will be interesting for the construction of a two-dimensional low-discrepancy *van der Corput–Kronecker-type sequence.* For such a two-dimensional van der Corput–Kronecker-type sequence choose $p \in \mathbb{P}$ and $\theta \in \mathbb{F}_p((X^{-1}))$ and combine the p-adic van der Corput sequence $(\varphi_p(n))_{n\geq 0}$ with the Kronecker-type sequence

$(\langle \theta n(X)\rangle\,|_p)_{n\geq 0}$. Levin [38] was the first who noted that a van der Corput–Kronecker-type sequence based on a counterexample to the X-adic Littlewood conjecture over $\mathbb{F}_p$ is a low-discrepancy sequence. Robertson [44] recently worked out a detailed proof for [44, Theorem 1.5] which covers the following Theorem 1.8. The result of Robertson [44, Theorem 1.5] is also valid for two-dimensional sequences where the second coordinate is the Kronecker-type sequence $(\langle \theta n(X)\rangle\,|_p)_{n\geq 0}$ based on a counterexample to the more general $P(X)$-adic Littlewood conjecture and where the first component is a so-called Halton-type sequence in the sense of [25] fitting to the polynomial $P(X)$ instead of X. Since the main result of this work is a lower bound on the discrepancy for van der Corput–Kronecker-type sequences (see Theorem 1.10), and the method of proof heavily uses the p-adic van der Corput component, we formulate and reprove the following theorem only for the special case in order to avoid a detailed definition of so-called Halton-type sequences (the interested reader is referred to [25, 26, 44]) and the method of proof of Theorem 1.8 also prepares parts of the proof of Theorem 1.10.

Theorem 1.8 ([44])**.** *Let $\theta \in \mathbb{F}_p((X^{-1}))$ be a counterexample to the X-adic Littlewood conjecture over $\mathbb{F}_p$, i.e.*

$$\inf_{r\geq 0, Q\in\mathbb{F}_p[X]\setminus\{\mathbf{0}\}} |Q|\cdot\|X^rQ\theta\| \geq c > 0.$$

Then the discrepancy of the two-dimensional van der Corput–Kronecker-type sequence satisfies

$$ND_N((\varphi_p(n), \langle\theta n(X)\rangle|_p)) = O(\log^2 N)$$

with an implied absolute constant only depending on θ and p.

Note that, similar to the other variants of the Littlewood conjecture, metrical results for weaker versions of the $P(X)$-adic Littlewood conjecture are known. See [45] and previously [26]. In [26] the metrical results therein were already applied to the discrepancy of so-called Halton–Kronecker-type sequences. In its most basic form such a sequence is a van der Corput–Kronecker-type sequence. For example in [26] the following result was obtained.

Theorem 1.9. *Let $\mathbb{F} = \mathbb{F}_p$ then for almost all θ, in the sense of Haar measure, for every $\varepsilon > 0$,*

$$ND_N((\varphi_p(n), \langle\theta n(X)\rangle|_p)) = O(\log^{2+\varepsilon}(N)),$$

with an implied constant only depending on θ, p, and ε.

In contrast to the fact that (1.8) over any finite field $\mathbb{F} = \mathbb{F}_p$ is still open, a recent counterexample to the X-adic Littlewood conjecture over $\mathbb{F} = \mathbb{F}_3$ was discovered by Adiceam, Nesharim, and Lunnon [2]. This counterexample is defined as $\theta = \sum_{i\geq 1} f_i X^{-i}$ over $\mathbb{F}_3$ with $(f_n)_{n\geq 1}$ is the paperfolding sequence in $\{0,1\}$, defined as

$$f_n = \begin{cases} 0 & k \equiv 1 \pmod 4 \\ 1 & k \equiv 3 \pmod 4 \end{cases},$$

with odd $k \in \mathbb{N}$ such that $n = 2^{\nu_2(n)}k$. Then

$$\inf_{r\geq 0, Q\in\mathbb{F}_3[X]\setminus\{\mathbf{0}\}} |Q| \cdot \|X^r Q\theta\| \geq 2^{-4} = c > 0.$$

For larger $p = 5, 7, 11$ further counterexample can be found in [23]. Note that the proofs in [2, 23] are computer assisted and therefore limited to small p. Adiceam and Badziahin [1] recently disproved the $P(X)$-adic Littlewood conjecture over any finite field with characteristic p where p is a prime numbers satisfying $p \equiv 3 \pmod 4$. It remains an open problem to find explicit counterexamples over every finite field with characteristic p satisfying $p \equiv 1 \pmod 4$. The case of $p = 2$ seems to be completely different. In [23, Conjecture 1.8] it's argued that the $P(X)$-adic Littlewood conjecture might be true in the case where $\mathbb{F} = \mathbb{F}_2$.

With the specific counterexamples to the X-adic Littlewood conjecture over a finite field $\mathbb{F}_p$, Theorem 1.8 gives new examples of low-discrepancy sequences. Having a new type of a low-discrepancy sequence in two dimensions, one is interested in a lower bound for its discrepancy. It should be emphasized once again at this point that for a concrete example of a low-discrepancy sequence in two-dimensions, a result of the form $ND_N = \Omega(\log^2 N)$ supports the well-established conjecture for the discrepancy while a smaller exact order of ND_N in N would bring a counterexample to this conjecture in uniform distribution. Such a counterexample would result in many new problems in the theory of uniform distribution. Our main theorem of this manuscript, Theorem 1.10, gives the desired lower bound for the discrepancy of the low-discrepancy sequences in Theorem 1.8.

Theorem 1.10. *Let $\theta \in \mathbb{F}_p((X^{-1}))$ be a counterexample to the X-adic Littlewood conjecture over $\mathbb{F}_p$, i.e.*

$$\inf_{r\geq 0, Q\in\mathbb{F}_p[X]\setminus\{\mathbf{0}\}} |Q| \cdot \|X^r Q\theta\| \geq c > 0. \tag{1.10}$$

Then the discrepancy of the two-dimensional van der Corput–Kronecker-type sequence satisfies

$$ND_N((\varphi_p(n), \langle\theta n(X)\rangle|_p)) = \Omega(\log^2 N)$$

with an implied constant only depending on θ and p.

It is a natural impulse to expand the two-dimensional sequence

$$((\varphi_p(n), \langle\theta n(X)\rangle|_p))_{n\geq 0}$$

to a three-dimensional one with the aim of preserving the low-discrepancy property. Two different approaches come to mind here. The first is to define a three dimensional sequence $((\varphi_p(n), \langle\theta_1 n(X)\rangle|_p, \langle\theta_2 n(X)\rangle|_p))_{n\geq 0}$ with $\theta_1, \theta_2 \in \mathbb{F}_p((X^{-1}))$ and study

$$\inf_{r\in\mathbb{N}_0,(Q_1,Q_2)\in\mathbb{F}[X]^2\setminus\{\mathbf{0}\}} \max\{1,|Q_1|\}\max\{1,|Q_2|\}\cdot\|(Q_1\theta_1+Q_2\theta_2)X^r\|. \tag{1.11}$$

Since it is open whether or not

$$\inf_{(Q_1,Q_2)\in\mathbb{F}[X]^2\setminus\{\mathbf{0}\}} \max\{1,|Q_1|\}\max\{1,|Q_2|\}\cdot\|(Q_1\theta_1+Q_2\theta_2)\| = 0$$

for every $\theta_1, \theta_2 \in \mathbb{F}_p((X^{-1}))$, a three dimensional generalization of this form is not the first choice.

As an alternative of (1.11) one can study whether or not

$$\inf_{r_1,r_2\in\mathbb{N}_0,\, Q\in\mathbb{F}[X]\setminus\{\mathbf{0}\}} |Q|\cdot\|X^{r_1}(X+1)^{r_2}Q\theta\| = 0 \tag{1.12}$$

for every $\theta \in \mathbb{F}_p((X^{-1}))$. While the corresponding variant of the Littlewood conjecture over the real numbers (1.7) holds true [19, Theorem 1.3], counterexamples to (1.12) are known for infinite fields $\mathbb{F}$, see [11, Theorem 1]. A counterexample over a finite field $\mathbb{F}_p$, i.e. $\theta \in \mathbb{F}_p((X^{-1}))$, such that $\inf_{r_1,r_2\in\mathbb{N}_0,\, Q\in\mathbb{F}[X]\setminus\{\mathbf{0}\}} |Q|\cdot\|X^{r_1}(X+1)^{r_2}Q\theta\| \geq c > 0$ is still an open problem. A counterexample over $\mathbb{F}_p$ would bring similarly to Theorem 1.8 a result of the form

$$ND_N((x_1, x_2, \langle\theta_1 n(X)\rangle|_p)) = O(\log^3 N),$$

where $(x_1,x_2)_{n\geq 0}$ is a two-dimensional Halton-type sequence in the sense of the author. To avoid further quite technical definitions in this manuscript we refer the interested reader to e.g. [24] and [26] for more information on Halton-type sequences. However, it can be stated here that this is not the end of research in the area of variants of Littlewood's conjecture and the uniform distribution properties of related sequences, but rather that it will give rise to further interesting problems.

The rest of the paper is devoted to prove our main Theorem 1.10 and to provide a second proof of Theorem 1.8. Section 2 collects the essential basic ideas of the concept of digital (t,s)-sequences in the sense of Niederreiter [40] as well as the most important properties of the simple continued fraction of an element in $\mathbb{F}((X^{-1}))$. We identify a van der Corput–Kronecker-type sequence based on a counterexample θ of the X-adic Littlewood conjecture over $\mathbb{F}_p$ as digital $(t,2)$-sequence, where t will be quantified depending on the degrees of the simple continued fraction coefficients of $\langle X^r\theta\rangle$. Within the concept of digital $(t,2)$-sequences the low-discrepancy upper bound in Theorem 1.8 then immediately follows. Finally, in Section 3 we develop the proof of the lower bound in Theorem 1.10, which is based on ideas from Levin [36].

2. Identifying van der Corput–Kronecker-type sequences as digital sequences and a short proof of Theorem 1.8

For some aspects of the class of van der Corput–Kronecker-type sequences, it is useful to interpret these sequences as digital (t, s)-sequences. In particular when investigating the discrepancy of a sequence the digital (t, s)-sequences concept, if applicable, is often the prefered method. We first define the digital method together with digital $(t, m, 3)$-nets and digital $(t, 2)$-sequences in the sense of Niederreiter (see e.g. [41]) as simply as possible to serve our purposes for Theorem 1.8 and Theorem 1.10. We do not distinguish between $\mathbb{F}_p$ and the set $\{0, 1, \dots, p-1\}$.

Definition 2.1. Let $p \in \mathbb{P}$ and $m \in \mathbb{N}$. Choose three $\mathbb{N} \times m$-matrices C_1, C_2, C_3 over $\mathbb{F}_p$, with $\mathbb{N}$ indicating that they have infinitely many rows indexed by $1, 2, 3, \dots$. We construct a so-called *digital pointset* $(\boldsymbol{x}_n)_{0 \le n < p^m}$ of p^m points in $[0, 1]^3$ as follows. Represent the nonnegative integer $n < p^m$ in base p, i.e.

$$n = n_0 + n_1 p + \dots + n_{m-1} p^{m-1} \quad \text{with } 0 \le n_j < p,$$

and set

$$\vec{n} := (n_0, \dots, n_{m-1})^T.$$

To generate the ith coordinate $x_n^{(i)}$ of nth term $\boldsymbol{x}_n$ of the pointset, compute

$$C_i \cdot \vec{n} =: (y_1^{(i)}, y_2^{(i)}, \dots)^T \pmod{p},$$

and set

$$x_n^{(i)} := \frac{y_1^{(i)}}{p} + \frac{y_2^{(i)}}{p^2} + \dots .$$

Let $t \in \mathbb{N}_0$, $t < m$. We call the digital pointset $(\boldsymbol{x}_n)_{0 \le n < p^m}$ a *digital* $(t, m, 3)$-*net over* $\mathbb{F}_p$ if the following holds: for all $d_1, d_2, d_3 \in \mathbb{N}_0$ with $d_1 + d_2 + d_3 \le m - t$ the $(d_1 + d_2 + d_3) \times m$-matrix consisting of the

- upper d_1 rows of C_1 together with the
- upper d_2 rows of C_2 together with the
- upper d_3 rows of C_3,

has full row rank $d_1 + d_2 + d_3$.

For a *digital sequence* $(\boldsymbol{x}_n)_{n \ge 0}$ in $[0, 1)^2$ we choose two $\mathbb{N} \times \mathbb{N}_0$-matrices C_1, C_2 over $\mathbb{F}_p$ with $\mathbb{N}_0$ indicating that they have infinitely many columns indexed by $0, 1, 2, 3, \dots$. To generate the ith coordinate $x_n^{(i)}$ of the nth term $\boldsymbol{x}_n$ of the sequence, represent the integer $n \ge 0$ in base p, i.e.

$$n = n_0 + n_1 p + \dots + n_r p^r \quad \text{with } 0 \le n_j < p,$$

and with $n_r \neq 0$, set

$$\vec{n} := (n_0, \dots, n_r, 0, 0, \dots)^T,$$

and
$$C_i \cdot \vec{n} =: (y_1^{(i)}, y_2^{(i)}, \dots)^T \pmod{p}.$$
Further
$$x_n^{(i)} := \frac{y_1^{(i)}}{p} + \frac{y_2^{(i)}}{p^2} + \dots .$$
Let $t \in \mathbb{N}_0$. We call the digital sequence $(\boldsymbol{x}_n)_{n\geq 0}$ a *digital* $(t,2)$*-sequence over* $\mathbb{F}_p$ if the following holds: for every $m \in \mathbb{N}, m > t$ and for all $d_1, d_2 \in \mathbb{N}_0$ with $d_1 + d_2 \leq m - t$ the $(d_1 + d_2) \times m$-matrix consisting of the

- left upper $d_1 \times m$-submatrix of C_1 together with the
- left upper $d_2 \times m$-submatrix of C_2

has full row rank $d_1 + d_2$.

Usually the construction for digital sequences and digital nets is described for a general dimension s and digital nets are based on $m \times m$ matrices instead of $\mathbb{N} \times m$ matrices. For our proof of Theorem 1.10 we need more exact digital information. It is well-known that digital (t,s)-sequences are low-discrepancy sequences (cf. e.g. [41, Theorem 4.17]). See for instance [20, 24, 29, 40, 42, 49, 50] for different constructions of (t,s)-sequences and [36, 37] for lower bounds of the discrepancy of the form $ND_N = \Omega(\log^s N)$ for many of them. The main challenge for a construction of matrices that are qualified to construct low-discrepancy sequences via a suitable generalization of Definition 2.1 for dimension s is to guarantee a quality parameter t. The constructions [20, 24, 29, 40, 42, 49, 50] guarantee the quality parameter t on e.g. the uniqueness of the partial fraction decomposition in $\mathbb{F}_p(X)$ or the uniqueness of the factorization into irreducible polynomials in $\mathbb{F}_p[x]$ or the so-called Riemann–Roch Theorem valid in a global function field.

The first step for proving Theorem 1.8 and Theorem 1.10 by applying methods developed for digital (t,s)-sequences is to recognize any van der Corput–Kronecker-type sequences over $\mathbb{F}_p$ as a digital sequence in $[0,1)^2$ due to Definition 2.1, see Lemma 2.2. One of the main reasons for reducing to the special case of the X-adic Littlewood conjecture in 1.9 instead of the $P(X)$-adic one, is that for this special case it is possible to identify the related van der Corput–Kronecker-type sequence as a digital $(t,2)$-sequence where the first generator matrix is the unit matrix. This unit matrix will be crucial in the proof of Theorem 1.10. Now suppose that a van der Corput–Kronecker-type sequences over $\mathbb{F}_p$ is based on a counterexample θ to the X-adic Littlewood conjecture over $\mathbb{F}_p$ as in Theorem 1.8 and in Theorem 1.10 our Corollary 2.5 identifies such a sequence as digital $(t,2)$-sequences over $\mathbb{F}_p$. We start with Lemma 2.2 which recognizes any van der Corput–Kronecker-type sequences as a digital sequence.

Lemma 2.2. *Let $p \in \mathbb{P}$ and $\theta = \sum_{i=j}^{\infty} a_i X^{-i} \in \mathbb{F}_p((X^{-1}))$. Let I denote the $\mathbb{N} \times \mathbb{N}_0$ unit matrix over $\mathbb{F}_p$. Let $H(\theta)$ be the* Hankel matrix *determined by the Laurent series θ defined as*

$$H(\theta) = \begin{pmatrix} a_1 & a_2 & a_3 & \dots \\ a_2 & a_3 & a_4 & \dots \\ a_3 & a_4 & a_5 & \dots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix} \in \mathbb{F}_p^{\mathbb{N}\times\mathbb{N}_0},$$

with the setting $a_1 = a_2 = \cdots = a_{j-1} = 0$ in the case where $j > 1$. Then the van der Corput–Kronecker-type sequence $((\varphi_p(n), \langle \theta n(X)\rangle|_p))_{n\geq 0}$ corresponds with the two-dimensional digital sequence $(\boldsymbol{x}_n)_{n\geq 0}$ generated by $C_1 = I$ and $C_2 = H(\theta)$.

Proof. Definition 2.1 immediately shows that a sequence $(\boldsymbol{x}_n)_{n\geq 0}$ constructed by $C_1 = I$ and $C_2 = H(\theta)$ satisfies that the first coordinate $x_n^{(1)}$ of the nth point $\boldsymbol{x}_n$ is of the form $x_n^{(1)} = \frac{n_0}{p} + \frac{n_1}{p^2} + \cdots$ which equals $\varphi_p(n)$ in Example 1.2 (1). For the second component $x_n^{(1)}$, note that

$$\begin{aligned} \langle \theta n(X)\rangle|_p &= \left\langle \left(\sum_{i=1}^{\infty} a_i X^{-i}\right)\left(\sum_{r=0}^{\infty} n_r X^r\right)\right\rangle\Bigg|_p \\ &= \sum_{j=1}^{\infty}\left(\sum_{l=0}^{\infty} a_{j+l} n_l \pmod p\right) p^{-j} \\ &= \sum_{j=1}^{\infty} \frac{y_j^{(2)}}{p^j} = x_n^{(2)}. \end{aligned}$$

□

The next step is to quantify a suitable t-value for a van der Corput–Kronecker-type sequence that is based on a counterexample to the X-adic Littlewood conjecture. Previous work related the t-value of a one-dimensional Kronecker-type sequence $(\langle \theta n(X)\rangle|_p)_{n\geq 0}$ to the degrees of the coefficients in the simple continued fraction expansion of θ (see e.g. [26] and [33]). Corollary 2.5 at the end of this section quantifies this t-value for van der Corput–Kronecker-type sequences, which are based on a counterexample to the X-adic Littlewood conjecture, via a uniform bound for the degrees of the coefficients in the simple continued fraction expansions of $\langle X^r\theta\rangle$, $r \geq 0$. For the sake of self-consistency, the basic properties of the simple continued fraction expansion of a formal Laurent series over $\mathbb{F}_p$ are summarized. (For concise information on continued fractions and convergents in $\mathbb{F}_p((X^{-1}))$ we refer the interested reader to e.g. the Appendix B of [41].) Here and in the rest of the paper we denote the simple continued fraction expansion of $\theta \in \mathbb{F}_p((X^{-1}))$ by $\theta = [A_0; A_1, A_2, \ldots]$ with $A_0 = \theta - \langle\theta\rangle$ and with polynomials $A_i \in \mathbb{F}_p[X]$, $i \geq 1$ of degrees ≥ 1.

For $h \geq 1$ the hth convergent P_h/Q_h of θ is a rational function in $\mathbb{F}_p(X)$ defined by

$$[A_0; A_1, \ldots, A_h] =: P_h/Q_h, \tag{2.1}$$

with $P_h, Q_h \in \mathbb{F}_p[X]$ satisfying $\gcd(P_h, Q_h) = 1$. The degree $\deg(Q_h)$ of Q_h is abbreviated to d_h and satisfies

$$d_h := \deg(Q_h) = \sum_{i=1}^{h} \deg(A_i). \tag{2.2}$$

Furthermore, $\deg(\theta - P_h/Q_h) = -d_h - d_{h+1}$ or equivalently

$$\deg(\langle Q_h\theta\rangle) = -d_{h+1} \tag{2.3}$$

for every $h \geq 1$. For all $k \in \mathbb{F}_p[X]$ with $d_h \leq \deg(k) < d_{h+1}$ we have

$$\deg(\theta - b/k) \geq \deg(\theta - P_h/Q_h) \text{ for all } b \in \mathbb{F}_p[X] \tag{2.4}$$

or equivalently

$$\deg(\langle k\theta\rangle) \geq \deg(\langle Q_h\theta\rangle) = -d_{h+1}. \tag{2.5}$$

In order to refer to the different magnitudes, as A_i, d_h, Q_h etc. in (2.1)–(2.5) for $\langle X^r\theta\rangle$ instead of θ we add a superscript (r). For example $[A_1^{(r)}, A_2^{(r)}, \ldots]$ denotes the simple continued fraction expansion of $\langle X^r\theta\rangle$ and $d_h^{(r)}$ equals $\sum_{i=1}^{h} \deg(A_i^{(r)})$.

Lemma 2.3 gives an important property about the rank structure of the Hankel matrix $H(\theta)$ determined by a Laurent series $\theta \in \mathbb{F}_p((X^{-1}))$. This property depends on the continued fraction of θ and, more exactly, on the degrees of the best approximation denomiators (2.2).

Lemma 2.3. *Let $\theta = \sum_{k=j}^{\infty} a_k X^{-k}$ be any formal Laurent series over $\mathbb{F}_p$. Let $m \in \mathbb{N}$. Then the upper left $m \times m$ submatrix of $H(\theta)$,*

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_m \\ a_2 & a_3 & \cdots & a_{m+1} \\ \vdots & \vdots & & \vdots \\ a_m & a_{m+1} & \cdots & a_{m+m-1} \end{pmatrix}$$

over $\mathbb{F}_p$ is regular if and only if $m = d_h = \deg(Q_h)$ defined in (2.2) *for an $h \in \mathbb{N}$.*

Proof. We observe for a linear combination of the columns of the matrix that

$$b_0 \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{pmatrix} + b_1 \begin{pmatrix} a_2 \\ a_3 \\ \vdots \\ a_{m+1} \end{pmatrix} + \cdots + b_{m-1} \begin{pmatrix} a_m \\ a_{m+1} \\ \vdots \\ a_{m+m-1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

with $b_0, b_1, \dots, b_{m-1} \in \mathbb{F}_p$ if and only if

$$\deg(\langle (b_0 + b_1 X + \cdots + b_{m-1} X^{m-1})\theta \rangle) < -m.$$

Let $m = d_h$. We know from (2.3) that $\deg(\langle Q_{h-1}\theta \rangle) = -d_h$ and from (2.5) that for all $P \in \mathbb{F}_p[X] \setminus \{0\}$ with $\deg(P) < d_h$ we have $\deg(\langle P\theta \rangle) \geq \deg(\langle Q_{h-1}\theta \rangle) = -d_h$. From the observation above we derive the linear independence of the columns for such m. Now if $d_{h-1} < m < d_h$ use $P = Q_{h-1}$ with $\deg P = d_{h-1} < m$ but $\deg(\{P\theta\}) = -d_h$. Hence the columns are linearly depending. $\square$

The following Lemma 2.4 relates the constant c in

$$\inf_{r \geq 0, Q \in \mathbb{F}_p[X] \setminus \{\mathbf{0}\}} |Q| \cdot \|X^r Q\theta\| \geq c > 0$$

with a uniform bound for the degrees of the coefficients in the simple continued fraction expansions of $\langle X^r\theta \rangle$, $r \geq 0$ and vice versa. The proof of Lemma 2.4 is straightforward and quite similar to the real case (cf. e.g. [30, Satz 23]), but as this lemma is crucial for our method of proof for Theorem 1.8 and for Theorem 1.10 we formulate a proof of it.

Lemma 2.4. *Let $\theta \in \mathbb{F}_p((X^{-1}))$. The series θ is a counterexample to the X-adic Littlewood conjecture in $\mathbb{F}_p((X^{-1}))$ if and only if the continued fraction expansion $[A_1^{(r)}, A_2^{(r)}, \dots]$ of $\langle X^r\theta \rangle$ satisfies $\deg(A_j^{(r)}) \leq D(\theta) + 1$ for every $j \geq 1$ and $r \geq 0$, where $D(\theta) \geq 0$ is a fixed constant in $\mathbb{N}_0$. We say then θ is of finite deficiency $D(\theta)$, and for such θ we have*

$$\inf_{r \geq 0, Q \in \mathbb{F}_p[X] \setminus \{\mathbf{0}\}} |Q| \cdot \|X^r Q\theta\| \geq 2^{-(D(\theta)+1)} > 0.$$

Proof. Let θ be a counterexample to the X-adic Littlewood conjecture, i.e.

$$\inf_{r \geq 0, Q \in \mathbb{F}_p[X] \setminus \{\mathbf{0}\}} |Q| \cdot \|X^r Q\theta\| \geq c > 0.$$

We fix $r \in \mathbb{N}_0$ and $Q = Q_h^{(r)}$ as a best approximation denominator of $\langle X^r\theta \rangle$ due to (2.1). Then from (2.2) and (2.3) we know $\deg(\langle Q_h^{(r)} X^r\theta \rangle) = -d_{h+1}^{(r)}$ and $\deg(Q_h^{(r)}) = d_h^{(r)}$. Therefore,

$$|Q| \cdot \|X^r Q\theta\| = 2^{d_h^{(r)} - d_{h+1}^{(r)}} = 2^{-\deg(A_{h+1}^{(r)})} \geq c.$$

Hence $\deg(A_h^{(r)} + 1) \leq \log_2(1/c)$ for every $r \in \mathbb{N}_0$ and $h \in \mathbb{N}_0$. Conversely, if $\deg(A_j^{(r)}) \leq D(\theta) + 1$ for every $j \in \mathbb{N}$ and $r \in \mathbb{N}_0$. Set $c := 2^{-(D(\theta)+1)}$. For any $r \geq 0$ and $Q \in \mathbb{F}_p[X] \setminus \{\mathbf{0}\}$ choose the unique $h \in \mathbb{N}_0$ such that $d_h^{(r)} \leq \deg(Q) < d_{h+1}^{(r)}$, then from (2.5) we derive

$$\deg(Q)\deg(\langle QX^r\theta \rangle) \geq -d_{h+1}^{(r)} + d_h^{(r)} = -\deg(A_h^{(r)} + 1) \geq -(D(\theta) + 1).$$

Thus

$$|Q| \cdot |\langle QX^r\theta\rangle| \geq 2^{-d_{h+1}^{(r)}+d_h^{(r)}} = 2^{-\deg(A_h^{(r)}+1)} \geq 2^{-(D(\theta)+1)} = c. \qquad \square$$

Proof of Theorem 1.8. We use for the counterexample θ to the X-adic Littlewood conjecture the finite deficiency $D(\theta)$ due to Lemma 2.4. Let $m \in \mathbb{N}$ satisfying $m > D(\theta)$. We make use of Lemma 2.2 and of Definition 2.1. We start with $d_1, d_2 \in \mathbb{N}_0$ such that $d_1 + d_2 = m - D(\theta)$. We focus on the $(m - D(\theta)) \times m$-matrix consisting of the

- left upper $d_1 \times m$-submatrix of $C_1 = I$ together with the
- left upper $d_2 \times m$-submatrix of $C_2 = H(\theta)$

and derive the linear independence of the $d_1 + d_2 = m - D(\theta)$ rows.

In the following we set $r = d_1$. As C_1 is the unit matrix, it is sufficient to prove the linear independence of the rows in the upper left $(m-D(\theta)-d_1)\times(m-d_1) = d_2\times(d_2+D(\theta))$ submatrix of the Hankel matrix $H(\langle X^r\theta\rangle)$ determined by $\langle X^r\theta\rangle$. Now choose $Q_{h-1}^{(r)}$ with $\deg(Q_{h-1}^{(r)}) = d_{h-1}^{(r)}$ such that $d_{h-1}^{(r)}$ satisfies $d_2 = m - d_1 - D(\theta) \leq d_{h-1}^{(r)} \leq m - d_1$. Such a $d_{h-1}^{(r)}$ exists as the finite deficiency $D(\theta)$ of θ is assumed. Indeed, if $d_{h-1}^{(r)} < m - d_1 - D(\theta)$ and $d_h^{(r)} > m - d_1$ then

$$\deg(A_h^{(r)}) = d_h^{(r)} - d_{h-1}^{(r)} \geq m - d_1 + 1 - (m - d_1 - D(\theta) - 1) = D(\theta) + 2$$

in contradiction to the finite deficiency $D(\theta)$. Note that finite deficiency $D(\theta)$ guarantees $\deg(A_h^{(r)}) \leq D(\theta)+1$ and therefore $d_h^{(r)} - d_{h-1}^{(r)} \leq D(\theta)+1$.

Lemma 2.3 ensures the regularity of the upper left $d_{h-1}^{(r)}\times d_{h-1}^{(r)}$ submatrix of $H(\langle X^r\theta\rangle)$. The linear independence of the $d_2 \leq d_{h-1}^{(r)}$ rows of length $\geq d_{h-1}^{(r)}$ then immediately follows from elementary linear algebra. The low-discrepancy bound follows e.g. from [41, Theorem 4.17]. $\square$

The above proof also covers the following corollary, which should definitely be noted here.

Corollary 2.5. *Let $\theta \in \mathbb{F}_p((X^{-1}))$ be a counterexample to the X-adic Littlewood conjecture. Let $D(\theta) \in \mathbb{N}_0$ be its finite deficiency due to Lemma 2.4. Then the matrices I and $H(\theta)$ over $\mathbb{F}_p$ are qualified to construct a digital $(D(\theta), 2)$-sequence over $\mathbb{F}_p$.*

3. Proof of Theorem 1.10

Following the ideas of Levin [36] we will derive a lower bound for the discrepancy of a suitable three dimensional digital $(D(\theta), m, 3)$-net over $\mathbb{F}_p$, $(\boldsymbol{y}_n)_{0\leq n<p^m}$, in order to obtain the lower bound for the discrepancy of the two dimensional digital $(\theta, 2)$-sequence over $\mathbb{F}_p$, $((\varphi_p(n), \langle\theta n(X)\rangle|_p))_{n\geq 0}$, in Theorem 1.10. Let $p \in \mathbb{P}$, $\theta \in \mathbb{F}_p((X^{-1}))$, and $m \in \mathbb{N}$. We regard the three

$\mathbb{N} \times m$ matrices $I^{(m)}, H^{(m)}(\theta)$ and $J^{(m)}$ over $\mathbb{F}_p$. Here $I^{(m)}$ are the first m columns of the $\mathbb{N} \times \mathbb{N}_0$ unit matrix I, $H^{(m)}(\theta)$ consists of the first m columns of the Hankelmatrix $H(\theta)$ defined in Lemma 2.3, and

$$J^{(m)} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \end{pmatrix}$$

is the $\mathbb{N} \times m$ *upper anti-diagonal matrix*. The following Lemma 3.1 follows from Corollary 2.5 using e.g. [15, Lemma 4.38].

Lemma 3.1. *Let $\theta \in \mathbb{F}_p((X^{-1}))$ be a counterexample to the X-adic Littlewood conjecture over $\mathbb{F}_p$ with finite deficiency $D(\theta)$ due to Lemma 2.4. Then for every $m > D(\theta)$ the three matrices $I^{(m)}, H^{(m)}(\theta)$ and $J^{(m)}$ are qualified to construct a digital $(D(\theta), m, 3)$-net over $\mathbb{F}_p$.*

We will derive a lower bound for the star-discrepancy of this digital $(D(\theta), m, 3)$-net which is stated in the subsequent Proposition 3.2.

Proposition 3.2. *Let $\theta \in \mathbb{F}_p((X^{-1}))$ be a counterexample to the X-adic Littlewood conjecture over $\mathbb{F}_p$ with finite deficiency $D(\theta)$ due to Lemma 2.4. Let m be a multiple of $24(D(\theta)+1)$ and large enough. Then the three-dimensional digital point set $(\boldsymbol{y}_n)_{0 \le n < p^m}$ constructed by $I^{(m)}, H^{(m)}(\theta)$ and $J^{(m)}$ satisfies*

$$p^m D^*_{p^m}(\boldsymbol{y}_n) \ge c\, m^2 \tag{3.1}$$

with a positive constant c depending on θ but independent of m.

Our Theorem 1.10 follows then from Proposition 3.2 together with the following Lemma 3.3 (cf. [41, Lemma 3.7]) and (1.1).

When reading Lemma 3.3, note that $J^{(m)}\vec{n} = (n_{m-1}, \dots, n_1, n_0, 0 \dots)^T$ and therefore $x_n^{(3)} = \frac{n_{m-1}}{p} + \cdots + \frac{n_1}{p^{m-1}} + \frac{n_0}{p^m} = \frac{n}{p^m}$.

Lemma 3.3. *Let $S = (\boldsymbol{x}_n)_{n \ge 0}$ be an arbitrary sequence in $[0,1)^s$. Let $N = p^m$ and $P = (\boldsymbol{y}_n)_{n=0}^{p^m-1}$ with $\boldsymbol{y}_n := (\boldsymbol{x}_n, n/p^m) \in [0,1)^{s+1}$. Then*

$$N D^*_N(\boldsymbol{y}_n) \le \max_{1 \le M \le N} M D^*_M(\boldsymbol{x}_n) + 1.$$

It remains to prove Proposition 3.2. Following ideas from Levin [36] we first study the so-called admissibility of the $(D(\theta), m, 3)$-net due to Lemma 3.1.

Definition 3.4. We say a digital $(t, m, 3)$-net over $\mathbb{F}_p$, $(\boldsymbol{y}_n)_{0\le n<p^m}$ in the sense of Definition 2.1 is d-admissible if

$$\min_{0\le k<n<p^m} \|\boldsymbol{y}_k \ominus \boldsymbol{y}_n\|_p > p^{-m-d}.$$

Here $\|\boldsymbol{y}_n\|_p = p^{-l}$ where $l = \sum_{i=1}^{3} \min\{j \in \mathbb{N} : y_{n,j}^{(i)} \neq 0\}$ with $y_n^{(i)} = \sum_{j=1}^{\infty} y_{n,j}^{(i)} p^{-j}$. The minimum of the empty set is considered to be ∞. Further $\ominus$ denotes the componentwise digit-wise subtraction, i.e. $y_k^{(i)} \ominus y_n^{(i)} =: \sum_{j=1}^{\infty} \Big(y_{k,j}^{(i)} - y_{n,j}^{(i)} \pmod p\Big) p^{-j}$.

In words the admissibility guarantees a digital minimum distance between two distinct points of a digital $(t, m, 3)$-net. The digital $(D(\theta), m, 3)$-nets described in Lemma 3.1 and relevant in this paper satisfy such a digital minimum distance.

Lemma 3.5. *Let $\theta \in \mathbb{F}_p((X^{-1}))$ be a counterexample to the X-adic Littlewood conjecture over $\mathbb{F}_p$ with finite deficiency $D(\theta)$ due to Lemma 2.4. Then for every $m > D(\theta)$ the digital $(D(\theta), m, 3)$-net over $\mathbb{F}_p$, $(\boldsymbol{y}_n)_{0\le n<p^m}$, constructed by $I^{(m)}, H^{(m)}(\theta)$ and $J^{(m)}$ is $D(\theta) + 3$ admissible.*

Proof. From the construction method in Definition 2.1 it is easy to see that it suffices to prove $\min_{0<n<p^m} \|\boldsymbol{y}_n\|_p > p^{-m-D(\theta)-3}$. Indeed, suppose there exists n such that $0 < n < p^m$ and

$$\|\boldsymbol{y}_n\|_p \le p^{-m-D(\theta)-3}. \tag{3.2}$$

We write

$$\|y_n^{(i)}\|_p =: p^{-l_i-1},\ i = 1, 2, 3. \tag{3.3}$$

Note then our assumption means

$$l_1 + l_2 + l_3 + 3 \ge m + D(\theta) + 3. \tag{3.4}$$

As $n \neq 0$ we easily see $y_n^{(1)} \neq 0$, $y_n^{(2)} \neq 0$, and $y_n^{(3)} \neq 0$. For the second, note that θ as a counterexample is non rational. From the matrices $I^{(m)}$ and $J^{(m)}$ we see that for such n satisfying (3.3) the vector $\vec{n}$ starts with l_1 zero entries and ends with l_3 zero entries and the in-between-part has to start and end with a nonzero entry and is of length at least one, i.e. $m - l_1 - l_3 \ge 1$. From $\|y_n^{(2)}\|_p =: p^{-l_2-1}$ we obtain under our assumption (3.4) a linear dependency in the upper $l_2 \times m$ submatrix of $H^{(m)}(\theta)$ using the observation at the beginning of the proof of Lemma 2.3. More exactly when deleting the first l_1 and the last l_3 columns of it, the columns in the remaining $l_2 \times (m - (l_1 + l_3))$ matrix are linearly depending. But from (3.4) we see $l_2 \ge m - (l_1 + l_3) + D(\theta)$. Similar as in the proof of Lemma 2.3 together with the assumption that θ is of bounded deficiency $D(\theta)$ we achieve linear independence of these columns, the desired contradiction. □

Proof of Proposition 3.2. We use the notation of Definition 2.1. For proving Proposition 3.2 assume m large enough and m to be a multiple of $24(D(\theta)+1)$. For technical details of the proof we write m as a multiple of $8v$ with $v=3(D(\theta)+1)$. We will construct an interval

$$J=[0,\gamma^{(1)})\times[0,\gamma^{(2)})\times[0,\gamma^{(3)})\subseteq[0,1)^3$$

such that

$$\#\{0\le n<p^m:\boldsymbol{y}_n\in J\}-p^m\lambda(J)\le -cm^2 \tag{3.5}$$

with a fixed constant c only depending on θ or $D(\theta)$ resp. which then implies (3.1).

For every $i\in\{1,2,3\}$ we write $\gamma^{(i)}=\sum_{j=1}^{r_i}\gamma_j^{(i)}p^{-j}$ in base p and identify it with the vector $\vec\gamma^{(i)}=(\gamma_1^{(i)},\dots,\gamma_{r_i}^{(i)})^T$. We define $\vec j_v=(0,\dots,0,1)$ of length v and set

$$\vec\gamma^{(1)}=(\underbrace{\vec j_v,\vec j_v,\dots,\vec j_v}_{m/(4v)\text{ many }\vec j_v\text{s.}})^T,\quad r_1=m/4,$$

$$\vec\gamma^{(2)}=(\underbrace{0,\dots,0,}_{v-u\text{ many 0s.}}\overbrace{\vec j_v,\dots,\vec j_v}^{m/(4v)-1\text{ many }\vec j_v\text{s.}})^T,\quad r_2=m/4-u$$

with u satisfying $D(\theta)\le u<2D(\theta)$ and which will be chosen later. Further we set

$$\vec\gamma^{(3)}=(\underbrace{0,\dots,0,}_{m/4+u\text{ many 0s.}}\overbrace{\dots\dots\dots,}^{m/4-u\text{ many fitting to }\gamma^{(2)}}\underbrace{\vec j_v,\dots,\vec j_v}_{m/(4v)\text{ many }\vec j_v\text{s.}})^T,\quad r_3=3m/4.$$

Here and henceforth, any vector occurring as a component of a larger vector is understood to be flattened; its entries become entries of the enclosing vector.

We introduce the truncation operator to $y=\sum_{j=1}^{\infty}y_jp^{-j}$ for $r\in\mathbb{N}$ as $[y]_r:=\sum_{j=1}^{r}y_jp^{-j}$.

First we ensure for a proper chosen u existence of exactly one $n\in\{0,1,\dots,p^m-1\}$ such that $[y_n^{(i)}]_{r_i}=\gamma^{(i)}$ for $i=1,2,3$. As $C_1=I^{(m)}$ we have $[x_n^{(1)}]_{r_1}=\gamma^{(1)}$ if and only if $(n_0,\dots,n_{r_1-1})^T=\vec\gamma^{(1)}$. Thus the first $r_1=m/4$ entries of $\vec n$ are uniquely determined. Now we use that $C_3=J^{(m)}$ and see that the first $m/4+u$ zeroes in $\vec\gamma^{(3)}$ and the $m/4$ entries in the $(\vec j_v,\dots,\vec j_v)$ string of $\vec\gamma^{(3)}$ uniquely determine the last $m/4+u$ entries of $\vec n$ as well as the $m/4$ entries of $\vec n$ indexed by $m/4,m/4+1,\dots,m/2-1$. So the remaining $m/4-u=r_2$ entries of $\vec n$ should guarantee $[y_n^{(2)}]_{r_2}=\gamma^{(2)}$. These remaining $m/4-u$ entries of $\vec n$ then uniquely determine the inner part of $\vec{\gamma}^{(3)}$, i.e., the $m/4-u$ entries in the vector $\vec{\gamma}^{(3)}$ that are chosen fitting to $\gamma^{(2)}$.

We summarize:

$$\vec{n}=(\underbrace{n_0,\dots,n_{\frac{m}{4}-1}}_{\text{determined by }\gamma^{(1)}},\overbrace{n_{\frac{m}{4}},\dots,n_{\frac{m}{2}-1}}^{\text{determined by }\gamma^{(3)}},\underbrace{\dots,\dots,\dots}_{\text{determined by }\gamma^{(2)}},\overbrace{n_{\frac{3m}{4}-u},\dots,n_{m-1}}^{\text{determined by }\gamma^{(3)}})^T$$

We guarantee through an appropriate choice of u that there is exactly one choice of the remaining $m/4-u$ entries of $\vec{n}$ such that $[y_n^{(2)}]_{r_2}=\gamma^{(2)}$. Since the first $m/2$ digits of n are fixed already, we reduce the unique solvability of the remaining $m/4-u$ entries of $\vec{n}$ such that $[y_n^{(2)}]_{r_2}=\gamma^{(2)}$ to the condition that $H^{(m/4-u)}(\langle X^{m/2}\theta\rangle)$ is regular. We can choose u satisfying $D(\theta)\leq u<2D(\theta)$ such that $H^{(m/4-u)}(\langle X^{m/2}\theta\rangle)$ is regular, as such a u always exists under the assumption that θ has finite deficiency $D(\theta)$ together with Lemma 2.3. These now uniquely determined $m/4-u$ remaining entries of $\vec{n}$ will be used to define the middle part of $\gamma^{(3)}$.

We summarize, after choosing a proper u and the remaining part of $\gamma^{(3)}$ fitting to $\gamma^{(2)}$ there is a uniquely determined $\bar{n}\in\{0,1,\dots,p^m-1\}$ such that $[y_{\bar{n}}^{i}]_{r_i}=\gamma^{(i)}$ for $i=1,2,3$ or equivalently $\boldsymbol{y}_{\bar{n}}\in[\gamma^{(1)},\gamma^{(1)}+1/p^{r_1})\times[\gamma^{(2)},\gamma^{(2)}+1/p^{r_2})\times[\gamma^{(3)},\gamma^{(3)}+1/p^{r_3})$. Note that we introduced the notation $\bar{n}$ for this unique integer in $\{0,1,\dots,p^m-1\}$ in order to make the rest of the proof easier to read.

Now we split up $J=[0,\gamma^{(1)})\times[0,\gamma^{(2)})\times[0,\gamma^{(3)})$ into a union of disjoint elementary intervals as follows. We write $\gamma^{(i)}=\sum_{j=1}^{r_i}\gamma_j^{(i)}p^{-j}$ and J as

$$J=\prod_{i=1}^{3}\bigcup_{j=1}^{r_i}\bigcup_{k=1}^{\gamma_j^{(i)}}\left[[\gamma^{(i)}]_{j-1}+(k-1)p^{-j},[\gamma^{(i)}]_{j-1}+kp^{-j}\right)$$

or equivalently

$$J=\bigcup_{j_1=1}^{r_1}\bigcup_{j_2=1}^{r_2}\bigcup_{j_3=1}^{r_3}\bigcup_{k_1=1}^{\gamma_{j_1}^{(1)}}\bigcup_{k_2=1}^{\gamma_{j_2}^{(2)}}\bigcup_{k_3=1}^{\gamma_{j_3}^{(3)}}I(j_1,k_1,j_2,k_2,j_3,k_3)$$

with

$$I(j_1,k_1,j_2,k_2,j_3,k_3):=\prod_{i=1}^{3}\left[[\gamma^{(i)}]_{j_i-1}+(k_i-1)p^{-j_i},[\gamma^{(i)}]_{j_i-1}+k_ip^{-j_i}\right).$$

Note that $I(j_1,k_1,j_2,k_2,j_3,k_3)$ is well-defined only if $\gamma_{j_1}^{(1)}\neq 0$, $\gamma_{j_2}^{(2)}\neq 0$, and $\gamma_{j_3}^{(3)}\neq 0$. The volume of $I(j_1,k_1,j_2,k_2,j_3,k_3)$ is then $p^{-(j_1+j_2+j_3)}$. We also say the elementary interval $I(j_1,k_1,j_2,k_2,j_3,k_3)$ has order $j_1+j_2+j_3$.

Now we distinguish between the orders of $I(j_1,k_1,j_2,k_2,j_3,k_3)$ as follows.

Case 1. If $j_1 + j_2 + j_3 \le m - D(\theta)$ then

$$\#\{0 \le n < p^m : \boldsymbol{x}_n \in I(j_1,k_1,j_2,k_2,j_3,k_3)\} = p^m \lambda(I(j_1,k_1,j_2,k_2,j_3,k_3)).$$

This follows from Definition 2.1 together with $d_1 = j_1$, $d_2 = j_2$, $d_3 = j_3$, and elementary linear algebra.

Case 2. If $j_1 + j_2 + j_3 > m - D(\theta)$ we want to show that

$$\#\{0 \le n < p^m : \boldsymbol{x}_n \in I(j_1,k_1,j_2,k_2,j_3,k_3)\} = 0$$

or in other words $I(j_1,k_1,j_2,k_2,j_3,k_3)$ remains empty. Here we make use of the $(D(\theta)+3)$-admissibilty of the $(D(\theta),m,3)$-net due to Lemma 3.5.

Note that always $j_1 + j_2 \le r_1 + r_2 = m/2 - u$. Thus in Case 2 for $j_1+j_2+j_3 > m-D(\theta) \ge m-u$ we need $j_3 > m/2$. In addition, we observe $\gamma^{(i)}_{j_i} \ne 0$ for $i = 1,2,3$ for j_1, j_2, j_3 satisfying $j_1+j_2+j_3 > m-D(\theta) \ge m-u$ if and only if

$$j_1 = \lambda_1 v,\ j_2 = v - u + \lambda_2 v,\ \text{and } j_3 = m/2 + \lambda_3 v \tag{3.6}$$

with $\lambda_1, \lambda_3 \in \{1,\dots,m/(4v)\}$, $\lambda_2 \in \{1,\dots,m/(4v)-1\}$ and $\lambda_1+\lambda_2+\lambda_3 \ge m/(2v)$. Thus we obtain the stronger bound $j_1+j_2+j_3 \ge m+v-u$ in this Case 2.

Suppose there exists $\widetilde{n} \in \{0,1,\dots,p^m-1\}$ such that

$$\boldsymbol{y}_{\tilde{n}} \in I(j_1,k_1,j_2,k_2,j_3,k_3)$$

for such j_1, j_2, j_3. Then

$$\|\boldsymbol{y}_{\tilde{n}} \ominus \boldsymbol{y}_{\overline{n}}\|_p \le p^{-(j_1+j_2+j_3)} \le p^{-(m+v-u)} \text{ and } \overline{n} \ne \widetilde{n}.$$

From Lemma 3.5 and Definition 3.4 we know that

$$\|\boldsymbol{y}_{\tilde{n}} \ominus \boldsymbol{y}_{\overline{n}}\|_p > p^{-(m+D(\theta)+3)}.$$

Since $v = 3(D(\theta)+1)$ and $D(\theta) \le u < 2D(\theta)$ we see

$$m+v-u > m+3D(\theta)+3-2D(\theta) = m+D(\theta)+3.$$

Therefore

$$p^{-(m+D(\theta)+3)} > p^{-(m+v-u)} \ge \|\boldsymbol{y}_{\tilde{n}} \ominus \boldsymbol{y}_{\overline{n}}\|_p > p^{-(m+D(\theta)+3)}$$

yields the desired contradiction.

Altogether

$$\begin{aligned}\#\{0 \le n < p^m : \boldsymbol{x}_n \in J\} - p^m\lambda(J) &\le -\sum_{j_1,j_2,j_3 \ge m+v-u} p^{m-(j_1+j_2+j_3)} \\ &\le -\sum_{\substack{j_1,j_2,j_3,\\ j_1+j_2+j_3=m+v-u}} p^{u-v}.\end{aligned}$$

It remains to ensure there are at least cm^2 (with $c > 0$) possible choices of such (j_1,j_2,j_3). Using (3.6) we know $j_1 = \lambda_1 v$, $j_2 = v - u + \lambda_2 v$, and

$j_3 = m/2 + \lambda_3 v$ with $\lambda_1, \lambda_3 \in \{1, \dots, m/(4v)\}$, $\lambda_2 \in \{1, \dots, m/(4v) - 1\}$ and $\lambda_1 + \lambda_2 + \lambda_3 \geq m/(2v)$ we easily see that for each choice of $(\lambda_1, \lambda_2) \in \{m/(8v), \dots, m/(4v)\} \times \{m/(8v), \dots, m/(4v) - 1\}$ there is exactly one $\lambda_3 \in \{1, \dots, m/(4v)\}$ such that $\lambda_1 + \lambda_2 + \lambda_3 = m/(2v)$. This completes the proof of Proposition 3.2. □

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