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Convolution and square in abelian groups II

par Yves BENOIST

Résumé. Une valeur λ est dite critique sur un groupe abélien d'ordre impair G si l'équation fonctionnelle $f\star f(2\,t)=\lambda f(t)^2$ sur G a une solution $f\neq 0$. Nous construisons de nombreuses valeurs critiques à l'aide des variétés abéliennes à multiplication complexe.

Abstract. A critical value on an abelian group G of odd order d is a value λ such that the functional equation $f\star f(2\,t)=\lambda f(t)^2$ on G has a nonzero solution f. We construct many critical values by using abelian varieties with complex multiplication.

1. Critical values

1.1. Introduction. This paper is the sequel of [3] in which we introduced the notion of d-critical values or critical values on the cyclic group $\mathbb{Z}/d\mathbb{Z}$, where d is an odd integer. Before recalling the precise definition of critical values in Section 1.2, I would like to sum up the content of this first paper. In [3] we reported some striking numerical experiments and explained that the value $\lambda:=\sqrt{a}+i\sqrt{b}$ is a d-critical value when $a+b=d$ and $a\equiv\frac{(d+1)^2}{4}\bmod 4$. It was surprising that Jacobi theta functions and elliptic curves with complex multiplication were needed to prove that these apparently very simple values are critical.

In the experimental list of critical values given in [3, Section 1.5], there still remained a few intriguing d-critical values that could not be explained by the techniques of [3] relying on Jacobi theta functions and elliptic curves. Those remaining values are shown in the insert below.

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The aim of the present paper is to explain a general construction of critical values λ by using torsion points on higher dimensional abelian varieties and Riemann theta functions (Theorem 4.4).

$$\begin{array}{ll} d=11 & \lambda = 1+\sqrt{5}+i\sqrt{5-2\sqrt{5}} \\ d=15 & \lambda = 1+\sqrt{5}+i\sqrt{9-2\sqrt{5}} \\ d=15 & \lambda = (\sqrt{3}+i\sqrt{2})\,(\sqrt{2}+i) \\ d=17 & \lambda = 1+2\sqrt{2}+2i\sqrt{2-\sqrt{2}} \end{array}$$

This construction gives all the d-critical values with $d \leq 11$. As an output, focusing on abelian varieties with complex multiplication, we will obtain new explicit critical values.

Note that the full list of critical values is known only when $d \leq 11$ and is given in [2, Section 4.4]. A few critical values for $d = 15$ and $d = 17$ that can not be explained by our technique relying on Riemann theta functions and abelian varieties have been found in [2, Section 4.4]. These partial lists have been recently extended by Ma Li (private communication) up to $d = 19$.

1.2. Constructing critical values. Before going on our introduction, we need to recall the definition of critical value from [3]. We let G be a finite abelian group of odd order d, but we keep in mind that the case where G is the cyclic group $G = \mathbb{Z}/d\mathbb{Z}$ is very important. A nonzero function $f : G \to \mathbb{C}$ satisfying the functional equation

$$\sum_{\ell \in G} f(k+\ell)\, f(k-\ell) \;=\; \lambda\, f(k)^2 \quad \text{for all } k \text{ in } G, \tag{1.1}$$

where $\lambda \in \mathbb{C}$ is a parameter, will be called a "λ-critical function on G", and a value λ for which such a function f exists will be called a "critical value on G", or a "d-critical value" when $G = \mathbb{Z}/d\mathbb{Z}$. This equation expresses a proportionality condition between the "convolution square" of f and its "multiplication square".

Our main result in this paper is a construction of explicit critical values. Beyond this precise list of critical values, the main interest and surprise is the fact that this naive question is related to abelian varieties with complex multiplication and to modular forms on the Siegel upper half-space.

We will explain the construction of these critical values from various points of view. First from the point of view of Riemann theta functions (Theorem 2.3). Then from the point of view of abelian varieties (Theorem 4.4). And finally we will apply this construction from the point of view of CM number fields (Theorem 5.4).

Even though the consequences of these theorems, such as Propositions 6.4, 6.5, and 7.15, have an elementary statement, I do not know an elementary proof. Indeed, our construction relies on the Riemann theta functions $\theta(z,\tau)$ for special values of the Riemann matrix τ, and the key point in the proof relies on modularity properties of these theta functions on the Siegel upper half-space which is due to Siegel (Lemma 2.6) and on its refinements due to Igusa (Corollary 2.9) and to Stark (Lemma 8.3). It also relies on a construction of principally polarized CM abelian varieties due to Shimura–Taniyama (Fact 7.1).

1.3. A few concrete examples of critical values. To give a flavor of the output of our method we just give here three concrete families of critical values.

Proposition 6.4. Let $d = d_1 d_2$ be a product of two coprime odd numbers. For $j = 1, 2$, let $d_j = a_j + b_j$ be all positive integers, $a_j - \frac{(d_j+1)^2}{4} \equiv 2 \bmod 4$, and $\varepsilon_j = \pm 1$. Then $\lambda := (\sqrt{a_1} + i\varepsilon_1\sqrt{b_1})(\sqrt{a_2} + i\varepsilon_2\sqrt{b_2})$ is d-critical.

Proposition 6.5. Let $d = a + b + c$ be positive integers with $b^2 - 4ac > 0$.

(A) Assume that $a \equiv 1 \bmod 4$ and $b \equiv c \equiv (0 \text{ or } 1) \bmod 4$.
Then the sum $\lambda = \sqrt{a} + \sqrt{c} + i\sqrt{b - 2\sqrt{ac}}$ is d-critical.

(B) Assume that $a \equiv 3 \bmod 4$ and $b \equiv c \equiv (0 \text{ or } 3) \bmod 4$.
Then the sum $\lambda = \sqrt{b - 2\sqrt{ac}} + i\sqrt{a} + i\sqrt{c}$ is d-critical.

Proposition 7.15. Let $n \geq 5$ be prime and $L_n := \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n$ be the n^{th} Lucas number. Then, for all choices of signs $\varepsilon_k = \pm 1$, the product $\lambda_n = \prod_{1 \leq k < n/2}(1 + 2i\varepsilon_k \sin(\frac{k\pi}{n}))$ or $-\lambda_n$ is L_n-critical.

To deal with Propositions 6.4 and 6.5, we will need to use abelian surfaces. To deal with Proposition 7.15, we will need to use an $(n-1)/2$-dimensional abelian variety.

1.4. Properties of critical values. It is important to keep in mind a few properties, proven in [3, Section 2.1].

Proposition 1.1. *Let G be an abelian group of odd order d. There exist only finitely many critical values on G. Let λ be a critical value on G.*

(i) *One has $|\lambda| \leq d$ with equality if and only if $\lambda = d$.*
(ii) *The ratio d/λ is also a critical value on G.*
(iii) *λ is an algebraic integer such that $\lambda \equiv 1 \bmod 2$.*
(iv) *All the Galois conjugates of λ are also critical values on G.*

The condition "$\lambda \equiv 1 \bmod 2$" means that the ratio $\frac{\lambda-1}{2}$ is an algebraic integer.

Since there are only finitely many critical values λ on G, it will be surprising to see that, for all the critical values λ we will construct in this paper, the variety of λ-critical functions f on G is higher dimensional.

1.5. Strategy, organization and main results. This paper is independent of [3], but it might be helpful for the reader to be familiar with the proof given in [3] that relies on Jacobi theta functions, and on elliptic curves with complex multiplication. The proof here will follow the same lines replacing the Jacobi theta functions by the Riemann theta functions, the elliptic curves by abelian varieties, and the imaginary quadratic fields by CM number fields.

In Section 2, we explain how one can use the Riemann theta functions for constructing critical functions on any finite abelian group G of odd order (Theorem 2.3). Such an abelian group G can be seen as a quotient $\mathbf{d}^{-1}\mathbb{Z}^g/\mathbb{Z}^g$ for an integral matrix $\mathbf{d}$ with $\mathbf{d} \equiv \mathbf{1} \bmod 2$. The Riemann theta functions $\theta_\tau(z)$ are $\mathbb{Z}^g$-periodic functions on $\mathbb{C}^g$ parametrized by a matrix τ that lives in the Siegel upper half space $\mathcal{H}_g$, i.e. τ is a complex symmetric matrix whose imaginary part is positive definite. The key point is a condition on the Riemann matrix τ (Lemma 2.7) that ensures that the restriction of any translate of the function $\theta_\tau(z)$ to the group G is critical.

When $g \geq 3$, this condition, which involves $2^g - 1$ equations while there are only $g(g+1)/2$ parameters, seems difficult to satisfy. However a nice fact due to Igusa (Lemma 2.9) helps us to construct solutions of these equations: it is sufficient (and conjecturally also necessary) that there exists an element σ of *the integral symplectic theta group* $\mathrm{Sp}^{\theta,2}_{g,\mathbb{Z}}$ *of level* 2 such that $\sigma\,\tau = {}^t\mathbf{d}\tau\mathbf{d}$. This group $\mathrm{Sp}^{\theta,2}_{g,\mathbb{Z}}$ is a normal finite index subgroup of the integral symplectic group $\mathrm{Sp}(g,\mathbb{Z})$. It acts naturally on the Siegel upper half space $\mathcal{H}_g$.

Theorem 2.3. Let $\tau \in \mathcal{H}_g$ and $\mathbf{d} \in \mathcal{M}(g,\mathbb{Z})$ with $\mathbf{d} \equiv \mathbf{1} \bmod 2$. Assume that there exists $\sigma = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \mathrm{Sp}^{\theta,2}_{g,\mathbb{Z}}$ such that $\sigma\tau = {}^t\mathbf{d}\tau\mathbf{d}$. Then the function θ_τ restricted to the group $G = \mathbf{d}^{-1}\mathbb{Z}^g/\mathbb{Z}^g$ is λ-critical for a critical value $\lambda := \kappa \det_{\mathbb{C}}(\gamma\tau + \delta)^{1/2}|G|$ with $\kappa^8 = 1$.

Here $\det_{\mathbb{C}}$ denotes the determinant of a complex endomorphism of $\mathbb{C}^g$.

In Section 3, we reinterpret our construction of critical values in terms of the integral symplectic theta subgroup of level 2 (Corollary 3.1). We also explain (Lemma 3.5) how to construct elements of the integral symplectic theta group of level 2 starting from an element of the rational symplectic theta subgroup $\mathrm{Sp}^{\theta,2}_{g,\mathbb{Q}}$ of level 2. This relies on the *symplectic adapted basis theorem* (Proposition 3.4). Finally we explain how to construct easily elements of the rational symplectic theta subgroup $\mathrm{Sp}^{\theta,2}_{g,\mathbb{Q}}$ of level 2 thanks to the Cayley transform (Lemma 3.8).

In Section 4, we are dealing with a principally polarized abelian variety $(A = \mathbb{C}^g/\Lambda, \omega)$, with its hermitian structure H on its Lie algebra $\mathbb{C}^g$ and with its integral symplectic form $\omega = \mathrm{Im}(H)$ on its lattice Λ. We reinterpret our general construction of critical functions and critical values from the point of view of abelian varieties in Theorem 4.4 which is the main theorem of this paper. Note that the Riemann theta functions do not occur in the statement of this theorem, but only in its proof. We refer to Section 4.2 for the precise definition of the notions involved in this theorem.

Theorem 4.4. Let $(A = \mathbb{C}^g/\Lambda, \omega)$ be a principally polarized abelian variety, ν be a unitary $\mathbb{Q}$-endomorphism of A preserving a theta structure of level 2, T_ν its tangent map, $G_\nu := \Lambda/(\Lambda \cap \nu\Lambda)$ and $d_\nu := |G_\nu|$. Then there exists a critical value $\lambda_\nu = \kappa_\nu\, d_\nu^{1/2} \det_{\mathbb{C}}(T_\nu)^{1/2}$ on the group G_ν with $\kappa_\nu^4 = 1$.

The square $\kappa_\nu^2 = \pm 1$ can be calculated from the condition $\lambda_\nu \equiv 1 \bmod 2$.

The abelian group G_ν depends not only on the tangent map $T_\nu \in \mathcal{M}(g, \mathbb{C})$ but also on the lattice Λ. It might be cyclic even when $g > 1$.

In Section 5, we are dealing with a CM number field K of degree $2g$ with a CM type $\Phi : K \to \mathbb{C}^g$ and with a lattice $\Lambda \subset K$, and we assume that the abelian variety is the CM abelian variety $A = \mathbb{C}^g/\Phi(\Lambda)$. We specialize our general construction of critical values to that case and express it from the point of view of CM number fields (Theorem 5.4):

- The symplectic form on Λ is given by a nonzero imaginary element t_0 of K thanks to the simple formula $\omega(x, x') = \mathrm{Tr}_{K/\mathbb{Q}}(\frac{x\overline{x'}}{t_0})$. A key point is to choose t_0 so that this symplectic form is integral with determinant 1 on Λ.
- The unitary $\mathbb{Q}$-endomorphisms ν are nothing but elements $\nu \in K$ of absolute value 1. On can construct such ν that preserve a theta structure of level 2 thanks to the Cayley transform (Lemma 5.3).
- The ratio λ_ν^2/d_ν is, up to sign, equal to the *reflex norm* $N_\Phi(\nu)$. In particular the critical value λ_ν is a d_ν-Weil number.

To get more examples we will allow in this section K to be a CM algebra instead of a CM number field.

In Section 6, we show on examples how to compute, at least up to sign, explicit critical values by using Theorem 5.4.

In Section 6.1 we discuss the case where K is an imaginary quadratic field and hence A is a CM elliptic curve. This case is the one we studied in [3].

In Section 6.2, we discuss the case where the CM algebra K is a product of two imaginary quadratic number fields or, equivalently, A is isogenous to the product of two elliptic curves with complex multiplication. This gives the 1st proposition of Section 1.3.

In Section 6.3, we discuss the case where K is a quartic CM field and hence A is a simple CM abelian surface. This gives the 2nd proposition of Section 1.3.

In Section 7, we come back to the general CM abelian varieties, but we specialize our Theorem 5.4 to the case where the lattice Λ is a fractional ideal $\mathfrak{m}$ of K, or equivalently, to the case where the abelian variety $A = \mathbb{C}^g/\Phi(\Lambda)$ has multiplication by $\mathcal{O}_K$ (Theorem 7.3). This case is particularly nice because of the following three reasons:

- A theorem of Shimura–Taniyama relying on class field theory tells us exactly, for which CM type Φ and for which ideal $\mathfrak{m}$ this abelian variety A is principally polarized (Fact 7.1).
- The group G_ν depends only on ν seen as an element of K and does not depend on the ideal $\mathfrak{m}$.
- Examples of unitary elements $\nu \in K$ that preserve a theta structure of level 2 are $\nu = \mu/\bar{\mu}$ with $\mu = 1 + s - \bar{s}$ where $s \in K$ has denominator prime to 2 and where the norm $N_{K/\mathbb{Q}}(\mu) \in \mathbb{Q}$ has an odd numerator (Lemma 7.2).

We quote below the part of Corollary 7.4 where the extension $K/(K\cap\mathbb{R})$ is *ramified* i.e. "ramified at a finite place".

Corollary 7.4.B. Let K be a CM field such that $K/(K\cap\mathbb{R})$ is ramified at a finite place. Let $s \in \mathcal{O}_K$, $\mu := 1+s-\bar{s}$ with $N_{K/\mathbb{Q}}(\mu)$ odd. Then, for all CM types Φ of K, the value $\lambda_\mu = N_\Phi(\mu)$ or $-\lambda_\mu$ is critical on $G_\mu := \mathcal{O}_K/\mu\mathcal{O}_K$.

Note that, the principally polarized CM abelian varieties and their theta structures of level 2 do not occur in the statement of this corollary, but only in its proof.

In Section 7.3, we focus on a very important case, when K is the cyclotomic field $K_n = \mathbb{Q}[\zeta_n]$. In this case, the conclusion of this corollary is always true even when K is unramified over its maximal real subfield $K\cap\mathbb{R}$.

In Section 7.4, we give an explicit example of application of Corollary 7.4.B: we take $s = \zeta_n$. This gives the 3rd proposition of Section 1.3.

In Section 8, we come back to the examples of Section 6, and explain how to remove the remaining ambiguity on the sign of the critical values. The key point is a precise formula for the theta cocycle $j(\sigma,\tau)$ in Lemma 8.3 which is due to Stark, combined with tricky explicit computations. This theta cocycle is the one that shows up in the transformation formula for the Riemann theta functions.

I became interested in the convolution equation (1.1) after computing experimentally the list of d-critical values for small values of d. On the one hand, one is looking for functions that have properties similar to those of the gaussian functions, and, on the other hand, the list is so striking that, as soon as one sees it, one wants to understand it theoretically.

This convolution equation can be seen as an analog of the equation satisfied by the equiangular configuration of d^2 lines in $\mathbb{C}^d$, whose experimental solutions only have a conjectural explanation: the Zauner conjecture. It can also be seen as an analog of the question raised by H. Cohn looking for functions on the cyclic group that have properties similar to those of the Dirichlet characters: such functions are constructed in [4].

2. Theta functions

The aim of this section is to explain (in Theorem 2.3) our general construction of critical values (Theorem 4.4) from the point of view of the Riemann theta functions.

2.1. Riemann matrices and symplectic group. We begin by preliminary classical notation and definition (see [1] or [8]).

A Riemann matrix τ is a complex symmetric matrix whose imaginary part is positive definite. For $g \geq 1$, let $\mathcal{H}_g$ be the Siegel upper half-space which is the space of Riemann matrices of size g,

$$\mathcal{H}_g \;=\; \Big\{\tau \in \mathcal{M}(g,\mathbb{C}) \,\Big|\, {}^t\tau = \tau,\;\; \mathrm{Im}\;\tau > 0\Big\}.$$

Let $\mathrm{Sp}(g,\mathbb{R}) := \{\sigma \in \mathrm{GL}(2g,\mathbb{R}) \,|\, {}^t\sigma J \sigma = J\}$, where $J = \left(\begin{smallmatrix} \mathbf{0} & \mathbb{1}_g \\ -\mathbb{1}_g & \mathbf{0}\end{smallmatrix}\right)$, be the real symplectic group. This group is the stabilizer of the symplectic form ω on $\mathbb{R}^{2g}$ given by

$$\omega(x,y) = {}^t x\, J\, y\,,$$

and seen as a group of 2 by 2 block matrices of size g, it is given by

$$\begin{aligned}\mathrm{Sp}(g,\mathbb{R}) &= \Big\{\sigma = \left(\begin{smallmatrix}\alpha & \beta \\ \gamma & \delta\end{smallmatrix}\right) \,\Big|\, {}^t\alpha\gamma = {}^t\gamma\alpha,\;\; {}^t\beta\delta = {}^t\delta\beta,\;\; {}^t\alpha\delta - {}^t\gamma\beta = \mathbb{1}_g\Big\},\\ &= \Big\{\sigma = \left(\begin{smallmatrix}\alpha & \beta \\ \gamma & \delta\end{smallmatrix}\right) \,\Big|\, \sigma^{-1} = \left(\begin{smallmatrix} {}^t\delta & -{}^t\beta \\ -{}^t\gamma & {}^t\alpha\end{smallmatrix}\right)\Big\}.\end{aligned} \tag{2.1}$$

The group $\mathrm{Sp}(g,\mathbb{R})$ acts transitively on the Siegel upper half-space $\mathcal{H}_g$,

$$\sigma\tau := (\alpha\tau + \beta)(\gamma\tau + \delta)^{-1}.$$

One cannot confuse this notation $\sigma\tau$ with the product of matrices since σ has size $2g$ while τ has size g. The stabilizer of the element $\tau_0 = i\mathbb{1}_g \in \mathcal{H}_g$ is the unitary group $U(g,\mathbb{R})$, so that $\mathcal{H}_g \simeq \mathrm{Sp}(g,\mathbb{R})/U(g,\mathbb{R})$.

For any subring R of $\mathbb{R}$, we let $\mathrm{Sp}(g,R) := \mathrm{GL}(2g,R) \cap \mathrm{Sp}(g,\mathbb{R})$. For $\ell \geq 1$, the following subgroups of the integral symplectic group $\mathrm{Sp}(g,\mathbb{Z})$ play an important role in the theory of theta functions. The first one is the *integral congruence symplectic group* $\mathrm{Sp}^{\ell}_{g,\mathbb{Z}}$ *of level* ℓ.

$$\mathrm{Sp}^{\ell}_{g,\mathbb{Z}} := \{\sigma \in \mathrm{Sp}(g,\mathbb{Z}) \,|\, \sigma \equiv \mathbb{1}_{2g} \bmod \ell\}.$$

The second one is the *integral symplectic theta group* $\mathrm{Sp}^{\theta,\ell}_{g,\mathbb{Z}}$ *of level* ℓ.

$$\mathrm{Sp}^{\theta,\ell}_{g,\mathbb{Z}} := \Big\{\sigma = \left(\begin{smallmatrix}\alpha & \beta \\ \gamma & \delta\end{smallmatrix}\right) \in \mathrm{Sp}^{\ell}_{g,\mathbb{Z}} \,\Big|\, ({}^t\alpha\gamma)_0 \equiv ({}^t\beta\delta)_0 \equiv 0 \bmod 2\ell\Big\},$$

where for a $g\times g$ symmetric matrix S, the notation S_0 means the diagonal of S. This group is sometimes called the *Igusa group of level* ℓ as in [11, p. 10] We will discuss in more details this symplectic theta group in Section 3.2. When $\ell = 1$ we just write $\mathrm{Sp}^{\theta}_{g,\mathbb{Z}}$ for $\mathrm{Sp}^{\theta,1}_{g,\mathbb{Z}}$

In this paper we will mainly need these groups $\mathrm{Sp}^{\ell}_{g,\mathbb{Z}}$ and $\mathrm{Sp}^{\theta,\ell}_{g,\mathbb{Z}}$ for $\ell = 2$. Indeed when $\ell = 2$, the modular variety $X_g^{\theta,\ell}$ of *theta structures of level* ℓ defined by $X_g^{\theta,\ell} := \mathrm{Sp}^{\theta,\ell}_{g,\mathbb{Z}}\backslash\mathcal{H}_g$ will play an important role in this paper.

2.2. Critical values and theta functions. We now recall the definition of the Riemann theta function:

$$\theta_\tau(z) = \theta(z,\tau) := \sum_{m\in\mathbb{Z}^g} e^{i\pi {}^t m\tau m} e^{2i\pi {}^t mz}, \quad \text{for } z\in\mathbb{C}^g \text{ and } \tau\in\mathcal{H}_g.$$

This function is a holomorphic function of z which is $\mathbb{Z}^g$-periodic. One has $\theta_\tau(z+q) = \theta_\tau(z)$ for all q in $\mathbb{Z}^g$. We can now explain our construction of λ-critical functions. The construction involves a matrix $\mathbf{d}$ with integer coefficients and $\det(\mathbf{d}) \neq 0$, and its associate group $G_{\mathbf{d}} := \mathbf{d}^{-1}\mathbb{Z}^g/\mathbb{Z}^g$ whose order $|G_{\mathbf{d}}|$ is equal to $|\det(\mathbf{d})|$. Very often, we will choose $\mathbf{d} = \mathrm{diag}(d_1,\dots,d_g)$ where each coefficient is positive and divides the next one: $d_1|d_2|\cdots|d_g$. Note that any finite abelian group is isomorphic to a unique group $G_{\mathbf{d}}$ with such a diagonal matrix $\mathbf{d}$.

Definition 2.1. Let $\tau\in\mathcal{H}_g$ and $\mathbf{d}\in\mathcal{M}(g,\mathbb{Z})$ with $\mathbf{d}\equiv\mathbf{1} \bmod 2$. We will say that the function θ_τ is $(\lambda,\mathbf{d})$-critical if, for all z in $\mathbb{C}^g$, the function $f_{z,\tau}:\ell\mapsto\theta(z+\ell,\tau)$ is λ-critical on the group $G_{\mathbf{d}} := \mathbf{d}^{-1}\mathbb{Z}^g/\mathbb{Z}^g$.

This means that, for all z in $\mathbb{C}^g$,

$$\sum_{\ell\in G_{\mathbf{d}}} \theta(z+\ell,\tau)\,\theta(z-\ell,\tau) \;=\; \lambda\,\theta(z,\tau)^2.$$

Remark 2.2. In particular the function $f_{0,\tau}:\ell\mapsto\theta(\ell,\tau)$ is a λ-critical function on $G_{\mathbf{d}}$ which is even, that is $f_{0,\tau}(-\ell) = f_{0,\tau}(\ell)$ for all ℓ in $G_{\mathbf{d}}$.

Note, when $\mathbf{d} = \mathrm{diag}(d_1,\dots,d_g)$ as above, that the group $G_{\mathbf{d}}$ has order $|G_{\mathbf{d}}| = d_1\cdots d_g$, and that this group $G_{\mathbf{d}}$ is cyclic of order d if and only if $1 = d_1 = \cdots = d_{g-1} \leq d_g = d$.

Here is our construction of critical values seen from the point of view of theta functions

Theorem 2.3. *Let $\tau\in\mathcal{H}_g$ and $\mathbf{d}\in\mathcal{M}(g,\mathbb{Z})$ with $\mathbf{d}\equiv\mathbf{1} \bmod 2$. Assume that there exists $\sigma = \left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right) \in \mathrm{Sp}^{\theta,2}_{g,\mathbb{Z}}$ such that $\sigma\tau = {}^t\mathbf{d}\tau\mathbf{d}$.*

(a) *There exists a unique $\lambda\in\mathbb{C}$ such that the function θ_τ is $(\lambda,\mathbf{d})$-critical.*

(b) *This λ satisfies $\lambda = \kappa\det_{\mathbb{C}}(\gamma\tau+\delta)^{1/2}|G_{\mathbf{d}}|$, where $\kappa^8 = 1$.*

Note that replacing σ by $-\sigma$ does not change the assumptions but changes the sign of the determinant when g is odd.

In (a) the converse is true for $g \leq 3$: if there exists λ such that the function θ_τ is $(\lambda, \mathbf{d})$-critical then there exists $\sigma \in \mathrm{Sp}_{g,\mathbb{Z}}^{\theta,2}$ such that $\sigma\tau = {}^t\mathbf{d}\tau\mathbf{d}$. This will follow from Remark 2.10. Moreover this converse is also expected to be true for all g.

A more precise formula for λ will be given as Formula (3.1). Notice that it is easy to determine the 8^{th} root of unity κ up to sign without using (3.1) by using instead Proposition 1.1 which says that λ is an algebraic integer satisfying $\lambda \equiv 1 \bmod 2$. Indeed, the only 8^{th} roots of unity which are equal to 1 mod 2 are ± 1.

2.3. Theta functions with characteristic. For the proof of Theorem 2.3 we will need to introduce the Riemann theta functions with characteristic (see [6]). We will also need three classical formulas satisfied by these functions, the "addition formula", the "isogeny formula", and the "transformation formula". We will only need special cases of these formulas that we state below.

The *theta functions with characteristic* a, b in $\mathbb{C}^g$, are defined by, for $z \in \mathbb{C}^g$ and $\tau \in \mathcal{H}_g$,

$$\theta\begin{bmatrix} a \\ b \end{bmatrix}(z,\tau) := \sum_{m\in\mathbb{Z}^g} e^{i\pi^t(m+a)\tau(m+a)} e^{2i\pi^t(m+a)(z+b)}.$$

Note that these functions satisfy the following periodicity when translating the characteristic by elements m, n in $\mathbb{Z}^g$,

$$\theta\begin{bmatrix} a+m \\ b+n \end{bmatrix}(z,\tau) = e^{2i\pi^t an}\, \theta\begin{bmatrix} a \\ b \end{bmatrix}(z,\tau). \tag{2.2}$$

In this paper, we will mainly use the following special cases of theta functions with characteristics. For $\xi \in \mathbb{Z}^g/2\mathbb{Z}^g$, seen as a subset of $\mathbb{Z}^g$, we define

$$\theta_{[\xi]}(z,\tau) = \theta\begin{bmatrix} \xi/2 \\ 0 \end{bmatrix}(2z,2\tau) := \sum_{m\in\xi} e^{i\pi^t m\frac{\tau}{2}m} e^{2i\pi^t mz}. \tag{2.3}$$

Note that one has the equalities:

$$\theta_{[0]}(z,\tau) = \theta(2z,2\tau) \quad \text{and} \quad \sum_{\xi\in\mathbb{Z}^g/2\mathbb{Z}^g} \theta_{[\xi]}(z,\tau) = \theta(z,\tau/2).$$

Here is the addition formula that we need.

Lemma 2.4. *For all z, w in $\mathbb{C}^g$, $\tau \in \mathcal{H}_g$, one has*

$$\theta(z+w,\tau)\,\theta(z-w,\tau) = \sum_{\xi\in\mathbb{Z}^g/2\mathbb{Z}^g} \theta_{[\xi]}(w,\tau)\,\theta_{[\xi]}(z,\tau). \tag{2.4}$$

Proof. Just write the left-hand side LHS as a double sum over m, n in $\mathbb{Z}^g$ and split this double sum according to the class $\xi \in \mathbb{Z}^g/2\mathbb{Z}^g$ in which $m-n$ lives, and note that one has the equivalence: $m-n \in \xi \iff m+n \in \xi$. Use then a change of variable $p := m-n$ and $q := m+n$, this gives

$$\begin{aligned} LHS &= \sum_{\xi \in \mathbb{Z}^g/2\mathbb{Z}^g} \sum_{p \in \xi} \sum_{q \in \xi} e^{i\pi^t p \frac{\tau}{2} p} e^{i\pi^t q \frac{\tau}{2} q} e^{2i\pi^t pw} e^{2i\pi^t qz} \\ &= \sum_{\xi \in \mathbb{Z}^g/2\mathbb{Z}^g} \theta_{[\xi]}(w,\tau)\, \theta_{[\xi]}(z,\tau), \end{aligned}$$

as required. □

The second formula is a simple but useful isogeny formula.

Lemma 2.5. *Let $\tau \in \mathcal{H}_g$ and $\mathbf{d} \in \mathcal{M}(g,\mathbb{Z})$ with $\mathbf{d} \equiv \mathbf{1} \bmod 2$. Set $G_{\mathbf{d}} := \mathbf{d}^{-1}\mathbb{Z}^g/\mathbb{Z}^g$. Then for all $\xi \in \mathbb{Z}^g/2\mathbb{Z}^g$, one has*

$$\sum_{\ell \in G_{\mathbf{d}}} \theta_{[\xi]}(\ell,\tau) = |G_{\mathbf{d}}|\ \theta_{[\xi]}(0, {}^t\mathbf{d}\tau\mathbf{d}).$$

Proof. Just write the left-hand side LHS as a double sum over m in $\mathbb{Z}^g$ and ℓ in $G_{\mathbf{d}}$ and notice that $\sum_{\ell \in G_{\mathbf{d}}} e^{2i\pi^t m\ell}$ is equal to the order $|G_{\mathbf{d}}|$ of the group $G_{\mathbf{d}}$ when m belongs to ${}^t\mathbf{d}\,\mathbb{Z}^g$ and is equal to 0 otherwise. Hence

$$\begin{aligned} LHS &= |G_{\mathbf{d}}| \sum_{m \in {}^t\mathbf{d}\mathbb{Z}^g \cap \xi} e^{i\pi^t m \frac{\tau}{2} m} \\ &= |G_{\mathbf{d}}|\ \theta_{[\xi]}(0, {}^t\mathbf{d}\tau\mathbf{d}). \end{aligned}$$

In the last equality we used $\mathbf{d} \equiv \mathbf{1} \bmod 2$ by writing $m = {}^t\mathbf{d}p$ with $p \in \xi$. □

2.4. The theta cocycle. The last formula is a transformation formula for the theta functions with characteristic. It deals with an element $\sigma = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \mathrm{Sp}(g,\mathbb{Z})$. This formula is particularly simple when σ belongs to the theta group and when it is expressed with the *modified* theta function

$$\widetilde{\theta}\begin{bmatrix} a \\ b \end{bmatrix}(z,\tau) = e^{-i\pi^t a(z+b)}\ \theta\begin{bmatrix} a \\ b \end{bmatrix}(z,\tau). \tag{2.5}$$

Note that there is no *modification* when $z = b = 0$.

Lemma 2.6. *Let $\tau \in \mathcal{H}_g$ and $\sigma \in \mathrm{Sp}^{\theta}_{g,\mathbb{Z}}$. Then, for a, b in $\mathbb{C}^g$, one has*

$$\widetilde{\theta}\begin{bmatrix} \delta a - \gamma b \\ -\beta a + \alpha b \end{bmatrix}(0,\sigma\tau) = j(\sigma,\tau)\ \widetilde{\theta}\begin{bmatrix} a \\ b \end{bmatrix}(0,\tau), \quad \textit{where} \tag{2.6}$$

$$j(\sigma,\tau) = \kappa(\sigma)\ \det{}_{\mathbb{C}}(\gamma\tau+\delta)^{\frac{1}{2}} \tag{2.7}$$

In this formula, $j(\sigma,\tau)$ is a cocycle on $\mathrm{Sp}^{\theta}_{g,\mathbb{Z}} \times \mathcal{H}_g$ called the *theta cocycle* which is analytic in τ: one has $j(\sigma_1\sigma_2,\tau) = j(\sigma_1,\sigma_2\tau)\, j(\sigma_2,\tau)$. The constant $\kappa(\sigma)$ is a eighth root of unity, $\kappa(\sigma)^8 = 1$, that depends only on σ for a

continuous choice of the square root $\det_{\mathbb{C}}(\gamma\tau+\delta)^{\frac{1}{2}}$ of the complex number $\det_{\mathbb{C}}(\gamma\tau+\delta)$. The precise value of $j(\sigma,\tau)$ will be explained in Section 8.2.

Proof. This is [11, Theorem 5.7] or [6, Section 8.6 p. 231]. The textbooks [10], [16], [20], or [22] also discuss this transformation formula. We recall the strategy of proof. One proves a more involved transformation formula, [11, Proposition 5.6, 5.7], for $\theta[\begin{smallmatrix}a\\b\end{smallmatrix}]$ valid for all σ in $\mathrm{Sp}(g,\mathbb{Z})$, by checking it on generators of $\mathrm{Sp}(g,\mathbb{Z})$. The first generators are translations by an integral symmetric matrix β,

$$\theta\begin{bmatrix}a\\-\beta a+b+\beta_0/2\end{bmatrix}(0,\tau+\beta)=e^{i\pi^t a(-\beta a+\beta_0)}\,\theta\begin{bmatrix}a\\b\end{bmatrix}(0,\tau), \tag{2.8}$$

where β_0 is the diagonal of β seen as an element of $\mathbb{Z}^g$.

The formula for the second generator is the Poisson formula,

$$\theta\begin{bmatrix}-b\\a\end{bmatrix}(0,-\tau^{-1})=\det_{\mathbb{C}}(-i\tau)^{\frac{1}{2}}\;e^{-2i\pi^t ab}\,\theta\begin{bmatrix}a\\b\end{bmatrix}(0,\tau), \tag{2.9}$$

where the square root is defined by holomorphic continuation in τ with the constraint that when $\tau=i\mathbb{1}$ it is equal to 1. One uses then the fact that the map $(\sigma,\tau)\mapsto\det_{\mathbb{C}}(\sigma\tau+\delta)$ is a cocycle on $\mathrm{Sp}(g,\mathbb{Z})\times\mathcal{H}_g$. □

2.5. The condition on theta constant. The first step in the proof of Theorem 2.3 is the following criterion on $\lambda,\tau,\mathbf{d}$ which ensures that the function θ_τ is $(\lambda,\mathbf{d})$-critical. This criterion is a relation between "theta constants", i.e. theta functions evaluated at $z=0$.

Lemma 2.7. *Let $\tau\in\mathcal{H}_g$, $\lambda\in\mathbb{C}$ and $\mathbf{d}\in\mathcal{M}(g,\mathbb{Z})$ with $\mathbf{d}\equiv\mathbf{1}\bmod 2$. The function θ_τ is $(\lambda,\mathbf{d})$-critical if and only if for all $\xi\in\mathbb{Z}^g/2\mathbb{Z}^g$ one has*

$$|G_{\mathbf{d}}|\;\theta_{[\xi]}(0,{}^t\mathbf{d}\tau\mathbf{d})\;=\;\lambda\,\theta_{[\xi]}(0,\tau).$$

Proof. For w in $\mathbb{C}^g$ we introduce the function on $\mathbb{C}^g$

$$z\longmapsto F_w(z)=F_w(z,\tau):=\theta(z+w,\tau)\,\theta(z-w,\tau).$$

We want to know when the two functions $\sum_{\ell\in G_{\mathbf{d}}}F_\ell$ and $F_0=\theta^2$ are proportional. The key point in the proof is that all these functions F_w live in a finite dimensional vector space with a convenient basis: $(\theta_{[\xi]})_{\xi\in\mathbb{Z}^g/2\mathbb{Z}^g}$. For the independence of this family, see [6, Proposition 7.3 in Chapter 8]. We only have to express that the coefficients of our two functions in this basis are proportional. These coefficients are given by the following calculation in

which we apply successively the addition formula and the isogeny formula,

$$\sum_{\ell\in G_{\mathbf{d}}} F_\ell(z,\tau) = \sum_{\ell\in G_{\mathbf{d}}} \sum_{\xi\in\mathbb{Z}^g/2\mathbb{Z}^g} \theta_{[\xi]}(\ell,\tau)\ \theta_{[\xi]}(z,\tau)$$
$$= |G_{\mathbf{d}}| \sum_{\xi\in\mathbb{Z}^g/2\mathbb{Z}^g} \theta_{[\xi]}(0,{}^t\mathbf{d}\tau\mathbf{d})\ \theta_{[\xi]}(z,\tau)$$
$$\text{and}\quad \theta(z,\tau)^2 = \sum_{\xi\in\mathbb{Z}^g/2\mathbb{Z}^g} \theta_{[\xi]}(0,\tau)\ \theta_{[\xi]}(z,\tau).$$

These two functions are proportional with proportionality factor λ if and only if one has,

$$\lambda = |G_{\mathbf{d}}|\ \frac{\theta_{[\xi]}(0,{}^t\mathbf{d}\tau\mathbf{d})}{\theta_{[\xi]}(0,\tau)},\quad \text{for all } \xi \text{ in } \mathbb{Z}^g/2\mathbb{Z}^g. \tag{2.10}$$

This is the criterion we were looking for. □

Remark 2.8. It might happen that the denominator $\theta_{[\xi]}(0,\tau)$ in (2.10) is zero for some ξ. In this case, this equality means that $\theta_{[\xi]}(0,{}^t\mathbf{d}\tau\mathbf{d})$ has to be zero too. Note that, for every τ in $\mathcal{H}_g$ and z in $\mathbb{C}^g$, there exists ξ in $\mathbb{Z}^g/2\mathbb{Z}^g$ such that $\theta_{[\xi]}(z,\tau)\neq 0$, see for instance [1, Sections 3.4–3.8].

2.6. The moduli variety of theta structures. In order to exploit the criterion (2.10), the following corollary of Lemma 2.6 will be very useful. It is due to Igusa (see [6, Lemma 9.2 p. 239]).

Corollary 2.9. *When $\sigma\in \mathrm{Sp}^{\theta,2}_{g,\mathbb{Z}}$ and $\tau\in\mathcal{H}_g$, for all $\xi,\eta\in\mathbb{Z}^g/2\mathbb{Z}^g$, one has*

$$\theta_{[\eta]}(0,\tau)\ \theta_{[\xi]}(0,\sigma\tau) = \theta_{[\xi]}(0,\tau)\ \theta_{[\eta]}(0,\sigma\tau)\,. \tag{2.11}$$

By Remark 2.8, both vectors $(\theta_{[\xi]}(0,\tau)_{\xi\in\mathbb{Z}^g/2\mathbb{Z}^g}$ and $(\theta_{[\xi]}(0,\sigma\tau)_{\xi\in\mathbb{Z}^g/2\mathbb{Z}^g}$ are non zero. Hence they both define a point in the projective space $\mathbb{P}(\mathbb{C}^{2^g})$ and Equality (2.11) should be interpreted as an equality between these two points.

Proof of Corollary 2.9. It will be useful to recall the proof of this corollary. Introduce $\sigma' := \left(\begin{smallmatrix}\alpha & 2\beta\\ \gamma/2 & \delta\end{smallmatrix}\right)$ so that $\sigma'(2\tau)=2\sigma\tau$. Since the matrix σ is in $\mathrm{Sp}^{\theta,2}_{g,\mathbb{Z}}$, the matrix σ' is in $\mathrm{Sp}^{\theta}_{g,\mathbb{Z}}$. We claim that, for all $\xi\in\mathbb{Z}^g/2\mathbb{Z}^g$,

$$\theta_{[\xi]}(0,\sigma\tau) = j(\sigma',2\tau)\ \theta_{[\xi]}(0,\tau). \tag{2.12}$$

Indeed, we compute remembering that, by assumption, the matrices $(\delta-1)/2$, $\beta/2$, the vector ξ and the scalar ${}^t\xi{}^t\delta\beta\xi/4$ are all integral,

$$\begin{aligned}
\theta_{[\xi]}(0,\sigma\tau) &= \theta\begin{bmatrix}\xi/2\\ 0\end{bmatrix}(0,\sigma'(2\tau)) && \text{by Definition (2.3),}\\
&= \theta\begin{bmatrix}\delta\xi/2\\ -\beta\xi\end{bmatrix}(0,\sigma'(2\tau)) && \text{by Property (2.2),}\\
&= \widetilde{\theta}\begin{bmatrix}\delta\xi/2\\ -\beta\xi\end{bmatrix}(0,\sigma'(2\tau)) && \text{by Definition (2.5).}
\end{aligned}$$

We now apply the transformation formula in Lemma 2.6 to the pair $(\sigma', 2\tau)$,

$$\begin{aligned}
\theta_{[\xi]}(0,\sigma\tau) &= j(\sigma',2\tau)\ \widetilde{\theta}\begin{bmatrix}\xi/2\\ 0\end{bmatrix}(0,2\tau)\\
&= j(\sigma',2\tau)\ \theta\begin{bmatrix}\xi/2\\ 0\end{bmatrix}(0,2\tau)\\
&= j(\sigma',2\tau)\ \theta_{[\xi]}(0,\tau).
\end{aligned}$$

This proves that the ratio $\frac{\theta_{[\xi]}(0,\sigma\tau)}{\theta_{[\xi]}(0,\tau)}$ does not depend on ξ as required. □

Corollary 2.9 tells us that the map Φ_2 given in homogeneous coordinates by

$$\begin{aligned}
\Phi_2 :\ \mathcal{H}_g &\longrightarrow \mathbb{P}(\mathbb{C}^{2^g})\\
\tau &\longmapsto [\dots, \theta_{[\xi]}(0,\tau), \dots]
\end{aligned}$$

induces a well defined holomorphic map

$$\varphi_2 : X_g^{\theta,2} \longrightarrow \mathbb{P}(\mathbb{C}^{2^g}).$$

Remark 2.10. These maps φ_2 and their analogs $\varphi_\ell : X_g^{\theta,\ell} \to \mathbb{P}(\mathbb{C}^{\ell^g})$, for $\ell \geq 2$, have a long history, as the quasi-projective realizations of the moduli varieties of theta structures of level ℓ. We will not need here the precise definition of these structures. But here are some comments that relate our computation to the existing literature.

For $\ell \geq 3$, according to successive works of Igusa, Mumford and Salvati Manni, these maps φ_ℓ are proven to be embeddings, see [15], [19], and [6, Section 8.10].

For $\ell = 2$, the situation is more delicate and has been studied in detail by Salvati Manni:

- He proves that the map φ_2 is generically injective, see [24, Proposition 1].
- He conjectures that the map φ_2 is injective, see [24, Theorem 3] where a tentative proof is given. See also [13, Theorem 3.6] and [21, Section 2] for more comments on this question.

- For $g \leq 3$, the map φ_2 is indeed injective, see [23].
- For $g \geq 4$, the map φ_2 is not a biholomorphism.

Proof of Theorem 2.3.

(a). Our assumptions and Corollary 2.9 tell us that

$$\Phi_2({}^t\mathbf{d}\tau\mathbf{d}) = \Phi_2(\tau).$$

This equality is nothing but the criterion of Lemma 2.7. Therefore the function θ_τ is $(\lambda, \mathbf{d})$-critical for a critical value λ.

(b). To compute the critical value λ, we use again Lemma 2.7 combined with Corollary 2.9, and more precisely with Formulas (2.12) and (2.7). We obtain, for all ξ in $\mathbb{Z}^g/2\mathbb{Z}^g$,

$$\begin{aligned} \lambda &= |G_{\mathbf{d}}| \, \frac{\theta_{[\xi]}(0, {}^t\mathbf{d}\tau\mathbf{d})}{\theta_{[\xi]}(0,\tau)} = |G_{\mathbf{d}}| \, \frac{\theta_{[\xi]}(0,\sigma\tau)}{\theta_{[\xi]}(0,\tau)}, \\ \lambda &= j(\sigma', 2\tau)\,|G_{\mathbf{d}}| \;\; = \;\; \kappa(\sigma') \;\det{}_{\mathbb{C}}(\gamma\tau+\delta)^{1/2}\,|G_{\mathbf{d}}|\,, \end{aligned} \tag{2.13}$$

where the matrix $\sigma' := \left(\begin{smallmatrix} \alpha & 2\beta \\ \gamma/2 & \delta \end{smallmatrix}\right)$ belongs to $\mathrm{Sp}^{\theta}_{g,\mathbb{Z}}$ and $\kappa(\sigma')^8 = 1$. □

Remark 2.11. One has $|\lambda| = |G_{\mathbf{d}}|^{1/2}$ and Equation (2.13) can be written as

$$\lambda = \kappa(\sigma') \det{}_{\mathbb{C}}(\gamma\overline{\tau} + \delta)^{-1/2}. \tag{2.14}$$

Indeed, let Λ_τ be the lattice $\Lambda_\tau := \tau\mathbb{Z}^g \oplus \mathbb{Z}^g$ of $\mathbb{C}^g$. Since $\sigma\tau = {}^t\mathbf{d}\tau\mathbf{d}$, the complex matrix $M := {}^t(\gamma\tau + \delta)^{-1} \in \mathrm{GL}(\mathbb{C}^g)$ satisfies the equality between $g \times 2g$ complex matrices:

$$M\,(\tau \;\; \mathbf{1}_g)\,{}^t\sigma = ({}^t\mathbf{d}\tau\mathbf{d} \;\; \mathbf{1}_g).$$

This equality implies that

$$M(\Lambda_\tau) = \Lambda_{{}^t\mathbf{d}\tau\mathbf{d}}.$$

Comparing the covolume of these lattices, one gets the equality

$$|\det{}_{\mathbb{C}}(M)| = |\det(\mathbf{d})| = |G_{\mathbf{d}}|,$$

which implies (2.14).

3. The symplectic group

In this section we first give in Corollary 3.1 a reformulation of Theorem 2.3 with a more precise formula for the critical value. We then explain various tools for studying the symplectic group like the symplectic adapted basis theorem and the Cayley transform that will be useful in the following sections.

3.1. Critical values and symplectic transformation. We recall that a matrix $h \in \mathrm{Sp}(g,\mathbb{R})$ is said to be *elliptic* if it is diagonalizable over $\mathbb{C}$ with all eigenvalues of modulus 1. It is equivalent to say that h belongs to a compact subgroup of $\mathrm{Sp}(g,\mathbb{R})$ or that h has a fixed point in the Riemannian symmetric space $\mathcal{H}_g$.

Corollary 3.1. *Let $\tau \in \mathcal{H}_g$, let $\mathbf{d} \in \mathcal{M}(g,\mathbb{Z})$ with $\mathbf{d} \equiv \mathbf{1} \bmod 2$, let $\sigma = \left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right) \in \mathrm{Sp}^{\theta,2}_{g,\mathbb{Z}}$ such that the sympletic matrix $h := \left(\begin{smallmatrix}{}^t\mathbf{d}^{-1}\alpha & {}^t\mathbf{d}^{-1}\beta\\ \mathbf{d}\gamma & \mathbf{d}\delta\end{smallmatrix}\right)$ is elliptic, let τ be a fixed point of h in $\mathcal{H}_g$ and let $\sigma' := \left(\begin{smallmatrix}\alpha & 2\beta\\ \gamma/2 & \delta\end{smallmatrix}\right)$. Then*

$$\lambda := j(\sigma', 2\tau)\,|\det(\mathbf{d})| \tag{3.1}$$

is a critical value on the group $\mathbf{d}^{-1}\mathbb{Z}^g/\mathbb{Z}^g$.

Proof of Corollary 3.1. This is a direct corollary of Theorem 2.3 with formula (2.13). Indeed the condition $h\tau = \tau$ is equivalent to $\sigma\tau = {}^t\mathbf{d}\tau\mathbf{d}$. A λ-critical function can be chosen to be the restriction of the $\mathbb{Z}^g$-periodic function θ_τ to the finite group $G_{\mathbf{d}} = \mathbf{d}^{-1}\mathbb{Z}^g/\mathbb{Z}^g$. □

3.2. Theta subgroup of level ℓ. In order to use efficiently Theorem 2.3, we recall equivalent definitions for the theta subgroup $\mathrm{Sp}^{\theta,2}_{g,\mathbb{Z}}$ (see [9, p. 177–182] for more on this topic).

Fact 3.2.

(a) *For $\ell \geq 1$, the group $\mathrm{Sp}^{\ell}_{g,\mathbb{Z}}$ is a normal subgroup of $\mathrm{Sp}(g,\mathbb{Z})$.*

(b) *When ℓ is even, one has the equalities*

$$\begin{aligned}\mathrm{Sp}^{\theta,\ell}_{g,\mathbb{Z}} &= \left\{\left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right) \in \mathrm{Sp}^{\ell}_{g,\mathbb{Z}} \,\middle|\, \beta_0 \equiv \gamma_0 \equiv 0 \bmod 2\ell\right\},\\ &= \left\{\sigma \in \mathrm{Sp}^{\ell}_{g,\mathbb{Z}} \,\middle|\, \omega(\sigma x, x) \equiv 0 \bmod 2\ell \text{ for all } x \text{ in } \mathbb{Z}^{2g}\right\}\end{aligned}$$

(c) *When ℓ is even, the group $\mathrm{Sp}^{\theta,\ell}_{g,\mathbb{Z}}$ is also a normal subgroup of $\mathrm{Sp}(g,\mathbb{Z})$.*

Proof. This is classical, see [15, Lemma 4 p. 177]. Here are some details.

(a). $\mathrm{Sp}^{\ell}_{g,\mathbb{Z}}$ is the kernel of the projection $\pi_\ell : \mathrm{Sp}(g,\mathbb{Z}) \to \mathrm{Sp}(g,\mathbb{Z}/\ell\mathbb{Z})$.

(b). It is enough to check that these three sets have the same image in the quotient $\mathrm{Sp}(g,\mathbb{Z}/2\ell\mathbb{Z})$. Since $\pi_{2\ell}$ is onto, and ℓ is even, the group

$$\pi_{2\ell}(\mathrm{Sp}^{\ell}_{g,\mathbb{Z}}) \simeq \left\{\sigma = \left(\begin{smallmatrix}\mathbb{1}+\ell a & \ell b\\ \ell c & \mathbb{1}+\ell d\end{smallmatrix}\right) \,\middle|\, b \equiv {}^t b\ ,\ c \equiv {}^t c \ \text{ and } \ d \equiv {}^t a \bmod 2\right\}$$

is abelian and is an $\mathbb{F}_2$-vector space $V_\ell \simeq \mathbb{F}_2^{(2g+1)g}$. The image in V_ℓ of each of these three sets is the $\mathbb{F}_2$-vector subspace V_ℓ^θ of codimension $2g$ in V_ℓ given by the equation $b_0 \equiv c_0 \equiv 0 \bmod 2$, where, as before, b_0 and c_0 are the diagonals of the symmetric matrices b and c.

(c). The quotient $\mathrm{Sp}(g,\mathbb{Z})/\,\mathrm{Sp}^2_{g,\mathbb{Z}}$ is the symplectic group $\mathrm{Sp}(g,\mathbb{F}_2)$ over the finite field $\mathbb{F}_2$, i.e. the stabilizer of the non-degenerate symmetric bilinear form ω seen on $\mathbb{F}_2^{2g}$. One easily checks that the action by conjugation of $\mathrm{Sp}(g,\mathbb{Z})$ on the quotient $V_\ell \simeq \mathrm{Sp}^\ell_{g,\mathbb{Z}}\,/\,\mathrm{Sp}^{2\ell}_{g,\mathbb{Z}}$ preserves the vector subspace V_ℓ^θ. Hence the group $\mathrm{Sp}^{\theta,\ell}_{g,\mathbb{Z}}$ is normal in $\mathrm{Sp}(g,\mathbb{Z})$. □

Remark 3.3. This also proves that the quotient $\widetilde{\mathrm{Sp}}(g,\mathbb{F}_2) := \mathrm{Sp}(g,\mathbb{Z})/\,\mathrm{Sp}^{\theta,2}_{g,\mathbb{Z}}$ is an extension of $\mathrm{Sp}(g,\mathbb{F}_2)$ by $\mathbb{F}_2^{2g}$:

$$1 \longrightarrow \mathbb{F}_2^{2g} \longrightarrow \widetilde{\mathrm{Sp}}(g,\mathbb{F}_2) \longrightarrow \mathrm{Sp}(g,\mathbb{F}_2) \longrightarrow 1. \tag{3.2}$$

3.3. The symplectic adapted basis. In this section we discuss the structure of the *rational symplectic group* $\mathrm{Sp}(g,\mathbb{Q}) := \mathrm{GL}(2g,\mathbb{Q}) \cap \mathrm{Sp}(g,\mathbb{R})$, and its relation with the *integral symplectic group* $\mathrm{Sp}(g,\mathbb{Z})$. We also introduce the rational symplectic theta group $\mathrm{Sp}^{\theta,2}_{g,\mathbb{Q}}$ of level 2.

The following proposition is a variation of the classical "adapted basis theorem" which takes into account the existence of a symplectic form.

Proposition 3.4. *Let $h \in \mathrm{Sp}(g,\mathbb{Q})$. Then there exists σ_1 and σ_2 in $\mathrm{Sp}(g,\mathbb{Z})$ and a diagonal matrix $\mathbf{d} = \mathrm{diag}(d_1,\dots,d_g)$ with $d_1|d_2|\dots|d_g$ integral and*

$$h = \sigma_1 \begin{pmatrix} {}^t\mathbf{d}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{d} \end{pmatrix} \sigma_2.$$

A proof of this proposition is given in [5].

For $\ell \geq 1$, let $\mathbb{Z}_{(\ell)}$ be the ring of rational numbers with denominator prime to ℓ. We introduce the *rational congruence symplectic group of level ℓ*

$$\mathrm{Sp}^\ell_{g,\mathbb{Q}} := \Big\{ h \in \mathrm{Sp}(g,\mathbb{Z}_{(\ell)}) \,\Big|\, \sigma \equiv \mathbb{1}_{2g} \bmod \ell \Big\},$$

and the *rational symplectic theta group of level* 2

$$\mathrm{Sp}^{\theta,2}_{g,\mathbb{Q}} := \Big\{ h = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \mathrm{Sp}^2_{g,\mathbb{Q}} \,\Big|\, ({}^t\alpha\gamma)_0 \equiv ({}^t\beta\delta)_0 \equiv 0 \bmod 4 \Big\}.$$

We will say that $h \in \mathrm{Sp}(g,\mathbb{Q})$ *preserves a theta structure of level* 2 if it belongs to $\mathrm{Sp}^{\theta,2}_{g,\mathbb{Q}}$. As in the integral case, the group $\mathrm{Sp}^{\theta,2}_{g,\mathbb{Q}}$ is a normal subgroup of the group $\mathrm{Sp}(g,\mathbb{Z}_{(2)})$, and one has the inclusions

$$\mathrm{Sp}^4_{g,\mathbb{Q}} \subset \mathrm{Sp}^{\theta,2}_{g,\mathbb{Q}} \subset \mathrm{Sp}^2_{g,\mathbb{Q}} \subset \mathrm{Sp}(g,\mathbb{Z}_{(2)}).$$

Indeed the reduction modulo 4 of the group $\mathrm{Sp}^{\theta,2}_{g,\mathbb{Q}}$ is the group $\widetilde{\mathrm{Sp}}(g,\mathbb{F}_2)$ which is a normal subgroup of the group $\mathrm{Sp}(g,\mathbb{Z}/4\mathbb{Z}) \simeq \mathrm{Sp}(g,\mathbb{Z}_{(2)})/\,\mathrm{Sp}^4_{g,\mathbb{Q}}$.

Lemma 3.5. *Let $h \in \mathrm{Sp}(g,\mathbb{Z}_{(2)})$ and write $h = \sigma_1 \left(\begin{smallmatrix} {}^t\mathbf{d}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{d} \end{smallmatrix}\right) \sigma_2$ with $\mathbf{d}$ in $\mathcal{M}(g,\mathbb{Z})$ and both σ_1 and σ_2 in $\mathrm{Sp}(g,\mathbb{Z})$. Then the following are equivalent:*

- *h preserves a theta structure of level 2, i.e. $h \in \mathrm{Sp}^{\theta,2}_{g,\mathbb{Q}}$.*
- *$\det(\mathbf{d})$ is odd and $\sigma_1\sigma_2 \in \mathrm{Sp}^{\theta,2}_{g,\mathbb{Z}}$.*

- $\det(\mathbf{d})$ *is odd and* $\sigma_2\sigma_1 \in \mathrm{Sp}_{g,\mathbb{Z}}^{\theta,2}$.

In most cases of Sections 8.4, 8.5 and 8.6, but not always, we will choose $\mathbf{d}$ to be diagonal, and both σ_1, σ_2 to be in $\mathrm{Sp}_{g,\mathbb{Q}}^{\theta,2}$ to ensure $\mathbf{d} \equiv 1 \bmod 2$.

Proof of Lemma 3.5. We first note the equivalence

$$h \in \mathrm{Sp}_{g,\mathbb{Q}}^{2} \Longleftrightarrow \det(\mathbf{d}) \text{ is odd and } \sigma_2\sigma_1 \in \mathrm{Sp}_{g,\mathbb{Q}}^{2}\,.$$

One conclude by noticing that $\mathrm{Sp}_{g,\mathbb{Q}}^{\theta,2}$ is a normal subgroup of $\mathrm{Sp}(g,\mathbb{Z}_{(2)})$. □

3.4. The Cayley transform. The Cayley transform is a convenient tool that allows us to construct symplectic transformations σ with rational coefficients (Lemma 3.6), and to recognize those that preserve a theta structure of level 2 (Lemma 3.8).

Let $\mathfrak{sp}(g,\mathbb{Q})$ be the "Lie algebra" of $\mathrm{Sp}(g,\mathbb{Q})$, that is,

$$\begin{aligned}\mathfrak{sp}(g,\mathbb{Q}) &= \Big\{X \in \mathcal{M}(2g,\mathbb{Q}) \,\Big|\, \omega(Xx,y)+\omega(x,Xy)=0 \text{ for all } x,y \text{ in } \mathbb{Q}^{2g}\Big\}.\\ &= \Big\{X = \left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right) \in \mathcal{M}(2g,\mathbb{Q}) \,\Big|\, b = {}^t b\,,\ c = {}^t c\,,\ d = -{}^t a\Big\},\\ &= \{X = \left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right) \in \mathcal{M}(2g,\mathbb{Q}) \,|\, X = -X^*\},\end{aligned}$$

where

$$X^* = J\,{}^t X J^{-1} = \left(\begin{smallmatrix} {}^t d & -{}^t b\\ -{}^t c & {}^t a\end{smallmatrix}\right).$$

The Cayley transform will give a bijection between the following two Zariski open subsets

$$\begin{aligned}\mathfrak{sp}_{g,\mathbb{Q}}^{\bullet} &:= \{X \in \mathfrak{sp}(g,\mathbb{Q}) \mid \det(\mathbb{1}-X) \neq 0\}\\ \text{and}\quad \mathrm{Sp}_{g,\mathbb{Q}}^{\bullet} &:= \{h \in \mathrm{Sp}(g,\mathbb{Q}) \mid \det(\mathbb{1}+h) \neq 0\}.\end{aligned}$$

Here is the construction of the Cayley transform.

Lemma 3.6. *The map* $X \mapsto C(X) := (\mathbb{1}+X)(\mathbb{1}-X)^{-1}$ *is a bijection from* $\mathfrak{sp}_{g,\mathbb{Q}}^{\bullet}$ *to* $\mathrm{Sp}_{g,\mathbb{Q}}^{\bullet}$ *with inverse map* $h \mapsto X := -(\mathbb{1}-h)(\mathbb{1}+h)^{-1}$.

Proof. These formulas define mutually inverse maps between the sets

$$\{X \in \mathcal{M}(g,\mathbb{Q}) \,|\, \det(\mathbb{1}-X) \neq 0\} \quad \text{and} \quad \{h \in \mathcal{M}(g,\mathbb{Q}) \,|\, \det(\mathbb{1}+h) \neq 0\}.$$

We only have to show that X is in $\mathfrak{sp}(g,\mathbb{Q})$ if and only if $h := C(X)$ is in $\mathrm{Sp}(g,\mathbb{Q})$. This follows from the equivalences:

$$\begin{aligned}X \in \mathfrak{sp}(g,\mathbb{Q}) &\Longleftrightarrow \omega(Xx,y)+\omega(x,Xy)=0 \text{ for all } x,y \text{ in } \mathbb{Q}^{2g}\\ &\Longleftrightarrow \omega((\mathbb{1}+X)x,(\mathbb{1}+X)y) = \omega((\mathbb{1}-X)x,(\mathbb{1}-X)y)\\ &\qquad\qquad \text{for all } x,y \text{ in } \mathbb{Q}^{2g}\\ &\Longleftrightarrow \omega(hx',hy') = \omega(x',y') \text{ for all } x',y' \text{ in } \mathbb{Q}^{2g}\\ &\Longleftrightarrow h \in \mathrm{Sp}(g,\mathbb{Q}).\end{aligned}$$

□

Let $\mathfrak{sp}^2_{g,\mathbb{Q}} = \mathfrak{sp}(g, \mathbb{Z}_{(2)})$ and

$$\mathfrak{sp}^{\theta,2}_{g,\mathbb{Q}} = \Big\{ X = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathfrak{sp}^2_{g,\mathbb{Q}} \,\Big|\, b_0 \equiv c_0 \equiv 0 \bmod 2 \Big\},$$
$$= \Big\{ X \in \mathfrak{sp}^2_{g,\mathbb{Q}} \,\Big|\, \omega(Xx, x) \equiv 0 \bmod 2 \text{ for all } x \text{ in } \mathbb{Z}^{2g}_{(2)} \Big\}.$$

where again b_0 and c_0 are the diagonals of the symmetric matrices b and c.

The equivalence between these two definitions follows from the fact that, for X in $\mathfrak{sp}^2_{g,\mathbb{Q}}$, the $\mathbb{Z}_{(2)}$-valued bilinear form $\omega(Xx, y)$ on $\mathbb{Z}^{2g}_{(2)}$ is symmetric.

By construction one has the inclusions

$$2\mathfrak{sp}^2_{g,\mathbb{Q}} \subset \mathfrak{sp}^{\theta,2}_{g,\mathbb{Q}} \subset \mathfrak{sp}^2_{g,\mathbb{Q}}.$$

Here is a useful interpretation of this intermediate $\mathbb{Z}_{(2)}$-module $\mathfrak{sp}^{\theta,2}_{g,\mathbb{Q}}$ using the involution $X \mapsto X^*$.

Lemma 3.7. *One has* $\mathfrak{sp}^{\theta,2}_{g,\mathbb{Q}} = \{X = Y - Y^* \mid Y \in \mathcal{M}(2g, \mathbb{Z}_{(2)})\}$

Proof. It is possible to write an element $X = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathfrak{sp}^2_{g,\mathbb{Q}}$ as a sum $X = \left(\begin{smallmatrix} \alpha - {}^t\delta & \beta + {}^t\beta \\ \gamma + {}^t\gamma & \delta - {}^t\alpha \end{smallmatrix}\right)$ with α, β, γ, δ in $\mathcal{M}(g, \mathbb{Z}_{[2]})$ if and only if the diagonals of b and c are even. □

The Cayley transform will induce a bijection between the two subsets

$$\mathfrak{sp}^{\bullet,2}_{g,\mathbb{Q}} := \Big\{ X \in \mathfrak{sp}^2_{g,\mathbb{Q}} \,\Big|\, \det(\mathbb{1} - X) \in \mathbb{Z}^*_{(2)} \Big\},$$
$$\mathrm{Sp}^{\bullet,2}_{g,\mathbb{Q}} := \Big\{ h \in \mathrm{Sp}^2_{g,\mathbb{Q}} \,\Big|\, \det(\tfrac{1}{2}(\mathbb{1} + h)) \in \mathbb{Z}^*_{(2)} \Big\}.$$

For a matrix $M \in \mathcal{M}(2g, \mathbb{Z}_{(2)})$, i.e. a matrix with odd denominator, the condition $\det(M) \in \mathbb{Z}^*_{(2)}$, means that the determinant is invertible in the ring $\mathbb{Z}_{(2)}$ and hence that the inverse M^{-1} exists and belongs to $\mathcal{M}(2g, \mathbb{Z}_{(2)})$, i.e. it also has odd denominator.

Lemma 3.8.

(a) *The Cayley transform* $X \mapsto C(X) = (\mathbb{1} + X)(\mathbb{1} - X)^{-1}$ *induces a bijection from* $\mathfrak{sp}^{\bullet,2}_{g,\mathbb{Q}}$ *onto* $\mathrm{Sp}^{\bullet,2}_{g,\mathbb{Q}}$.

(b) *For* $X \in \mathfrak{sp}^{\bullet,2}_{g,\mathbb{Q}}$ *and* $h = C(X)$, *one has the equivalence:*

$$X \in \mathfrak{sp}^{\theta,2}_{g,\mathbb{Q}} \iff h \in \mathrm{Sp}^{\theta,2}_{g,\mathbb{Q}} \ .$$

Proof.

(a). Recall that the equality $h = C(X)$ means $\frac{1}{2}(\mathbb{1} + h)(\mathbb{1} - X) = \mathbb{1}$. For such a pair (X, h) in $\mathcal{M}(2g, \mathbb{Q})$, one has the equivalence

$$\det(\mathbb{1} - X) \in \mathbb{Z}^*_{(2)} \iff \det(\frac{1}{2}(\mathbb{1} + h)) \in \mathbb{Z}^*_{(2)}$$

and, in that case,

$$X \in \mathcal{M}(2g, \mathbb{Z}_{(2)}) \Longleftrightarrow \frac{1}{2}(\mathbb{1} + h) \in \mathcal{M}(2g, \mathbb{Z}_{(2)}).$$

Hence one has the equivalence:

$$X \in \mathfrak{sp}^{\bullet,2}_{g,\mathbb{Q}} \Longleftrightarrow h \in \mathrm{Sp}^{\bullet,2}_{g,\mathbb{Q}}\,.$$

(b). In that case, one has the equivalences:

$$\begin{aligned} X \in \mathfrak{sp}^{\theta,2}_{g,\mathbb{Q}} &\Longleftrightarrow \omega(Xx, x) \equiv 0 \bmod 2 && \text{for all } x \text{ in } \mathbb{Z}^{2g}_{(2)} \\ &\Longleftrightarrow \omega((\mathbb{1} + X)x, (\mathbb{1} - X)x) \equiv 0 \bmod 4 && \text{for all } x \text{ in } \mathbb{Z}^{2g}_{(2)} \\ &\Longleftrightarrow \omega(hx', x') \equiv 0 \bmod 4 && \text{for all } x' \text{ in } \mathbb{Z}^{2g}_{(2)} \\ &\Longleftrightarrow h \in \mathrm{Sp}^{\theta,2}_{g,\mathbb{Q}}\,. \end{aligned}$$

□

4. Abelian varieties

The aim of this section is to interpret our general construction of critical values from the point of view of abelian varieties (Theorem 4.4).

4.1. Principally polarized abelian varieties. In this section and the next one we fix our choice of notation and definition.

Let $(A = V/\Lambda, \omega)$ be a *polarized abelian variety*. This means that A is a complex torus, with $V = \mathbb{C}^g$, that Λ is a lattice in V, and that $\omega : V \times V \to \mathbb{R}$ is a *real symplectic form* on V satisfying the following two conditions:

(i) the symplectic form ω takes integral values on $\Lambda \times \Lambda$, and

(ii) ω is the imaginary part $\mathrm{Im}(H)$ of a positive hermitian form H on V.

Assumption (ii) implies that $\omega(iv_1, iv_2) = \omega(v_1, v_2)$ for all v_1, v_2 in V and the hermitian form H can be recovered as $H(v_1, v_2) = \omega(iv_1, v_2) + i\,\omega(v_1, v_2)$. Our convention is that $H(v_1, v_2)$ is linear in v_1 and antilinear in v_2.

We will always assume that the *polarization is principal*, i.e. that the restriction of ω to $\Lambda \times \Lambda$ has determinant 1. At first glance, this assumption looks harmless for us since every polarized abelian variety is isogenous to a principally polarized abelian variety. The problem is that changing Λ might change the group G_ν in Theorem 5.4 and it will be a delicate issue to choose Λ so that G_ν is cyclic.

When the polarization is principal, there exists a symplectic basis $(f_1, \ldots, f_g, e_1, \ldots, e_g)$ of the lattice Λ, i.e. a $\mathbb{Z}$-basis such that

$$\omega(e_j, e_k) = \omega(f_j, f_k) = \omega(f_j, e_k) - \delta_{jk} = 0 \quad \text{for all } j, k.$$

The family $(e_1, \ldots, e_g)$ is then a basis of $\mathbb{C}^g$. We denote by τ the $g \times g$ matrix given by $(f_1, \ldots, f_g) = (e_1, \ldots, e_g)\tau$. This matrix τ is a symmetric

complex matrix with positive definite imaginary part, that is τ belongs to $\mathcal{H}_g$. Moreover this map $(A,\omega)\to\tau$ gives a bijection

$$\left\{\begin{matrix}\text{principally polarized}\\ \text{abelian varieties}\end{matrix}\right\} \longleftrightarrow \ \mathrm{Sp}(g,\mathbb{Z})\backslash\mathcal{H}_g. \tag{4.1}$$

More precisely, for τ in $\mathcal{H}_g$, we introduce the lattice $\Lambda_\tau := \tau\mathbb{Z}^g\oplus\mathbb{Z}^g$ of $\mathbb{C}^g$, the quotient torus $A_\tau := \mathbb{C}^g/\Lambda_\tau$, the hermitian form H_τ on $\mathbb{C}^g$ whose matrix is $(\mathrm{Im}\,\tau)^{-1}$ in the canonical basis $(e_1,\dots,e_g)$ and the imaginary part ω_τ of H_τ. The pair (A_τ,ω_τ) is then a principally polarized abelian variety, and the map $\tau\mapsto(A_\tau,\omega_\tau)$ is the inverse map of (4.1).

4.2. Unitary $\mathbb{Q}$-endomorphisms. Let $(A=V/\Lambda,\omega)$ be a principally polarized abelian variety.

We denote by $\mathrm{End}(A)$ the ring of endomorphisms $\mu: A\to A$, and by $\mathrm{End}_{\mathbb{Q}}(A) := \mathrm{End}(A)\otimes_{\mathbb{Z}}\mathbb{Q}$ the $\mathbb{Q}$-algebra of $\mathbb{Q}$-endomorphisms ν of A. An *isogeny of* A is an endomorphism of A which is invertible in $\mathrm{End}_{\mathbb{Q}}(A)$, i.e. an endomorphism μ whose kernel $K_\mu\subset A$ is a finite subgroup.

To each $\mathbb{Q}$-endomorphism $\nu\in\mathrm{End}_{\mathbb{Q}}(A)$ is associated

- a *tangent map* $T_\nu\in\mathrm{End}_{\mathbb{C}}(V)\simeq\mathcal{M}(g,\mathbb{C})$,
- a *holonomy map* $h_\nu\in\mathrm{End}_{\mathbb{Q}}(\Lambda_{\mathbb{Q}})\simeq\mathcal{M}(2g,\mathbb{Q})$, where $\Lambda_{\mathbb{Q}} := \Lambda\otimes_{\mathbb{Z}}\mathbb{Q}$.

The map h_ν is the restriction of T_ν to $\Lambda_{\mathbb{Q}}$. More precisely, an endomorphism (resp. $\mathbb{Q}$-endomorphism) ν of A is nothing but a $\mathbb{C}$-endomorphism of V that preserves Λ (resp. $\Lambda_{\mathbb{Q}}$). This is why one sometimes writes abusively ν instead of T_ν or h_ν. But it is useful to keep the two notations because, in coordinates, T_ν is a $g\times g$ complex matrix while h_ν is a $2g\times 2g$ rational matrix.

The *Rosati antiinvolution* $\nu\mapsto\nu^*$ is the antiinvolution of the $\mathbb{Q}$-algebra $\mathrm{End}_{\mathbb{Q}}(A)$ defined by one of the two equivalent properties:

- T_{ν^*} is the adjoint of T_ν for the hermitian form H on V.
- h_{ν^*} is the adjoint of h_ν for the symplectic form ω on $\Lambda_{\mathbb{Q}}$.

Definition 4.1. A *similarity* of A of ratio k is an isogeny μ of A such that $\mu\mu^*=k^2\mathbb{1}$. An isogeny of A is *primitive* if it is not an integral multiple of an isogeny. A *unitary* $\mathbb{Q}$-endomorphism of A is a $\mathbb{Q}$-endomorphism ν such that $\nu\nu^*=\mathbb{1}$.

The following lemma is nothing but a useful remark.

Lemma 4.2. *Let (A,ω) be a principally polarized abelian variety, and μ be an isogeny of A. When $k\geq 1$ is an integer, the following are equivalent:*

- *the isogeny μ is a similarity of ratio k.*
- *the $\mathbb{Q}$-endomorphism $\nu := \frac{1}{k}\mu$ of A is unitary.*
- *the $\mathbb{Q}$-linear map $h_\nu=\frac{1}{k}h_\mu$ belongs to $\mathrm{Sp}(\Lambda_{\mathbb{Q}},\omega)$.*

Definition 4.3. We say that the *unitary* $\mathbb{Q}$-endomorphism ν of A *preserves a theta structure of level* 2 if the holonomy h_ν belongs to the rational symplectic theta subgroup $\mathrm{Sp}_{g,\mathbb{Q}}^{\theta,2}$ of level 2 in a symplectic basis of (Λ, ω). This condition does not depend on the choice of the symplectic basis of Λ.

4.3. Critical values and abelian varieties. We can now explain our construction of critical values from the point of view of abelian varieties. In Section 5, we will specialize this theorem to the case of abelian varieties with complex multiplication.

Theorem 4.4. *Let $(A = V/\Lambda, \omega)$ be a principally polarized abelian variety, ν be a unitary $\mathbb{Q}$-endomorphism of A preserving a theta structure of level* 2*, T_ν be its tangent map, $G_\nu := \Lambda/(\Lambda \cap \nu\Lambda)$ and $d_\nu := |G_\nu|$. Then there exists a critical value $\lambda_\nu = \kappa_\nu\, d_\nu^{1/2} \det_{\mathbb{C}}(T_\nu)^{1/2}$ on the group G_ν with $\kappa_\nu^4 = 1$.*

Remark 4.5. The group G_ν is naturally isomorphic to the torsion subgroup $G_\nu \simeq (\nu^{-1}\Lambda + \Lambda)/\Lambda \subset A$ and the order d_ν of G_ν is a divisor of k^g where $k > 0$ is chosen so that $\mu := k\nu$ is a $\mathbb{Z}$-endomorphism of A. In particular, since by assumption, one can choose k odd, the order d_ν is odd.

Note also that, since ν is unitary, the order d_ν of the group G_ν and the absolute value of the critical value are related by the equality

$$|\lambda_\nu| = d_\nu^{1/2}.$$

Note finally that, once the universal cover $V = \mathbb{C}^g$ and the tangent map $T_\nu \in \mathcal{M}(g, \mathbb{C})$ is fixed, except for a fourth root of unity, the ratio $\lambda_\nu/|\lambda_\nu|$ does not depend on the lattice $\Lambda \subset \mathbb{C}^g$ in its commensurability class. Equivalently it does not depend on A in its isogeny class. Indeed, changing Λ in its commensurability class may change d_ν and the group G_ν but it does not change the tangent map T_ν. We will see examples already in Section 6.1.

The easiest way to determine the square $\kappa_\nu^2 = \pm 1$ is to remember that by Proposition 1.1 one has $\lambda_\nu \equiv 1 \bmod 2$. We will see in Section 6.1 that both signs $\pm$ can occur.

Proof of Theorem 4.4. The key point is the interrelation between the tangent map T_ν and the holonomy h_ν, together with the use of Proposition 3.4. We fix a symplectic $\mathbb{Z}$-basis $(f_1, \dots, f_g, e_1, \dots, e_g)$ of Λ so that $\omega = \sum f_j^* \wedge e_j^*$.

$$\begin{aligned} &\text{Let } m_\nu \in \mathrm{Sp}_{g,\mathbb{Q}}^{\theta,2} \text{ be the matrix of } h_\nu^{-1} \text{ in} \\ &\text{the symplectic basis } (e_1, \dots, e_g, -f_1, \dots, -f_g) \end{aligned} \tag{4.2}$$

so that, by (2.1), one has $J m_\nu^{-1} J^{-1} = {}^t m_\nu$ and hence the equality in V^{2g}

$$(T_\nu f_1, \dots, T_\nu f_g, T_\nu e_1, \dots, T_\nu e_g) = (f_1, \dots, f_g, e_1, \dots, e_g)\, {}^t m_\nu. \tag{4.3}$$

By the adapted symplectic basis in Proposition 3.4, there exist σ_1, σ_2 in $\mathrm{Sp}(g,\mathbb{Z})$ and an integral matrix $\mathbf{d}$ with $\det(\mathbf{d})\neq 0$ such that

$$m_\nu = \sigma_1\, D\sigma_2 \quad \text{with} \quad D := \begin{pmatrix} {}^t\mathbf{d}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{d} \end{pmatrix}. \tag{4.4}$$

The matrix $\mathbf{d}$ can be chosen to be a diagonal matrix $\mathrm{diag}(d_1,\dots,d_g)$ with positive integer coefficients $d_1|d_2|\dots|d_g$, but we will not use that fact.

We introduce two new symplectic $\mathbb{Z}$ basis of Λ.

$$\begin{aligned}(F_1,\dots,F_g,E_1,\dots,E_g) &:= (f_1,\dots,f_g,e_1,\dots,e_g)\,{}^t\sigma_1^{-1},\\ (F'_1,\dots,F'_g,E'_1,\dots,E'_g) &:= (f_1,\dots,f_g,e_1,\dots,e_g)\,{}^t\sigma_2.\end{aligned} \tag{4.5}$$

Combining (4.3), (4.4) and (4.5), one gets the equalities in V^g,

$$\begin{aligned}(T_\nu F_1,\dots,T_\nu F_g) &= (F'_1,\dots,F'_g)\,\mathbf{d}^{-1}\,,\\ (T_\nu E_1,\dots,T_\nu E_g) &= (E'_1,\dots,E'_g)\,{}^t\mathbf{d}\,.\end{aligned} \tag{4.6}$$

Note that the group $G_\nu \simeq \Lambda/(\Lambda\cap T_\nu\Lambda) \simeq \mathbb{Z}^g/\mathbf{d}\mathbb{Z}^g$ has order $d_\nu = |\det_{\mathbb{C}}(\mathbf{d})|$.

Now we continue our analysis of the unitary $\mathbb{Q}$-endomorphism ν. We denote by τ, ρ and ρ' the $g\times g$ complex matrices that give the basis changes defined by the equalities in V^g,

$$\begin{aligned}(F_1,\dots,F_g) &= (E_1,\dots,E_g)\,\tau,\\ (E'_1,\dots,E'_g) &= (E_1,\dots,E_g)\,\rho,\\ (F'_1,\dots,F'_g) &= (E_1,\dots,E_g)\,\rho'.\end{aligned} \tag{4.7}$$

Remember that the matrix τ is a Riemann matrix, that is $\tau\in\mathcal{H}_g$.

Let M_ν be the $g\times g$ complex matrix that expresses T_ν in the basis $(E_1,\dots,E_g)$ of $\mathbb{C}^g$ so that one has the equalities

$$\begin{aligned}(T_\nu F_1,\dots,T_\nu F_g) &= (F_1,\dots,F_g)\,\tau^{-1}M_\nu\tau,\\ (T_\nu E_1,\dots,T_\nu E_g) &= (E_1,\dots,E_g)\,M_\nu,\end{aligned}$$

and Equalities (4.6) can be rewritten as equalities in $\mathcal{M}(g,\mathbb{C})$:

$$\begin{aligned}M_\nu^{-1}\rho' &= \tau\mathbf{d},\\ M_\nu^{-1}\rho &= {}^t\mathbf{d}^{-1}.\end{aligned}$$

Let $\sigma := \sigma_2\sigma_1 = \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} \in \mathrm{Sp}(g,\mathbb{Z})$ so that by (4.5), one has

$$(F'_1,\dots,F'_g,E'_1,\dots,E'_g) = (F_1,\dots,F_g,E_1,\dots,E_g)\,{}^t\sigma,$$

or, equivalently,

$$\begin{aligned}\rho' &= \tau\,{}^t\alpha + {}^t\beta,\\ \rho &= \tau\,{}^t\gamma + {}^t\delta.\end{aligned}$$

From these four equalities, one gets

$$\tau\mathbf{d} = M_\nu^{-1}(\tau\,{}^t\alpha + {}^t\beta), \tag{4.8}$$

$${}^t\mathbf{d}^{-1} = M_\nu^{-1}(\tau\,{}^t\gamma + {}^t\delta). \tag{4.9}$$

Hence one has, taking into account that τ is a symmetric matrix,

$$\begin{aligned} {}^t\mathbf{d}\tau\mathbf{d} &= (\tau\,{}^t\gamma + {}^t\delta)^{-1}\,(\tau\,{}^t\alpha + {}^t\beta), \\ &= (\alpha\tau + \beta)\,(\gamma\tau + \delta)^{-1}. \end{aligned}$$

This can be rewritten as

$$\sigma\,\tau \;=\; {}^t\mathbf{d}\tau\mathbf{d}. \tag{4.10}$$

Since h_ν preserves a theta structure of level 2, by Lemma 3.5, the symplectic matrix σ belongs to $\mathrm{Sp}_{g,\mathbb{Z}}^{\theta,2}$. Therefore by Theorem 2.3, the value

$$\lambda_\nu = j(\sigma', 2\tau)|\det(\mathbf{d})| = \kappa\,\det_{\mathbb{C}}(\gamma\tau + \delta)^{1/2}|\det(\mathbf{d})| \tag{4.11}$$

is critical on the group $G_{\mathbf{d}}$, where $\sigma' := \left(\begin{smallmatrix} \alpha & 2\beta \\ \gamma/2 & \delta \end{smallmatrix}\right)$ and where κ is a 8^{th} root of unity. Using (4.9) one computes,

$$\det_{\mathbb{C}}(\gamma\tau + \delta) = \det_{\mathbb{C}}(\mathbf{d})^{-1}\det_{\mathbb{C}}(M_\nu). \tag{4.12}$$

Plugging this into (4.11), we obtain the equality $\lambda_\nu = \kappa'\,d_\nu^{1/2}\det_{\mathbb{C}}(T_\nu)^{1/2}$.

It remains to explain why the 8^{th} root of unity κ' is a 4^{th} root of unity. This follows from Proposition 1.1 and the claim below.

Claim. *There exists an odd integer k such that $k^g\det_{\mathbb{C}}(T_\nu)^{-1} \equiv 1 \bmod 2$.*

This claim is true because there exists an odd integer k such that the complex matrix $(kT_\nu - 1)/2$ preserves a lattice in $\mathbb{C}^g$. Hence all the eigenvalues of kT_ν are algebraic integers which are equal to 1 mod 2. Their product too. □

Remark 4.6. When we deal with the sign issue in Section 8, we will not only need the statement of Theorem 4.4 but also the precise way m_ν, σ and τ are constructed in its proof. Note that, thanks to our choices (4.3), (4.5) and (4.7), the Riemann matrix $\theta \in \mathcal{H}_g$ defined by $(f_1, \ldots, f_g) = (e_1, \ldots, e_g)\,\theta$ satisfies the equalities $m_\nu\theta = \theta$ and $\theta = \sigma_1\tau$.

Remark 4.7. As we have seen, the group G_ν is not cyclic in general but, even when $g > 1$, it may be cyclic of order d. In the next sections, we will construct many examples of such endomorphisms ν with G_ν cyclic when A has complex multiplication by a CM field. See for instance Remark 7.5.

5. CM algebras

The aim of this section is to specialize our general construction of critical values to the case of CM abelian varieties and to express it (Theorem 5.4) from the point of view of symplectic CM algebras (K, ω) and their autodual lattices Λ.

We will first recall in Sections 5.1, and 5.2 a few classical definitions and facts. See [26], [18, Chapter 1] and [28, Chapter 1] for details.

In order to state this theorem, we will introduce in Section 5.3 the theta subgroup $U^{\theta,2}_{K,\Lambda}$ of level 2 of the unitary subgroup U_K of K, and we will describe it thanks to the Cayley transform (Lemma 5.3).

5.1. CM algebras, CM types and reflex norms. A number field $K \subset \mathbb{C}$ is said to have complex multiplication or to be a *CM number field* if K is a totally imaginary quadratic extension of a totally real number field K_0. More generally a *CM algebra* K is a product of CM fields $K = \prod_j K_j$. It has even dimension $2g := \dim_{\mathbb{Q}} K$. We denote by $x \mapsto \bar{x}$ the *complex conjugation* of K, that is the automorphism of K equal to the complex conjugation on each K_j.

An algebraic number x, or an element $x \in K$, is said to be *totally real* (resp. *totally imaginary*, resp. *totally unitary*) if all its Galois conjugates in $\mathbb{C}$ are real (resp. imaginary, resp. unitary). For instance $x = \sqrt{2}$ (resp. $x = i\sqrt{2}$, resp. $x = \frac{3+4i}{5}$). We denote by K_0 (resp. $\mathcal{I}_K$, resp. U_K) the subset of totally real (resp. totally imaginary, resp. totally unitary) elements of K.

Let $m \geq 1$ be an integer. An algebraic integer μ is said to be a *m-Weil number* if all its complex Galois conjugates have modulus equal to $\sqrt{m}$.

Those notions are strongly related:

- In a CM field $K \subset \mathbb{C}$, every real (resp. imaginary or unitary) element is totally real (resp. totally imaginary or totally unitary).
- When t is a nonzero totally imaginary algebraic number, the field $\mathbb{Q}[t]$ is CM. Conversely, when x is in a CM algebra, the difference $t := x - \bar{x}$ is totally imaginary.
- When $\nu \neq \pm 1$ is a totally unitary algebraic number, the field $\mathbb{Q}[\nu]$ is CM. Conversely, when x is invertible in a CM algebra, the ratio $\nu := x/\bar{x}$ is totally unitary.
- Let $k > 0$ be an integer. When μ is a k-Weil number, the ratio $\nu := \mu/k^{1/2}$ is totally unitary. Conversely, when $\nu \in \mathbb{C}$ is totally unitary, if the multiple $\mu := k^{1/2}\nu$ is integral, it is a k-Weil number.

The following fact implies that the totally unitary algebraic numbers $\nu \neq -1$ are exactly the images of totally imaginary numbers by the Cayley

transform. For a CM algebra K, we set

$$\mathcal{I}_K^\bullet := \{t \in \mathcal{I}_K \,|\, 1-t \in K^*\},$$
$$U_K^\bullet := \{\nu \in U_K \,|\, \nu+1 \in K^*\},$$

where K^* is the group of invertible elements of K.

Fact 5.1. *Let K be a CM algebra. The maps $t \mapsto \nu = \frac{1+t}{1-t}$ and $\nu \mapsto t = \frac{\nu-1}{\nu+1}$ are inverse bijections between $\mathcal{I}_K^\bullet$ and $U_K^\bullet$.*

Proof. Indeed the condition $\bar{\nu} = \nu^{-1}$ is equivalent to $\bar{t} = -t$. □

Let $K = \prod_j K_j$ be a CM algebra of dimension $2g$. All algebras morphisms $\rho : K \to \mathbb{C}$ are implicitely assumed to satisfy $\rho(1) = 1$. They are embeddings for exactly one of the field factors K_j and they are zero on the other factors.

A *CM*-type Φ of K is a tuple $\Phi = (\rho_1, \ldots, \rho_g)$ of distinct algebras morphisms from K to $\mathbb{C}$ no two of which are complex conjugate. We will call (K, Φ) a CM pair and we will think of Φ as an algebra morphism $\Phi : K \mapsto \mathbb{C}^g$.

A CM pair (K, Φ) is said to be *primitive*, if K is a field and the set of restrictions of the elements of Φ to any proper CM subfield is not a CM type.

For an element μ in K, we denote by

$$N_\Phi(\mu) := \rho_1(\mu) \cdots \rho_g(\mu)$$

its *reflex norm* or *type norm*. It lives in the reflex field $K^r \subset \mathbb{C}$. More precisely, the *reflex field* K^r is the subfield of $\mathbb{C}$ spanned by the reflex norms of the elements of K. The field K^r is always a CM field.

5.2. CM abelian varieties. An abelian variety is *simple* if it does not contain proper abelian subvarieties. Every abelian variety is isogenous to a product of simple abelian varieties.

A *CM abelian variety* is an abelian variety A such that $\mathrm{End}_{\mathbb{Q}}(A)$ contains a CM algebra K_A of $\mathbb{Q}$-dimension $2\dim(A)$. An abelian variety that is isogenous to a CM abelian variety is also CM. The action of K_A on the tangent space TA of A gives a CM type $\Phi_A : K_A \to \mathbb{C}^g$.

A CM abelian variety is *simple* if and only if the CM pair (K_A, Φ_A) is primitive, and the map

$$A \longmapsto (K_A, \Phi_A) \tag{5.1}$$

is a bijection between the set of isogeny classes of simple CM abelian varieties and the set of isomorphism classes of primitive CM pairs.

We now recall the inverse map to (5.1), i.e. the construction of the abelian variety A starting from a CM pair (K, Φ). The most interesting case is when K is a CM number field and Φ is primitive. But, in some examples

(as Propositions 6.4 and 6.5) it will be useful to deal with a more general CM pair (K, Φ).

Let $K = \prod K_j$ be a CM algebra. Let $\mathcal{O}_K = \prod \mathcal{O}_{K_j}$, where $\mathcal{O}_{K_j}$ is the *ring of integers of* K_j. A lattice Λ of K is an additive subgroup of K that is commensurable to $\mathcal{O}_K$. It can be seen, via any CM type $\Phi : K \to \mathbb{C}^g$, as a lattice in $\mathbb{C}^g$. When there is no possible confusion on the choice of Φ, we will also write Λ for $\Phi(\Lambda)$. The following construction due to Shimura is very useful, since it tells us that the complex torus $A = \mathbb{C}^g/\Phi(\Lambda)$ is an abelian variety. To state it we need more notation.

Given $t_0 \in K$ totally imaginary and invertible, we introduce the $\mathbb{Q}$-symplectic form $\omega_0 = \omega_{t_0}$ of the $2g$-dimensional $\mathbb{Q}$-vector space K

$$\omega_{t_0}(x, x') = \mathrm{Tr}_{K/\mathbb{Q}}\left(\frac{x\overline{x'}}{t_0}\right) \quad \text{for all } x, x' \text{ in } K. \tag{5.2}$$

Such a pair (K, ω_{t_0}) will be called a *symplectic CM algebra*. Replacing Λ by a multiple, we can assume that $\Lambda \subset \mathcal{O}_K$ and $\omega_{t_0}(\Lambda, \Lambda) \subset \mathbb{Z}$. We also denote by ω_{t_0} the $\mathbb{R}$-linear extension of ω_{t_0} to the $2g$-dimensional $\mathbb{R}$-vector space $\mathbb{C}^g$. We introduce then the CM type $\Phi_0 = \Phi_{\omega_0} = \Phi_{t_0}$ of K given by

$$\Phi_{t_0} = \{\rho \in \mathrm{Hom}_{\mathrm{alg}}(K, \mathbb{C}) \,|\, \mathrm{Im}(\rho(t_0)) > 0\}. \tag{5.3}$$

Note that, for any CM type Φ of K, one can find an invertible imaginary element t_0 with $\Phi_{t_0} = \Phi$. By Definition (5.3), this symplectic form ω_{t_0} is a polarization on $A = \mathbb{C}^g/\Phi_{t_0}(\Lambda)$, and hence (A, ω_{t_0}) is a polarized abelian variety. For our construction we will need ω_{t_0} to have determinant 1 on Λ so that Λ is autodual with respect to ω_{t_0}, which means that (A, ω_{t_0}) is principally polarized.

5.3. Multiplicative theta subgroup of level 2. We define now the unitary theta subgroup of K^* of level 2.

We recall that $\mathbb{Z}_{(2)}$ is the ring of rational numbers with odd denominator. For Λ lattice in K we set $\Lambda_{(2)} = \mathbb{Z}_{(2)}\Lambda$. For instance $\mathcal{O}_{K,(2)}$ is the set of ratios of elements of $\mathcal{O}_K$ with a denominator prime to 2.

Let Λ be an autodual lattice in a symplectic CM algebra (K, ω_0) as above. We introduce two subgroups of the unitary group $U_K = \{\nu \in K \mid \nu\overline{\nu} = 1\}$.

Definition 5.2. The congruence subgroup $U^2_{K,\Lambda}$ of level 2 and the theta subgroup $U^{\theta,2}_{K,\Lambda}$ of level 2 are

$$\begin{aligned} U^2_{K,\Lambda} &:= \left\{\nu \in U_K \,\middle|\, (\nu - 1)(\Lambda_{(2)}) \subset 2\Lambda_{(2)}\right\} \\ U^{\theta,2}_{K,\Lambda} &:= \left\{\nu \in U^2_{K,\Lambda} \,\middle|\, \omega_0(\nu x, x) \in 4\,\mathbb{Z}_{(2)} \ \text{ for all } x \in \Lambda_{(2)}\right\} \end{aligned} \tag{5.4}$$

One can also define $U^2_{K,\Lambda}$ as the set of $\nu \in U_K$ for which there exists an odd integer d such that $d\nu(\Lambda) \subset \Lambda$ and $d\nu$ acts trivially on the quotient

$\Lambda/2\Lambda$. Note that the elements ν of $U^2_{K,\Lambda}$ preserve $\Lambda_{(2)}$ and act trivially on $\Lambda_{(2)}/2\Lambda_{(2)}$.

Let us fix a symplectic basis of Λ. For ν in K, set $m_\nu \in \mathcal{M}(2g,\mathbb{Q})$ for the transpose of the matrix of the multiplication by ν in this basis. When the element ν is in U_K this matrix is symplectic: $m_\nu \in \mathrm{Sp}(g,\mathbb{Q})$. Here is an equivalent definition

$$U^2_{K,\Lambda} = \left\{\nu \in U_K \,\middle|\, m_\nu \in \mathrm{Sp}^2_{g,\mathbb{Q}}\right\}$$
$$U^{\theta,2}_{K,\Lambda} = \{\nu \in U_K \mid m_\nu \in \mathrm{Sp}^{\theta,2}_{g,\mathbb{Q}}\}$$

This definition does not depend on the choice of the symplectic basis of Λ.

We also define the analogous subsets in the imaginary elements of K:

$$\begin{aligned} \mathcal{I}^2_{K,\Lambda} &:= \left\{t \in \mathcal{I}_K \,\middle|\, t\Lambda_{(2)} \subset \Lambda_{(2)}\right\} \\ \mathcal{I}^{\theta,2}_{K,\Lambda} &:= \left\{t \in \mathcal{I}_{K,\Lambda}(2) \,\middle|\, \omega_0(tx,x) \in 2\,\mathbb{Z}_{(2)} \ \text{ for all } x \in \Lambda_{(2)}\right\} \end{aligned} \tag{5.5}$$

or, equivalently,

$$\mathcal{I}^2_{K,\Lambda} = \left\{t \in \mathcal{I}_K \,\middle|\, m_t \in \mathfrak{sp}^2_{g,\mathbb{Q}}\right\}$$
$$\mathcal{I}^{\theta,2}_{K,\Lambda} = \left\{t \in \mathcal{I}_K \,\middle|\, m_t \in \mathfrak{sp}^{\theta,2}_{g,\mathbb{Q}}\right\}$$

The following lemma is an analog of Lemma 3.8 using the Cayley transform of Fact 5.1. It will give a bijection between the following two subsets.

$$\mathcal{I}^{\bullet,2}_{K,\Lambda} := \left\{t \in \mathcal{I}^2_{K,\Lambda} \,\middle|\, N_{K/\mathbb{Q}}(1-t) \in \mathbb{Z}^*_{(2)}\right\},$$
$$U^{\bullet,2}_{K,\Lambda} := \left\{\nu \in U^2_{K,\Lambda} \,\middle|\, N_{K/\mathbb{Q}}((\nu+1)/2) \in \mathbb{Z}^*_{(2)}\right\}.$$

For an element $x \in K$ preserving $\Lambda_{(2)}$, the condition $N_{K/\mathbb{Q}}(x) \in \mathbb{Z}^*_{(2)}$, tells us that the inverse x^{-1} exists and also preserves $\Lambda_{(2)}$.

When K is a CM field, this condition $N_{K/\mathbb{Q}}(x) \in \mathbb{Z}^*_{(2)}$ means that x is a ratio of two elements of $\mathcal{O}_K$ which are prime to 2.

Lemma 5.3. *Let (K,ω_0) be a symplectic CM algebra, $\Lambda \subset K$ an autodual lattice.*

(a) *The Cayley transform $t \mapsto \frac{1+t}{1-t}$ is a bijection from $\mathcal{I}^{\bullet,2}_{K,\Lambda}$ onto $U^{\bullet,2}_{K,\Lambda}$.*

(b) *For $t \in \mathcal{I}^{\bullet,2}_{K,\Lambda}$ and $\nu = \frac{1+t}{1-t}$, one has the equivalence $t \in \mathcal{I}^{\theta,2}_{K,\Lambda} \iff \nu \in U^{\theta,2}_{K,\Lambda}$.*

Proof. The proof is similar to the proof of Lemma 3.8.

(a). The equality $\nu = \frac{1+t}{1-t}$ can be rewritten as $(1-t)(1+\nu)/2 = 1$. For such a pair (t,ν), one has the equivalence

$$N_{K/\mathbb{Q}}(1-t) \in \mathbb{Z}^*_{(2)} \iff N_{K/\mathbb{Q}}((1+\nu)/2) \in \mathbb{Z}^*_{(2)}$$

and, in that case,

$$t\Lambda_{(2)} \subset \Lambda_{(2)} \Longleftrightarrow (1+\nu)\Lambda_{(2)} \subset 2\Lambda_{(2)}.$$

Hence one has the equivalence:

$$t \in \mathcal{I}^{\bullet,2}_{K,\Lambda} \Longleftrightarrow \nu \in U^{\bullet,2}_{K,\Lambda}.$$

(b). In that case, one has the equivalences:

$$\begin{aligned} t \in \mathcal{I}^{\theta,2}_{K,\Lambda} &\Longleftrightarrow \omega_0(tx,x) \equiv 0 \bmod 2 && \text{for all } x \text{ in } \Lambda_{(2)} \\ &\Longleftrightarrow \omega_0((1+t)x,(1-t)x) \equiv 0 \bmod 4 && \text{for all } x \text{ in } \Lambda_{(2)} \\ &\Longleftrightarrow \omega_0(\nu x', x') \equiv 0 \bmod 4 && \text{for all } x' \text{ in } \Lambda_{(2)} \\ &\Longleftrightarrow \nu \in U^{\theta,2}_{K,\Lambda}. \end{aligned}$$

□

5.4. Critical values and CM field. All the examples of critical values found in the lists of [3, Section 1.5] can be explained thanks to the following theorem. This theorem is a special case of our main theorem 4.4 for CM abelian varieties and is expressed from the point of view of CM number fields.

Theorem 5.4. *Let K be a CM algebra of degree $2g$, $t_0 \in \mathcal{I}_K$ invertible, ω_{t_0} be the symplectic form as in* (5.2) *and $\Phi = \Phi_{t_0}$ be the CM type as in* (5.3).

Let $\Lambda \subset K$ be an autodual lattice and $\nu \in K$ be a unitary element that belongs to the unitary theta subgroup $U^{\theta,2}_{K,\Lambda} \subset U_K$ of level 2.

Let $G_\nu := \Lambda/(\Lambda \cap \nu\Lambda)$ and $d_\nu := |G_\nu|$. Then there exists a critical value $\lambda_\nu = \kappa_\nu\, d_\nu^{1/2}\, N_\Phi(\nu)^{1/2}$ on G_ν with $\kappa_\nu^4 = 1$.

Note that this critical value λ_ν on G_ν is a d_ν-Weil number since $N_\Phi(\nu)$ is a totally unitary algebraic number.

This gives a critical value λ_ν modulo a fourth root of unity κ_ν. We can determine this root of unity by using Corollary 3.1 but note that replacing ν by $-\nu$ changes the sign of $N_\Phi(\nu)$ when g is odd. It is easier to determine κ_ν up to sign by using the fact that $(\lambda_\nu - 1)/2$ is an algebraic integer.

Proof of Theorem 5.4. As seen in Section 5.2, the pair $(A = \mathbb{C}^g/\Phi(\Lambda), \omega_{t_0})$ is a principally polarized abelian variety. The element $\nu \in K$ induces a unitary $\mathbb{Q}$-endomorphism of A that preserves a theta structure of level 2 on A. Theorem 4.4 gives us a critical value $\lambda_\nu := \kappa_\nu\, d_\nu^{1/2}\, \det_{\mathbb{C}}(T_\nu)^{1/2}$ on G_ν. We conclude thanks to the equality $N_\Phi(\nu) = \det_{\mathbb{C}}(T_\nu)$. □

6. Examples

In this section we show with three examples how to compute explicit critical values by using Theorem 5.4. In each of these examples, we follow always the same strategy, keeping the notation (4.2) as in the proof of Theorem 4.4.

(a) We choose the CM pair (K, Φ), and its symplectic form ω_0.
(b) We choose a first lattice $\Lambda = \Lambda_0$ in K that is autodual for ω_0.
(c) We choose ν unitary in K and compute the group $G_{0,\nu} := \Lambda_0/(\Lambda_0 \cap \nu\Lambda_0)$.
(d) We modify slightly Λ and ω_0 so that ν belongs to $U^{\theta,2}_{K,\Lambda}$.
(e) We compute the square root of the reflex norm $N_\Phi(\nu)$.
(f) We comment on the 4^{th}-root of unity involved in the formula for λ_ν. In this section, we will determine this 4^{th}-root of unity only up to sign. The determination of this sign will be given in Section 8.

In these examples, the CM algebra K is not always a number field and the lattice Λ is not always a fractional ideal of K.

6.1. Imaginary quadratic fields. We begin by the case where K is an imaginary quadratic field and hence A is a CM elliptic curve. This case, which has already been worked out in [3], will help the reader to understand the above strategy.

Proposition 6.1. *Let $d = a + b$ with a, b positive integers and $a \equiv \frac{(d+1)^2}{4} \bmod 4$. Then the complex number $\lambda := \sqrt{a} + i\sqrt{b}$ is a d-critical value.*

Proof. Note that the congruence relation on a is equivalent to

$$a - b \equiv 1 \bmod 4 \quad \text{and} \quad ab \equiv 0 \bmod 4. \tag{6.1}$$

(a). We choose for CM algebra the field $K = \mathbb{Q}[\alpha]$ with $\alpha = i\sqrt{ab}$. We introduce the $\mathbb{Q}$-basis $(e_{0,1}, f_{0,1})$ of K where $e_{0,1} = 1$ and $f_{0,1} = \alpha$.

We choose the symplectic form ω_0 on the $\mathbb{Q}$-vector space K to be

$$\omega_0(x, x') = \mathrm{Tr}_{K/\mathbb{Q}}\left(\frac{x\bar{x}'}{2\alpha}\right)$$

so that $\omega_0 = f^*_{0,1} \wedge e^*_{0,1}$, i.e. $\omega_0(f_{0,1}, e_{0,1}) = 1$. Since $\mathrm{Im}(\alpha) > 0$, the CM type defined by (5.3) is the singleton $\Phi = \{\rho\}$ where ρ is the injection of K in $\mathbb{C}$.

(b). We first introduce the lattice $\Lambda_0 = \mathbb{Z}[\alpha] = \mathbb{Z}e_{0,1} \oplus \mathbb{Z}f_{0,1}$. This lattice is autodual for the symplectic form ω_0.

(c). We choose the totally unitary element ν of K to be

$$\nu := \frac{\sqrt{a} + i\sqrt{b}}{\sqrt{a} - i\sqrt{b}} = \frac{1}{d}(a - b + 2i\sqrt{ab}). \tag{6.2}$$

The matrix $m_{0,\nu}$ of multiplication by ν^{-1} in the basis $(e_{0,1}, -f_{0,1})$ is

$$m_{0,\nu} = \frac{1}{d}\begin{pmatrix} a-b & -2ab \\ 2 & a-b \end{pmatrix}.$$

Since the coefficients of the matrix $dm_{0,\nu}$ are integral with gcd equal to 1, the group $G_{0,\nu} = \Lambda_0/(\Lambda_0 \cap \nu\Lambda_0)$ is isomorphic to $\mathbb{Z}/d\mathbb{Z}$. The fixed point of $m_{0,\nu}$ in $\mathcal{H}_1$ is $\theta_0 := i\sqrt{ab}$.

(d). The matrix $m_{0,\nu}$ is symplectic but does not belong to the symplectic theta subgroup $\mathrm{Sp}^{\theta,2}_{2,\mathbb{Q}}$ because the lower-left coefficient of $m_{0,\nu}$ is not equal to 0 mod 4. This is why we introduce the basis e_1, f_1 with $e_1 = 2e_{0,1}$ and $f_1 = f_{0,1}$ and the lattice $\Lambda := \mathbb{Z}e_1 \oplus \mathbb{Z}f_1$. This lattice Λ is a sublattice of Λ_0 of index 2 that is autodual for $\frac{1}{2}\omega_0$. Since d is odd, one still has

$$G_\nu = \Lambda/(\Lambda \cap \nu\Lambda) \simeq \Lambda_0/(\Lambda_0 \cap \nu\Lambda_0) = G_{0,\nu} \simeq \mathbb{Z}/d\mathbb{Z},$$

The matrix m_ν of multiplication by ν^{-1} in the basis $(e_1, -f_1)$ is

$$m_\nu = \frac{1}{d}\begin{pmatrix} a-b & -ab \\ 4 & a-b \end{pmatrix}.$$

Since $ab \equiv 0 \bmod 4$, the element ν belongs to the unitary theta subgroup $U^{\theta,2}_{K,\Lambda}$ of level 2. The fixed point of m_ν in $\mathcal{H}_1$ is $\theta := i\sqrt{ab}/2$.

(e). Hence by Theorem 5.4, one has a d-critical value

$$\lambda_\nu = \kappa_\nu d^{1/2} N_\Phi(\nu)^{1/2} = \kappa_\nu(\sqrt{a} + i\sqrt{b}) \quad \text{with } \kappa_\nu^4 = 1.$$

(f). Since by Proposition 1.1, the ratio $(\lambda_\nu - 1)/2$ is an algebraic integer, one has $\kappa_\nu = \pm 1$. The precise sign $\pm$ in κ_ν will be computed in Section 8.4. □

Remark 6.2. Note that the conclusion of Proposition 6.1 does not hold when $a - \frac{(d+1)^2}{4} \equiv 2 \bmod 4$. This explains why, in the assumptions of Theorem 5.4, the unitary theta subgroup $U^{\theta,2}_{K,\Lambda}$ of level 2 can not be replaced by the congruence unitary subgroup $U^2_{K,\Lambda}$ of level 2.

Remark 6.3. Note that in this proof, when a and b are not relatively prime, we can not choose Λ to be the whole ring of integers $\mathcal{O}_K$.

Note also that if we fix $d_0 = a_0 + b_0$ with $a_0 \equiv \frac{(d_0+1)^2}{4} \bmod 4$ and let $d = md_0$, $a = ma_0$, $b = mb_0$ where m varies among the positive integers $m \equiv 1 \bmod 4$, the value of the totally unitary element ν in (6.2) does not depend on m, but the field K, the lattice Λ, the group $G_\nu \simeq \mathbb{Z}/d\mathbb{Z}$ and the critical value λ_ν do depend on m.

6.2. Products of imaginary quadratic fields. The new critical values that we obtain in this section (Proposition 6.4) are obtained by using products of imaginary fields.

Let d be the product $d = d_1 d_2$ of two odd integers. When λ_1 is a d_1-critical value and λ_2 is a d_2-critical value, then the product $\lambda = \lambda_1\lambda_2$ is a critical value on the product $\mathbb{Z}/d_1\mathbb{Z} \times \mathbb{Z}/d_2\mathbb{Z}$ (see [3, Section 1.4]). In particular, when d_1 and d_2 are coprime, this value λ is d-critical.

In the list of [3, Section 1.5], we found a d-critical value with $d = 15$,

$$\lambda = (\sqrt{3} + i\sqrt{2})(\sqrt{2} + i).$$

This d-critical value is of the form $\lambda = \lambda_1\lambda_2$ but is not obtained by the above process. Indeed, one can not write $\lambda = \lambda_1'\lambda_2'$ with λ_1' in the list of 3-critical values and λ_2' in the list of 5-critical values that are given in [3, Section 1.5].

The following proposition explains why this value λ is d-critical. The proof uses an abelian surface that is isogenous but not isomorphic to a product of two elliptic curves.

Proposition 6.4. *Let $d = d_1 d_2$ be a product of two coprime odd numbers. For $j = 1, 2$, let $d_j = a_j + b_j$ all positive, $a_j - \frac{(d_j+1)^2}{4} \equiv 2 \bmod 4$, and $\varepsilon_j = \pm 1$. Then $\lambda := (\sqrt{a_1} + i\varepsilon_1\sqrt{b_1})(\sqrt{a_2} + i\varepsilon_2\sqrt{b_2})$ is d-critical.*

For instance:

- $\lambda = (\sqrt{2}+i)(\sqrt{3}+i\sqrt{2})$ and $\lambda = (\sqrt{2}+i)(\sqrt{3}-i\sqrt{2})$ are 15-critical.
- $\lambda = (\sqrt{2}+i)(\sqrt{6}+i)$ and $\lambda = (\sqrt{2}+i)(\sqrt{6}-i)$ are 21-critical.

Notice that the first two λ's are Galois conjugate but that the last two are not. This last example emphasizes that we have to pay attention to the signs ε_j occuring in the formula for λ.

Proof. Note that the congruence relation on a_j, b_j is equivalent to

$$a_j - b_j \equiv 1 \bmod 4 \quad \text{and} \quad a_j b_j \equiv 2 \bmod 4.$$

(a). We choose for CM algebra the product $K = K_1 \times K_2$ of the quadratic fields $K_j = \mathbb{Q}[\alpha_j]$ with $\alpha_j = i\sqrt{a_j b_j}$. We introduce the $\mathbb{Q}$-basis $e_{0,1}$, $e_{0,2}$, $f_{0,1}$, $f_{0,2}$ of K where $e_{0,1} = (1, 0)$, $e_{0,2} = (0, 1)$, $f_{0,1} = (\alpha_1, 0)$ and $f_{0,2} = (0, \alpha_2)$. Note that the element $\alpha := (\alpha_1, \alpha_2) \in K$ is totally imaginary and invertible.

We define the symplectic form ω_0 on K by

$$\omega_0(x, x') = \mathrm{Tr}_{K/\mathbb{Q}}\left(\frac{x\overline{x'}}{4\alpha}\right)$$

so that $\omega_0 = f_{0,1}^* \wedge e_{0,1}^* + f_{0,2}^* \wedge e_{0,2}^*$. Since both $\mathrm{Im}(\alpha_j) > 0$, the CM type defined by (5.3) is $\Phi = \{\rho_1, \rho_2\}$, where $\rho_j : K \to \mathbb{C}$ is the projection on the factor $K_j \subset \mathbb{C}$.

(b). We choose for lattice $\Lambda_0 = \mathbb{Z}e_{0,1} \oplus \mathbb{Z}e_{0,2} \oplus \mathbb{Z}f_{0,1} \oplus \mathbb{Z}f_{0,2}$. This lattice is autodual for the symplectic form ω_0.

(c). We set $d'_j = \pm d_j$ so that $d'_j \equiv 1 \bmod 4$. We choose the unitary element

$$\nu := (\nu_1, \nu_2) \in K \quad \text{with} \quad \nu_j = \frac{1}{d'_j}\Big(a_j - b_j + 2i\varepsilon_j\sqrt{a_j b_j}\Big) \in K_j.$$

The same computation as the one given in Section 6.1, proves that the group $G_{0,\nu}$ is isomorphic to $\mathbb{Z}/d\mathbb{Z}$, Indeed,

$$G_{0,\nu} = \Lambda_0/(\Lambda_0 \cap \nu\Lambda_0) \simeq (\mathbb{Z}/d_1\mathbb{Z}) \times (\mathbb{Z}/d_2\mathbb{Z}) \simeq \mathbb{Z}/d\mathbb{Z}$$

The matrix of multiplication by ν^{-1} in the basis $e_{0,1}$, $e_{0,2}$, $-f_{0,1}$, $-f_{0,2}$ is

$$m_{0,\nu} = \begin{pmatrix} u_1 & 0 & \varepsilon_1 v_1 & 0 \\ 0 & u_2 & 0 & \varepsilon_2 v_2 \\ \varepsilon_1 w_1 & 0 & u_1 & 0 \\ 0 & \varepsilon_2 w_2 & 0 & u_2 \end{pmatrix}, \tag{6.3}$$

with

$$\begin{aligned} u_j &:= \frac{a_j - b_j}{d'_j} && \equiv 1 \bmod 4, \\ v_j &:= \frac{-2a_j b_j}{d'_j} && \equiv 4 \bmod 8, \\ w_j &:= \frac{2}{d'_j} && \equiv 2 \bmod 8. \end{aligned} \tag{6.4}$$

For $j = 1, 2$ set $\theta_j := i\sqrt{a_j b_j}$. The fixed point of $m_{0,\nu}$ in $\mathcal{H}_2$ is $\theta_0 := \left(\begin{smallmatrix} \theta_1 & 0 \\ 0 & \theta_2 \end{smallmatrix}\right)$.

(d). Since ε_1 and ε_2 are odd the element ν is not in $\mathcal{U}^{\theta,2}_{K,\Lambda_0}$. This is why we have to choose another basis given by a matrix P to be chosen later,

$$(e_1, e_2, -f_1, -f_2) = (e_{0,1}, e_{0,2}, -f_{0,1}, -f_{0,2})\, P \tag{6.5}$$

and another lattice $\Lambda = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \oplus \mathbb{Z}f_1 \oplus \mathbb{Z}f_2$. This lattice Λ will be a sublattice of index 4 in Λ_0 that will be autodual for $\frac{1}{2}\omega_0$, and such that ν belongs to $U^{\theta,2}_{K,\Lambda}$. Since d is odd, one will still have

$$G_\nu = \Lambda/(\Lambda \cap \nu\Lambda) \simeq \Lambda_0/(\Lambda_0 \cap \nu\Lambda_0) = G_{0,\nu} \simeq \mathbb{Z}/d\mathbb{Z}.$$

To explain precisely this choice of Λ we need to distinguish two subcases that we call (d1) and (d2). With no loss of generality, we assume $\varepsilon_1 = 1$.

(d1). We assume that $\varepsilon_1 = -\varepsilon_2 = 1$.

We choose the new basis (6.5) to be given by the basis change matrix $P = \left(\begin{smallmatrix} \mathbf{p} & \mathbf{0} \\ \mathbf{0} & \mathbf{p} \end{smallmatrix}\right)$ with $\mathbf{p} = \left(\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right)$ so that

$$(e_1, e_2, -f_1, -f_2) = (e_{0,1} + e_{0,2},\ e_{0,1} - e_{0,2},\ -f_{0,1} - f_{0,2},\ -f_{0,1} + f_{0,2})$$

This basis is symplectic for the form $\frac{1}{2}\omega_0$.

The matrix m_ν of multiplication by ν^{-1} in the basis e_1, e_2, $-f_1$, $-f_2$ is

$$m_\nu = P^{-1} m_{0,\nu} P = \begin{pmatrix} u_+ & u_- & v_- & v_+ \\ u_- & u_+ & v_+ & v_- \\ w_- & w_+ & u_+ & u_- \\ w_+ & w_- & u_- & u_+ \end{pmatrix}, \tag{6.6}$$

with $u_\pm = (u_1 \pm u_2)/2$, $\quad v_\pm = (v_1 \pm v_2)/2$ and $w_\pm = (w_1 \pm w_2)/2$.

The congruence conditions (6.4) imply that $m_\nu \equiv \mathbf{1} \bmod 2$ and that the diagonal coefficients of both the upper right and lower left blocks satisfy

$$v_- \equiv w_- \equiv 0 \bmod 4.$$

This implies that the element ν belongs to $U^{\theta,2}_{K,\Lambda}$. Set $\theta_\pm := (\theta_1 \pm \theta_2)/2$. The fixed point of m_ν in $\mathcal{H}_2$ is $\theta = P^{-1}\theta_0 = \left(\begin{smallmatrix} \theta_+ & \theta_- \\ \theta_- & \theta_+ \end{smallmatrix}\right)$.

(d2). We assume that $\varepsilon_1 = \varepsilon_2 = 1$.

We choose the new basis (6.5) to be given by the basis change matrix $P = \left(\begin{smallmatrix} \mathbf{2} & \mathbf{p} \\ \mathbf{0} & \mathbf{1} \end{smallmatrix}\right)$ with $\mathbf{p} = \left(\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right)$ so that

$$(e_1, e_2, -f_1, -f_2) = (2e_{0,1},\ 2e_{0,2},\ e_{0,1} + e_{0,2} - f_{0,1},\ e_{0,1} - e_{0,2} - f_{0,2}).$$

This basis is symplectic for the form $\frac{1}{2}\omega_0$.

The matrix m_ν of multiplication by ν^{-1} in the basis e_1, e_2, $-f_1$, $-f_2$ is

$$\begin{aligned} m_\nu &= P^{-1} m_{0,\nu} P \\ &= \begin{pmatrix} u_1 - w_1 & -w_2 & (v_1 - w_1 - w_2)/2 & (u_1 - u_2 - w_1 + w_2)/2 \\ -w_1 & u_2 + w_2 & (u_2 - u_1 + w_2 - w_1)/2 & (v_2 - w_1 - w_2)/2 \\ 2w_1 & 0 & u_1 + w_1 & w_1 \\ 0 & 2w_2 & w_2 & u_2 - w_2 \end{pmatrix}. \end{aligned} \tag{6.7}$$

The congruence conditions (6.4) imply that $m_\nu \equiv \mathbf{1} \bmod 2$ and that the diagonal coefficients of both the upper right and lower left blocks satisfy

$$(v_1 - w_1 - w_2)/2 \equiv (v_2 - w_1 - w_2)/2 \equiv 2w_1 \equiv 2w_2 \equiv 0 \bmod 4.$$

This implies that the element ν belongs to $U^{\theta,2}_{K,\Lambda}$. The fixed point of m_ν in $\mathcal{H}_2$ is $\theta = P^{-1}\theta_0 = \frac{1}{2}\left(\begin{smallmatrix} \theta_1 - 1 & -1 \\ -1 & \theta_2 + 1 \end{smallmatrix}\right)$.

(e). Hence, in all cases, by Theorem 5.4, one has a d-critical value

$$\begin{aligned} \lambda_\nu &= \kappa_\nu d^{1/2} N_\Phi(\nu)^{1/2} = \kappa_\nu (d_1\nu_1)^{1/2} (d_2\nu_2)^{1/2} \\ &= \kappa'_\nu (\sqrt{a_1} + i\varepsilon_1\sqrt{b_1})(\sqrt{a_2} + i\varepsilon_2\sqrt{b_2}), \end{aligned}$$

with ${\kappa'_\nu}^4 = 1$.

(f). Since by Proposition 1.1, the ratio $(\lambda_\nu - 1)/2$ is an algebraic integer, one must have $\kappa'_\nu = \pm 1$. The precise sign κ'_ν will be computed in Section 8.5. $\square$

6.3. Quartic CM algebras. In this Section we find new critical values by using quartic CM algebras (Proposition 6.5). Among them we will find the last four critical values that were pointed out in Section 1.1.

Proposition 6.5. *Let $d = a + b + c$ be positive integers with $b^2 - 4ac > 0$.*

(A) *Assume that $a \equiv 1 \bmod 4$ and $b \equiv c \equiv (0 \text{ or } 1) \bmod 4$.*
Then the sum $\lambda = \sqrt{a} + \sqrt{c} + i\sqrt{b - 2\sqrt{ac}}$ is d-critical.

(B) *Assume that $a \equiv 3 \bmod 4$ and $b \equiv c \equiv (0 \text{ or } 3) \bmod 4$.*
Then the sum $\lambda = \sqrt{b - 2\sqrt{ac}} + i\sqrt{a} + i\sqrt{c}$ is d-critical.

For instance

- $\lambda = 1 + \sqrt{5} + i\sqrt{5 - 2\sqrt{5}}$ is 11-critical.
- $\lambda = 1 + \sqrt{5} + i\sqrt{9 - 2\sqrt{5}}$ is 15-critical.
- $\lambda = 2\sqrt{2 - \sqrt{3}} + i\sqrt{3} + 2i = (\sqrt{3} + i\sqrt{2})\,(\sqrt{2} + i)$ is 15-critical.
- $\lambda = 1 + 2\sqrt{2} + 2i\sqrt{2 - \sqrt{2}}$ is 17-critical.
- $\lambda = \sqrt{11 - 2\sqrt{21}} + i\sqrt{3} + i\sqrt{7}$ is 21-critical.

Remark 6.6. Note that λ satisfies the condition $\lambda \equiv 1 \bmod 2$ from Proposition 1.1. Indeed, checking for instance the case when $a \equiv b \equiv c \equiv 1 \bmod 4$, one writes $\lambda = \lambda_1 + \lambda_2 + \lambda_3$ with $\lambda_1 = \sqrt{a}$, $\lambda_2 = \sqrt{c}$ and $\lambda_3 = i\sqrt{b - 2\sqrt{ac}}$, and one has $\lambda_1 \equiv \lambda_2 \equiv \lambda_3 \equiv 1 \bmod 2$ because $\lambda_1^2 \equiv \lambda_2^2 \equiv \lambda_3^2 \equiv 1 \bmod 4$.

The key remark that relates these values λ and the reflex norm as in Theorem 5.4 is the following old-fashioned factorization.

Lemma 6.7. *Let $d = a+b+c$ be positive integers with positive discriminant $\Delta := b^2 - 4ac > 0$. Then one has the equality*

$$\sqrt{a} + \sqrt{c} + i\sqrt{b - 2\sqrt{ac}} = \sqrt{a}\,\left(1 + i\sqrt{\frac{b + \sqrt{\Delta}}{2a}}\right)\,\left(1 - i\sqrt{\frac{b - \sqrt{\Delta}}{2a}}\right).$$

The proof of this factorization is left to the reader. For instance, one has

$$1 + \sqrt{5} + i\sqrt{9 - 2\sqrt{5}} = \left(1 + i\sqrt{\frac{9 + \sqrt{61}}{2}}\right)\,\left(1 - i\sqrt{\frac{9 - \sqrt{61}}{2}}\right).$$

Proof of Proposition 6.5. We will prove simultaneously (A) and (B). The only difference between the two proofs will occur in the last step (f) when computing the fourth root of unity κ_ν.

(a). We choose for CM algebra the quartic $\mathbb{Q}$-algebra $K = \mathbb{Q}[\alpha]/(F(\alpha))$ where

$$F(\alpha) = a\alpha^4 + b\alpha^2 + c,$$

is a polynomial with positive integral coefficients a, b, c and with positive discriminant $\Delta = b^2 - 4ac$. This CM algebra is a CM field if and only if the polynomial F is irreducible. We set

$$\delta := 2a\alpha^2 + b \in K$$

so that $\delta^2 = \Delta$ and let K_0 be the totally real subalgebra $K_0 := \mathbb{Q}[\delta]$. One has $K_0 = \mathbb{Q}1 \oplus \mathbb{Q}\delta$ where 1 is the unity element of K. We introduce the $\mathbb{Q}$-basis of K

$$e_{0,1} := 1, \quad e_{0,2} := \delta, \quad f_{0,1} := \alpha\delta, \quad f_{0,2} := \alpha.$$

We define the symplectic form ω_0 on K by

$$\omega_0(x, x') = \mathrm{Tr}_{K/\mathbb{Q}}\left(\frac{x\overline{x}'}{4\alpha\delta}\right).$$

so that, by direct computation, one has $\omega_0 = f_{0,1}^* \wedge e_{0,1}^* + f_{0,2}^* \wedge e_{0,2}^*$. We introduce the morphisms $\rho_\pm : K \to \mathbb{C}$ defined by

$$\rho_+(\alpha) = \alpha_+ := i\sqrt{(b - \sqrt{\Delta})/2a} \quad \text{and} \quad \rho_-(\alpha) = \alpha_- := -i\sqrt{(b + \sqrt{\Delta})/2a}.$$

The signs have been chosen carefully so that both $\mathrm{Im}(\rho_\pm(\alpha\delta)) > 0$. Therefore the CM type defined by (5.3) is $\Phi = \{\rho_+, \rho_-\}$.

(b). We first introduce the lattice $\Lambda_0 = \mathbb{Z}e_{0,1} \oplus \mathbb{Z}e_{0,2} \oplus \mathbb{Z}f_{0,1} \oplus \mathbb{Z}f_{0,2}$. This lattice is autodual for the symplectic form ω_0.

(c). We set $t = \alpha$. We choose the totally unitary element

$$\nu := \frac{1+t}{1-t} = 1 + \frac{2}{d}(a\alpha^3 + a\alpha^2 + (a+b)\alpha - c) \in K. \tag{6.8}$$

In the basis $e_{0,1}, e_{0,2}, -f_{0,1}, -f_{0,2}$, the matrix $m_{0,\nu}$ of multiplication by ν^{-1} is

$$m_{0,\nu} = \frac{1}{d}\begin{pmatrix} a-c & \Delta & \Delta & -b-2c \\ 1 & a-c & -b-2c & 1 \\ 1 & 2a+b & a-c & 1 \\ 2a+b & \Delta & \Delta & a-c \end{pmatrix}. \tag{6.9}$$

By Proposition 3.4, there exists $\sigma_{0,1}$ and $\sigma_{0,2}$ in $\mathrm{Sp}(g, \mathbb{Z})$ and a diagonal matrix $\mathbf{d}_0 = \mathrm{diag}(d_1, d_2)$ with $d_1 | d_2$ such that the symplectic matrix $m_{0,\nu}$ can be written as $m_{0,\nu} = \sigma_{0,1} \begin{pmatrix} {}^t\mathbf{d}_0^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{d}_0 \end{pmatrix} \sigma_{0,2}$.

Note that, by the above computation, the matrix $dm_{0,\nu}$ has integer coefficients, the gcd of its coefficients is equal to 1 and the gcd of all its 2×2 minor determinants is equal to d. Therefore one can choose $d_1 = 1$ and $d_2 = d$. This proves that the group $G_{0,\nu} = \Lambda_0/(\Lambda_0 \cap \nu\Lambda_0)$ is isomorphic to $\mathbb{Z}/d\mathbb{Z}$.

(d). The matrix $m_{0,\nu}$ is symplectic but does not belong to the symplectic theta subgroup $\mathrm{Sp}^{\theta,2}_{2,\mathbb{Q}}$ because the diagonal of the lower-left block of $dm_{0,\nu}$, which is $(1,\Delta)$ is not equal to 0 mod 4. This is why we again have to choose another basis

$$(e_1,e_2,-f_1,-f_2)=(e_{0,1},e_{0,2},-f_{0,1},-f_{0,2})\,P \tag{6.10}$$

and another lattice $\Lambda=\mathbb{Z}e_1\oplus\mathbb{Z}e_2\oplus\mathbb{Z}f_1\oplus\mathbb{Z}f_2$. This lattice Λ will be sublattice of Λ_0 of index 16 that will be autodual for $\frac{1}{4}\omega_0$, and such that ν belongs to $U^{\theta,2}_{K,\Lambda}$, Therefore, since d is odd, one will still have

$$G_\nu=\Lambda/(\Lambda\cap\nu\Lambda)\simeq\Lambda_0/(\Lambda_0\cap\nu\Lambda_0)=G_{0,\nu}\simeq\mathbb{Z}/d\mathbb{Z},$$

and ν will belong to the unitary theta subgroup $U^{\theta,2}_{K,\Lambda}$ of level 2.

To explain precisely this choice of Λ we again need to distinguish two subcases that we call (d1) and (d2).

(d1). We assume that a is odd and $b\equiv c\equiv 0$ mod 4.

We choose the new basis (6.10) to be given by the basis change matrix $P=\left(\begin{smallmatrix}2\mathbf{p} & \mathbf{0}\\ \mathbf{0} & 2\,{}^t\mathbf{p}^{-1}\end{smallmatrix}\right)$ with $\mathbf{p}=\left(\begin{smallmatrix}2 & b/2\\ 0 & 1/2\end{smallmatrix}\right)$, so that

$$(e_1,e_2,-f_1,-f_2)=(4e_{0,1},\ b\,e_{0,1}+e_{0,2},\ -f_{0,1}+b\,f_{0,2},\ -4f_{0,2}).$$

The matrix m_ν of multiplication by ν^{-1} in the basis e_1, e_2, $-f_1$, $-f_2$ is

$$\begin{aligned} m_\nu&=P^{-1}m_{0,\nu}P\\ &=\frac{1}{d}\begin{pmatrix}2a-d & -ac & b^2+bc-ac & -2b-2c\\ 4 & d-2c & -2b-2c & 4\\ 4 & 2a+2b & 2a-d & 4\\ 2a+2b & b^2+ab-ac & -ac & d-2c\end{pmatrix}.\end{aligned} \tag{6.11}$$

The element ν belongs to $U^{\theta,2}_{K,\Lambda}$ because $m_\nu\equiv\mathbf{1}$ mod 2, and both the diagonal of the upper-right block and of the lower-left block of the matrix $d\,m_\nu$ satisfy

$$b^2+bc-ac\equiv 4\equiv 4\equiv b^2+ab-ac\equiv 0 \bmod 4.$$

(d2). We assume that a is odd and $b\equiv c\equiv a$ mod 4.

We choose the new basis (6.10) to be given by the basis change matrix $P=\left(\begin{smallmatrix}2\mathbf{p} & \mathbf{p}\\ \mathbf{0} & 2\,{}^t\mathbf{p}^{-1}\end{smallmatrix}\right)$ with $\mathbf{p}=\left(\begin{smallmatrix}2 & b/2\\ 0 & 1/2\end{smallmatrix}\right)$, so that

$$\begin{aligned}&(e_1,e_2,-f_1,-f_2)\\ &\quad=(4e_{0,1},\ 2b\,e_{0,1}+2e_{0,2},\ 2e_{0,1}-f_{0,1}+b\,f_{0,2},\ b\,e_{0,1}+e_{0,2}-2f_{0,2}).\end{aligned}$$

The matrix m_ν of multiplication by ν^{-1} in the basis e_1, e_2, $-f_1$, $-f_2$ is

$$\begin{aligned} m_\nu &= P^{-1} m_{0,\nu} P \\ &= \frac{1}{d}\begin{pmatrix} 2a-d-2 & -2ac-2a-2b & b^2+bc-ac-1 & -ac-d-b-1 \\ 2-2a-2b & d-2c-2f & ac-d-b+1 & 1-f \\ 4 & 4a+4b & 2a-d+2 & 2a+2b+2 \\ 4a+4b & 4f & 2a+2b-2ac & d-2c+2f \end{pmatrix}, \end{aligned} \tag{6.12}$$

where $f := b^2 + ab - ac$. The element ν belongs to $U^{\theta,2}_{K,\Lambda}$ because $m_\nu \equiv \mathbf{1} \bmod 2$, and both the diagonal of the upper-right and lower-left block of the matrix $d\, m_\nu$ satisfy

$$b^2 + bc - ac - 1 \equiv 4 \equiv 4 \equiv 4f \equiv 0 \bmod 4.$$

(e). We compute the critical value λ_ν given by Theorem 5.4. We first notice that $N_{K/\mathbb{Q}}(1+\alpha) = \frac{d}{a}$ because this norm is the sum of the coefficients of the polynomial F/a, and hence that

$$N_\Phi(\nu) = \frac{N_\Phi(1+\alpha)}{N_\Phi(1-\alpha)} = \frac{N_\Phi(1+\alpha)^2}{N_{K/\mathbb{Q}}(1+\alpha)} = \frac{a}{d}\, N_\Phi(1+\alpha)^2.$$

Therefore by Theorem 5.4 combined with Lemma 6.7, one gets

$$\begin{aligned} \lambda_\nu &= \kappa_\nu\, d^{1/2} N_\Phi(\nu)^{1/2} = \kappa_\nu\, a^{1/2} N_\Phi(1+\alpha) \\ &= \kappa_\nu\, a^{1/2}(1+\alpha_+)(1+\alpha_-) = \kappa_\nu\Big(\sqrt{a} + \sqrt{c} - i\sqrt{b - 2\sqrt{ac}}\Big). \end{aligned}$$

(f). Since by Proposition 1.1, the ratio $(\lambda - 1)/2$ is an algebraic integer, arguing as in Remark 6.6, one must have

$$\lambda_\nu = \begin{cases} \pm\Big(\sqrt{a} + \sqrt{c} - i\sqrt{b - 2\sqrt{ac}}\Big) & \text{when } a \equiv 1 \bmod 4, \quad (6.13) \\ \pm\Big(\sqrt{b - 2\sqrt{ac}} + i\sqrt{a} + i\sqrt{c}\Big) & \text{when } a \equiv 3 \bmod 4. \quad (6.14) \end{cases}$$

The precise sign $\pm$ will be computed in Section 8.6. □

Note that the complex conjugation will give another critical value with opposite imaginary part. However, it is important to remember for latter use that the critical values λ_ν we have constructed in this section satisfy:

$$\frac{\mathrm{Im}(\lambda_\nu)}{\mathrm{Re}(\lambda_\nu)} \begin{cases} < 0 & \text{in Case (6.13)} \\ > 0 & \text{in Case (6.14).} \end{cases}$$

7. Ideals in CM fields

In this section, we come back to the general CM abelian varieties, but we specialize our Theorem 5.4 to the case when the lattice Λ is a fractional ideal $\mathfrak{m}$ of K, or, equivalently, the abelian variety $A = \mathbb{C}^g/\Phi(\Lambda)$ has multiplication by $\mathcal{O}_K$ (Theorem 7.3). In this case a result of Shimura–Taniyama tells us when this abelian variety A is principally polarized (Fact 7.1).

An important case is when K is the cyclotomic field $\mathbb{Q}[\zeta_n]$ (Proposition 7.9). In this case, for all CM types Φ of K one can find an ideal $\mathfrak{m}$ of K such that the abelian variety $\mathbb{C}^g/\Phi(\mathfrak{m})$ is principally polarized (Corollary 7.8).

In the last Section 7.4, we give another concrete family of d-critical values where $d = L_n$ is the n^{th} Lucas number with $n \geq 5$ prime (Proposition 7.15).

7.1. Using class field theory. When K is a CM number field, we would like to find $t_0 \in \mathcal{I}_K$ and an ideal $\mathfrak{m}$ of K such that the symplectic form ω_{t_0} restricted to $\mathfrak{m}$ takes values in $\mathbb{Z}$ and has determinant equal to 1, because this means that the polarized abelian variety $(\mathbb{C}^g/\Phi_{t_0}(\mathfrak{m}), \omega_{t_0})$ is principally polarized. Taniyama and Shimura explained when this is possible: in concrete words, they show that,

(A) such a construction is possible, if we allow the CM type Φ to vary,
(B) if the extension K/K_0 is ramified, one can prescribe the CM type Φ,
(C) if the extension K/K_0 is unramified, they describe the possible CM types.

Here is their precise result in which $\mathcal{D}_K$ and $\mathcal{D}_{K/K_0}$ are the differents.

Fact 7.1 (Shimura, Taniyama)**.** *Let K be a CM field.*

(1) *There exist an ideal $\mathfrak{m}$ of $\mathcal{O}_K$ and an imaginary element $t_0 \in \mathcal{I}_K$ such that the polarized abelian variety $(\mathbb{C}^g/\Phi_{t_0}(\mathfrak{m}), \omega_{t_0})$ is principal.*
(2) *If $\mathcal{D}_{K/K_0} \neq \mathcal{O}_K$, for all CM types Φ of K, there exist an ideal $\mathfrak{m}$ of $\mathcal{O}_K$ and $t_0 \in \mathcal{I}_K$ such that $\Phi_{t_0} = \Phi$ and the polarized abelian variety $(\mathbb{C}^g/\Phi_{t_0}(\mathfrak{m}), \omega_{t_0})$ is principal.*
(3) *If $\mathcal{D}_{K/K_0} = \mathcal{O}_K$, let $t_{00} \in \mathcal{I}_K$ and $\mathfrak{n} := t_{00}\mathcal{D}_K \cap K_0$. One has* (i) $\Leftrightarrow$ (ii)*:*
   (i) *There exist an ideal $\mathfrak{m}$ of $\mathcal{O}_K$ and $t_0 \in \mathcal{I}_K$ such that $\Phi_{t_0} = \Phi_{t_{00}}$ and the polarized abelian variety $(\mathbb{C}^g/\Phi_{t_0}(\mathfrak{m}), \omega_{t_0})$ is principal.*
   (ii) *The element $\mathrm{Art}_{K/K_0}(\mathfrak{n}) \in \mathrm{Gal}(K/K_0)$ is trivial.*

Condition (ii) involves the image of the ideal $\mathfrak{n}$ of K_0 by the Artin map and its restriction $\mathrm{Art}_{K/K_0}(\mathfrak{n})$ to the unramified extension K of K_0. This restriction belongs to the Galois group $\mathrm{Gal}(K/K_0)$ which has order 2.

Condition (ii) is automatically satisfied when the ideal $\mathfrak{n}$ is principal. We will see in Corollary 7.8 that this is always the case for cyclotomic fields.

Note that for the cyclotomic fields $K = \mathbb{Q}[\zeta_n]$, both cases (B) and (C) can occur (see Lemma 7.6),

Note that, the extension K/K_0 is always ramified when $[K : \mathbb{Q}] = 2$ or when $[K : \mathbb{Q}] = 4$ and K does not contain an imaginary quadratic subfield (see [7, Proof of Theorem 3.1]).

Proof of Fact 7.1. This is [25, Proposition 1], but it will be useful to point out how class field theory is used in the proof. We will discuss successively the points (C), (B) and (A).

We first note that the relative different $\mathcal{D}_{K/K_0}$ is generated by imaginary elements. Therefore, given a nonzero t_{00} in $\mathcal{I}_K$, there exists a fractional ideal $\mathfrak{n}$ of K_0 such that $t_{00}^{-1}\mathcal{D}_K = \mathfrak{n}^{-1}\mathcal{O}_K$. We want to know if there exist a totally positive element s_0 in K_0 and an ideal $\mathfrak{m}$ of $\mathcal{O}_K$ which is autodual for ω_{t_0} where $t_0 := s_0 t_{00}$. This means that

$$t_0^{-1}\mathcal{D}_K \mathfrak{m}\overline{\mathfrak{m}} = \mathcal{O}_K \quad \text{or, equivalently,} \quad \mathfrak{m}\overline{\mathfrak{m}} = s_0\mathfrak{n}\mathcal{O}_K.$$

This exactly means that the class $[\mathfrak{n}]$ in the narrow class group $\mathcal{C}\ell^+(K_0)$ is the image by the norm map N_{K/K_0} of a class $[\mathfrak{m}]$ of the narrow class group $\mathcal{C}\ell^+(K)$. Let L_0^+ be the maximal unramified (at all finite places) abelian extension of K_0 and L^+ the maximal unramified abelian extension of K. By class field theory, the Artin maps Art_K and Art_{K_0} give a commutative diagram with horizontal isomorphisms as in [31, p. 400],

$$\begin{array}{ccc} \mathcal{C}\ell^+(K) & \xrightarrow{\mathrm{Art}_K} & \mathrm{Gal}(L^+/K) \\ {\scriptstyle N_{K/K_0}}\downarrow & & \downarrow{\scriptstyle \mathrm{Res}_{L_0^+}} \\ \mathcal{C}\ell^+(K_0) & \xrightarrow{\mathrm{Art}_{K_0}} & \mathrm{Gal}(L_0^+/K_0) \end{array} \tag{7.1}$$

The question is whether the element $\sigma := \mathrm{Art}_{K_0}([\mathfrak{n}]) \in \mathrm{Gal}(L_0^+/K_0)$ is in the image of the restriction map $\mathrm{Res}_{L_0^+}$. Equivalently the question is whether the restriction of σ to $L_0^+ \cap K$ is trivial.

In case (C), the field K is included in L_0^+ and the criterion is the triviality of σ in K.

In case (B), the intersection $L_0^+ \cap K$ is equal to K_0. Therefore the criterion is always true.

In case (A), we do not worry about the total positivity of s_0 therefore we introduce the Hilbert class field L_0 of K_0 which is also the maximal totally real subfield of L_0^+, and the Hilbert class field L of K which is equal to L^+. Class field theory gives a similar commutative diagram, with horizontal

isomorphisms Art_K and Art_{K_0},

$$\begin{array}{ccc} \mathcal{C}\ell(K) & \xrightarrow{\mathrm{Art}_K} & \mathrm{Gal}(L/K) \\ {\scriptstyle N_{K/K_0}}\downarrow & & \downarrow{\scriptstyle \mathrm{Res}_{L_0}} \\ \mathcal{C}\ell(K_0) & \xrightarrow{\mathrm{Art}_{K_0}} & \mathrm{Gal}(L_0/K_0) \end{array} \tag{7.2}$$

Since L_0 is totally real, one has $K \cap L_0 = K_0$ and the restriction map Res_{L_0} is onto. The norm map N_{K/K_0} is also onto. □

7.2. Critical values and ideals in CM field. We now give a consequence of Theorem 5.4 applied to the case where the lattice Λ is an ideal $\mathfrak{m}$ of $\mathcal{O}_K$, combined with Shimura Taniyama's Fact 7.1.

We begin by an explicit construction of elements of the group $U^{\theta,2}_{K,\mathfrak{m}}$ when $\mathfrak{m}$ is an ideal. This construction does not depend on $\mathfrak{m}$.

Lemma 7.2. *Let K be a CM number field, ω_{t_0} a symplectic form on K as in* (5.2) *and $\mathfrak{m}$ be an autodual fractional ideal of K.*

(a) *For all s in K with denominator prime to 2, the difference $t := s - \bar{s}$ belongs to $\mathcal{I}^{\theta,2}_{K,\mathfrak{m}}$.*

(b) *If moreover $N_{K/\mathbb{Q}}(1+t)$ has an odd numerator, then $\nu := \frac{1+t}{1-t}$ is in $U^{\theta,2}_{K,\mathfrak{m}}$.*

Proof.

(a). Indeed, one has $\omega_{t_0}(tx, x) = 2\,\omega_{t_0}(sx, x) \in 2\mathbb{Z}_{(2)}$, for all x in $\mathfrak{m}$.

(b). Since $N_{K/\mathbb{Q}}(1-t) = N_{K/\mathbb{Q}}(1+t)$, this follows from Point (a) and from Lemma 5.3. □

The three parts of the following Theorem 7.3 correspond to the three parts of Shimura Taniyama's Fact 7.1.

Theorem 7.3. *Let K be a CM field and $K_0 := K \cap \mathbb{R}$. Let $\nu = \mu/\bar{\mu}$ with $\mu = 1 + s - \bar{s}$ where $s \in K$ has denominator prime to 2 and $N_{K/\mathbb{Q}}(\mu)$ has odd numerator. Set $G_\nu := \mathcal{O}_K/(\mathcal{O}_K \cap \nu\mathcal{O}_K)$ and $d_\nu := |G_\nu|$.*

(A) *There exists a CM type Φ of K such that the following holds:*

$$\begin{array}{c} \textit{One of the values } \lambda_\nu = \kappa_\nu d_\nu^{1/2} N_\Phi(\nu)^{1/2} \\ \textit{with } \kappa_\nu^4 = 1 \textit{ is critical on } G_\nu. \end{array} \tag{7.3}$$

(B) *If $\mathcal{D}_{K/K_0} \neq \mathcal{O}_K$, then for all CM types Φ of K,* (7.3) *holds.*

(C) *If $\mathcal{D}_{K/K_0} = \mathcal{O}_K$, let $t_{00} \in \mathcal{I}_K$ and set $\Phi = \Phi_{t_{00}}$ and $\mathfrak{n} := t_{00}\mathcal{D}_K \cap K_0$. Assume that the element $\mathrm{Art}_{K/K_0}(\mathfrak{n}) \in \mathrm{Gal}(K/K_0)$ is trivial, then* (7.3) *holds with this Φ.*

Corollary 7.4. *Moreover, in all these cases, if s belongs to $\mathcal{O}_K$, one has*

$$\lambda_\nu = \pm N_\Phi(\mu) \quad \textit{and} \quad G_\nu = \mathcal{O}_K/\mu\mathcal{O}_K. \tag{7.4}$$

We recall that the expression *s has denominator prime to* 2, means that the prime divisors of the ideal $2\mathcal{O}_K$ never occur in the prime decomposition of the fractional ideal $s\mathcal{O}_K$ with a negative power.

Proof of Theorem 7.3. By Shimura Taniyama Fact 7.1, we know that in these three cases, there exists $t_0 \in \mathcal{I}_K$ with $\Phi_{t_0} = \Phi$ and a fractional ideal $\mathfrak{m}$ of K such that the polarized abelian variety $(\mathbb{C}^g/\Phi(\mathfrak{m}), \omega_{t_0})$ is principal. We will choose $\Lambda = \mathfrak{m}$. By Lemma 7.2, our unitary element $\nu \in K$ belongs to the unitary theta subgroup $U^{\theta,2}_{K,\mathfrak{m}}$ of level 2.

Therefore Theorem 5.4 tells us that λ_ν is a critical value on the group $G_\nu = \mathfrak{m}/(\mathfrak{m} \cap \nu\mathfrak{m})$. Since we are dealing with a finite quotient of $\mathfrak{m}$, we can replace $\mathfrak{m}$ by $\mathcal{O}_K$ so that $G_\nu \simeq \mathcal{O}_K/(\mathcal{O}_K \cap \nu\mathcal{O}_K)$. □

Proof of Corollary 7.4. We first show that $\kappa_\nu = \pm 1$. According to Proposition 1.1, it is enough to check that $\frac{1}{2}(N_\Phi(\mu) - 1)$ is an algebraic integer. We set $\mu_0 := 1 + s + \bar{s}$. Since $\mu - \mu_0 \in 2\mathcal{O}_K$, it is enough to check that $\frac{1}{2}(N_\Phi(\mu_0) - 1)$ is an algebraic integer. But since the integer $N_{K/\mathbb{Q}}(\mu_0) \equiv N_{K/\mathbb{Q}}(\mu) \bmod 2$ is odd, the integer $N_\Phi(\mu_0) = N_{K_0/\mathbb{Q}}(\mu_0)$ is odd too.

We now compute G_ν. When μ is in $\mathcal{O}_K$, since $\mu + \bar{\mu} = 2$ and $N_{K/\mathbb{Q}}(\mu)$ is odd, the integers μ and $\bar{\mu}$ are relatively prime and one has $\mu\mathcal{O}_K \cap \bar{\mu}\mathcal{O}_K = \mu\bar{\mu}\mathcal{O}_K$ so that $G_\nu \simeq \mathcal{O}_K/\bar{\mu}^{-1}(\mu\mathcal{O}_K \cap \bar{\mu}\mathcal{O}_K) = \mathcal{O}_K/\mu\mathcal{O}_K$ as required. □

Remark 7.5. Note that in Theorem 7.3, it is easy to determine the structure of the abelian group G_ν by decomposing the ideal $\mathcal{J}_\nu := \mathcal{O}_K \cap \nu\mathcal{O}_K$ as a product $\prod_j \mathfrak{p}_j^{n_j}$ of prime ideals $\mathfrak{p}_j$ of $\mathcal{O}_K$ so that $G_\nu \simeq \prod_j \mathcal{O}_K/\mathfrak{p}_j^{n_j}$. In particular, the group G_ν is cyclic if and only if the prime ideals $\mathfrak{p}_j$ that divide $\mathcal{J}_\nu$ are over different primes p_j of $\mathbb{Z}$ and have inertia degree equal to 1.

7.3. Critical values and cyclotomic fields. In this section we apply Theorem 7.3 to the cyclotomic fields $\mathbb{Q}[\zeta_n]$.

We first recall notation related to cyclotomic fields. Let ζ_n be the primitive n^{th}-root of unity $\zeta_n := e^{2i\pi/n}$, let Φ_n be the n^{th} cyclotomic polynomial, let $K_n := \mathbb{Q}[\zeta_n]$ be the cyclotomic field, let $\varphi(n)$ be the Euler number $\varphi(n) := [K_n : \mathbb{Q}]$ which is the degree of Φ_n, and let $K_{n,0} := \mathbb{Q}[\zeta_n + \zeta_n^{-1}]$ be the real cyclotomic field. Their rings of integers are $\mathcal{O}_{K_n} = \mathbb{Z}[\zeta_n]$ and $\mathcal{O}_{K_{n,0}} = \mathbb{Z}[\zeta_n + \zeta_n^{-1}]$. See [31, Chapter 2].

Lemma 7.6. *Let $n \not\equiv 2 \bmod 4$ and $K_n = \mathbb{Q}[\zeta_n]$.*

(a) *The different $\mathcal{D}_{K_n}$ is a principal ideal. More precisely, there exists an imaginary element $t_n \in \mathcal{O}_{K_n}$ such that $\mathcal{D}_{K_n} = t_n \mathcal{O}_{K_n}$.*

(b) *The extension $K_n/K_{n,0}$ is ramified if and only if n is a prime power.*

Since $\mathbb{Q}[\zeta_m] = \mathbb{Q}[\zeta_{2m}]$ for m odd, the assumption on n is innocuous.

Proof. This is classical. We just sketch the argument.

(a). Let $\xi_n = \prod_{p|n}(1-\zeta_p)$ where the product is over the prime divisors of n. We first claim that $\mathcal{D}_{K_n} = n\xi_n^{-1}\mathcal{O}_{K_n}$. We write $n = \prod p^{r_p}$. For all integers k_p one has

$$\mathrm{Tr}_{K_n/\mathbb{Q}}\left(\xi_n \prod_p \zeta_{p^{r_p}}^{k_p}\right) = \prod_p \mathrm{Tr}_{K_{p^{r_p}}/\mathbb{Q}}\Big((1-\zeta_p)\zeta_{p^{r_p}}^{k_p}\Big) \equiv 0 \bmod n$$

By definition of the different, this proves the inclusion $\mathcal{D}_{K_n} \subset n\xi_n^{-1}\mathcal{O}_{K_n}$. By [31, Proposition 2.7] the absolute value of the discriminant d_{K_n} is equal to $n^{\varphi(n)}/\prod_{p|n} p^{\varphi(n)/(p-1)}$. Since this number is also equal to $N_{K_n/\mathbb{Q}}(n\xi_n^{-1})$, this proves the equality $\mathcal{D}_{K_n} = n\xi_n^{-1}\mathcal{O}_{K_n}$.

To conclude, we need to find a unit $u_n \in \mathcal{O}_{K_n}^*$ such that $u_n\xi_n$ is imaginary.

We first notice that for p odd the ratio $u_p := \frac{\zeta_p - \zeta_p^{-1}}{1-\zeta_p}$ is a unit and that for $p = 2$ the ratio $u_p := \frac{2\zeta_4}{1-\zeta_2} = i$ is also a unit. Therefore we set $u'_n := \prod_{p|n} u_p$, so that the product $u'_n\xi_n$ is imaginary when n has an odd number of prime factors, and is real when n has an even number of prime factors.

In the first case, we are done with $u_n = u'_n$.

In the second case, n is not a prime power and by Lemma 7.7 below, there exists an imaginary unit u''_n in $\mathbb{Q}[\zeta_n]$ and we are done with $u_n := u'_n u''_n$.

(b). We will not use this fact which is proven in [31, Proposition 2.15]. It explains why we have to pay attention to the case (C) in Theorem 7.3. □

In the proof we have used the following lemma.

Lemma 7.7. *Let $n \not\equiv 2 \bmod 4$. If n is not a prime power, then the ring $\mathbb{Z}[\zeta_n]$ contains an imaginary unit.*

Proof. If n is even, the ring $\mathbb{Z}[\zeta_n]$ contains the imaginary unit $i = \zeta_4$.

If n is odd, it admits two distinct odd prime factors p, q and $\mathbb{Q}[\zeta_n]$ contains the imaginary unit $\zeta_{pq} - \zeta_{pq}^{-1}$. □

Combining Lemma 7.6 and Shimura–Taniyama Fact 7.1, we now deduce that there exists a principally polarized abelian variety with multiplication by $\mathcal{O}_{K_n}$ and with any prescribed CM type.

Corollary 7.8. *Let $K_n = \mathbb{Q}[\zeta_n]$, let $g = \varphi(n)/2$ and let $t_{00} \in \mathcal{I}_{K_n}$ nonzero.*

(a) *The fractional ideal $\mathfrak{n} := t_{00}\mathcal{D}_{K_n} \cap K_{n,0}$ is principal.*

(b) *There exist an ideal $\mathfrak{m}$ of $\mathcal{O}_{K_n}$ and $t_0 \in \mathcal{I}_{K_n}$ such that $\Phi_{t_0} = \Phi_{t_{00}}$ and such that $(\mathbb{C}^g/\Phi_{t_0}(\mathfrak{m}), \omega_{t_0})$ is a principally polarized abelian variety.*

Proof.

(a). According to Lemma 7.6 (a), the different $\mathcal{D}_{K_n}$ is a principal ideal of $\mathcal{O}_{K_n}$ generated by an imaginary element $t_n \in \mathcal{I}_{K_n}$. Therefore, the element $t_n t_{00}$ is in $K_{n,0}$ and one has

$$\mathfrak{n} = t_n t_{00} \mathcal{O}_{K_{n,0}}.$$

This proves that $\mathfrak{n}$ is a principal ideal of $K_{n,0}$.

(b). This follows from Fact 7.1 (3), since, by (a), the ideal $\mathfrak{n}$ is principal. □

Thanks to Corollary 7.8, the statement of Theorem 7.3 is cleaner for the cyclotomic field. Here it is:

Proposition 7.9. *Let $K_n := \mathbb{Q}[\zeta_n]$ and $\nu = \mu/\bar{\mu}$ with $\mu = 1 + s - \bar{s}$ where $s \in K_n$ has denominator prime to 2 and $N_{K_n/\mathbb{Q}}(\mu)$ has odd numerator. Let $G_\nu := \mathcal{O}_{K_n}/(\mathcal{O}_{K_n} \cap \nu\mathcal{O}_{K_n})$ and $d_\nu = |G_\nu|$.*

(i) *Then for all CM types Φ of K_n, there exists a critical value*

$$\lambda_\nu = \kappa_\nu \, d_\nu^{1/2} N_\Phi(\nu)^{1/2} \text{ on } G_\nu \text{ with } \kappa_\nu^4 = 1.$$

(ii) *Moreover, if s is in $\mathcal{O}_{K_n}$, one has*

$$\lambda_\nu = \pm N_\Phi(\mu) \text{ and } G_\nu = \mathcal{O}_{K_n}/\mu\mathcal{O}_{K_n}.$$

Remark 7.10. Point (ii) does not apply to $\mu = 1 + \zeta_3 - \overline{\zeta_3} = 1 + i\sqrt{3}$ because the norm of this μ is even. Indeed in this case $G_\nu = \{1\}$.

The group G_ν has order d_ν but is not always cyclic. In the next section, we will give a very simple example where one can check that G_ν is cyclic.

Proof of Proposition 7.9.

(i). We distinguish two cases.

In case the extension $K_n/K_{n,0}$ is ramified, we apply Theorem 7.3 (B).

In case the extension $K_n/K_{n,0}$ is unramified, we apply Theorem 7.3 (C), remembering that, by Corollary 7.8, the ideal $\mathfrak{n} := t_{00}\mathcal{D}_{K_n} \cap K_{n,0}$ is principal, for any t_{00} in $\mathcal{I}_{K_n}$.

(ii). Apply Corollary 7.4. □

7.4. Fibonacci and Lucas numbers. As an application of Proposition 7.9, we construct in Proposition 7.15 new d-critical values where the integers d are Lucas numbers L_n with n prime.

These Lucas numbers are intimately related with the Fibonacci numbers. We recall that the Fibonacci numbers $(F_n)_{n\geq 0} = (0, 1, 1, 2, 3, 5, 8, \dots)$ are defined by their first two terms and the recurrence relation $F_{n+1} = F_n + F_{n-1}$. They can also be defined as the coefficients of a matrix n^{th}-power

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}. \tag{7.5}$$

The Lucas numbers $(L_n)_{n\geq 0} = (2, 1, 3, 4, 7, 11, 18, \dots)$ are also defined by their first two terms and the same recurrence relation $L_{n+1} = L_n + L_{n-1}$. Hence they are equal to the trace of the matrix (7.5): $L_n = F_{n+1} + F_{n-1}$. Together they can be defined by

$$\left(\frac{1+\sqrt{5}}{2}\right)^n = \frac{L_n + F_n\sqrt{5}}{2}. \tag{7.6}$$

We will need a few properties of Fibonacci and Lucas numbers.

Lemma 7.11.

(a) *One has* $F_n = \prod_{1\leq k<n/2}(1 + 4\cos^2(\frac{k\pi}{n}))$, *and*

(b) $F_n = \prod_{d|n} F'_d$ *where* F'_n *is the integer* $F'_n := \prod_{\substack{1\leq k<n/2 \\ k\wedge n=1}}(1+4\cos^2(\frac{k\pi}{n}))$.

Proof of Lemma 7.11. All this is classical. Here are a few hints.

(a). The Fibonacci polynomials $P_n(X)$ are defined by $P_1 = 1$, $P_2 = X$ and $P_{n+1} = XP_n + P_{n-1}$. They satisfy the equality $F_n = P_n(1)$ and are related to the Chebychev polynomial so that the zeros of P_n are the $2i\cos(\frac{k\pi}{n})$ for $1 \leq k \leq n$.

(b). These numbers F'_n are both algebraic integers and invariant by Galois. Hence they are integers. □

Lemma 7.12. *Assume that* m *and* n *are odd.*

(a) *One has* $L_n = \prod_{1\leq k<n/2}(1 + 4\sin^2(\frac{k\pi}{n}))$, *and*

(b) $L_n = \prod_{d|n} L'_d$ *where* L'_n *is the integer* $L'_n := \prod_{\substack{1\leq k<n/2 \\ k\wedge n=1}}(1+4\sin^2(\frac{k\pi}{n}))$.

(c) *These integers* L'_n *are odd except* $L'_3 = 4$.

For instance, one has $L'_5 = 11$, $L'_7 = 29$, $L'_9 = 19, \dots$.

Proof of Lemma 7.12.

(a) and (b). From Formula (7.6), one gets the equality $F_{2n} = F_n L_n$. Our claims together with $L'_n = F'_{2n}$ follow from Lemma 7.11.

(c). Just compute the sequence $(L_n \bmod 8)_{n\geq 0}$ and notice that it has period 12 with pattern $[2,1,3,4,7,3,2,5,7,4,3,7] \bmod 8$. □

The following lemma will be useful.

Lemma 7.13. *For $n \geq 5$ odd, set $\mu_n := 1 + \zeta_n - \zeta_n^{-1}$. Then the quotient ring $\mathbb{Z}[\zeta_n]/\mu_n\mathbb{Z}[\zeta_n]$ is cyclic and isomorphic to $\mathbb{Z}/L'_n\mathbb{Z}$.*

Proof of Lemma 7.13. Denote by x_n the image of ζ_n^{-1} in the quotient ring $A_n = \mathbb{Z}[\zeta_n]/\mu_n\mathbb{Z}[\zeta_n]$. By definition L'_n is equal to the norm of μ_n, that is $L'_n := N_{K_n/\mathbb{Q}}(\mu_n)$. Therefore the ring A_n has order L'_n. The element x_n is invertible in A_n and satisfies the equality $x_n^2 = x_n + 1$. Therefore by (7.6), one has for all integer $k \geq 1$,

$$2\,x_n^k = L_k + (2x_n - 1)F_k.$$

We apply this equality with $k = n$, remembering that, since L'_n divides L_n, one has $L_n \equiv 0$ in A_n. One gets

$$(2x_n - 1)F_n = 2 \quad \text{in } A_n.$$

Since, $n \neq 3$, by Lemma 7.12, L'_n is odd, and the element 2 is invertible in A_n. Therefore x_n is a multiple of 1 in A_n and the group A_n is cyclic. □

Proposition 7.14. *For all $n \geq 5$ odd, and all signs $\varepsilon_k = \pm 1$, the product $\lambda'_n = \prod_{\substack{k<n/2 \\ k\wedge n=1}} (1 + 2i\varepsilon_k \sin(\frac{k\pi}{n}))$ or $-\lambda'_n$ is L'_n-critical.*

Proof of Proposition 7.14. This is a consequence of Proposition 7.9 combined with Lemma 7.13. We work with the cyclotomic field $K = K_n$ and choose the CM type $\Phi := \{\rho_k \,|\, k \in (\mathbb{Z}/n\mathbb{Z})^*\}$ where $\rho_k(\zeta_n) = \zeta_n^{\varepsilon_k k}$, and choose $\mu = \mu_n := 1 + \zeta_n - \zeta_n^{-1}$ so that one has $\lambda'_n := N_\Phi(\mu_n)$. According to Proposition 7.9, λ'_n or $-\lambda'_n$ is a critical value on the group $\mathcal{O}_{K_n}/\mu_n\mathcal{O}_{K_n}$ for a 4^{th} root of unity κ_n. According to Lemma 7.13, this group is cyclic of order L'_n. □

The precise sign κ_n involved with λ'_n could be computed thanks to the exact transformation formula for the Riemann theta function, and the conclusion should be that the value λ'_n (with a plus sign) is always L'_n-critical.

The following is a special case of Proposition 7.14.

Proposition 7.15. *For all $n \geq 5$ prime, and all signs $\varepsilon_k = \pm 1$, the product $\lambda_n = \prod_{1\leq k<n/2} (1 + 2i\varepsilon_k \sin(\frac{k\pi}{n}))$ or $-\lambda_n$ is L_n-critical.*

Remark 7.16. Assume now that the integer n is coprime to 6 but is not a prime number. Since λ_n is the product of λ'_d for d divisors of n, the value $\pm\lambda_n$ is critical on the group G_n product of the cyclic groups $\mathbb{Z}/L'_d\mathbb{Z}$. The product of these integers L'_d is equal to L_n. but, since these integers L'_d are not always coprime, this group G_n is not always the cyclic group $\mathbb{Z}/L_n\mathbb{Z}$.

To be concrete, we give three examples, the first ones being the conjugates of $\lambda_5 = 1+\sqrt{5}+i\sqrt{5-2\sqrt{5}}$ that already occured in [3, Section 1.5], and that was also part of Proposition 6.5:

- $n=5 : \lambda_5 := (1 \pm 2i\sin(\frac{\pi}{5}))(1 \pm 2i\sin(\frac{2\pi}{5})) : 11$-critical.
- $n=7 : \lambda_7 := (1 \pm 2i\sin(\frac{\pi}{7}))(1 \pm 2i\sin(\frac{2\pi}{7}))(1 \pm 2i\sin(\frac{3\pi}{7})) : 29$-critical.
- $n=9 : \lambda'_9 := (1 \pm 2i\sin(\frac{\pi}{9}))(1 \pm 2i\sin(\frac{2\pi}{9}))(1 \pm 2i\sin(\frac{4\pi}{9})) : 19$-critical.

8. On the sign of the critical values

This aim of this section is to explain how to prove that the values λ given in the examples of Section 6 are indeed d-critical. The arguments given in Section 6 prove that either λ or $-\lambda$ is d-critical. We will complete here these arguments and prove that λ is indeed d-critical. This will rely on extra computations together with a formula due to Stark for the theta cocycle $j(\sigma,\tau)$ that we give in Section 8.2. Since this sign is often equal to the plus sign, these computations give a cleaner statement for our examples in Section 6.

8.1. Square root of determinant of Riemann matrices. We begin by two simple lemmas on the determinant of a Riemann matrix. For τ in $\mathcal{H}_g$, we define the function

$$h(\tau) = \det\left(\frac{\tau}{i}\right)^{1/2} \tag{8.1}$$

as the continuous square root of $\det(\frac{\tau}{i})$ such that $h(i\mathbf{1}_g) = 1$.

Lemma 8.1. *Let τ in $\mathcal{H}_g$.*

(a) *One has* $\det_{\mathbb{C}}(\tau/i) \neq 0$.
(b) *All the eigenvalues λ of τ/i have positive real part:* $\mathrm{Re}(\lambda) > 0$.
(c) *The function $h(\tau)$ is the product of the square roots $\lambda^{1/2}$ of the eigenvalues λ of τ/i with* $\arg(\lambda^{1/2}) \in (-\frac{\pi}{4}, \frac{\pi}{4})$.
(d) *If $g=2$, the function $h(\tau)$ has positive real part:* $\mathrm{Re}(h(\tau)) > 0$.

Proof.

(a). If $z \in \mathbb{C}^g$ is in the kernel of τ, one has $\mathrm{Im}({}^t\bar{z}\tau z) = 0$. Since τ is symmetric, this can be rewritten as ${}^t\bar{z}\,\mathrm{Im}(\tau)z = 0$. Since $\mathrm{Im}(\tau)$ is positive this gives $z=0$. And hence τ is invertible.

(b). Indeed, if $\mathrm{Re}(\lambda) \leq 0$, the matrix $\tau - i\lambda\mathbf{1}_g$ is in $\mathcal{H}_g$ and $\det(\tau - i\lambda\mathbf{1}_g) \neq 0$.

(c). This follows from (b) by analytic continuation.

(d). This follows from (c). □

When S is a real $g \times g$ symmetric matrix, its signature s is the number of positive eigenvalues minus the number of negative eigenvalues.

Lemma 8.2. *Let τ in $\mathcal{H}_g$ and S be an invertible real $g \times g$ symmetric matrix and s be its signature. Then one has*

$$h(\tau)\, h(-\tau^{-1} - S) \;=\; e^{is\pi/4}\, h(\tau + S^{-1})\, |\det(S)|^{1/2}. \tag{8.2}$$

Proof. The squares are equal and one checks the sign by looking at the limit when $\tau = it\mathbf{1}_g$ with t going to 0. □

8.2. The sign in the transformation formula. The following lemma give precise formulas for the theta cocycle $j(\sigma, \tau)$ that we introduced in Lemma 2.6. We recall that, for $\sigma = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \mathrm{Sp}^{\theta}_{g,\mathbb{Z}}$ and $\tau \in \mathcal{H}_g$, the theta cocycle is defined by the equality

$$\theta(0, \sigma\tau) = j(\sigma, \tau)\ \theta(0, \tau). \tag{8.3}$$

To compute this cocycle we need to introduce a function $J_{1/2}(\sigma, \tau)$ which is a continuous square root of the tangent cocycle $J(\sigma, \tau) := \det_{\mathbb{C}}(\gamma\tau + \delta)$. The function $J_{1/2}(\sigma, \tau)$, for $\det(\delta) \neq 0$ is given by the formula

$$J_{1/2}(\sigma, \tau) := |\det(\delta)|^{1/2}\, h(\tau)\, h(-\tau^{-1} - \delta^{-1}\gamma), \tag{8.4}$$

where $h(\tau)$ is the function defined in (8.1).

Lemma 8.3 (Stark)**.** *Let $\sigma \in \mathrm{Sp}^{\theta}_{g,\mathbb{Z}}$ with $\det(\delta) \neq 0$ and $\tau \in \mathcal{H}_g$.*

(a) *Then, one has*

$$j(\sigma, \tau) = \kappa(\sigma)\ J_{1/2}(\sigma, \tau) \tag{8.5}$$

where $\kappa(\sigma)$ is the 8^{th} root of unity given by the normalized Gauss sum

$$\kappa(\sigma) = \frac{1}{|\det(\delta)|^{1/2}} \sum_{x \in \delta^{-1}\mathbb{Z}^g/\mathbb{Z}^g} e^{i\pi^t x^t \delta\beta x} \tag{8.6}$$

(b) *If $\det(\delta) = \pm d_0$ with $d_0 > 0$ odd, square-free and $\det(\gamma) \neq 0$, one has*

$$j(\sigma, \tau) = e^{is\pi/4}\, \varepsilon_{d_0} \left(\frac{c}{d_0}\right) |\det(\delta^{-1}\gamma)|^{1/2}\, h(\tau + \gamma^{-1}\delta), \tag{8.7}$$

where

- *s is the signature of the real symmetric matrix $S := \delta^{-1}\gamma$,*
- *$\varepsilon_{d_0} = 1$ when $d_0 \equiv 1 \bmod 4$ and $\varepsilon_{d_0} = -i$ when $d_0 \equiv 3 \bmod 4$,*
- *$c = 2\, d_0\, {}^t m \delta^{-1}\gamma m$ where m is any primitive vector of $\mathbb{Z}^g$,*
- *$(\frac{c}{d_0})$ is a Jacobi symbol.*

This precise determination of the theta cocycle has a long history.

For $g = 1$, these formulas which are due to Hecke were useful in [3].

When $g > 1$, the first formulas (8.5) and (8.6) were known to Siegel and Igusa, see [14, p. 228]. They can be explicitly found in [17, 2.2.26 p. 169], [27, p. 7] or [11, p. 26–27]. There also exists a very simple formula for $\kappa(\sigma)^2$

due to Igusa in [15, p. 182]. It says that the map $\sigma \mapsto \kappa(\sigma)^2$ is a character of $\mathrm{Sp}^{\theta}_{g,\mathbb{Z}}$ and when σ is in $\mathrm{Sp}^{2}_{g,\mathbb{Z}}$, one has $\kappa(\sigma)^2 = e^{i\frac{\pi}{2}\,\mathrm{Tr}(\delta - \mathbb{1}_g)}$.

The second formula (8.7) is due to Stark in [27] for an odd prime d_0 and was extended by Styer in [30, Theorem 2 p. 660] to a square-free d_0, even allowing a degenerate block γ. The proof of (8.7) relies on (8.2), on the first formulas (8.5) and (8.6), and on the classical formula for Gauss sums on cyclic groups. An extension of (8.7) where d_0 is not square-free can be found in [29].

The strong relationship between the transformation formula (8.3) and the Weil representation was discovered by A. Weil. See [12], [17] and [22].

We will only need Lemma 8.3 when $d_0 = 1$. This case is easier because the Gauss sum and the Jacobi symbol are equal to 1.

Corollary 8.4. *Let $\sigma \in \mathrm{Sp}^{\theta}_{g,\mathbb{Z}}$, $\tau \in \mathcal{H}_g$. If $\det(\delta) = \pm 1$, $\det(\gamma) \neq 0$, one has*

$$j(\sigma,\tau) = e^{is\pi/4}\,|\det(\delta^{-1}\gamma)|^{1/2}\,h(\tau + \gamma^{-1}\delta). \tag{8.8}$$

8.3. Finding the sign of critical values. We now explain the strategy to compute sign of the critical value λ_ν in the examples of Section 6. This strategy follows the proof of Theorem 4.4.

We recall that $(A = \mathbb{C}^g/\Lambda, \omega)$ is a principally polarized abelian variety and that ν is a unitary $\mathbb{Q}$-endomorphism of A preserving a theta structure of level 2, that G_ν is the finite abelian group $G_\nu := \Lambda/(\Lambda \cap \nu\Lambda)$. Remember that ν can be thought both as a unitary transformation T_ν of $\mathbb{C}^n$ or as a symplectic transformation h_ν of $(\Lambda_{\mathbb{Q}}, \omega)$, and that, as in (4.2), in a symplectic basis $e_1, \dots, e_g, -f_1, \dots, -f_g$ of Λ, the multiplication by ν^{-1} is given by a symplectic rational matrix $m_\nu \in \mathrm{Sp}^{\theta,2}_{g,\mathbb{Q}}$ preserving a theta structure of level 2. Since it is elliptic, this matrix m_ν has a fixed point $\theta \in \mathcal{H}_g$:

$$m_\nu \theta = \theta$$

By Proposition 3.4, there exists σ_1 and σ_2 in $\mathrm{Sp}(g,\mathbb{Z})$ and an integral matrix $\mathbf{d}$ such that the symplectic matrix m_ν can be written as

$$m_\nu = \sigma_1\, D\, \sigma_2 \quad \text{with} \quad D = \begin{pmatrix} {}^t\mathbf{d}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{d} \end{pmatrix}. \tag{8.9}$$

When $g = 2$, we will often but not always choose $\mathbf{d} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & d \end{smallmatrix}\right)$.

By Lemma 3.5 the matrix $\sigma := \sigma_2\sigma_1$ belongs to $\mathrm{Sp}^{\theta,2}_{g,\mathbb{Z}}$. This matrix $\sigma = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)$, and the element $\tau =: \sigma_1^{-1}\theta \in \mathcal{H}_g$ are related by the equality $\sigma\tau = D\tau$ that is:

$$\sigma\tau = {}^t\mathbf{d}\tau\mathbf{d}.$$

Therefore one can apply Lemma 8.3 to compute the critical value λ_ν given by Corollary 3.1. In all the examples below, since we have already computed λ_ν^2 in Section 6, we will be able to avoid a few computations thanks to the following corollary:

Corollary 8.5. *Assume* $\det(\gamma) \neq 0$. *Let* s *be the signature of* $\delta^{-1}\gamma$.

(a) *If* $g = 1$ *and* $|\delta| = 1$, *then one has* $\mathrm{Re}(\lambda_\nu) > 0$.

(b) *If* $g = 2$ *and* $|\det(\delta)| = 1$, *then one has* $\mathrm{Re}(e^{-is\pi/4}\lambda_\nu) > 0$.

Proof. By Corollary 3.1, one has $\lambda_\nu = j(\sigma', 2\tau)\,|G_\nu|$. By Formula (8.8), one has $\lambda_\nu = e^{is\pi/4}\,h(\tau + \gamma^{-1}\delta)\,|\det(\gamma)|^{1/2}\,|G_\nu|$. We now use Lemma 8.1.

When $g = 1$, one has $s = \pm 1$, and $-\frac{\pi}{4} < \arg(h(\tau + \gamma^{-1}\delta)) < \frac{\pi}{4}$.

When $g = 2$, one has $\mathrm{Re}(h(\tau + \gamma^{-1}\delta)) > 0$. □

Hence, for computing precisely λ_ν, the remaining two steps beyond the steps (a) to (f) in Section 6 are :

(g) We compute a decomposition $m_\nu = \sigma_1 D \sigma_2$ with σ_1, σ_2 in $\mathrm{Sp}(g, \mathbb{Z})$.

(h) We write $\sigma = \sigma_2\sigma_1 = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)$ and conclude thanks to Corollary 8.5.

8.4. The sign for imaginary quadratic fields. In this section we check the sign of the d-critical values from Proposition 6.1. We will use freely the notation of Sections 6.1 and 8.3. Remember that, from Part (f) of Section 6.1, we know that $\lambda_\nu = \pm(\sqrt{a} + i\sqrt{b})$.

(g). We recall that in the basis $e_1, -f_1$, one has $m_\nu = \frac{1}{d}\left(\begin{smallmatrix} a-b & -ab \\ 4 & a-b \end{smallmatrix}\right)$ and that $\theta := i\sqrt{ab}/2 \in \mathcal{H}_1$ is fixed by m_ν. One checks that $m_\nu = \sigma_1 D \sigma_2$ with $\sigma_1 = \left(\begin{smallmatrix} a-b & \frac{a-b-1}{4} \\ 4 & 1 \end{smallmatrix}\right)$, $D = \left(\begin{smallmatrix} d^{-1} & 0 \\ 0 & d \end{smallmatrix}\right)$, and $\sigma_2 = \left(\begin{smallmatrix} 1 & \frac{a-b-d^2}{4} \\ 0 & 1 \end{smallmatrix}\right)$.

Hence one has $\sigma\tau = d^2\sigma$ with $\sigma = \sigma_2\sigma_1 = \left(\begin{smallmatrix} 2a-2b-d^2 & a-(d+1)^2/4 \\ 4 & 1 \end{smallmatrix}\right)$ and with $\tau = \sigma_1^{-1}\,\theta = \frac{(a-b-d^2)+2i\sqrt{ab}}{4d^2}$. By (6.1) this matrix σ belongs to the theta subgroup $\mathrm{Sp}^{\theta,2}_{1,\mathbb{Z}}$ of level 2.

(h). One has $g = 1$ and $\delta = 1$. Hence by Corollary 8.5, the real part of the critical value λ_ν is positive. Hence, using Part (f) of Section 6.1, we conclude that $\lambda_\nu = \sqrt{a} + i\sqrt{b}$. And this value is always d-critical.

8.5. The sign for products of imaginary quadratic fields. In this section we check the sign of the d-critical values from Proposition 6.4. We will use freely the notation of Sections 6.2 and 8.3. Remember that, from Part (f) of Section 6.2, we know that $\lambda_\nu = \pm(\sqrt{a_1} + i\varepsilon_1\sqrt{b_1})(\sqrt{a_2} + i\varepsilon_2\sqrt{b_2})$.

To simplify the calculation we will begin by the simpler symplectic basis $e_{0,1}, \ldots, e_{0,g}, -f_{0,1}, \ldots, -f_{0,g}$ for which the matrix $m_{0,\nu} \in \mathrm{Sp}(g, \mathbb{Q})$ of multiplication by ν^{-1} has been computed in (6.3)

$$m_{0,\nu} = \begin{pmatrix} u_1 & 0 & \varepsilon_1 v_1 & 0 \\ 0 & u_2 & 0 & \varepsilon_2 v_2 \\ \varepsilon_1 w_1 & 0 & u_1 & 0 \\ 0 & \varepsilon_2 w_2 & 0 & u_2 \end{pmatrix},$$

with

$$u_j := \frac{a_j - b_j}{d'_j} \equiv 1 \bmod 4,$$
$$v_j := \frac{-2a_jb_j}{d'_j} \equiv 4 \bmod 8,$$
$$w_j := \frac{2}{d'_j} \equiv 2 \bmod 8.$$

One finds an explicit decomposition $m_{0,\nu} = \sigma_{0,1} D_0 \sigma_{0,2}$ with

$$\sigma_{0,1} = \begin{pmatrix} a_1-b_1 & 0 & \varepsilon_1(a_1-b_1-1)/2 & 0 \\ 0 & a_2-b_2 & 0 & \varepsilon_2(a_2-b_2-1)/2 \\ 2\varepsilon_1 & 0 & 1 & 0 \\ 0 & 2\varepsilon_2 & 0 & 1 \end{pmatrix}$$
$$\sigma_{0,2} = \begin{pmatrix} 1 & 0 & \varepsilon_1(a_1-b_1-d_1^2)/2 & 0 \\ 0 & 1 & 0 & \varepsilon_2(a_2-b_2-d_2^2)/2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

and

$$D_0 = \begin{pmatrix} {}^t\mathbf{d}_0^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{d}_0 \end{pmatrix} \quad \text{where } \mathbf{d}_0 = \begin{pmatrix} d'_1 & 0 \\ 0 & d'_2 \end{pmatrix}.$$

The product $\sigma_0 = \sigma_{0,2}\sigma_{0,1}$ can be written as $\sigma_0 = \left(\begin{smallmatrix} \alpha_0 & \beta_0 \\ \gamma_0 & \delta_0 \end{smallmatrix}\right)$ where

$$\alpha_0 = \begin{pmatrix} 2a_1-2b_1-d_1^2 & 0 \\ 0 & 2a_2-2b_2-d_2^2 \end{pmatrix},$$
$$\beta_0 = \begin{pmatrix} \varepsilon_1(4a_1-(d_1+1)^2)/2 & 0 \\ 0 & \varepsilon_2(4a_2-(d_2+1)^2)/2 \end{pmatrix} \tag{8.10}$$
$$\gamma_0 = \begin{pmatrix} 2\varepsilon_1 & 0 \\ 0 & 2\varepsilon_2 \end{pmatrix}, \quad \delta_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

As we have seen in Section 6.2 we need to introduce a new lattice Λ with basis $(e_1, e_2, -f_1, -f_2) = (e_{0,1}, e_{0,2}, -f_{0,1}, -f_{0,2})\, P$ where P is a basis change matrix. This matrix P is a scalar multiple of an element of $\mathrm{Sp}(g, \mathbb{R})$. We need to distinguish two cases. A key difference between these two cases will be the value of the parameter s, when we will apply Corollary 8.5.

First case: We assume that $\varepsilon_1 = -\varepsilon_2 = 1$. As in Section 6.2 (d1), we choose $P = \left(\begin{smallmatrix} \mathbf{p} & \mathbf{0} \\ \mathbf{0} & \mathbf{p} \end{smallmatrix}\right)$ with $\mathbf{p} = \left(\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right)$.

(g1). In this new basis, the matrix $m_\nu = P^{-1} m_{0,\nu} P$ of multiplication by ν^{-1} belongs to $\mathrm{Sp}^{\theta,2}_{2,\mathbb{Q}}$. One can write $m_\nu = \sigma_1 D \sigma_2$ with

$$\sigma_1 := P^{-1}\sigma_{0,1}P\ , \quad D = P^{-1}D_0P\ , \quad \sigma_2 = P^{-1}\sigma_{0,2}P.$$

Since the matrix P is block diagonal, the matrix D is too, and one has

$$D = \begin{pmatrix} {}^t\mathbf{d}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{d} \end{pmatrix} \quad \text{with} \quad \mathbf{d} = \mathbf{p}^{-1}\mathbf{d}_0\mathbf{p} = \begin{pmatrix} d'_+ & d'_- \\ d'_- & d'_+ \end{pmatrix} \text{ where } d'_\pm = \frac{d'_1 \pm d'_2}{2}.$$

Both σ_1 and σ_2 belong to $\mathrm{Sp}(g, \mathbb{Z})$ and one has

$$\sigma = \sigma_1\sigma_2 = P^{-1}\sigma_0 P = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \mathbf{p}^{-1}\alpha_0\mathbf{p} & \mathbf{p}^{-1}\beta_0\mathbf{p} \\ \mathbf{p}^{-1}\gamma_0\mathbf{p} & \mathbf{p}^{-1}\delta_0\mathbf{p} \end{pmatrix}.$$

By Lemma 3.5, this matrix σ belongs to the theta subgroup $\mathrm{Sp}^{\theta,2}_{2,\mathbb{Z}}$ of level 2.

(h1). Using (8.10), one checks that $\det(\delta) = 1$ and that the symmetric matrix $\gamma^{-1}\delta$ has signature $s = 0$. Hence by Corollary 8.5, the real part of the critical value λ_ν is positive. Therefore, using Part (f) of Section 6.2, we conclude that $\lambda_\nu = (\sqrt{a_1} + i\sqrt{b_1})(\sqrt{a_2} - i\sqrt{b_2})$ and that this value is always d-critical.

Second case: We assume that $\varepsilon_1 = \varepsilon_2 = 1$. As in Section 6.2 (d2), we choose $P = \left(\begin{smallmatrix} \mathbf{2} & \mathbf{p} \\ \mathbf{0} & \mathbf{1} \end{smallmatrix}\right)$ with $\mathbf{p} = \left(\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right)$.

(g2). In this new basis, the matrix $m_\nu = P^{-1}m_{0,\nu}P$ of multiplication by ν^{-1} also belongs to $\mathrm{Sp}^{\theta,2}_{2,\mathbb{Q}}$. Since the matrix P is not block diagonal, neither is the matrix $P^{-1}DP$. This is why we introduce the commutator

$$\sigma_{0,3} = PD_0^{-1}P^{-1}D_0 = \left(\begin{smallmatrix} \mathbf{1} & \mathbf{x} \\ \mathbf{0} & \mathbf{1} \end{smallmatrix}\right) \quad \text{where} \quad \mathbf{x} = \mathbf{p} - \mathbf{d}_0\mathbf{p}\mathbf{d}_0.$$

One can write $m_\nu = \sigma_1 D \sigma_2$ with

$$\sigma_1 := P^{-1}\sigma_{0,1}P\ , \quad D := D_0\ , \quad \sigma_2 := P^{-1}\sigma_{0,3}\sigma_{0,2}P.$$

Both σ_1 and σ_2 belong to $\mathrm{Sp}(g, \mathbb{Z})$ and one has $\sigma = \sigma_2\sigma_1 = P^{-1}\widetilde{\sigma}_0 P$ where

$$\widetilde{\sigma}_0 := \sigma_{0,3}\sigma_0 = \begin{pmatrix} \alpha_0 + \mathbf{x}\gamma_0 & \beta_0 + \mathbf{x}\delta_0 \\ \gamma_0 & \delta_0 \end{pmatrix}.$$

By Lemma 3.5, this matrix σ belongs to the theta subgroup $\mathrm{Sp}^{\theta,2}_{2,\mathbb{Z}}$ of level 2. This matrix $\sigma = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)$ has lower blocks $\gamma = 2\gamma_0$ and $\delta = \delta_0$

(h2). The end of the argument is as in (h1). One checks that $\det(\delta) = 1$. This time the symmetric matrix $\gamma^{-1}\delta$ is positive and hence has signature $s = 2$. Therefore by Corollary 8.5, the imaginary part of the critical value λ_ν is positive. Hence, using again Part (f) of Section 6.2, we conclude that $\lambda_\nu = (\sqrt{a_1} + i\sqrt{b_1})(\sqrt{a_2} + i\sqrt{b_2})$. And this value is always d-critical.

8.6. The sign for quartic CM fields. In this section we check the sign of the d-critical values from Proposition 6.5. We will use freely the notation of Sections 6.3 and 8.3. Remember from Part (f) of Section 6.3, we know that $\lambda_\nu = \pm i^{(a-1)/2}(\sqrt{a}+\sqrt{c}-i\sqrt{b-2\sqrt{ac}})$.

As in Section 6.3, we distinguish two cases, where $\varepsilon = \pm 1$.

First case: We assume that $a \equiv \varepsilon \bmod 4$, and $b \equiv c \equiv 0 \bmod 4$. As in Section 6.3(d1), we choose $P = \left(\begin{smallmatrix} 2\mathbf{p} & \mathbf{0} \\ \mathbf{0} & 2\,{}^t\mathbf{p}^{-1} \end{smallmatrix}\right)$ with $\mathbf{p} = \left(\begin{smallmatrix} 2 & b/2 \\ 0 & 1/2 \end{smallmatrix}\right)$.

(g1). We have computed in (6.11) the matrix m_ν of multiplication by ν^{-1} in the corresponding basis e_1, e_2, $-f_1$, $-f_2$. This matrix m_ν belongs to $\mathrm{Sp}^{\theta,2}_{2,\mathbb{Q}}$. Using the method of elimination by symplectic matrices, we find explicit matrices $\mathbf{d}$, σ_1 and σ_2 satisfying (8.9). They are given by $\mathbf{d} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & d \end{smallmatrix}\right)$,

$$\sigma_1 = \begin{pmatrix} 1 & (\varepsilon d-1)(b+c)/2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & (\varepsilon d(d-2a)+d-2c)/4 & (1-\varepsilon d)(b+c)/2 & 1 \end{pmatrix},$$

$$\sigma_2 = \begin{pmatrix} 1-2\varepsilon(b+c) & \varepsilon(b+c)(2c-d+\varepsilon)/2-c & \varepsilon(b+c)^2-4c & -2\varepsilon(b+c) \\ 4 & d-2c & -2b-2c & 4 \\ 0 & 1 & 1 & 0 \\ \varepsilon & (\varepsilon(d+2b)-1)/4 & 0 & \varepsilon \end{pmatrix}.$$

Since $d \equiv \varepsilon \bmod 4$ and $b \equiv c \equiv 0 \bmod 4$, these two matrices have integer coefficients. Hence, by Lemma 3.5, the matrix $\sigma := \sigma_2\sigma_1$ belongs to $\mathrm{Sp}^{\theta,2}_{g,\mathbb{Z}}$.

(h1). If we write $\sigma = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)$, the 2×2 lower blocks are given by

$$\gamma = \begin{pmatrix} 0 & 2 \\ 2\varepsilon & x \end{pmatrix} \quad \text{and} \quad \delta = \begin{pmatrix} 1 & 0 \\ y & \varepsilon \end{pmatrix},$$

where x and y are integers. This proves that $\det(\delta) = \varepsilon = \pm 1$ and the symmetric matrix $\delta^{-1}\gamma = \left(\begin{smallmatrix} 0 & 2 \\ 2 & \varepsilon(x-2y) \end{smallmatrix}\right)$, has signature $s = 0$.

Therefore by Corollary 8.5, the real part of the critical value λ_ν is positive. Hence, using again Part (f) of Section 6.3, we conclude that

$$\lambda_\nu = \begin{cases} \sqrt{a}+\sqrt{c}-i\sqrt{b-2\sqrt{ac}} & \text{when } \varepsilon = 1, \\ \sqrt{b-2\sqrt{ac}}+i\sqrt{a}+i\sqrt{c} & \text{when } \varepsilon = -1. \end{cases}$$

And these values and their complex conjugate are always d-critical.

Second case: We assume that $a \equiv b \equiv c \equiv \varepsilon \bmod 4$. As in Section 6.3(d2), we choose $P = \left(\begin{smallmatrix} 2\mathbf{p} & \mathbf{p} \\ \mathbf{0} & 2\,{}^t\mathbf{p}^{-1} \end{smallmatrix}\right)$ with $\mathbf{p} = \left(\begin{smallmatrix} 2 & b/2 \\ 0 & 1/2 \end{smallmatrix}\right)$.

(g2). We have computed in (6.12) the matrix m_ν of multiplication by ν^{-1} in the corresponding basis e_1, e_2, $-f_1$, $-f_2$. This matrix m_ν belongs to $\mathrm{Sp}^{\theta,2}_{2,\mathbb{Q}}$. Using the method of elimination by symplectic matrices, we find explicit matrices $\mathbf{d}$, σ_1 and σ_2 satisfying (8.9). They are given by $\mathbf{d} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & d \end{smallmatrix}\right)$,

$$\sigma_1 = \begin{pmatrix} 1 & 2b+2c+2 & 0 & 0 \\ -1 & d-2c-2 & (b+c+1)h/2 & -h/4 \\ 0 & -4 & 1-2\varepsilon b-2\varepsilon c-2\varepsilon & \varepsilon \\ 2 & -2d+4c & -(b+c+1)(h+2\varepsilon) & (h+2\varepsilon)/2 \end{pmatrix},$$

$$\sigma_2 = \begin{pmatrix} 1 & -F & \varepsilon(b+c+1)^2-c & -F/2 \\ -1 & c+df/2 & (b+c+1)(1-\varepsilon d)/2-1 & (2c+df-2)/4 \\ 0 & 2 & 1 & 1 \\ 0 & 2 & 0 & 1 \end{pmatrix},$$

where $h := \varepsilon d + 2\varepsilon b - 1$, $f := (2a-d-2)\varepsilon - 1$ and $F = (b+c+1)f + 2c + 2$.

Since $a \equiv b \equiv c \equiv \varepsilon \bmod 4$, these two matrices have integer coefficients. Hence, by Lemma 3.5, the matrix $\sigma := \sigma_2\sigma_1$ belongs to $\mathrm{Sp}^{\theta,2}_{g,\mathbb{Z}}$.

(h2). If we write $\sigma = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)$, the 2×2 lower blocks are given by

$$\gamma = \begin{pmatrix} 0 & -8 \\ 0 & -4 \end{pmatrix} \quad \text{and} \quad \delta = \begin{pmatrix} 1-4\varepsilon(b+c+1) & 2\varepsilon \\ -2\varepsilon(b+c+1) & \varepsilon \end{pmatrix},$$

where x and y are integers. This proves that $\det(\delta) = \varepsilon = \pm 1$. We can not apply our Corollary 8.5 because the symmetric matrix $\delta^{-1}\gamma = \left(\begin{smallmatrix} 0 & 0 \\ 0 & -4\varepsilon \end{smallmatrix}\right)$ is degenerate. Instead we use Formulas (8.5) and compute

$$j(\sigma', 2\tau) = h(\tau)\, h(-\tau^{-1} - \delta^{-1}\gamma) = \det(\mathbf{1} + \delta^{-1}\gamma\tau)^{1/2} = (1 - 4\varepsilon\tau_{2,2})^{1/2}$$

which always has positive real part. Hence, λ_ν also has positive real part and, using again Part (f) of Section 6.3, we conclude that

$$\lambda_\nu = \begin{cases} \sqrt{a} + \sqrt{c} - i\sqrt{b - 2\sqrt{ac}} & \text{when } \varepsilon = 1, \\ \sqrt{b - 2\sqrt{ac}} + i\sqrt{a} + i\sqrt{c} & \text{when } \varepsilon = -1. \end{cases}$$

And these values and their complex conjugate are always d-critical.

8.7. Conclusion. In this paper we have constructed many d-critical values λ. However, there remain many explicit challenges (see also [3, Section 2.3]):

(q1) What is the list of d-critical values for $d = 17$?
(q2) Is the value $\lambda := 7 + 2\sqrt{5}$ d-critical for $d = 29$?
(q3) Is the value $\lambda := -1 + 6i$ d-critical for $d = 37$?
(q4) Is the value $\lambda := -7$ d-critical for $d = 49$?

I would like to conclude by explaining why the theory we developed in this paper is so delicate for such elementary statements as in Section 1.3. In short the answer is that the experimental list of d-critical values is subtle and the method to justify it must contain these subtleties. More precisely, there are three main reasons.

The first reason is the congruence condition "$\lambda \equiv 1 \bmod 2$" that the d-critical values λ must satisfy. For instance the value $\lambda := 1+\sqrt{5}+i\sqrt{9-2\sqrt{5}}$ is 15-critical while the value $\lambda := 1+\sqrt{3}+i\sqrt{11-2\sqrt{3}}$ is not 15-critical. Our theory has to be general and to see this congruence condition. From the point of view of abelian varieties this condition follows from a deep understanding of the theta structure of level 2 due to Igusa.

The second reason is the sign issue. For instance, the value $\lambda := 1+2i$ is 5-critical but the opposite value $\lambda := -1-2i$ is not 5-critical. Similarly, the value $\lambda := 1+\sqrt{5}+i\sqrt{9-2\sqrt{5}}$ is 15-critical, while the opposite value $\lambda := -1-\sqrt{5}-i\sqrt{9-2\sqrt{5}}$ is most probably not critical -at least one can check that there does not exist even λ-critical functions. Our theory has to be general and to see this sign issue. From the point of view of abelian varieties this sign issue follows from a deep understanding of the theta cocycle due to Stark.

The third reason is that one has to deduce very explicit values that are d-critical. Our theory has to be general and to be explicit. This is possible because of a deep construction of abelian varieties with complex multiplication by the ring of integers of a CM number fields due to Shimura and Taniyama.

Beyond these three theorical reasons, one has to verify explicitly that the theta structure of level 2 is preserved and that the theta cocycle has the expected sign.

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