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A note on the number of irrational odd zeta values II

par Li LAI

Résumé. Nous prouvons qu'il y a au moins $1.284 \cdot \sqrt{s/\log s}$ nombres irrationnels parmi $\zeta(3), \zeta(5), \zeta(7), \ldots, \zeta(s-1)$ pour tout entier pair s suffisamment grand. Ce résultat améliore la constante d'un travail antérieur. La démonstration combine la technique d'élimination de Fischler–Sprang–Zudilin (2019) avec la méthode du facteur Φ_n de Zudilin (2001).

Abstract. We prove that there are at least $1.284 \cdot \sqrt{s/\log s}$ irrational numbers among $\zeta(3), \zeta(5), \zeta(7), \ldots, \zeta(s-1)$ for any sufficiently large even integer s. This result improves the constant of a previous work. The proof combines the elimination technique of Fischler–Sprang–Zudilin (2019) with the Φ_n factor method of Zudilin (2001).

1. Introduction

Let $\zeta(s) := \sum_{m=1}^{\infty} m^{-s}$ (for $\mathrm{Re}(s) > 1$) be the Riemann zeta function. We are interested in the special values $\zeta(s)$ for odd integers $s > 1$ (referred to as *odd zeta values* for brevity). The aim of this note is to prove the following:

Theorem 1.1. *For any sufficiently large positive integer s, we have*

$$\#\{\textit{odd } i \in [3, s] \mid \zeta(i) \notin \mathbb{Q}\} \geqslant 1.284579 \cdot \sqrt{\frac{s}{\log s}}.$$

This theorem represents a small progress on the irrationality of odd zeta values. Let us briefly review a few previously known results on this topic. In 1979, Apéry [1] proved that $\zeta(3)$ is irrational. The next significant development was made by Rivoal [8] in 2000 (see also Ball and Rivoal [2]), who showed that the dimension of the $\mathbb{Q}$-linear span of $1, \zeta(3), \zeta(5), \ldots, \zeta(s)$ is at least $(1 - o(1))(\log s)/(1 + \log 2)$ as the odd integer $s \to \infty$. So far, the irrationality of $\zeta(5)$ remains open, although Zudilin [10] in 2001 proved that at least one of $\zeta(5), \zeta(7), \zeta(9), \zeta(11)$ is irrational. A key ingredient of Zudilin's theorem is the consideration of certain arithmetic factors, usually denoted by Φ_n. We refer to it as the "Φ_n factor method" in this note.

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During 2018–2020, following a series of developments in [4, 6, 9, 13], it was proved that the number of irrationals among $\zeta(3), \zeta(5), \ldots, \zeta(s)$ is at least $1.192507 \cdot \sqrt{s/\log s}$ for any sufficiently large odd integer s (see [6, Thm. 1.1]). A key ingredient of this result is an elimination technique, whose effectiveness was first demonstrated by Fischler, Sprang, and Zudilin in [4]. Roughly speaking, the elimination technique of Fischler, Sprang, and Zudilin is useful for removing undesirable terms in the linear combinations of 1 and odd zeta values.

Recently, the author [5] incorporated Zudilin's Φ_n factor method into the original proof of the Ball–Rivoal theorem [2] and improved the constant $1/(1+\log 2)$ to $1.119/(1+\log 2)$. However, this result in [5] is subsumed by a much stronger result of Fischler [3], where it was proved that the dimension of the $\mathbb{Q}$-linear span of $1, \zeta(3), \zeta(5), \ldots, \zeta(s)$ is at least $0.21 \cdot \sqrt{s/\log s}$ for any sufficiently large odd integer s.

In this note, we combine the elimination technique of Fischler, Sprang, and Zudilin with Zudilin's Φ_n factor method to prove Theorem 1.1.

The structure of this note is as follows: In Section 2, we construct rational functions and linear forms. In Section 3, we study the arithmetic properties of the linear forms. In Section 4, we estimate the linear forms. In Section 5, we utilize the elimination technique to transform our task into a computational problem. Finally, in Section 6, we present the computational results and provide a proof for Theorem 1.1.

2. Rational functions and linear forms

2.1. Parameters. We fix a positive integer M and a finite collection of non-negative integers $\{\delta_j\}_{j=1}^{J}$ such that

$$0 \leqslant \delta_1 \leqslant \delta_2 \leqslant \cdots \leqslant \delta_J < \frac{M}{2}.$$

Here, $J \geqslant 1$ is the number of δ_j's.

Fix a positive rational number r. Denote by $\mathrm{den}(r)$ the denominator of r in its reduced form. Eventually, we will take r to be sufficiently close to a real number r_0 determined by the parameters $M, \delta_1, \ldots, \delta_J$.

Let B be a positive real number, and let s be an integral multiple of $2J$. In the sequel, we always assume that:

$$\text{Both } s \text{ and } B \text{ are sufficiently large, and } s \geqslant 10(2r+1)MB^2. \tag{2.1}$$

Eventually, we will take $B = c \cdot \sqrt{s/\log s}$ for some constant $c > 0$ depending only on the parameters $M, \delta_1, \ldots, \delta_J$. Thus, the condition $s \geqslant 10(2r+1)MB^2$ is automatically satisfied when s is sufficiently large.

2.2. Normalization factors. As usual, we denote Euler's totient function by $\varphi(\,\cdot\,)$.

Definition 2.1. We define the following two sets depending only on B:

$$\Psi_B := \{b \in \mathbb{N} \mid \varphi(b) \leqslant B\},$$
$$\mathcal{Z}_B := \left\{ \frac{a}{b} \in \mathbb{Q} \cap (0,1] \;\middle|\; b \in \Psi_B, 1 \leqslant a \leqslant b, \text{ and } \gcd(a,b) = 1 \right\}.$$

Moreover, we define the following two numbers $A_1(B)$ and $A_2(B)$ depending only on M, r, and B:

$$A_1(B) := \prod_{b \in \Psi_B} b^{(2r+1)M\varphi(b)},$$
$$A_2(B) := \prod_{b \in \Psi_B} \prod_{\substack{p \mid b \\ p \text{ prime}}} p^{(2r+1)M\varphi(b)/(p-1)}.$$

By [6, Prop. 2.2], we have the following properties for the sets Ψ_B and $\mathcal{Z}_B$:

$$|\Psi_B| = \left(\frac{\zeta(2)\zeta(3)}{\zeta(6)} + o_{B \to +\infty}(1) \right) B, \tag{2.2}$$

$$\left\{ \frac{1}{b}, \frac{2}{b}, \ldots, \frac{b}{b} \right\} \subset \mathcal{Z}_B \text{ for any } b \in \Psi_B, \tag{2.3}$$

$$|\mathcal{Z}_B| \leqslant B^2 \text{ for any sufficiently large } B. \tag{2.4}$$

By [6, Lem. 2.4], we have the following estimates for $A_1(B)$ and $A_2(B)$:

$$A_1(B) = \exp\left(\left(\frac{1}{2} \frac{\zeta(2)\zeta(3)}{\zeta(6)} + o_{B \to +\infty}(1) \right) (2r+1)MB^2 \log B \right), \tag{2.5}$$

$$A_2(B) \leqslant \exp\left(10(2r+1)MB^2 (\log\log B)^2 \right) \text{ for any sufficiently large } B. \tag{2.6}$$

2.3. Rational functions. Define the integer

$$P_{B,\mathrm{den}(r)} := 2\,\mathrm{den}(r) \cdot \mathrm{LCM}_{\substack{b \in \Psi_B \\ \text{prime } p \mid b}} \{p-1\},$$

where LCM denotes the least common multiple. Note that for any $n \in P_{B,\mathrm{den}(r)}\mathbb{N}$, we have $A_1(B)^n \in \mathbb{N}$ and $A_2(B)^n \in \mathbb{N}$. For any positive integer k, we denote by

$$(x)_k := x(x+1)\ldots(x+k-1)$$

the Pochhammer symbol.

Definition 2.2. For any $n \in P_{B,\mathrm{den}(r)}\mathbb{N}$, we define the rational function $R_n(t) \in \mathbb{Q}(t)$ as follows:

$$R_n(t) := A_1(B)^n A_2(B)^n \cdot \frac{\prod_{j=1}^{J} ((M-2\delta_j)n)!^{s/J}}{(n/\mathrm{den}(r))!^{\mathrm{den}(r)(2r+1)M|\mathcal{Z}_B| - \mathrm{den}(r)M}} \cdot (2t+Mn)$$
$$\times \frac{(t-rMn)_{rMn}(t+Mn+1)_{rMn} \prod_{\theta \in \mathcal{Z}_B \setminus \{1\}} (t-rMn+\theta)_{(2r+1)Mn}}{\prod_{j=1}^{J} (t+\delta_j n)_{(M-2\delta_j)n+1}^{s/J}}.$$

Note that the degree of the denominator of $R_n(t)$ is

$$\frac{s}{J}\sum_{j=1}^{J}((M-2\delta_j)n+1) \geqslant \frac{s}{J}\sum_{j=1}^{J}(n+1) = (n+1)s.$$

Meanwhile, the degree of the numerator of $R_n(t)$ is

$$1+((2r+1)M|\mathcal{Z}_B|-M)n \leqslant (2r+1)MB^2 n$$

by (2.4). Under our assumption (2.1), we conclude that

$$\deg R_n(t) \leqslant -2$$

for any $n \in P_{B,\mathrm{den}(r)}\mathbb{N}$. Thus, the rational function $R_n(t)$ has the unique partial-fraction decomposition of the following form.

Definition 2.3. For any $n \in P_{B,\mathrm{den}(r)}\mathbb{N}$, we define $a_{n,i,k} \in \mathbb{Q}$ ($1 \leqslant i \leqslant s$, $\delta_1 n \leqslant k \leqslant (M-\delta_1)n$) as the coefficients appearing in the partial-fraction decomposition of $R_n(t)$:

$$R_n(t) =: \sum_{i=1}^{s}\sum_{k=\delta_1 n}^{(M-\delta_1)n}\frac{a_{n,i,k}}{(t+k)^i}. \tag{2.7}$$

2.4. Linear forms.

Definition 2.4. For any $n \in P_{B,\mathrm{den}(r)}\mathbb{N}$ and any $\theta \in \mathcal{Z}_B$, we define

$$S_{n,\theta} := \sum_{m=0}^{\infty} R_n(m+\theta). \tag{2.8}$$

It is easy to check that the set $\mathcal{Z}_B \setminus \{1\}$ is invariant under the transform $x \mapsto 1-x$. Since s is a multiple of $2J$ and n is a multiple of $P_{B,\mathrm{den}(r)}$, all of s/J, rMn, and $(2r+1)Mn$ are even integers. Thus, the rational function $R_n(t)$ has the symmetry

$$R_n(-t-Mn) = -R_n(t).$$

Using similar arguments as in [4, Lem. 1], we can express $S_{n,\theta}$ as a linear form in 1 and Hurwitz zeta values. Recall that the Hurwitz zeta values are defined by

$$\zeta(i,\alpha) := \sum_{m=0}^{\infty}\frac{1}{(m+\alpha)^i}$$

for any $i \in \mathbb{Z}_{\geqslant 2}$ and $\alpha > 0$.

Lemma 2.5. *For any $n \in P_{B,\mathrm{den}(r)}\mathbb{N}$ and any $\theta \in \mathcal{Z}_B$, we have*

$$S_{n,\theta} = \rho_{n,0,\theta} + \sum_{\substack{3\leqslant i\leqslant s-1 \\ i\ \mathit{odd}}} \rho_{n,i}\zeta(i,\theta),$$

where the rational coefficients are given by

$$\rho_{n,i} = \sum_{k=\delta_1 n}^{(M-\delta_1)n} a_{n,i,k} \quad (3 \leqslant i \leqslant s-1,\ i \text{ odd}), \tag{2.9}$$

$$\rho_{n,0,\theta} = -\sum_{k=\delta_1 n}^{(M-\delta_1)n} \sum_{\ell=0}^{k-1} \sum_{i=1}^{s} \frac{a_{n,i,k}}{(\ell+\theta)^i}. \tag{2.10}$$

Note that the coefficients $\rho_{n,i}$ $(3 \leqslant i \leqslant s-1$, i odd$)$ do not depend on $\theta \in \mathcal{Z}_B$.

3. Arithmetic properties

As usual, we denote by

$$D_m = \mathrm{LCM}\{1, 2, 3, \ldots, m\}$$

the least commom multiple of the numbers $1, 2, 3 \ldots, m$ for any positive integer m. We denote by $f^{(\lambda)}(t)$ the λ-th order derivative of a function $f(t)$ for any non-negative integer λ. For a prime number p, we denote by $v_p(x)$ the p-adic order of a rational number x, with the convention $v_p(0) = +\infty$.

We first state two basic lemmas.

Lemma 3.1. *Let a, b, a_0, b_0 be integers such that $a_0 \leqslant a \leqslant b \leqslant b_0$ and $b_0 > a_0$. Consider the rational function*

$$G(t) = \frac{(b-a)!}{(t+a)_{b-a+1}}.$$

Then, we have

$$D_{b_0-a_0}^{\lambda} \cdot \frac{1}{\lambda!}\left(G(t)(t+k)\right)^{(\lambda)}\big|_{t=-k} \in \mathbb{Z} \tag{3.1}$$

for any integer $k \in [a_0, b_0] \cap \mathbb{Z}$ and any non-negative integer λ.

Moreover, for any prime number $p > \sqrt{b_0 - a_0}$, any integer $k \in [a_0, b_0] \cap \mathbb{Z}$, and any non-negative integer λ, we have

$$v_p\Big(\left(G(t)(t+k)\right)^{(\lambda)}\big|_{t=-k}\Big) \geqslant -\lambda + \left\lfloor \frac{b-a}{p} \right\rfloor - \left\lfloor \frac{k-a}{p} \right\rfloor - \left\lfloor \frac{b-k}{p} \right\rfloor. \tag{3.2}$$

Proof. See [12, Lem. 16, 18]. We have replaced b and b_0 in [12, Lem. 16, 18] by $b+1$ and b_0+1, respectively. □

Lemma 3.2 ([5, Lem. 4.2])**.** *Let a, b, c, m be integers with $m > 0$ and $b > 0$. Let*

$$\mu_m(b) = b^m \prod_{\substack{p \mid b \\ p \text{ prime}}} p^{\lfloor m/(p-1) \rfloor},$$

with the convention $\mu_m(1) = 1$. *Consider the polynomial*

$$F(t) = \mu_m(b) \cdot \frac{(ct + a/b)_m}{m!}.$$

Then, we have

$$D_m^\lambda \cdot \frac{1}{\lambda!} F^{(\lambda)}(t)\big|_{t=-k} \in \mathbb{Z}$$

for any integer $k \in \mathbb{Z}$ *and any non-negative integer* λ.

In the rest of this section, we will first establish the arithmetic properties of the coefficients $a_{n,i,k}$ (defined by (2.7)). Following this, we will address the coefficients $\rho_{n,i}$ (defined by (2.9)) and $\rho_{n,0,\theta}$ (defined by (2.10)).

Lemma 3.3. *For any* $n \in P_{B,\mathrm{den}(r)}\mathbb{N}$ *with* $n > s^2$, *we have*

$$\Phi_n^{-s/J} D_{(M-2\delta_1)n}^{s-i} \cdot a_{n,i,k} \in \mathbb{Z} \quad (1 \leqslant i \leqslant s,\ \delta_1 n \leqslant k \leqslant (M-\delta_1)n),$$

where the factor Φ_n *is a product of certain primes:*

$$\Phi_n := \prod_{\substack{\sqrt{Mn} < p \leqslant (M-2\delta_1)n \\ p \text{ prime}}} p^{\omega(n/p)}, \tag{3.3}$$

and the function $\omega(\,\cdot\,)$ *is defined by*

$$\omega(x) = \min_{0 \leqslant y < 1} \sum_{j=1}^{J} \left(\lfloor (M - 2\delta_j)x \rfloor - \lfloor y - \delta_j x \rfloor - \lfloor (M-\delta_j)x - y \rfloor \right). \tag{3.4}$$

Proof. Fix any $i \in \{1, 2, \ldots, s\}$ and any $k \in [\delta_1 n,\ (M-\delta_1)n] \cap \mathbb{Z}$. By (2.7), we have

$$a_{n,i,k} = \frac{1}{(s-i)!} \left(R_n(t)(t+k)^s \right)^{(s-i)}\Big|_{t=-k}. \tag{3.5}$$

Define the following rational functions (we use the notation $\mu_m(b)$ as defined in Lemma 3.2):

$$
\begin{aligned}
F_0(t) &:= 2t + Mn,\\
F_{1,-,\nu}(t) &:= \frac{(t - rMn + (\nu-1)n/\operatorname{den}(r))_{n/\operatorname{den}(r)}}{(n/\operatorname{den}(r))!}\\
&\quad \text{for } \nu = 1,2,\dots,\operatorname{den}(r)rM,\\
F_{1,+,\nu}(t) &:= \frac{(t + Mn + 1 + (\nu-1)n/\operatorname{den}(r))_{n/\operatorname{den}(r)}}{(n/\operatorname{den}(r))!}\\
&\quad \text{for } \nu = 1,2,\dots,\operatorname{den}(r)rM,\\
F_{\theta,\nu}(t) &:= \mu_{n/\operatorname{den}(r)}(\operatorname{den}(\theta)) \cdot \frac{(t - rMn + \theta + (\nu-1)n/\operatorname{den}(r))_{n/\operatorname{den}(r)}}{(n/\operatorname{den}(r))!}\\
&\quad \text{for } \theta \in \mathcal{Z}_B \setminus \{1\} \text{ and } \ \nu = 1,2,\dots,\operatorname{den}(r)(2r+1)M,\\
G_j(t) &:= \frac{((M-2\delta_j)n)!}{(t+\delta_j n)_{(M-2\delta_j)n+1}} \quad \text{for } j = 1,2,\dots,J.
\end{aligned}
$$

Since n is an integral multiple of $P_{B,\operatorname{den}(r)}$, the exponent $(n/\operatorname{den}(r))/(p-1)$ of any prime p appearing in

$$
\mu_{n/\operatorname{den}(r)}(\operatorname{den}(\theta)) = \operatorname{den}(\theta)^{n/\operatorname{den}(r)} \prod_{\substack{p \mid \operatorname{den}(\theta)\\ p \text{ prime}}} p^{(n/\operatorname{den}(r))/(p-1)}
$$

is an integer. This eliminates the need for the integer-part symbol in $\mu_{n/\operatorname{den}(r)}(\operatorname{den}(\theta))$. One can verify that

$$
\prod_{\theta \in \mathcal{Z}_B \setminus \{1\}} \Big(\mu_{n/\operatorname{den}(r)}(\operatorname{den}(\theta))\Big)^{\operatorname{den}(r)(2r+1)M} = A_1(B)^n A_2(B)^n.
$$

Therefore, by Definition 2.2, we have

$$
\begin{aligned}
(3.6) \quad R_n(t)(t+k)^s = F_0(t) &\prod_{\nu=1}^{\operatorname{den}(r)rM} (F_{1,-,\nu}(t) F_{1,+,\nu}(t))\\
&\times \prod_{\theta \in \mathcal{Z}_B \setminus \{1\}} \prod_{\nu=1}^{\operatorname{den}(r)(2r+1)M} F_{\theta,\nu}(t) \times \prod_{j=1}^{J} (G_j(t)(t+k))^{s/J}.
\end{aligned}
$$

By Lemma 3.2 (with the convention $\mu_m(1) = 1$), for any polynomial $F(t)$ of the form $F_0(t)$, $F_{1,\pm,\nu}(t)$, $F_{\theta,\nu}(t)$, we have

$$
(3.7) \qquad D_{n/\operatorname{den}(r)}^{\lambda} \cdot \frac{1}{\lambda!} F^{(\lambda)}(t)\big|_{t=-k} \in \mathbb{Z}
$$

for any non-negative integer λ. By (3.1) of Lemma 3.1 (with $a_0 = \delta_1 n$ and $b_0 = (M - \delta_1)n$), we have

$$D^{\lambda}_{(M-2\delta_1)n} \cdot \frac{1}{\lambda!}(G_j(t)(t+k))^{(\lambda)}\big|_{t=-k} \in \mathbb{Z} \tag{3.8}$$

for any $j = 1, 2, \ldots, J$ and any non-negative integer λ. Now, substituting (3.6) into (3.5), applying the Leibniz rule, and using (3.7) and (3.8), we obtain

$$D^{s-i}_{(M-2\delta_1)n} \cdot a_{n,i,k} \in \mathbb{Z}. \tag{3.9}$$

Moreover, for any prime p such that $\sqrt{Mn} < p \leqslant (M - 2\delta_1)n$, taking also (3.2) of Lemma 3.1 into consideration, and noting that $p > s$ (because $n > s^2$), we obtain

$$\begin{aligned} v_p(a_{n,i,k}) &= v_p\Big((R_n(t)(t+k)^s)^{(s-i)}\Big|_{t=-k}\Big) \\ &\geqslant -(s-i) + \frac{s}{J} \cdot \sum_{j=1}^{J}\left(\left\lfloor \frac{(M-2\delta_j)n}{p} \right\rfloor - \left\lfloor \frac{k-\delta_j n}{p} \right\rfloor - \left\lfloor \frac{(M-\delta_j)n-k}{p} \right\rfloor\right) \\ &\geqslant -(s-i) + \frac{s}{J} \cdot \omega\left(\frac{n}{p}\right), \end{aligned} \tag{3.10}$$

where the function $\omega(\,\cdot\,)$ is given by (3.4). Combining (3.9) and (3.10), we obtain the desired conclusion:

$$\Phi_n^{-s/J} D^{s-i}_{(M-2\delta_1)n} \cdot a_{n,i,k} \in \mathbb{Z} \quad (1 \leqslant i \leqslant s,\ \delta_1 n \leqslant k \leqslant (M-\delta_1)n). \qquad \square$$

Lemma 3.4. *For any $n \in P_{B,\mathrm{den}(r)}\mathbb{N}$ with $n > s^2$, we have*

$$\Phi_n^{-s/J} D^{s-i}_{(M-2\delta_1)n} \cdot \rho_{n,i} \in \mathbb{Z} \quad (3 \leqslant i \leqslant s-1,\ i \text{ odd}).$$

Proof. It follows directly from (2.9) and Lemma 3.3. $\square$

Lemma 3.5. *For any $n \in P_{B,\mathrm{den}(r)}\mathbb{N}$ with $n > s^2$, and any $\theta \in \mathcal{Z}_B \setminus \{1\}$, we have*

$$\Phi_n^{-s/J} D^{s}_{(M-2\delta_1)n} \cdot \rho_{n,0,\theta} \in \mathbb{Z}.$$

Proof. We prove by contradiction. Let $\theta \in \mathcal{Z}_B \setminus \{1\}$ and suppose that

$$\Phi_n^{-s/J} D^{s}_{(M-2\delta_1)n} \cdot \rho_{n,0,\theta} \notin \mathbb{Z}.$$

By substituting (2.10) into above equation, we obtain that, there exist integers k_0, ℓ_0 such that $\delta_1 n \leqslant k_0 \leqslant (M - \delta_1)n$, $0 \leqslant \ell_0 < k_0$, and

$$\Phi_n^{-s/J} D^{s}_{(M-2\delta_1)n} \sum_{i=1}^{s} \frac{a_{n,i,k_0}}{(\ell_0 + \theta)^i} \notin \mathbb{Z}.$$

Note that $1-\theta \in \mathcal{Z}_B \setminus \{1\}$ and $t+k_0-\ell_0-\theta$ is a factor of $(t-rMn+1-\theta)_{(2r+1)Mn}$. Thus, $t+k_0-\ell_0-\theta$ is a factor of the numerator of $R_n(t)$ (see Definition 2.2) and

$$R_n(-k_0+\ell_0+\theta)=0.$$

Then, by (2.7) we have

$$\Phi_n^{-s/J} D_{(M-2\delta_1)n}^s \sum_{i=1}^{s} \frac{a_{n,i,k_0}}{(\ell_0+\theta)^i} = -\Phi_n^{-s/J} D_{(M-2\delta_1)n}^s \sum_{i=1}^{s} \sum_{\substack{k=\delta_1 n\\ k\neq k_0}}^{(M-\delta_1)n} \frac{a_{n,i,k}}{(k-k_0+\ell_0+\theta)^i} \notin \mathbb{Z}.$$

Therefore, there exist a prime p, two indices $i_0, i_1 \in \{1,2,\ldots,s\}$, and an index $k_1 \in \{\delta_1 n, \ldots, (M-\delta_1)n\}$ with $k_1 \neq k_0$ such that

$$v_p\Big(\Phi_n^{-s/J} D_{(M-2\delta_1)n}^s \cdot \frac{a_{n,i_0,k_0}}{(\ell_0+\theta)^{i_0}}\Big) < 0$$
$$\text{and } v_p\Big(\Phi_n^{-s/J} D_{(M-2\delta_1)n}^s \cdot \frac{a_{n,i_1,k_1}}{(k_1-k_0+\ell_0+\theta)^{i_1}}\Big) < 0.$$

By Lemma 3.3, we have

$$v_p\Big(\Phi_n^{-s/J} D_{(M-2\delta_1)n}^{s-i_0} \cdot a_{n,i_0,k_0}\Big) \geqslant 0 \text{ and } v_p\Big(\Phi_n^{-s/J} D_{(M-2\delta_1)n}^{s-i_1} \cdot a_{n,i_1,k_1}\Big) \geqslant 0.$$

Thus,

$$v_p(\ell_0+\theta) > v_p(D_{(M-2\delta_1)n}) \text{ and } v_p(k_1-k_0+\ell_0+\theta) > v_p(D_{(M-2\delta_1)n}).$$

It follows that $v_p(k_1-k_0) > v_p(D_{(M-2\delta_1)n})$, which is absurd because $0 < |k_1-k_0| \leqslant (M-2\delta_1)n$. $\square$

Remark 3.6. For $\theta=1$, we can prove that

$$\Phi_n^{-s/J} \prod_{j=1}^{J} D_{M_j n}^{s/J} \cdot \rho_{n,0,1} \in \mathbb{Z},$$

where $M_j = \max\{M-2\delta_1, M-\delta_j\}$. The proof is the same as that of [5, Lem. 4.4]. In some sense, the arithmetic property of $\rho_{n,0,1}$ is the worst among the coefficients $\rho_{n,0,\theta}$ $(\theta \in \mathcal{Z}_B)$. We will eliminate $\rho_{n,0,1}$ in our final linear forms (see the proof of Lemma 5.1).

4. Asymptotic estimates

Recall that $S_{n,\theta}$ $(\theta \in \mathcal{Z}_B)$ are defined by (2.8). We have the following asymptotic estimates for $S_{n,\theta}$ $(\theta \in \mathcal{Z}_B)$ as $n \in P_{B,\mathrm{den}(r)}\mathbb{N}$ and $n \to \infty$.

Lemma 4.1. *For $\theta = 1 \in \mathcal{Z}_B$, we have*

$$\lim_{n\to\infty} S_{n,1}^{1/n} = g(x_0),$$

where

$$g(x) := A_1(B)A_2(B)\operatorname{den}(r)^{(2r+1)M|\mathcal{Z}_B|-M}\left(\prod_{j=1}^{J}(M-2\delta_j)^{M-2\delta_j}\right)^{s/J} \times ((2r+1)M+x)^{(2r+1)M|\mathcal{Z}_B|}\cdot\frac{(rM+x)^{rM}}{((r+1)M+x)^{(r+1)M}} \times \prod_{j=1}^{J}\left(\frac{(rM+\delta_j+x)^{rM+\delta_j}}{((r+1)M-\delta_j+x)^{(r+1)M-\delta_j}}\right)^{s/J},$$

and x_0 is the unique positive real solution of the equation $f(x)=1$ with

$$f(x) := \left(\frac{(2r+1)M+x}{x}\right)^{|\mathcal{Z}_B|}\frac{rM+x}{(r+1)M+x}\times\prod_{j=1}^{J}\left(\frac{rM+\delta_j+x}{(r+1)M-\delta_j+x}\right)^{s/J}.$$

Moreover, for any $\theta \in \mathcal{Z}_B$, we have

$$\lim_{n\to\infty}\frac{S_{n,1}}{S_{n,\theta}} = 1.$$

Proof. We first prove that the equation $f(x)=1$ has a unique positive real solution $x=x_0$. A straightforward calculation for the logarithmic derivative of $f(x)$ yields

$$\frac{f'(x)}{f(x)} = \frac{u(x)}{x((2r+1)M+x)},\quad x\in(0,+\infty),$$

where the function $u(x)$ is defined on $(0,+\infty)$ by

$$u(x) := -(2r+1)M|\mathcal{Z}_B| + M\left(1-\frac{(rM)((r+1)M)}{(rM+x)((r+1)M+x)}\right) + \frac{s}{J}\sum_{j=1}^{J}(M-2\delta_j)\left(1-\frac{(rM+\delta_j)((r+1)M-\delta_j)}{(rM+\delta_j+x)((r+1)M-\delta_j+x)}\right). \tag{4.1}$$

Noting that $M-2\delta_j \geqslant 1$ and

$$1-\frac{(rM+\delta_j)((r+1)M-\delta_j)}{(rM+\delta_j+x)((r+1)M-\delta_j+x)} = \frac{x((2r+1)M+x)}{(rM+\delta_j+x)((r+1)M-\delta_j+x)} > \frac{x}{(2r+1)M+x}\qquad\text{for } j=1,2,\ldots,J,$$

we have

$$(4.2)\qquad u(x) > -(2r+1)M|\mathcal{Z}_B| + s\cdot\frac{x}{(2r+1)M+x},\quad x\in(0,+\infty).$$

Let

$$x_2 := \frac{(2r+1)^2M^2|\mathcal{Z}_B|}{s-(2r+1)M|\mathcal{Z}_B|},\quad \text{(we have } x_2>0 \text{ by (2.1) and (2.4))}$$

which is the root of the right-hand side of (4.2). Then $u(x_2) > 0$. On the other hand, by (4.1), the function $u(x)$ is increasing on $(0,+\infty)$ and $u(0^+) = -(2r+1)M|\mathcal{Z}_B| < 0$. Thus, there exists a unique $x_1 \in (0,x_2)$ such that $u(x_1)=0$. Therefore, $f(x)$ is decreasing on $(0,x_1)$ and increasing on $(x_1,+\infty)$. Since $f(0^+) = +\infty$ and $f(+\infty) = 1$, there exists a unique $x_0 \in (0,x_1)$ such that $f(x_0)=1$. Moreover, $f(x) > 1$ for $x\in(0,x_0)$ and $f(x) < 1$ for $x \in (x_0,+\infty)$. By (2.4), we have

$$(4.3)\qquad x_0 < x_2 = \frac{(2r+1)^2M^2|\mathcal{Z}_B|}{s-(2r+1)M|\mathcal{Z}_B|} \leqslant \frac{(2r+1)^2M^2B^2}{s-(2r+1)MB^2}.$$

The rest of the proof is similar to that of [6, Lem. 4.1]. We only give a sketch. Since $R_n(m+\theta)=0$ for $m=0,1,\dots,rMn-1$ and $\theta\in\mathcal{Z}_B$, we have

$$S_{n,\theta} = \sum_{k=0}^{\infty} R_n(rMn+k+\theta).$$

We can express $R_n(rMn+k+\theta)$ almost as a ratio of Gamma functions. Specifically, we have

$$\begin{aligned}
&R_n(rMn+k+\theta)\\
&\quad = A_1(B)^nA_2(B)^n\cdot\frac{\prod_{j=1}^{J}((M-2\delta_j)n)!^{s/J}}{(n/\operatorname{den}(r))!^{\operatorname{den}(r)(2r+1)M|\mathcal{Z}_B|-\operatorname{den}(r)M}}\\
&\qquad\times((2r+1)Mn+2k+2\theta)\cdot\prod_{\theta'\in\mathcal{Z}_B}\frac{\Gamma((2r+1)Mn+k+\theta+\theta')}{\Gamma(k+\theta+\theta')}\\
&\qquad\times\frac{\Gamma(rMn+k+\theta)}{\Gamma((r+1)Mn+k+1+\theta)}\\
&\qquad\times\prod_{j=1}^{J}\left(\frac{\Gamma((rM+\delta_j)n+k+\theta)}{\Gamma(((r+1)M-\delta_j)n+k+1+\theta)}\right)^{s/J}.
\end{aligned}$$

Suppose for the moment that $k=\kappa n$ for some constant $\kappa>0$, then by Stirling's formula we have (as $n\to\infty$)

$$\begin{aligned}
&R_n(rMn+\kappa n+\theta)\\
&\quad= n^{O(1)}\cdot A_1(B)^n A_2(B)^n \operatorname{den}(r)^{((2r+1)M|\mathcal{Z}_B|-M)n}\\
&\qquad\times\left(\prod_{j=1}^{J}(M-2\delta_j)^{(M-2\delta_j)n}\right)^{s/J}\cdot\left(\frac{((2r+1)M+\kappa)^{((2r+1)M+\kappa)n}}{\kappa^{\kappa n}}\right)^{|\mathcal{Z}_B|}\\
&\qquad\times\frac{(rM+\kappa)^{(rM+\kappa)n}}{((r+1)M+\kappa)^{((r+1)M+\kappa)n}}\\
&\qquad\times\prod_{j=1}^{J}\left(\frac{(rM+\delta_j+\kappa)^{(rM+\delta_j+\kappa)n}}{((r+1)M-\delta_j+\kappa)^{((r+1)M-\delta_j+\kappa)n}}\right)^{s/J}\\
&\quad= n^{O(1)}\cdot(f(\kappa)^{\kappa}g(\kappa))^n.
\end{aligned}$$

Define the function $h(x):=f(x)^x g(x)$ on $(0,+\infty)$. A direct calculation yields $h'(x)/h(x)=\log f(x)$. Thus, $h(x)$ is increasing on $(0,x_0)$ and decreasing on $(x_0,+\infty)$. Based on this observation, one can argue as in the proof of [6, Lem. 4.1] to show that $S_{n,\theta}=n^{O(1)}h(x_0)^n$ for any $\theta\in\mathcal{Z}_B$. Consequently, we have

$$\lim_{n\to\infty}S_{n,\theta}^{1/n}=h(x_0)=g(x_0).$$

Moreover, for any fixed small $\varepsilon_0>0$ and any $\theta\in\mathcal{Z}_B$, we have

$$\begin{aligned}
S_{n,1}&=(1+o(1))\sum_{(x_0-\varepsilon_0)n\leqslant k\leqslant(x_0+\varepsilon_0)n}R_n(m_1n+k+1),\\
S_{n,\theta}&=(1+o(1))\sum_{(x_0-\varepsilon_0)n\leqslant k\leqslant(x_0+\varepsilon_0)n}R_n(m_1n+k+\theta).
\end{aligned}$$

Using the relation $\Gamma(x+1-\theta)/\Gamma(x)=(1+o_{x\to+\infty}(1))x^{1-\theta}$, we obtain uniformly for $(x_0-\varepsilon_0)n\leqslant k\leqslant(x_0+\varepsilon_0)n$ that

$$\frac{R_n(rMn+k+1)}{R_n(rMn+k+\theta)}=(1+o(1))f(k/n)^{1-\theta}\quad\text{as } n\to\infty.$$

Therefore,

$$f(x_0+\varepsilon_0)^{1-\theta}\leqslant\liminf_{n\to\infty}\frac{S_{n,1}}{S_{n,\theta}}\leqslant\limsup_{n\to\infty}\frac{S_{n,1}}{S_{n,\theta}}\leqslant f(x_0-\varepsilon_0)^{1-\theta}.$$

Letting $\varepsilon_0\to0^+$, we obtain

$$\lim_{n\to\infty}\frac{S_{n,1}}{S_{n,\theta}}=1. \qquad\square$$

Next, we estimate the factor Φ_n (defined by (3.3)). It is well known that the prime number theorem implies

$$D_m = e^{m+o(m)} \quad \text{as } m \to \infty. \tag{4.4}$$

The following lemma is also a corollary of the prime number theorem. For details, we refer the reader to [11, Lem. 4.4].

Lemma 4.2. *We have*

$$\Phi_n = \exp(\varpi n + o(n)) \quad \textit{as } n \to \infty,$$

where the constant

$$\varpi = \int_0^1 \omega(x)\ \mathrm{d}\psi(x) - \int_0^{1/(M-2\delta_1)} \omega(x)\ \frac{\mathrm{d}x}{x^2}. \tag{4.5}$$

The function $\psi(\,\cdot\,)$ is the digamma function, and the function $\omega(\,\cdot\,)$ is given by (3.4).

5. Elimination procedure

We use the elimination technique of Fischler–Sprang–Zudilin [4] to prove the following.

Lemma 5.1. *If*

$$\log(g(x_0)) + s \cdot \left(-\frac{\varpi}{J} + (M - 2\delta_1)\right) < 0, \tag{5.1}$$

then there are at least $|\Psi_B| - 1$ irrational numbers among $\zeta(3)$, $\zeta(5), \dots,$ $\zeta(s-1)$.

Proof. Suppose that the number of irrationals among $\zeta(3), \zeta(5), \dots, \zeta(s-1)$ is less than $|\Psi_B| - 1$; then we can take a subset $I \subset \{3, 5, \dots, s-1\}$ with $|I| = |\Psi_B| - 2$ such that $\zeta(i) \in \mathbb{Q}$ for all $i \in \{3, 5, \dots, s-1\} \setminus I$. Let $A \in \mathbb{N}$ be a common denominator of these rational zeta values; that is, $A \cdot \zeta(i) \in \mathbb{Z}$ for all $i \in \{3, 5, \dots, s-1\} \setminus I$.

Since the generalized Vandermonde matrix

$$\left[b^i\right]_{b \in \Psi_B,\ i \in \{0,1\} \cup I}$$

is invertible (see [4, §4]), there exist integers $w_b \in \mathbb{Z}$ ($b \in \Psi_B$) such that

$$\sum_{b \in \Psi_B} w_b b^i = 0 \quad \text{for any } i \in \{0\} \cup I, \text{ and}$$
$$\sum_{b \in \Psi_B} w_b b \neq 0.$$

By (2.3), we have $k/b \in \mathcal{Z}_B$ for any $b \in \Psi_B$ and any $k = 1, 2, \ldots, b$; thus $S_{n,k/b}$ is defined by Definition 2.8. For any $n \in P_{B,\mathrm{den}(r)}\mathbb{N}$ with $n > s^2$, we define

$$\widetilde{S}_n := \sum_{b \in \Psi_B} w_b \sum_{k=1}^{b} S_{n,k/b}.$$

By Lemma 2.5 and the fact

$$\sum_{k=1}^{b} \zeta(i, k/b) = b^i \zeta(i),$$

we have

$$\begin{aligned}\widetilde{S}_n &= \sum_{b \in \Psi_B} w_b \sum_{k=1}^{b} \rho_{n,0,k/b} + \sum_{i \in \{3,5,\ldots,s-1\}} \left(\sum_{b \in \Psi_B} w_b b^i \right) \rho_{n,i} \zeta(i) \\ &= \sum_{b \in \Psi_B} w_b \sum_{k=1}^{b-1} \rho_{n,0,k/b} + \sum_{i \in \{3,5,\ldots,s-1\} \setminus I} \left(\sum_{b \in \Psi_B} w_b b^i \right) \rho_{n,i} \zeta(i).\end{aligned}$$

By Lemma 4.1, we have

$$\widetilde{S}_n = \left(\sum_{b \in \Psi_B} w_b b + o(1) \right) S_{n,1} \quad \text{as } n \to \infty.$$

By Lemma 3.4 and Lemma 3.5, we have

$$A \cdot \Phi_n^{-s/J} D_{(M-2\delta_1)n}^{s} \cdot \widetilde{S}_n \in \mathbb{Z}.$$

On the other hand, by Eq. (4.4), Lemma 4.2, Lemma 4.1, and Eq. (5.1), we have

$$\begin{aligned}0 &< \lim_{n \to \infty} \left| A \cdot \Phi_n^{-s/J} D_{(M-2\delta_1)n}^{s} \cdot \widetilde{S}_n \right|^{1/n} \\ &= g(x_0) \exp\left(s \cdot \left(-\frac{\varpi}{J} + (M - 2\delta_1) \right) \right) < 1,\end{aligned}$$

a contradiction. □

The next theorem reduces our task to a computational problem.

Theorem 5.2. *Fix any positive integers M and J, and any non-negative integers $\delta_1, \delta_2, \ldots, \delta_J$ such that*

$$0 \leqslant \delta_1 \leqslant \delta_2 \leqslant \cdots \leqslant \delta_J < \frac{M}{2}.$$

Let ϖ be the real number determined by $M, \delta_1, \delta_2, \ldots, \delta_J$ as follows:

$$\varpi = \int_0^1 \omega(x)\ \mathrm{d}\psi(x) - \int_0^{1/(M-2\delta_1)} \omega(x)\ \frac{\mathrm{d}x}{x^2},$$

where $\psi(x) = \Gamma'(x)/\Gamma(x)$ is the digamma function, and

$$\omega(x) = \min_{0 \leqslant y < 1} \sum_{j=1}^{J} \left(\lfloor (M - 2\delta_j)x \rfloor - \lfloor y - \delta_j x \rfloor - \lfloor (M - \delta_j)x - y \rfloor \right).$$

Let $r_0 \in (-\delta_1/M, +\infty)$ be the real number determined by $M, \delta_1, \delta_2, \ldots, \delta_J$ as follows:

$$\sum_{j=1}^{J} (M - 2\delta_j)\Big(\log((r_0 + 1)M - \delta_j) + \log(r_0 M + \delta_j)\Big) = -2\varpi + 2J(M - 2\delta_1) + 2\sum_{j=1}^{J} (M - 2\delta_j) \log(M - 2\delta_j). \tag{5.2}$$

If $r_0 > 0$, then we have

$$\#\{i \in \{3, 5, \ldots, s - 1\} \mid \zeta(i) \notin \mathbb{Q}\} \geqslant (C_0 - o(1)) \cdot \sqrt{\frac{s}{\log s}}$$

as the even integer $s \to \infty$, where the constant

$$C_0 = \sqrt{\frac{2\zeta(2)\zeta(3)}{\zeta(6)} \cdot \frac{1}{J} \sum_{j=1}^{J} \log \frac{(r_0 + 1)M - \delta_j}{r_0 M + \delta_j}}.$$

Proof. Let s be a sufficiently large multiple of $2J$. Take $B = c\sqrt{s/\log s}$ for some constant $c > 0$, and let $r \in \mathbb{Q}_{>0}$. By the definition of $g(x_0)$ (see Lemma 2.5) and using Eqs. (2.4), (2.5), (2.6), and (4.3), we have

$$\begin{aligned}
&\lim_{\substack{s \to \infty \\ B = c\sqrt{s/\log s}}} \frac{\log(g(x_0))}{s} \\
&= \frac{1}{2} \frac{\zeta(2)\zeta(3)}{\zeta(6)} (2r + 1)M \cdot \frac{c^2}{2} + \frac{1}{J} \sum_{j=1}^{J} (M - 2\delta_j) \log(M - 2\delta_j) \\
&\quad + \frac{1}{J} \sum_{j=1}^{J} \Big((rM + \delta_j) \log(rM + \delta_j) - ((r+1)M - \delta_j) \log\big((r+1)M - \delta_j\big) \Big).
\end{aligned}$$

If the constant c satisfies

$$c^2 < \frac{4\zeta(6)}{\zeta(2)\zeta(3)} \cdot F(r), \tag{5.3}$$

where

$$F(r) := \frac{1}{(2r+1)MJ} \cdot \Bigg(\sum_{j=1}^{J} ((r+1)M - \delta_j) \log((r+1)M - \delta_j) \\ - \sum_{j=1}^{J} (rM + \delta_j) \log(rM + \delta_j) \\ - \sum_{j=1}^{J} (M - 2\delta_j) \log(M - 2\delta_j) \\ + \varpi - J(M - 2\delta_1) \Bigg),$$

then (5.1) is satisfied when s is sufficiently large, and hence

$$\#\{i \in \{3,5,\ldots,s-1\} \mid \zeta(i) \notin \mathbb{Q}\} \geqslant (1+o(1)) \frac{\zeta(2)\zeta(3)}{\zeta(6)} \cdot c \cdot \sqrt{\frac{s}{\log s}} \tag{5.4}$$

as $s \to \infty$, by Lemma 5.1 and Eq. (2.2).

Therefore, we want to maximize $F(r)$. Note that

$$F'(r) = \frac{1}{(2r+1)^2 MJ} \cdot G(r),$$

where

$$G(r) := -2\varpi + 2J(M - 2\delta_1) + 2 \sum_{j=1}^{J} (M - 2\delta_j) \log(M - 2\delta_j) \\ - \sum_{j=1}^{J} (M - 2\delta_j) \Big(\log((r+1)M - \delta_j) + \log(rM + \delta_j) \Big).$$

Obviously, $G(r)$ is decreasing on $(-\delta_1/M, +\infty)$. Since $G((-\delta_1/M)^+) = +\infty$ and $G(+\infty) = -\infty$, there exists a unique $r_0 \in (-\delta_1/M, +\infty)$ satisfying (5.2); that is, $G(r_0) = 0$. Thus, $F(r_0)$ is the maximum value for $F(r)$. We have $F(r_0) > 0$ because $F(r)$ is decreasing on $(r_0, +\infty)$ and $F(+\infty) = 0$.

Fix any small $\varepsilon > 0$. If $r_0 > 0$, then we can take $r \in \mathbb{Q}_{>0}$ sufficiently close to r_0 such that $F(r) > (1 - \varepsilon/2) F(r_0)$. Take

$$c = \sqrt{\left(1 - \frac{\varepsilon}{2}\right) \cdot \frac{4\zeta(6)}{\zeta(2)\zeta(3)} \cdot F(r_0)},$$

then (5.3) is satisfied, and hence by (5.4) we have

$$\begin{aligned}\#\{i \in \{3,5,\ldots,s-1\} \mid \zeta(i) \notin \mathbb{Q}\} &\geqslant \left(1-\frac{\varepsilon}{2}\right)\sqrt{\frac{4\zeta(2)\zeta(3)}{\zeta(6)}F(r_0)} \cdot \sqrt{\frac{s}{\log s}} \\ &= \left(1-\frac{\varepsilon}{2}\right)C_0 \cdot \sqrt{\frac{s}{\log s}}\end{aligned}$$

for any sufficiently large s which is a multiple of $2J$. Thus,

$$\#\{i \in \{3,5,\ldots,s-1\} \mid \zeta(i) \notin \mathbb{Q}\} \geqslant (1-\varepsilon)C_0 \cdot \sqrt{\frac{s}{\log s}}$$

for any sufficiently large even integer s. □

6. Computations

In this section, we will take explicit parameters $M, \delta_1, \delta_2, \ldots, \delta_J$ in Theorem 5.2 and calculate the corresponding ϖ, r_0, and C_0. We refer the reader to [7, §5] for the method of calculating ϖ using `MATLAB`.

If we take $M = 1$, $J = 1$, and $\delta_1 = 0$, then we find that $\varpi = 0$, $r_0 = 2.263884\ldots$, and $C_0 = 1.192507\ldots$. Thus, we rediscover the result in [6]: for any sufficiently large even integer s,

$$\#\{i \in \{3,5,\ldots,s-1\} \mid \zeta(i) \notin \mathbb{Q}\} \geqslant 1.192507 \cdot \sqrt{\frac{s}{\log s}}.$$

The simplest parameters to improve the above result are: $M = 7$, $J = 2$, $\delta_1 = 0$, and $\delta_2 = 1$. For this collection of parameters, we have $\varpi = 2.284309\ldots$, $r_0 = 1.850170\ldots$, and

$$C_0 = 1.197980\ldots.$$

Take $M = 57$, $J = 18$, and

$$(\delta_1, \delta_2, \ldots, \delta_{18}) = (1,1,1,1,1,2,2,3,3,5,5,6,6,7,8,9,10,12).$$

Then, we find that $\varpi = 238.966249\ldots$, $r_0 = 1.573948\ldots$, and

$$C_0 = 1.262672\ldots.$$

Proof of Theorem 1.1. Take $M = 563$, $J = 76$, and $\delta_1, \delta_2, \ldots, \delta_{76}$ as in Table 6.1.

Using `MATLAB`, we find that $\varpi = 12694.987927\ldots$, $r_0 = 1.502726\ldots$, and $C_0 = 1.284579\ldots$. By Theorem 5.2, we obtain

$$\#\{i \in \{3,5,\ldots,s-1\} \mid \zeta(i) \notin \mathbb{Q}\} \geqslant 1.284579 \cdot \sqrt{\frac{s}{\log s}}$$

for any sufficiently large even integer s. □

Table 6.1. The choice of $\delta_1, \delta_2, \ldots, \delta_{76}$

$\delta_1 = 1$	$\delta_{12} = 5$	$\delta_{23} = 16$	$\delta_{34} = 30$	$\delta_{45} = 52$	$\delta_{56} = 74$	$\delta_{67} = 96$
$\delta_2 = 1$	$\delta_{13} = 6$	$\delta_{24} = 17$	$\delta_{35} = 32$	$\delta_{46} = 54$	$\delta_{57} = 76$	$\delta_{68} = 98$
$\delta_3 = 1$	$\delta_{14} = 7$	$\delta_{25} = 18$	$\delta_{36} = 34$	$\delta_{47} = 56$	$\delta_{58} = 78$	$\delta_{69} = 100$
$\delta_4 = 1$	$\delta_{15} = 8$	$\delta_{26} = 19$	$\delta_{37} = 36$	$\delta_{48} = 58$	$\delta_{59} = 80$	$\delta_{70} = 102$
$\delta_5 = 1$	$\delta_{16} = 9$	$\delta_{27} = 20$	$\delta_{38} = 38$	$\delta_{49} = 60$	$\delta_{60} = 82$	$\delta_{71} = 104$
$\delta_6 = 2$	$\delta_{17} = 10$	$\delta_{28} = 21$	$\delta_{39} = 40$	$\delta_{50} = 62$	$\delta_{61} = 84$	$\delta_{72} = 108$
$\delta_7 = 2$	$\delta_{18} = 11$	$\delta_{29} = 22$	$\delta_{40} = 42$	$\delta_{51} = 64$	$\delta_{62} = 86$	$\delta_{73} = 112$
$\delta_8 = 3$	$\delta_{19} = 12$	$\delta_{30} = 23$	$\delta_{41} = 44$	$\delta_{52} = 66$	$\delta_{63} = 88$	$\delta_{74} = 116$
$\delta_9 = 3$	$\delta_{20} = 13$	$\delta_{31} = 24$	$\delta_{42} = 46$	$\delta_{53} = 68$	$\delta_{64} = 90$	$\delta_{75} = 120$
$\delta_{10} = 4$	$\delta_{21} = 14$	$\delta_{32} = 26$	$\delta_{42} = 48$	$\delta_{54} = 70$	$\delta_{65} = 92$	$\delta_{76} = 124$
$\delta_{11} = 4$	$\delta_{22} = 15$	$\delta_{33} = 28$	$\delta_{44} = 50$	$\delta_{55} = 72$	$\delta_{66} = 94$	

Remark 6.1. The δ_j values from Table 6.1 were initially found in [5] (with a different M) for a highly related problem. These parameters are not theoretically special; they are merely empirically effective. They were obtained through a random search and several manual adjustments, such as fixing all δ_j values and varying M, or fixing all other parameters while varying, adding, or deleting one δ_j.

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References

[1] R. Apéry, "Irrationalité de $\zeta(2)$ et $\zeta(3)$", in *Journées Arithmétiques (Luminy, 1978)*, Astérisque, vol. 61, Société Mathématique de France, 1979, p. 11-13.

[2] K. Ball & T. Rivoal, "Irrationalité d'une infinité de valeurs de la fonction zêta aux entiers impairs", *Invent. Math.* **146** (2001), no. 1, p. 193-207.

[3] S. Fischler, "Linear independence of odd zeta values using Siegel's lemma", *J. Lond. Math. Soc. (2)* **113** (2026), no. 4, article no. e70535 (42 pages).

[4] S. Fischler, J. Sprang & W. Zudilin, "Many odd zeta values are irrational", *Compos. Math.* **155** (2019), no. 5, p. 938-952.

[5] L. Lai, "Small improvements to the Ball–Rivoal theorem and its p-adic variant", 2024, `https://arxiv.org/abs/2407.14236v2`.

[6] L. Lai & P. Yu, "A note on the number of irrational odd zeta values", *Compos. Math.* **156** (2020), no. 8, p. 1699-1717.

[7] L. Lai & L. Zhou, "At least two of $\zeta(5), \zeta(7), \ldots, \zeta(35)$ are irrational", *Publ. Math. Debr.* **101** (2022), no. 3-4, p. 353-372.

[8] T. Rivoal, "La fonction zêta de Riemann prend une infinité de valeurs irrationnelles aux entiers impairs", *C. R. Acad. Sci. Paris* **331** (2000), no. 4, p. 267-270.

[9] J. Sprang, "Infinitely many odd zeta values are irrational. By elementary means", 2018, `https://arxiv.org/abs/1802.09410`.

[10] W. Zudilin, "One of the numbers $\zeta(5)$, $\zeta(7)$, $\zeta(9)$, $\zeta(11)$ is irrational", *Usp. Mat. Nauk* **56** (2001), no. 4, p. 149-150, translation in *Russ. Math. Surv.* **56** (2001), no. 4, p. 774-776.

[11] ———, "Irrationality of values of the Riemann zeta function", *Izv. Ross. Akad. Nauk, Ser. Mat.* **66** (2002), no. 3, p. 49-102, translation in *Izv. Math.* **66** (2002), no. 3, p. 489-542.

[12] ———, "Arithmetic of linear forms involving odd zeta values", *J. Théor. Nombres Bordeaux* **16** (2004), no. 1, p. 251-291.

[13] ———, "One of the odd zeta values from $\zeta(5)$ to $\zeta(25)$ is irrational. By elementary means", *SIGMA, Symmetry Integrability Geom. Methods Appl.* **14** (2018), article no. 028 (8 pages).

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