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On Grothendieck's section conjecture for curves of index 1

par Giulio BRESCIANI

Résumé. Dans tous les cas où la conjecture de la section de Grothendieck a été démontrée, la courbe ne possède aucun point rationnel et, de plus, son indice est différent de 1. Nous fournissons de nombreux nouveaux exemples de courbes satisfaisant la conjecture ; en particulier, nous démontrons que les exemples d'indice 1 sont très fréquents.

Étant donné un nombre premier impair p, nous démontrons que toute courbe hyperbolique munie d'une action fidèle d'un p-groupe non cyclique admet une forme tordue d'indice 1 qui satisfait la conjecture de la section de Grothendieck. De plus, nous montrons que, pour toute courbe hyperbolique S définie sur un corps k de type fini sur $\mathbb{Q}$, il existe une extension finie K/k et un revêtement étale fini $C \to S_K$ tels que C vérifie la conjecture.

Abstract. Each curve for which Grothendieck's section conjecture has been proved has no rational points, and additionally it has index different from 1. We provide many new examples of curves satisfying the conjecture; in particular, we prove that examples of index 1 are very common.

Given an odd prime p, we prove that every hyperbolic curve with a faithful action of a non-cyclic p-group has a twisted form of index 1 which satisfies Grothendieck's section conjecture. Furthermore, we prove that for every hyperbolic curve S over a field k finitely generated over $\mathbb{Q}$ there exists a finite extension K/k and a finite étale cover $C \to S_K$ such that C satisfies the conjecture.

1. Introduction

Curves are always smooth, projective and geometrically connected over a base field k.

Given a hyperbolic curve C over a field k finitely generated over $\mathbb{Q}$, there is a short exact sequence of étale fundamental groups

$$1 \longrightarrow \pi_1(C_{\bar{k}}) \longrightarrow \pi_1(C) \longrightarrow \mathrm{Gal}(\bar{k}/k) \longrightarrow 1.$$

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Denote by $\Pi_{C/k}(k)$ the set of sections of this short exact sequence modulo the action by conjugation of $\pi_1(C_{\bar{k}})$. There is a natural map

$$C(k) \longrightarrow \Pi_{C/k}(k).$$

In 1983, in a famous letter to Faltings [12], Grothendieck conjectured that this map is bijective; this statement is known as the *section conjecture*. Injectivity was already known to Grothendieck and follows from the Mordell–Weil theorem; the hard part is surjectivity. Some cases of the conjecture are known; these are all curves without rational points. Let us briefly recall them.

- The conjecture holds if k has an embedding $k \subset \mathbb{R}$ such that $C(\mathbb{R}) = \emptyset$. The literature contains several proofs starting with Huisman in 2001 [16, Proposition 4.2 (7.)], [19, Corollary 3.13], [24, Appendix A], [20], [28]. The result was probably known even before Huisman, though; exact attribution is unclear.
- In 2008, J. Stix proved it in the case in which $k = \mathbb{Q}$ and C maps to a non-trivial Brauer–Severi variety [24] (see [5] for a generalization). More generally, Stix proved the conjecture if k is a number field and C maps to a Brauer–Severi variety P such that, for some place $\mathfrak{p}$ of characteristic p, the order of $[P_{k_\mathfrak{p}}] \in \mathrm{Br}(k_\mathfrak{p})$ is not a power of p [24, Section 7.4].
- Later in 2008, D. Harari and T. Szamuely proved it in the case in which the cohomology class of $\underline{\mathrm{Pic}}^1_C$ in $\mathrm{H}^1(k, \underline{\mathrm{Pic}}^0_C)$ is not in the maximal divisible subgroup [14].
- In 2010, R. Hain proved it for the generic fiber of the universal curve over $\mathscr{M}_g$, $g \geq 5$, over a field finitely generated over $\mathbb{Q}$ [13].

Recall that the index of a curve C is the greatest common divisor of the degrees of the residue fields of the closed points of C. All the results above share a common feature: they are all about curves of index larger than 1. This is obvious for curves without real points and for the results of Stix and Harari-Szamuely. For Hain's result, it follows from the Franchetta conjecture — proved by Beauville (unpublished) and Arbarello–Cornalba [1] — which states that the index of the generic curve is $2g - 2 = \deg \omega$ for $g \geq 3$.

This is not by chance: at least in the first three cases listed above, it can be checked that the authors prove that the *abelianized* fundamental group (that is, the étale fundamental group of $\underline{\mathrm{Pic}}^1_C$) has no Galois sections [25, Chapter 3], even if they do not state it explicitly. If a curve has index 1, then $\underline{\mathrm{Pic}}^1_C(k) \neq \emptyset$ and hence the abelianized fundamental group has a Galois section.

While A. Tamagawa showed that it is sufficient to prove the conjecture for curves without rational points [25, Corollary 101], curves of index larger than 1 are not sufficient. In order to make progress, we need to deal with curves of index 1 and hence to rely more on the anabelian nature of the fundamental group.

We use a non-abelian method to prove two results. The first is the existence and abundance of curves of index 1 satisfying the conjecture. The second is that, up to a finite extension of the base field, every hyperbolic curve has a finite étale cover which satisfies the conjecture.

Curves of index 1. We prove that every hyperbolic curve with a faithful action of a non-cyclic p-group for p an odd prime has a twisted form of index 1 which satisfies the conjecture.

Theorem 1. *Let C be a smooth, projective, geometrically connected hyperbolic curve over a field k finitely generated over $\mathbb{Q}$ with a faithful action of a finite, non-cyclic p-group G for some odd prime p.*

There exists a finitely generated extension K/k and a twisted form $\mathfrak{C}$ of C_K such that $\mathfrak{C}$ has index 1 and $\Pi_{\mathfrak{C}/K}(K) = \emptyset$. In particular, Grothendieck's section conjecture holds for $\mathfrak{C}$.

The theorem essentially holds for $p = 2$, too, but in this case there are a few exceptions that we have to rule out, see Theorem 18.

There is a birational version of the section conjecture which is arguably more approachable, and more is known about it [9, 11, 15, 17, 21, 26]. In particular, H. Esnault and O. Wittenberg have proved a result about curves of index 1, let us describe it.

Let C be a curve over a number field k. Denote by $G^{[\mathrm{ab}]}_{k(C)}$ the Galois group of $\bar{k}(C)^{\mathrm{ab}}/k(C)$, where $\bar{k}(C)^{\mathrm{ab}}$ is the maximal abelian subextension of $\overline{k(C)}/\bar{k}(C)$. There is a natural projection $G^{[\mathrm{ab}]}_{k(C)} \to \mathrm{Gal}(\bar{k}/k)$; we might think of it as a birational, abelianized version of $\pi_1(C) \to \mathrm{Gal}(\bar{k}/k)$. Assuming that the Tate-Shafarevich group of the Jacobian of C is finite, H. Esnault and O. Wittenberg have proved that C has index 1 if and only if $G^{[\mathrm{ab}]}_{k(C)} \to \mathrm{Gal}(\bar{k}/k)$ is split.

While at first glance Theorem 1 might seem in conflict with this result, they are actually perfectly compatible. In fact, if we believe in Grothendieck's anabelian philosophy and C is a hyperbolic curve of index 1 with $C(k) = \emptyset$, then we should expect $\pi_1(C) \to \mathrm{Gal}(\bar{k}/k)$ to be non-split even though the *abelianized* fundamental group has a splitting induced by some point of $\underline{\mathrm{Pic}}^1_C(k) \neq \emptyset$ (ignoring the "birational" side of the story for clarity's sake).

Étale covers. Recall that, if $C \to S$ is a finite étale cover and the section conjecture holds for C_K *for all finite extensions* K/k, then the section conjecture holds for S [25, Proposition 111]. Using the same methods as in Theorem 1, we prove that every hyperbolic curve, after a finite extension of the base field, has a finite étale cover which satisfies the conjecture.

Theorem 2. *Let S be a hyperbolic curve over a field k and ℓ a prime. Assume either that k is finitely generated over $\mathbb{Q}$, or that it is a finite extension of $\mathbb{Q}_p$ with $p \neq \ell$. There exists a finite extension K/k and a finite étale $(\mathbb{Z}/\ell)^2$-cover $\mathfrak{C} \to S_K$ such that $\Pi_{\mathfrak{C}/K}(K) = \emptyset$. In particular, Grothendieck's section conjecture holds for $\mathfrak{C}$.*

Notice that in the statement of Theorem 2 K/k is a *finite* extension, while in Theorem 1 K/k is *finitely generated*: we obtain a better control over K/k in exchange for losing control over the index of $\mathfrak{C}$.

Étale fundamental gerbes. We use the formalism of *étale fundamental gerbes* introduced by N. Borne and A. Vistoli [2, Section 8], [4, Appendix A]: it is an alternative point of view on the theory of étale fundamental groups which is particularly well-suited for studying the section conjecture. From this point of view, the main object is *not* the fundamental group, but *the space of Galois sections itself*, constructed directly without passing through fundamental groups.

If S is a geometrically connected scheme (or algebraic stack) over a field k, the étale fundamental gerbe of S is a pro-finite étale gerbe $\Pi_{S/k}$ over k with a structural morphism $S \to \Pi_{S/k}$ which is universal for this property: for every morphism $S \to \Phi$ where Φ is a pro-finite étale gerbe, there exists a factorization $S \to \Pi_{S/k} \to \Phi$ unique up to a unique isomorphism.

The space of Galois sections of S is in natural bijection with the set of isomorphism classes of the groupoid $\Pi_{S/k}(k)$. If S is a hyperbolic curve over a field finitely generated over $\mathbb{Q}$, then it is known that $\Pi_{S/k}(k)$ is already a set (as opposed to a groupoid), namely its elements have no non-trivial automorphisms [25, Proposition 104], [4, Lemma 2.6].

2. An obstruction for Galois sections of twisted forms

We give an obstruction for the existence of Galois sections of twisted forms. We developed the obstruction for completely different purposes, namely proving the existence of certain cycles on $\mathbb{P}^2$ not defined over their field of moduli [8, Theorem 4], [6], [7], but the technique is completely general and, as A. Vistoli pointed out to us, can be applied to the study of the section conjecture.

As J. Stix pointed out to us, an obstruction involving twisted forms has already been used by M. Stoll [27, Lemma 5.5] (see also [25, Theorem 142]). It is not clear to us whether the two obstructions are directly linked, though. One thing is sure: Stoll's obstruction relies crucially on twisted forms which are geometrically disconnected, while all of our twisted forms are geometrically connected. If a link exists, it is not an obvious one.

Let S be a geometrically connected algebraic stack of finite type over a field k. Our main example is the case in which k is finitely generated over $\mathbb{Q}$ and S is a smooth, projective, hyperbolic orbifold curve, but the obstruction is completely general.

We stress the fact that allowing S to be an algebraic stack, rather than a scheme, is an important assumption. While we could work with schemes, we would obtain significantly weaker results. For instance, restricting ourselves to schemes would force us to assume that the action is free in Theorem 1.

Consider a short exact sequence of group schemes

$$1 \longrightarrow N \longrightarrow E \longrightarrow G \longrightarrow 1$$

pro-finite étale over k. Suppose that we have a geometrically connected E-torsor $U \to S$, i.e. an E-covering; denote by C the quotient stack $[U/N]$, it is a G-covering.

$$\begin{tikzcd}
U \arrow[d, "N"] \arrow[dd, bend right=60, "E"'] \\
C \arrow[d, "G"] \\
S
\end{tikzcd}$$

Example 3. If E is a *constant* finite group scheme, i.e. just a finite group, then U is simply a Galois E-covering, while $C = [U/N]$ is a Galois G-covering.

Example 4. Consider $s : \operatorname{Spec} k \to \Pi_{S/k}$ a Galois section and define U as the fibre product

$$U \overset{\text{def}}{=} \operatorname{Spec} k \times_{\Pi_{S/k}} S.$$

The morphism $U \to S$ is the projective limit of all the étale neighbourhoods of s: it is a k-form of the universal covering of S, namely $U_{k^s} \to S$ is the universal covering, where k^s/k is a separable closure. We have that U is an $E = \underline{\operatorname{Aut}}_{\Pi_{S/k}}(s)$-torsor where E is the group scheme of automorphisms of $s \in \Pi_{S/k}(k)$ and $C = [U/N]$ is some étale neighbourhood of s. If s is associated with a rational point $p \in S(k)$, then $E = \underline{\pi}_1(S, p)$ is the étale fundamental group scheme of (S, p).

The E-torsor $U \to S$ induces morphisms

$$S \longrightarrow \mathscr{B}E \longrightarrow \mathscr{B}G,$$

where $\mathscr{B}E$ and $\mathscr{B}G$ are the classifying stacks of E and G respectively. Recall that the set of rational points modulo isomorphism $\mathscr{B}G(k)/\sim$ of $\mathscr{B}G$ is the set of G-torsors over $\operatorname{Spec} k$ modulo isomorphism, which coincides with the non-abelian cohomology set $\mathrm{H}^1(k, G)$, and similarly for E.

Choose $T \to \operatorname{Spec} k$ a G-torsor. We may use T to define a twisted form $\mathfrak{C}$ of C, namely

$$\mathfrak{C} \overset{\text{def}}{=} C \times^G T = (C \times T)/G$$

where G acts on both sides of $C \times T$ simultaneously. Equivalently, we may define $\mathfrak{C}$ as the fibre product

$$\begin{tikzcd}
\mathfrak{C} \arrow[r] \arrow[d] & \operatorname{Spec} k \arrow[d] \\
S \arrow[r] & \mathscr{B}G,
\end{tikzcd}$$

where the map $\operatorname{Spec} k \to \mathscr{B}G$ is the one given by T.

Proposition 5. *If T does not lift to E, i.e. its class is not in the image of $\mathrm{H}^1(k, E) \to \mathrm{H}^1(k, G)$, then $\Pi_{\mathfrak{C}/k}(k) = \emptyset$.*

Proof. Consider the fibre product

$$\begin{tikzcd}
\Phi_T \arrow[r] \arrow[d] & \operatorname{Spec} k \arrow[d, "T"] \\
\mathscr{B}E \arrow[r] & \mathscr{B}G
\end{tikzcd}$$

where Φ_T is by definition the gerbe of liftings of T to E. Since the two compositions $\mathfrak{C} \to \operatorname{Spec} k \to \mathscr{B}G$ and $\mathfrak{C} \to S \to \mathscr{B}E \to \mathscr{B}G$ are equivalent by construction, we get an induced map $\mathfrak{C} \to \Phi_T = \operatorname{Spec} k \times_{\mathscr{B}G} \mathscr{B}E$.

Since $\mathscr{B}E$, $\mathscr{B}G$ are pro-finite étale gerbes and $E \to G$ is surjective, then Φ_T is a pro-finite étale gerbe over $\operatorname{Spec} k$ as well, and hence we have an induced map $\Pi_{\mathfrak{C}/k} \to \Phi_T$ by the universal property of the étale fundamental gerbe. By construction, $\Phi_T(k)/\sim$ is the set of E-torsors over k which lift T, by hypothesis it is empty. It follows that $\Pi_{\mathfrak{C}/k}(k)$ is empty as well. $\square$

Remark 6. Let us sketch an alternative proof of Proposition 5 in terms of étale fundamental groups, for which I thank J. Stix. Let k^s be a separable closure of k and $\Gamma = \operatorname{Gal}(k^s/k)$. The torsors $U \to C \to S$ and $T \to \operatorname{Spec} k$ correspond to homomorphisms

$$\pi_1(S) \longrightarrow E(k^s) \rtimes \Gamma \longrightarrow G(k^s) \rtimes \Gamma,$$

$$\Gamma \longrightarrow G(k^s) \rtimes \Gamma$$

over Γ, and $\pi_1(\mathfrak{C})$ identifies with the fibre product of $\pi_1(S)$ and Γ over $G(k^s) \rtimes \Gamma$. A Galois section $\Gamma \to \pi_1(\mathfrak{C})$ induces by composition a homomorphism $\Gamma \to \pi_1(\mathfrak{C}) \to \pi_1(S) \to E(k^s) \rtimes \Gamma$. Since $\pi_1(\mathfrak{C})$ is a fibre product over $G(k^s) \rtimes \Gamma$, this gives a lifting $\Gamma \to E(k^s) \rtimes \Gamma$ of the homomorphism $\Gamma \to G(k^s) \rtimes \Gamma$ corresponding to T, which is absurd.

Remark 7. In the particular case in which U is the pro-finite limit of étale neighbourhoods of a Galois section of S, and hence $\Pi_{C/k} = \mathscr{B}N$, it is not difficult to see that the gerbe of liftings $\Phi_T \simeq \mathscr{B}N \times^G T = \Pi_{C/k} \times^G T$ coincides with the étale fundamental gerbe of the twist $\mathfrak{C}$.

3. About the étale fundamental group of orbifold surfaces

Let S be an orientable, complete orbifold surface, that is a connected, complete orbifold whose charts are analytic open subsets of the orbifolds $[\mathbb{C}/\mu_\gamma]$ for varying γ, with $\gamma = 1$ for all but finitely many charts. Equivalently, we may consider S as a 1-dimensional smooth, connected, proper Deligne–Mumford stack over $\mathbb{C}$ with non-empty schematic locus.

Let π be the orbifold fundamental group from topology of S; so π is a classical group, not a profinite one. Let $\widehat{\pi}$ be the profinite completion. As we will see, the notion of hyperbolic surface generalizes naturally to hyperbolic orbifold surface. The aim of this section is to prove that, if S is hyperbolic, a pro-p group of finite derived length with a closed embedding in $\widehat{\pi}$ is pro-cyclic (with one exception if $p = 2$), see Proposition 9.

While this is an essential tool for us, it is only used at the very last step of the proof of Theorems 1 and 18, and its purpose might not seem clear before reaching that point. Furthermore, the proof of Proposition 9 is quite long. Because of this, depending on taste the reader might want to read the proof of Theorems 1 and 18 in Section 4 first.

We work over $\mathbb{C}$ for simplicity. Since the étale fundamental group is invariant under base change of algebraically closed fields, the results hold for smooth, proper Deligne–Mumford stacks with non-empty schematic locus over any algebraically closed field of characteristic 0.

Recall that the rational Euler characteristic of S is defined as

$$\chi(S) = 2 - 2g - \sum_s \frac{\gamma_s - 1}{\gamma_s},$$

where the sum runs over non-trivial orbifold points s of degree γ_s, namely S is étale locally isomorphic to $[\mathbb{C}/\mu_{\gamma_s}]$ around s. The orbifold surface S is hyperbolic if $\chi(S) < 0$. The following is well known to experts, but we couldn't find a reference.

Lemma 8. *If $f : S' \to S$ is a finite étale cover of degree d of complete orbifold surfaces, then $\chi(S') = d\chi(S)$.*

Proof. Denote by g, g' the genus of S, S' respectively. If $s' \in S'$, denote by $e_{s'}$ the ramification index at s' of the underlying morphism of coarse moduli spaces. By Riemann–Hurwitz, we get

$$2g' - 2 = d(2g - 2) + \sum_{s' \in S'} (e_{s'} - 1),$$

where the sum is finite since $e_{s'} = 1$ if s' is not a ramification point. Hence,

$$\chi(S') = d(2 - 2g) - \sum_{s' \in S'} e_{s'} - 1 + \frac{\gamma_{s'} - 1}{\gamma_{s'}}.$$

Again, the sum is finite since $\gamma_{s'} = 1$ if s' is not an orbifold point. Since $S' \to S$ is étale, then $\gamma_{f(s')} = e_{s'}\gamma_{s'}$. We thus get

$$\begin{aligned}\chi(S') &= d(2g-2) + \sum_{s'\in S'} \frac{\gamma_{f(s')} - \gamma_{s'} + \gamma_{s'} - 1}{\gamma_{s'}} \\ &= d(2g-2) + \sum_{s'\in S'} e_{s'}\frac{\gamma_{f(s')} - 1}{\gamma_{f(s')}} = d(2g-2) + d\sum_{s\in S}\frac{\gamma_s - 1}{\gamma_s}\end{aligned}$$

where the last passage follows from the equality $\sum_{f(s')=s} e_{s'} = d$. ☐

Consider the action by -1 of $\mathbb{Z}/2$ on $\mathbb{Z}_2$, and write $D_{2^\infty} = \mathbb{Z}_2 \rtimes \mathbb{Z}/2$ for the infinite, 2-adic dihedral group. Here is the main result of this section.

Proposition 9. *Assume that S is hyperbolic, and let $G \subset \widehat{\pi}$ be a closed subgroup which is a p-group for some prime p.*

If G has finite derived length, it is either pro-cyclic or isomorphic to D_{2^∞}. Furthermore, if $G \simeq D_{2^\infty}$, then S has at least one orbifold point of even degree.

While the statement of Proposition 9 is rather intuitive, the only proof we have found is rather complicated. The starting point is the following well-known result.

Lemma 10. *A homomorphism $G \to H$ of pro-p-groups is surjective if and only if $G \to H^{\mathrm{ab}}/p$ is surjective, where H^{ab} is the abelianization.*

Proof. This is a direct consequence of Burnside's basis theorem. See also [23, Section 4.2]. ☐

Before entering into details, let us sketch the idea of the proof of Proposition 9. If G^{ab} is pro-cyclic, then G is pro-cyclic by Lemma 10. Hence, we may assume that G^{ab} is not pro-cyclic.

We want to find a homomorphism $\pi \to \pi'$, where π' is the fundamental group of a genus 0 orbifold surface S' with $\chi(S') \leq 0$, such that the composition $G^{\mathrm{ab}}/p \to \pi^{\mathrm{ab}}/p \to \pi'^{\mathrm{ab}}/p$ is surjective. Up to passing to a finite index subgroup of π containing G, this can be done in most cases with G^{ab} not pro-cyclic.

Assume that we can find such a homomorphism, and let π'' be the fundamental group of the maximal abelian cover $S'' \to S'$. It's not difficult to impose that S'' is a surface and that the degree of $S'' \to S'$ is a power of p, so let us make this assumption. Since $\chi(S') \leq 0$, we get that $\chi(S'') \leq 0$ as well, and hence S'' has genus $g \geq 1$.

Since $G^{\mathrm{ab}}/p \to \pi'^{\mathrm{ab}}/p$ is surjective, G maps surjectively on the p-adic completion of π' by Lemma 10. This implies that we get a surjective homomorphism

$$[G, G] \longrightarrow \mathbb{Z}_p^{2g},$$

where $\mathbb{Z}_p^{2g}$ is the p-adic completion of π''^{ab}. As a consequence, $[G, G]$ is neither pro-cyclic nor isomorphic to D_{2^∞}. Now it's easy to conclude by induction on the derived length, because some element in the derived series of G must be trivial.

The only case in which the plan above does not work is if $G^{\text{ab}} = \mathbb{Z}/2 \times \mathbb{Z}/2$, because this forces $\chi(S') > 0$. A similar argument can then be implemented to show that $G \simeq D_{2^\infty}$ in this case.

Turning this plan into an actual proof requires a certain amount of work. We break it into several steps.

3.1. On the 2-adic dihedral group. First, let us study D_{2^∞}. Clearly, D_{2^∞} is the 2-adic completion of the infinite dihedral group $D_\infty = \mathbb{Z} \rtimes \mathbb{Z}/2$, and there is a section $D_{2^\infty} \subset \widehat{D_\infty} = \widehat{\mathbb{Z}} \rtimes \mathbb{Z}/2$ of $\widehat{D_\infty} \to D_{2^\infty}$. Moreover, $D_{2^\infty}^{\text{ab}} \simeq (\mathbb{Z}/2)^2$ and $[D_{2^\infty}, D_{2^\infty}] \simeq \mathbb{Z}_2$.

The following fact, along with its proof, was kindly told to us by J. Stix.

Lemma 11. *The free product* $\mathbb{Z}/2 * \mathbb{Z}/2$ *is isomorphic to the infinite dihedral group* D_∞.

Proof. Write $D_\infty = \langle r, s \mid s^2 = (rs)^2 = 1 \rangle$. Consider the action of D_∞ on $\mathbb{R}$ where r acts as $x \mapsto x + 2$ and s acts as $x \mapsto -x$, the statement then follows from [22, Section I.4, Theorem 6]. Alternatively, if $\mathbb{Z}/2 * \mathbb{Z}/2 = \langle a, b \mid a^2 = b^2 = 1 \rangle$, the homomorphisms $a \mapsto s$, $b \mapsto rs$ and $r \mapsto ba$, $s \mapsto a$ are inverses. ☐

It follows that D_{2^∞} is the 2-adic completion of the fundamental group $\mathbb{Z}/2 * \mathbb{Z}/2$ of the disk with two orbifold points of order 2. We can now show that D_{2^∞} can actually appear as a subgroup of the profinite completion of the fundamental group of an hyperbolic orbifold surface.

Example 12. Let M be the unit disk, and consider a complete surface S of arbitrary genus with a *continuous* (not holomorphic) map to the disk $S \to M$ such that for some open subdisk $U \subset M$ the restriction $S_U \to U$ is a disjoint union of copies of U. Define $M_{2,2} \to M$ as an orbifold whose coarse moduli space is the disk M and which has two orbifold points of order 2 over points of $U \subset M$, and let $U_{2,2} \subset M_{2,2}$ be the inverse image of U.

Define S' as the fibre product $S \times_M M_{2,2}$, then S' is an orbifold surface whose coarse moduli space is S with maps $S' \to M_{2,2}$, $U_{2,2} \to S'$ such that the composition $U_{2,2} \to M_{2,2}$ is the given embedding $U_{2,2} \subset M_{2,2}$ and thus defines an isomorphism $\pi_1(U_{2,2}) \simeq \pi_1(M_{2,2})$. Since $D_{2^\infty} \subset \widehat{D_\infty} = \widehat{\pi_1(M_{2,2})}$, then $\widehat{\pi_1(S')}$ has a closed subgroup isomorphic to D_{2^∞}.

3.2. Reduction to genus $\geq$ 2. Next, we show that we can focus on the case of orbifold surfaces of genus ≥ 2.

In a previous version of this article, the following Lemma 14 required $d \geq n(n+1)/2 + r + 1$. I thank an anonymous referee for the improved estimate.

Lemma 13. *Let r, n, d be positive integers with $d \geq n + r + 1$, and k a field. Consider the standard scalar product on k^d.*

For every vector subspace $V \subset k^d$ of rank $\leq r$, there exists a vector orthogonal to V and with at least $n+1$ non-zero coordinates.

Proof. Write $\delta \geq d - r \geq n+1$ for the dimension of the orthogonal $V^{\perp}$, and let $e_1, \ldots, e_d$ be the standard basis of k^d, and choose any basis $f_1, \ldots, f_\delta$ of $V^{\perp}$. We can choose $d - \delta$ elements of $\{e_1, \ldots, e_d\}$ which, together with $f_1, \ldots, f_\delta$, form a basis of k^d. Up to reordering, we may then assume that $f_1, \ldots, f_\delta, e_{\delta+1}, \ldots, e_d$ is a basis. It follows that the projection $V^{\perp} \to k^\delta$ on the first δ coordinates is an isomorphism, hence $V^{\perp}$ contains a vector whose first $\delta \geq n+1$ coordinates are not 0. $\square$

Lemma 14. *Let S be a complete hyperbolic orbifold surface with fundamental group π and $G \subset \widehat{\pi}$ an infinite index closed subgroup such that G^{ab} has finite topological rank. For every N, there exists a finite index subgroup $\pi' \subset \pi$ such that $\widehat{\pi}' \subset \widehat{\pi}$ contains G and such that the associated covering $S' \to S$ has genus at least N.*

Proof. Assume by contradiction that the genus of every covering whose fundamental group contains G is strictly smaller than N.

Notice that, if $S' \to S$ is unramified, the degrees of the orbifold points of S' divide the degrees of the orbifold points of S. Because of this, as long as every cover is unramified, only a fixed finite number of primes can divide the degrees of orbifold points.

Since G has infinite index and $\chi(S) < 0$, up to passing to a finite index subgroup of π we might assume that $-\chi(S)$ is arbitrarily large thanks to Lemma 8. Thanks to the observation above about primes, and since the coverings have bounded genus, we might then assume that there exists a single prime p with $M \geq 2N + r + 3$ orbifold points of degree divisible by p, where r is the rank of G^{ab}.

Now that we have a large number of orbifold points of degree divisible by p, the idea is to find a single $\mathbb{Z}/p$ covering satisfying the condition on G and which ramifies over many of these points, thus making the genus explode.

Let $\gamma_1, \ldots, \gamma_M$ be small closed loops around the orbifold points of degree divisible by p. Consider the standard presentation

$$\pi = \left\langle \begin{matrix} a_1, b_1, \ldots, a_g, b_g, \\ \gamma_1, \ldots, \gamma_M, \eta_1, \ldots, \eta_c \end{matrix} \;\middle|\; \begin{matrix} [a_1, b_1] \cdots [a_g, b_g] \cdot \gamma_1 \cdots \gamma_M \cdot \eta_1 \cdots \eta_c \\ \hfill = \gamma_i^{c_i} = \eta_j^{d_j} = 1 \end{matrix} \right\rangle$$

where the η_i are closed loops around the other orbifold points. We can construct a homomorphism

$$f : \pi \longrightarrow H = \Big(\mathbb{F}_p^M = \langle e_1, \ldots, e_M\rangle\Big)/(e_1 + \cdots + e_M)$$

mapping γ_i to the vector e_i for $i = 1, \ldots, M$ and all the other standard generators to 0. The relation $\eta_j^{d_j}$ clearly maps to 0, $\gamma_i^{c_i}$ maps to $c_i e_i = 0$ because $p \mid c_i$, and the other relation maps to $e_1 + \cdots + e_M$ which is 0 in H.

Let $V \subset \mathbb{F}_p^M$ be the inverse image of $f(G)$, it is a vector space of dimension at most $r + 1$. Since $M \geq 2N + r + 3$, by Lemma 13 with $n = 2N + 1$ and $k = \mathbb{F}_p$ we get a vector $w \in \mathbb{F}_p^M$ which is orthogonal to V, and which has at least $2N + 2$ non-zero coordinates.

Since $e_1 + \cdots + e_M \in V$, then $w \cdot (e_1 + \ldots e_M) = 0$. Hence, scalar product with w defines a homomorphism

$$H \longrightarrow \mathbb{F}_p,$$

and by composition a homomorphism $\pi \to \mathbb{F}_p$. Notice that G maps to $0 \in \mathbb{F}_p$, because w is orthogonal to V which is the inverse image of $f(G)$.

Furthermore, γ_i maps to $w_i \in \mathbb{F}_p$, where $w = \sum_i w_i e_i$. In particular, at least $2N + 2$ loops γ_i do not map to 0 in $\mathbb{F}_p$, and hence the corresponding degree p covering $S' \to S$ has at least $2N + 2$ points with ramification index p when we pass to coarse moduli spaces.

If g is the genus of S', by Riemann–Hurwitz we get

$$\begin{gathered} 2g - 2 \geq -2p + (2N + 2)(p - 1), \\ g \geq N(p - 1) \geq N, \end{gathered}$$

which gives a contradiction because $G \subset \widehat{\pi}' \subset \widehat{\pi}$, where π' is the fundamental group of S'. □

3.3. Constructing some homomorphisms from $\boldsymbol{\pi}$.

In order to prove Proposition 9, we need to construct several homomorphisms from π to various other groups. Here we detail how to do it.

Lemma 15. *Let p be a prime number and*

$$\pi = \Big\langle a_1, b_1, \ldots, a_g, b_g \,\Big|\, \prod [a_i, b_i] = 1 \Big\rangle$$

the fundamental group of a complete, orientable topological surface of genus $g \geq 2$. Let r be either 1 or 2, and $H \subset \pi^{\mathrm{ab}}/p$ a subgroup of rank $\geq r$.

Let Q be a free group of rank r, and $Q \to Q^{\mathrm{ab}} \simeq \mathbb{Z}^r$ the abelianization. There exists a surjective homomorphism

$$q : \pi \longrightarrow Q$$

such that the composition $H \subset \pi^{\mathrm{ab}}/p \to Q^{\mathrm{ab}}/p \simeq (\mathbb{Z}/p)^r$ is surjective.

Proof. For ease of notation, rename the generators $a_1, b_1, \dots, a_g, b_g$ as $x_1, \dots, x_{2g}$ respectively.

If $r = 1$ it is enough to define $\pi \to Q = \mathbb{Z}$ by mapping one suitably chosen generator x_i to $1 \in \mathbb{Z}$ and all the others to 0.

Assume $r = 2$, write $Q = \mathbb{Z} * \mathbb{Z} = \langle \alpha, \beta \rangle$. Since $H \subset \pi^{\mathrm{ab}}/p = \mathbb{F}_p x_1 \oplus \cdots \oplus \mathbb{F}_p x_{2g}$ has rank ≥ 2, we may choose two linearly independent vectors $(u_i)_i, (v_i)_i \in H$.

Assume first that there are two indexes $1 \leq j < j' \leq 2g$ which are *not* of the form $j = 2m - 1$, $j' = 2m$ and such that the 2×2 matrix $\left(\begin{smallmatrix} u_j & v_j \\ u_{j'} & v_{j'} \end{smallmatrix}\right)$ has rank 2. We may then define $\pi \to Q = \langle \alpha, \beta \rangle$ by

$$x_i \longmapsto 1 \text{ if } i \neq j, j',$$

$$x_j \longmapsto \alpha, \; x_{j'} \longmapsto \beta.$$

It is clear that the relation $[x_1, x_2] \cdots [x_{2g-1}, x_{2g}]$ maps to $1 \in \langle \alpha, \beta \rangle$, so the homomorphism is well defined.

If such indexes do not exist, then there exists $1 \leq m \leq g$ such that $\left(\begin{smallmatrix} u_{2m-1} & v_{2m-1} \\ u_{2m} & v_{2m} \end{smallmatrix}\right)$ has rank 2 *and* for every $i \neq 2m-1, 2m$ we have $u_i = v_i = 0$, since (u_i, v_i) must be a multiple of both (u_{2m-1}, v_{2m-1}) and (u_{2m}, v_{2m}). If $m \leq g - 1$, define $\pi \to Q = \langle \alpha, \beta \rangle$ by

$$x_{2m}, x_{2m+1} \longmapsto \alpha, \quad x_{2m+2}, x_{2m-1} \longmapsto \beta, \quad x_i \longmapsto 1 \text{ otherwise }.$$

The relation $[x_1, x_2] \cdots [x_{2g-1}, x_{2g}]$ maps to

$$[\beta, \alpha] \cdot [\alpha, \beta] = 1,$$

hence the homomorphism is well defined. The case $m = g$ is analogous. $\square$

Lemma 16. *Let p be a prime, and*

$$\pi = \left\langle \gamma_0, \dots, \gamma_n \,\middle|\, \gamma_i^{c_i} = 1, \; \prod_i \gamma_i = 1 \right\rangle,$$

$$\pi' = \left\langle x_0, \dots, x_m \,\middle|\, x_i^{p} = 1, \; \prod_i x_i = 1 \right\rangle.$$

Assume there is a subgroup $H \subset \pi^{\mathrm{ab}}/p$ of rank $\geq m$. Then there exists a homomorphism $\pi \to \pi'$ such that the composition $H \to \pi^{\mathrm{ab}}/p \to \pi'^{\mathrm{ab}}$ is surjective.

Proof. If η is the quotient of π where we add the relations $\gamma_i^p = 1$ for every i, then $\eta^{\mathrm{ab}}/p = \pi^{\mathrm{ab}}/p$. Hence, up to replacing π with η, we may assume that $c_i = p$ for every i.

For $i = 0, \dots, n$, write η_i for the quotient of π where we add the relation $\gamma_i = 1$, and H_i for the image of H in η_i^{ab}/p. If $\operatorname{rank} H_i \geq m$, we replace π with η_i. By recursion, we can repeat the process untill $\operatorname{rank} H_i < m$ for every i, and $\operatorname{rank} H = m$.

Let e_i be the image of γ_i in π^{ab}/p, we have $\pi^{\mathrm{ab}}/p = \langle e_0, \dots, e_n \rangle / (1, \dots, 1)$. Since $\operatorname{rank} H_i < m$ for every i, we get that $e_i \in H$ for every i, and hence $n = \operatorname{rank} \pi^{\mathrm{ab}}/p = \operatorname{rank} H = m$. We may then identify $\pi = \pi'$, and conclude the proof. □

Lemma 17. *Let*

$$\pi = \left\langle \gamma_0, \dots, \gamma_n \,\middle|\, \gamma_i^{c_i} = 1,\ \prod_i \gamma_i = 1 \right\rangle,$$

$$\pi' = \left\langle x_0, x_1, x_2 \,\middle|\, x_0^4 = x_1^4 = x_2^2 = x_0 x_1 x_2 = 1 \right\rangle.$$

Assume there is a subgroup $H \subseteq \pi^{\mathrm{ab}}/4$ *of exponent* 4 *such that* $H/2$ *has rank* ≥ 2. *Then there exists a homomorphism* $\pi \to \pi'$ *such that the composition* $H \to \pi^{\mathrm{ab}}/4 \to \pi'^{\mathrm{ab}}$ *is surjective.*

Proof. We say that a quotient $\pi \to \eta$ is an *easy quotient* if η is defined with the same generators and relations as π, except that we replace c_i with a divisor of c_i for a single index i. By recursion we may assume that, for every easy quotient $f : \pi \to \eta$, either $f(H) \subseteq \eta^{\mathrm{ab}}/4$ has exponent < 4, or $f(H)/2$ has rank ≤ 1. In particular, c_i is either 2 or 4 for every i.

Furthermore, we may assume that $c_i \geq c_{i+1}$: if we think of π has the fundamental group of a genus 0 orbifold surface, this amounts to re-numbering the orbifold points. In particular, $c_0 = c_1 = 4$, because otherwise $\pi^{\mathrm{ab}}/4$ has exponent 2.

Notice that $c_2 \geq 2$, otherwise $\pi^{\mathrm{ab}}/2$ has rank 1 (which is impossible because $H/2$ has rank ≥ 2). If $c_2 = 4$, we can find an easy quotient $f : \pi \to \eta$ such that $f(H) \subseteq \eta^{\mathrm{ab}}/4$ has exponent 4 by replacing either c_0, c_1 or c_2 with 2. Since $\pi^{\mathrm{ab}}/2 = \eta^{\mathrm{ab}}/2$, then $H/2 = f(H)/2$, and we get a contradiction with our assumption on easy quotients. We have then proved that $c_2 = 2$.

We may identify

$$\pi^{\mathrm{ab}} = (\mathbb{Z}/4 \times \mathbb{Z}/4 \times \mathbb{Z}/2 \times \cdots \times \mathbb{Z}/2)/(1, \dots, 1).$$

The fact that H has exponent 4 implies that H contains a vector of the form $(0, 1, a_2, \dots, a_n)$. The fact that $H/2$ has rank ≥ 2 implies that H contains a non-zero vector of the form $(0, 0, b_2, \dots, b_n)$. If $b_2 = 0$, we may define an easy quotient $\pi \to \eta$ replacing c_2 with 1 and such that the image of H in $\eta^{\mathrm{ab}}/4$ has exponent 4, and the image in $\eta^{\mathrm{ab}}/2$ has rank ≥ 2. This contradicts our assumption, hence $b_2 = 1$.

The homomorphism $\pi \to \pi'$ mapping γ_i to x_i for $i = 0, 1, 2$ and to 1 otherwise is the desired one, since $\pi'^{\mathrm{ab}} = (\mathbb{Z}/4 \times \mathbb{Z}/4 \times \mathbb{Z}/2)/(1, 1, 1)$ is generated by the images of $(0, 1, a_2, \dots, a_n)$ and $(0, 0, b_2, \dots, b_n)$. □

3.4. Proof of Proposition 9. If $D_{2^\infty} \subset \widehat{\pi}$, then up to passing to an étale cover of S we may assume that $\widehat{\pi}^{\mathrm{ab}} \supset (\mathbb{Z}/2)^2 = D_{2^\infty}^{\mathrm{ab}}$ has non-trivial 2-torsion, hence S has non-trivial orbifold points of even degree.

Let us now prove the main statement. It is enough to prove the following.

- $[G, G]$ is not isomorphic to D_{2^∞}.
- $[G, G]$ pro-cyclic implies that G is either pro-cyclic or isomorphic to D_{2^∞}.

These are sufficient by induction on the derived length. If the derived length of G is 0 then $G = 1$ and there is nothing to prove. Otherwise, the commutator subgroup $[G, G]$ has smaller derived length and therefore by induction is either pro-cyclic or D_{2^∞}. Since $[G, G]$ cannot be isomorphic to D_{2^∞}, then it is pro-cyclic, and hence G is either pro-cyclic or isomorphic to D_{2^∞}.

Up to passing to a finite index subgroup of π, thanks to Lemma 14 we may assume that the genus g of S is ≥ 2. Consider the usual presentation

$$\pi = \left\langle a_1, b_1, \dots, a_g, b_g, c_1, \dots, c_n \,\middle|\, c_i^{\gamma_i} = 1,\ \prod [a_i, b_i] \cdot \prod c_i = 1 \right\rangle.$$

Furthermore, consider the auxiliary quotients

$$\tau = \left\langle a_1, b_1, \dots, a_g, b_g \,\middle|\, \prod [a_i, b_i] = 1 \right\rangle,$$
$$\tau' = \left\langle c_1, \dots, c_n \,\middle|\, c_i^{\gamma_i} = 1,\ \prod c_i = 1 \right\rangle$$

of π. There is an induced surjective map $\pi \to \tau * \tau'$ whose abelianization is an isomorphism.

We break down the analysis in four cases: one for $p \neq 2$, and three for $p = 2$.

Case $p \neq 2$. If $p \neq 2$, we only need to prove that $[G, G]$ pro-cyclic implies G pro-cyclic.

If G^{ab} has p-rank 1, then G is pro-cyclic by Lemma 10. Assume by contradiction that G^{ab} has p-rank ≥ 2. Up to passing to a finite index subgroup of π, we might also assume that the image of G in $\widehat{\pi}^{\mathrm{ab}}/p$ has rank at least 2.

Let us check that there exists a map

$$\pi \longrightarrow \pi' = \langle x_1, x_2, x_3 \mid x_1^p = x_2^p = x_3^p = x_1 x_2 x_3 = 1 \rangle$$

such that the composition $G \to \pi'^{\mathrm{ab}} = \widehat{\pi}'^{\mathrm{ab}} = \mathbb{Z}/p \times \mathbb{Z}/p$ is surjective. Notice that π' is a quotient of $\mathbb{Z}/p * \mathbb{Z}/p$.

If the image of G in τ^{ab}/p has rank ≥ 2, by Lemma 15 we get a map $\tau \to \mathbb{Z} * \mathbb{Z}$ such that $G \to \widehat{\pi} \to \widehat{\tau} \to \mathbb{Z}/p \times \mathbb{Z}/p$ is surjective, so the composition $\pi \to \tau \to \mathbb{Z} * \mathbb{Z} \to \pi'$ works.

If the image of G in τ'^{ab}/p has rank ≥ 2, by Lemma 16 we get a map $\tau' \to \pi'$ such that $G \to \pi'^{\mathrm{ab}} = \mathbb{Z}/p \times \mathbb{Z}/p$ is surjective, so the composition $\pi \to \tau' \to \pi'$ works.

The only remaining case is the one in which the images of G in τ^{ab}/p and τ'^{ab}/p have both rank 1. With Lemma 15 again, we may find a quotient

$\tau \to \mathbb{Z}/p$ such that the composition $G \to \widehat{\tau} \to \mathbb{Z}/p$ is surjective. The kernel G_0 of $G \to \tau^{\mathrm{ab}}/p$ has image of rank 1 in τ'^{ab}/p, by Lemma 16 we get a map $\tau' \to \mathbb{Z}/p$ such that $G_0 \to \widehat{\tau}' \to \mathbb{Z}/p$ is surjective. The composition $\pi \to \tau * \tau' \to \mathbb{Z}/p * \mathbb{Z}/p \to \pi'$ is the desired homomorphism, since G surjects on $\mathbb{Z}/p \times \mathbb{Z}/p$ by construction.

The group π' is the fundamental group of a complete orbifold surface S' of genus 0 with three orbifold points of degree p. The $(\mathbb{Z}/p)^2$ Galois covering $S'' \to S'$ associated with the abelianization morphism $\pi' \to (\mathbb{Z}/p)^2$ defines a surface S'' of genus g'' without orbifold points, because the loops around orbifold points of S' map to non-zero elements of $(\mathbb{Z}/p)^2$.

Because of this, the pro-p completion Q of $\pi'/[[\pi',\pi'],[\pi',\pi']]$ is a pro-p-group with $Q^{\mathrm{ab}} \simeq (\mathbb{Z}/p)^2$ and $[Q,Q] = \mathbb{Z}_p^{2g''}$. By Lemma 10, $G \to Q$ is surjective. Since $p \geq 3$ then $\chi(S'') = p^2\chi(S') \leq 0$, hence $g'' \geq 1$ and $[Q,Q]$ has rank ≥ 2. This is in contradiction with the fact that $[G,G]$ is pro-cyclic, since there is a surjective map $[G,G] \to [Q,Q]$.

This concludes the case $p \neq 2$. We subdivide the case $p = 2$ in three subcases.

Case $p = 2$*,* G^{ab} *has exponent* > 2*.* Clearly, G is not isomorphic to D_{2^∞}, because $D_{2^\infty}^{\mathrm{ab}} = \mathbb{Z}/2 \times \mathbb{Z}/2$ has exponent 2. If G is pro-cyclic, there is nothing to prove. It is then enough to show that G not pro-cyclic implies both $[G,G]$ not pro-cyclic and $[G,G] \not\simeq D_{2^\infty}$.

Assume that G is not pro-cyclic; by Lemma 10, G^{ab} is not pro-cyclic as well. Since G^{ab} is not pro-cyclic and has exponent > 2, up to passing to a finite index subgroup we may assume that there exists a quotient $\pi \to \mathbb{Z}/4 \times \mathbb{Z}/2$ such that the composition $G \to \widehat{\pi} \to \mathbb{Z}/4 \times \mathbb{Z}/2$ is surjective. In particular, the image of G in $\pi^{\mathrm{ab}}/4$ has exponent 4, and the image in $\pi^{\mathrm{ab}}/2$ has rank ≥ 2.

Let us check that there exists a map

$$\pi \longrightarrow \pi' = \left\langle x_1, x_2, x_3 \,\middle|\, x_1^4 = x_2^4 = x_3^2 = x_1x_2x_3 = 1 \right\rangle.$$

such that $G \to \pi'^{\mathrm{ab}}/2 \simeq \mathbb{Z}/2 \times \mathbb{Z}/2$ is surjective. We notice that π' is a quotient of $\mathbb{Z}/4 * \mathbb{Z}/2$, so finding a map $\pi \to \mathbb{Z}/4 * \mathbb{Z}/2$ is sufficient.

If G has image of rank 2 in $\tau^{\mathrm{ab}}/2$, Lemma 15 gives a homomorphism $\tau \to \mathbb{Z} * \mathbb{Z}$ such that $G \to \mathbb{Z}/2 \times \mathbb{Z}/2$ is surjective.

If G has image of rank 1 in $\tau^{\mathrm{ab}}/2$, Lemma 15 gives a homomorphism $\tau \to \mathbb{Z}$ such that $G \to \mathbb{Z}/2$ is surjective. Let G_0 be the kernel of $G \to \tau^{\mathrm{ab}}/2$, it has image of rank ≥ 1 in $\tau'^{\mathrm{ab}}/2$. We can then choose $\tau'^{\mathrm{ab}}/2 \to \mathbb{Z}/2$ such that $G_0 \to \widehat{\tau}' \to \mathbb{Z}/2$ is surjective, and thus get a homomorphism $\pi \to \mathbb{Z} * (\mathbb{Z}/2)$ such that $G \to \mathbb{Z} * (\mathbb{Z}/2) \to \mathbb{Z}/2 \times \mathbb{Z}/2$ is surjective.

If G has trivial image in $\tau^{\mathrm{ab}}/2$, then its image in $\tau'^{\mathrm{ab}}/4$ has exponent 4, and its image in $\tau'^{\mathrm{ab}}/2$ has rank ≥ 2. By Lemma 17, we get the desired map $\tau' \to \pi'$, and may define $\pi \to \pi'$ by composition.

The group π' is the fundamental group of an orbifold surface S' with $\chi(S') = 0$. The abelianization of π' is a finite 2-group and defines a Galois cover which is a surface of genus 1. Let Q be the pro-2 completion of $\pi'/[[\pi', \pi'], [\pi', \pi']]$. As in the case $p \neq 2$, we have a surjective homomorphism $[G, G] \to [Q, Q] = \mathbb{Z}_2^2$. This clearly implies that $[G, G]$ is neither pro-cyclic nor D_{2^∞}.

Case $p = 2$, G^{ab} has exponent 2 and rank ≥ 3. We have that $G \not\simeq D_{2^\infty}$, because G^{ab} has rank ≥ 3. For the same reason, G is not pro-cyclic. It is enough to prove that $[G, G]$ is neither pro-cyclic nor D_{2^∞}.

By passing to a finite index subgroup, we may assume that the image of G in $\pi^{\mathrm{ab}}/2$ has rank ≥ 3.

Since τ^{ab} is torsion free and $\pi^{\mathrm{ab}} = \tau^{\mathrm{ab}} \oplus \tau'^{\mathrm{ab}}$, the image of G in $\tau'^{\mathrm{ab}}/2$ has rank ≥ 3 as well. By Lemma 16 we can define a homomorphism

$$\tau' \longrightarrow \pi' = \left\langle x_1, x_2, x_3, x_4 \,\middle|\, \forall\, i : x_i^2 = 1,\ x_1 x_2 x_3 x_4 = 1 \right\rangle$$

such that the composition $G \to \pi'^{\mathrm{ab}} = (\mathbb{Z}/2)^3$ is surjective.

The group π' is the fundamental group of an orbifold surface S' with $\chi(S') = 0$. The abelianization of π' is a finite 2-group and defines a Galois cover which is a surface of genus 1. We conclude as in the previous case.

Case $p = 2$, $G^{\mathrm{ab}} \simeq (\mathbb{Z}/2)^2$. In this case we prove that $[G, G] \not\simeq D_{2^\infty}$, and that $[G, G]$ pro-cyclic implies $G \simeq D_{2^\infty}$. These will both follow from the existence of a surjective homomorphism $G \to D_{2^\infty}$.

Up to passing to a finite index subgroup of π we might assume that $G^{\mathrm{ab}} \subset \pi^{\mathrm{ab}}$. In particular, S has at least 3 orbifold points of even degree. Since the genus is positive and G^{ab} is torsion, the pullback of any abelian, non-trivial étale cover of the coarse moduli space of S defines an étale cover of S whose étale fundamental group contains G and which has at least 6 orbifold points of even degree. Hence, we may reduce to the case in which S has at least 4 orbifold points of even degree.

If S has at least 4 orbifold points of even degree and since $G^{\mathrm{ab}} \subset \pi^{\mathrm{ab}}$, there exists a quotient π' of π of the form

$$\left\langle x_1, x_2, x_3, x_4 \,\middle|\, \forall\, i : x_i^2 = 1,\ x_1 x_2 x_3 x_4 = 1 \right\rangle$$

such that G^{ab} maps injectively in $\widehat{\pi}'^{\mathrm{ab}} = \pi'^{\mathrm{ab}} = (\mathbb{Z}/2)^4/(1, 1, 1, 1) \simeq (\mathbb{Z}/2)^3$. The group π' is the fundamental group of an orbifold surface S' of genus 0 with 4 orbifold points of degree 2.

Consider the homomorphism $\pi' \to \mathbb{Z}/2$ which maps x_i to 1 for every i, the associated covering $S'' \to S'$ is a torus with fundamental group

$\pi'' = \mathbb{Z}^2 \subset \pi'$. If the image of G is contained in $\widehat{\pi}'' \subset \widehat{\pi}'$, then $G^{\mathrm{ab}} \simeq (\mathbb{Z}/2)^2$ embeds into $\widehat{\pi}'' = \widehat{\mathbb{Z}}^2$, which is absurd since the latter is torsion free. Because of this,

$$\ker\Big((\mathbb{Z}/2)^4/(1,1,1,1) \longrightarrow \mathbb{Z}/2\Big) \cap G^{\mathrm{ab}} \subset (\mathbb{Z}/2)^4/(1,1,1,1)$$

has rank 1.

Assume for simplicity that $(1,1,0,0) \in \ker\Big((\mathbb{Z}/2)^4/(1,1,1,1) \to \mathbb{Z}/2\Big)$ is not in G^{ab}. If that's not the case, then $(0,1,1,0)$ is not in G^{ab}; the rest of the proof is analogous.

Consider the homomorphism $\pi' \to \mathbb{Z}/2 * \mathbb{Z}/2$ which maps x_1, x_2 to the first generator and x_3, x_4 to the second generator (if we work with $(0,1,1,0)$ instead, x_1, x_4 map to the first generator, x_2, x_3 to the second). The kernel of the induced map $(\mathbb{Z}/2)^4/(1,1,1,1) \to (\mathbb{Z}/2)^2$ between abelianizations is generated by $(1,1,0,0)$ (resp. $(0,1,1,0)$), which is not contained in the image of G. Because of this, the composition $G^{\mathrm{ab}} = (\mathbb{Z}/2)^2 \hookrightarrow (\mathbb{Z}/2)^4/(1,1,1,1) \to (\mathbb{Z}/2)^2$ is injective and thus bijective, which in turn implies that the composition $G \to \widehat{\mathbb{Z}/2 * \mathbb{Z}/2} \to D_{2^\infty}$ is surjective and induces an isomorphism $G^{\mathrm{ab}} \simeq D_{2^\infty}^{\mathrm{ab}}$.

This shows that $[G,G] \not\simeq D_{2^\infty}$, because otherwise we would get a surjective homomorphism $[G,G] = D_{2^\infty} \to [D_{2^\infty}, D_{2^\infty}] = \mathbb{Z}_2$, which is absurd since $D_{2^\infty}^{\mathrm{ab}} = \mathbb{Z}/2 \times \mathbb{Z}/2$.

If $[G,G]$ is pro-cyclic, the induced surjective homomorphism $[G,G] \to [D_{2^\infty}, D_{2^\infty}] \simeq \mathbb{Z}_2$ must be an isomorphism; since $G^{\mathrm{ab}} \to D_{2^\infty}^{\mathrm{ab}}$ is an isomorphism, we get that $G \to D_{2^\infty}$ is an isomorphism as well.

4. Proof of Theorem 1

We state a version of Theorem 1 for $p = 2$. We will prove the two statements at the same time.

Theorem 18. *Let C be a smooth, projective, geometrically hyperbolic curve over a field k finitely generated over $\mathbb{Q}$ with a faithful action of a finite, non-cyclic* 2*-group G. If the action is not free and G has a cyclic subgroup of index* 2*, furthermore assume that G is not dihedral and that C has odd genus.*

There exists a finitely generated extension K/k and a twisted form $\mathfrak{C}$ of C_K such that $\mathfrak{C}$ has index 1 *and $\Pi_{\mathfrak{C}/K}(K) = \emptyset$.*

Let us now prove Theorems 1 and 18. Denote by S the quotient stack $[C/G]$. Up to passing to a finite extension of k, we might assume that $S(k)$ is non-empty.

4.1. Definition of $\mathfrak{C}$. Recall that, by assumption, all curves are smooth, projective and geometrically connected. Let $P = \underline{\mathrm{Pic}}^1_C$ be the Picard scheme of line bundles of degree 1 on C, the action of G on C induces an action on P.

Denote by X the quotient stack $[P/G]$. Since the action on $C \subset P$ is faithful then the action on P is generically free, hence X is generically a scheme. Let $K = k(X)$ be its function field, then $k(P)/K$ is a Galois extension with Galois group G and $\operatorname{Spec} k(P) \to \operatorname{Spec} K$ is a G-torsor associated with a morphism $\operatorname{Spec} K \to \mathscr{B}G$. As in Section 2, define $\mathfrak{C}$ as the twist of C_K by this torsor, namely

$$\mathfrak{C} \overset{\text{def}}{=} C_K \times^G \operatorname{Spec} k(P) = (C_K \times_K \operatorname{Spec} k(P))/G.$$

4.2. The index of $\mathfrak{C}$. We want to show that $\mathfrak{C}$ has index 1. By construction, $\underline{\operatorname{Pic}}^1_{\mathfrak{C}} = P_K \times^G \operatorname{Spec} k(P)$ has a K-rational point and hence $\mathfrak{C}$ has period 1. Let I be the index of $\mathfrak{C}$. By Lichtenbaum's theorem [18, Theorem 8], we have either $I = 1$ or $I = 2$, and if $I = 2$ then $g_C - 1$ is odd, where g_C is the genus of C. Since $S(k) \neq \emptyset$, the index divides $|G|$, hence if $p \neq 2$ then $I = 1$.

Assume $p = 2$, it is sufficient to show that $g_C - 1$ is even. If by contradiction $g_C - 1$ is odd, by the assumption in Theorem 18 either the action is free or G has no cyclic subgroups of index 2. In both cases 4 divides both $|G|$ and the cardinality of each orbit (since geometric stabilizers are cyclic), hence by Riemann–Hurwitz 4 divides $2g_C - 2$ as well, which is absurd.

4.3. Galois sections of $\mathfrak{C}$. It remains to prove that $\Pi_{\mathfrak{C}/K}(K) = \emptyset$. By Proposition 5, it is enough to show that $T = \operatorname{Spec} k(P) \to \operatorname{Spec} K$ does not lift to a $\underline{\pi}_1(S,p)$-torsor, where $p \in S(k)$ is some rational point. Equivalently, we want to show that $\operatorname{Spec} K \to \mathscr{B}G$ does not lift to $\Pi_{S/k} = \mathscr{B}\underline{\pi}_1(S,p)$.

Assume by contradiction that there exists a section $\operatorname{Spec} K \to \Pi_{S/k}$ which lifts $\operatorname{Spec} K \to \mathscr{B}G$. We would like to show that $\operatorname{Spec} K \to \Pi_{S/k}$ extends to a morphism $X \to \Pi_{S/k}$; if X and S were schemes this would follow from a well-known weight argument. Let us show how to reduce to this case.

We have a 2-cartesian diagram

$$\begin{tikzcd}
\operatorname{Spec} k(P) \arrow[r] \arrow[d] & \Pi_{C/k} \arrow[r] \arrow[d] & \operatorname{Spec} k \arrow[d] \\
\operatorname{Spec} K \arrow[r] & \Pi_{S/k} \arrow[r] & \mathscr{B}G.
\end{tikzcd}$$

By the aforementioned weight argument (see e.g. [3, Corollary A.11], [10, Lemma 3.3, Corollary 3.4], [25, Section 8.2, Proposition 91]) $\operatorname{Spec} k(P) \to \Pi_{C/k}$ extends to a morphism $P \to \Pi_{C/k}$. By descent theory, this induces a morphism $X \to \Pi_{S/k}$. It is possible to avoid using descent theory as follows.

Let $R \subset \mathrm{Gal}(\overline{K}/k(P))$ be the kernel of $\mathrm{Gal}(\overline{K}/k(P)) \to \pi_1(P)$. We have a commutative diagram of short exact sequences

$$\begin{tikzcd}
1 \arrow[r] & \mathrm{Gal}(\overline{K}/k(P)) \arrow[r] \arrow[d] & \mathrm{Gal}(\overline{K}/k(X)) \arrow[r] \arrow[d] & G \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & \pi_1(P) \arrow[r] & \pi_1(X) \arrow[r] & G \arrow[r] & 1,
\end{tikzcd}$$

hence $\pi_1(X) = \mathrm{Gal}(\overline{K}/k(X))/R$. Since $\mathrm{Spec}\, k(P) \to \Pi_{S/k}$ extends to P, the associated homomorphism $\mathrm{Gal}(\overline{K}/k(P)) \to \pi_1(S)$ maps R to the identity and hence we get a factorization

$$\mathrm{Gal}(\overline{K}/k(X)) \longrightarrow \pi_1(X) \longrightarrow \pi_1(S)$$

of the homomorphism $\mathrm{Gal}(\overline{K}/k(X)) \to \pi_1(S)$ associated with the section $\mathrm{Spec}\, K \to \Pi_{S/k}$.

Since $\mathrm{Spec}\, K \to \Pi_{S/k}$ lifts $\mathrm{Spec}\, K \to \mathscr{B}G$, the composition

$$\pi_1(X_{\bar{k}}) \longrightarrow \pi_1(S_{\bar{k}}) \longrightarrow G$$

is a factorization of the surjective homomorphism $\pi_1(X_{\bar{k}}) \to G$ induced by the G-covering $P \to X$.

Since $\pi_1(P_{\bar{k}})$ is abelian and G is a finite p-group, the image $H \subset \pi_1(S_{\bar{k}})$ of a p-Sylow of $\pi_1(X_{\bar{k}})$ is an extension of G by an abelian pro-p-group; in particular H has finite derived length. By Proposition 9, such a subgroup of $\pi_1(S_{\bar{k}})$ is either pro-cyclic or isomorphic to D_{2^∞}. If H is pro-cyclic, then G is cyclic, giving a contradiction. If $H \simeq D_{2^\infty}$, then $p = 2$ and G is dihedral. By hypothesis this implies that the action is free, which is contradiction with Proposition 9.

Remark 19. If the section conjecture holds for $C_{k(P)}$ and the action is free, we only need to assume that G is non-trivial to obtain that the curve $\mathfrak{C}$ constructed above satisfies $\Pi_{\mathfrak{C}/K}(K) = \emptyset$. In fact, if $\mathrm{Spec}\, K \to \mathscr{B}G$ lifts to $\Pi_{S/k}$, the induced morphism $\mathrm{Spec}\, k(P) \to \Pi_{C/k}$ is associated with a point $\mathrm{Spec}\, k(P) \to C$ which factorizes through $\mathrm{Spec}\, k$ since C is hyperbolic and P is an abelian variety. With notation as above, this implies that $H \simeq G$ is finite, which is absurd since $\pi_1(S_{\bar{k}})$ is torsion free.

On the other hand, this is false if the action is not free and G is cyclic. For instance, if G is cyclic and fixes a rational point $c \in C(k)$, then c induces a K-rational point of $\mathfrak{C}$.

5. Proof of Theorem 2

Consider the action ϕ of $\mathbb{Z}/\ell$ on $\mathbb{Z}/\ell^2$ given by $1 \mapsto 1 + \ell$, there exists a surjective homomorphism $\mathbb{Z}/\ell^2 \rtimes_\phi \mathbb{Z}/\ell \to (\mathbb{Z}/\ell)^2$.

Lemma 20. *Let k, ℓ be as in Theorem 2. There exists a finite extension K/k with a surjective homomorphism $\mathrm{Gal}\left(\overline{K}/K\right) \to (\mathbb{Z}/\ell)^2$ which does not lift to $\mathbb{Z}/\ell^2 \rtimes_\phi \mathbb{Z}/\ell$.*

Proof. Up to a finite extension, we may assume that k contains a primitive ℓ^2-th root of 1.

Assume first that k is a finite extension of $\mathbb{Q}_p$, $p \neq \ell$. Since $p \neq \ell$, the maximal ℓ-adic quotient of $\mathrm{Gal}(\bar{k}/k)$ is $\mathbb{Z}_\ell \rtimes_\rho \mathbb{Z}_\ell$, where $\mathbb{Z}_\ell$ acts on itself with the cyclotomic character ρ. Since k contains an ℓ^2-th root of 1, the action of $\mathbb{Z}_\ell$ on the quotient $\mathbb{Z}/\ell^2$ of $\mathbb{Z}_\ell$ is trivial, hence there exists a surjective homomorphism $\mathbb{Z}_\ell \rtimes_\rho \mathbb{Z}_\ell \to \left(\mathbb{Z}/\ell^2\right)^2$; furthermore, every finite quotient of $\mathbb{Z}_\ell \rtimes_\rho \mathbb{Z}_\ell$ of exponent ℓ^2 is a quotient of $\left(\mathbb{Z}/\ell^2\right)^2$.

It follows that there are no surjective homomorphisms $\mathbb{Z}_\ell \rtimes_\rho \mathbb{Z}_\ell \to \mathbb{Z}/\ell^2 \rtimes_\phi \mathbb{Z}/\ell$. By Lemma 10, this implies that there are no liftings $\mathbb{Z}_\ell \rtimes_\rho \mathbb{Z}_\ell \to \mathbb{Z}/\ell^2 \rtimes_\phi \mathbb{Z}/\ell$ of the composition $\mathbb{Z}_\ell \rtimes_\rho \mathbb{Z}_\ell \to \left(\mathbb{Z}/\ell^2\right)^2 \to (\mathbb{Z}/\ell)^2$. This concludes the proof for local fields.

If k is finitely generated over $\mathbb{Q}$, let $h \subset k$ be the algebraic closure of $\mathbb{Q}$ in k. I claim that there exists a finite extension h'/h with a section $\mathrm{Gal}(\bar{h}/h') \to \mathrm{Gal}(\bar{k}/k)$. By induction, it is enough to do the case in which k has transcendence degree 1 over h. Let V be a smooth curve over h whose fraction field is k, and $v \in V$ a closed point with residue field h'. Let k_v be the completion of k at the place defined by v, the residue field of k_v is h' and hence we get the desired splitting $\mathrm{Gal}(\bar{h}/h') \to \mathrm{Gal}(\bar{k}_v/k_v) \to \mathrm{Gal}(\bar{k}/k)$. Up to replacing h with h' and k with $h' \otimes_h k$, we might then assume that $\mathrm{Gal}(\bar{k}/k) \to \mathrm{Gal}(\bar{h}/h)$ is split.

Let h_ν be the completion of h at a prime ν dividing $p \neq \ell$. Since $\mathrm{Gal}(\bar{k}/k) \to \mathrm{Gal}(\bar{h}/h)$ is split and $\mathrm{Gal}(\bar{h}_\nu/h_\nu) \to \mathrm{Gal}(\bar{h}/h)$ is injective, then $\mathrm{Gal}(\bar{k}/k)$ has a closed subgroup isomorphic to $\mathrm{Gal}(\bar{h}_\nu/h_\nu)$. Let $\mathrm{Gal}(\bar{h}_\nu/h_\nu) \to (\mathbb{Z}/\ell)^2$ be the only surjective homomorphism to $(\mathbb{Z}/\ell)^2$, since $\mathrm{Gal}(\bar{h}_\nu/h_\nu)$ is a closed subgroup of $\mathrm{Gal}(\bar{k}/k)$ there exists a finite index subgroup $\Gamma \subset \mathrm{Gal}(\bar{k}/k)$ containing $\mathrm{Gal}(\bar{h}_\nu/h_\nu)$ with a factorization $\mathrm{Gal}(\bar{h}_\nu/h_\nu) \to \Gamma \to (\mathbb{Z}/\ell)^2$. Since k, and hence h_ν, contains a primitive ℓ^2-rooth of 1, then $\Gamma \to (\mathbb{Z}/\ell)^2$ does not lift to $\mathbb{Z}/\ell^2 \rtimes_\phi \mathbb{Z}/\ell$ and we may choose K/k as the field fixed by Γ. □

Let us now prove Theorem 2. Since S is hyperbolic, up to a finite extension of k by Lemma 15 with $r = 2$ we may assume that there exists a geometrically connected, finite étale Galois covering $U \to S$ with Galois group $\mathbb{Z}/\ell^2 \rtimes_\phi \mathbb{Z}/\ell$; denote by $C \to S$ the induced $(\mathbb{Z}/\ell)^2$-cover.

Let K/k, $\mathrm{Gal}(\overline{K}/K) \to (\mathbb{Z}/\ell)^2$ be as in Lemma 20, the homomorphism corresponds to a $(\mathbb{Z}/\ell)^2$-torsor $T \to \operatorname{Spec} K$. The twist $\mathfrak{C} = C_K \times^{(\mathbb{Z}/\ell)^2} T \to S_K$ is a torsor for the group scheme $\underline{\mathrm{Aut}}(T)$ of $(\mathbb{Z}/\ell)^2$-equivariant automorphisms of T. Since $(\mathbb{Z}/\ell)^2$ is abelian, then $\underline{\mathrm{Aut}}(T) \simeq (\mathbb{Z}/\ell)^2$ and hence $\mathfrak{C} \to S_K$ is a Galois $(\mathbb{Z}/\ell)^2$-cover. By Proposition 5 the twist $\mathfrak{C}$ satisfies the section conjecture.

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