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# Additive Ramsey theory over Piatetski-Shapiro numbers

par JONATHAN CHAPMAN, SAM CHOW et PHILIPPA HOLDRIDGE

RÉSUMÉ. Nous caractérisons la régularité de partitions pour les équations linéaires en nombres de Piatetski-Shapiro  $[n^c]$  lorsque  $1 < c < c^\dagger(s)$ , où  $s \geq 3$  est le nombre de variables. Ici  $c^\dagger(3) = 12/11$  et  $c^\dagger(4) = 7/6$ , tandis que  $c^\dagger(s) = 2$  pour  $s \geq 5$ . Nous établissons également des résultats de densité avec des bornes quantitatives. Suite aux développements récents, nous saisissons cette occasion pour mettre à jour la version de Browning et Prendiville du principe de transfert analytique de Fourier de Green, en renforçant ainsi sa conclusion.

ABSTRACT. We characterise partition regularity for linear equations over the Piatetski-Shapiro numbers  $[n^c]$  when  $1 < c < c^\dagger(s)$ , where  $s \geq 3$  is the number of variables. Here  $c^\dagger(3) = 12/11$  and  $c^\dagger(4) = 7/6$ , while  $c^\dagger(s) = 2$  for  $s \geq 5$ . We also establish density results with quantitative bounds. Following recent developments, we take this opportunity to update Browning and Prendiville's version of Green's Fourier-analytic transference principle, strengthening its conclusion.

## 1. Introduction

**1.1. Monochromatic Schur triples.** A *Schur triple* is  $(x, y, z) \in \mathbb{N}^3$  such that  $x + y = z$ . Schur [37] showed that if  $\mathbb{N}$  is partitioned into finitely many colour classes then there exists a monochromatic Schur triple. We state Schur's result below, writing  $[r]$  for the set of positive integers  $n \leq r$ .

**Theorem 1.1** (Schur [37]). *Suppose  $\mathbb{N} = C_1 \cup \dots \cup C_r$ . Then there exist  $k \in [r]$  and  $x, y, z \in C_k$  such that  $x + y = z$ .*

This was generalised by Rado [33] to systems of linear equations. A system of equations is *partition regular* if, for any partition of  $\mathbb{N}$  into finitely many colour classes, it contains a non-trivial solution for which all of the

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*Mots-clefs.* Arithmetic combinatorics, Ramsey theory, Diophantine equations, Hardy-Littlewood method, exponential sums, partition regularity, restriction theory.

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variables lie in the same colour class. A solution is *non-trivial* if the variables are pairwise distinct. We now state Rado's criterion in the case of a single equation.

**Theorem 1.2** (Rado [33]). *Let  $s \geq 3$  and  $c_1, \dots, c_s$  be non-zero integers. Then the equation  $\mathbf{c} \cdot \mathbf{x} = 0$  is partition regular if and only if  $\sum_{i \in I} c_i = 0$  for some non-empty  $I$ .*

Erdős and Graham [16, 17] asked whether there are monochromatic Schur triples in the squares — i.e. monochromatic Pythagorean triples — offering \$250 for an answer.

**Problem A.** Prove or disprove that if  $\mathbb{N} = C_1 \cup \dots \cup C_r$  then there exist  $k \in [r]$  and  $x, y, z \in C_k$  such that  $x^2 + y^2 = z^2$ .

A computer-assisted proof was found in the case of two colours [22] which, at the time, was the largest mathematical proof ever [25]. Partition regularity for generalised Pythagorean triples in five or more variables, i.e. solutions to

$$x_1^2 + \dots + x_{s-1}^2 = x_s^2$$

where  $s \geq 5$ , was demonstrated by Chow, Lindqvist and Prendiville [11]. A higher-degree analogue of Rado's criterion was also provided therein, and the result was generalised in [7, 8, 9]. Equations that are quadratic in some variables and linear in others have also received significant attention [20, 28, 29, 32]. Recently, it was shown in [14] that Pythagorean triples exist in which two of the variables have the same colour.

**Theorem 1.3** (Frantzikinakis–Klurman–Moreira [14]). *If*

$$\mathbb{N} = C_1 \cup \dots \cup C_r$$

*then:*

- (i) *There exists  $u \in [r]$  such that  $x^2 + y^2 = z^2$  for some  $x, y \in C_u$  and some  $z \in \mathbb{N}$ .*
- (ii) *There exists  $v \in [r]$  such that  $x^2 + y^2 = z^2$  for some  $x, z \in C_v$  and some  $y \in \mathbb{N}$ .*

The *Piatetski-Shapiro numbers* with parameter  $c > 1$  form a set

$$\text{PS}_c = \{\lfloor n^c \rfloor : n \in \mathbb{N}\}.$$

Note that  $\text{PS}_2$  is the set of positive squares. When  $1 < c < 2$ , the Piatetski-Shapiro numbers behave similarly to the squares in terms of growth and separation, but the squares are arithmetically richer. We solve a version of Problem A for Piatetski-Shapiro numbers. For  $s \geq 3$  an integer, let

$$(1.1) \quad c^*(s) = \frac{2s+6}{s+8}, \quad c^\dagger(s) = \begin{cases} c^*(s), & \text{if } s = 3, 4 \\ 2, & \text{if } s \geq 5. \end{cases}$$

For  $c > 0$  and  $N \geq 1$ , let

$$\text{PS}_c(N) = \{n \in \text{PS}_c : n \leq N\}.$$

**Theorem 1.4.** *Let  $s \geq 3$  be an integer, and let  $1 < c < c^\dagger(s)$ . Let  $c_1, \dots, c_s$  be non-zero integers with  $\sum_{i \in I} c_i = 0$ , for some non-empty  $I$ . Let  $N \in \mathbb{N}$  be large, and suppose*

$$\text{PS}_c(N) = C_1 \cup \dots \cup C_r.$$

*Then there exist  $k \in [r]$  and  $\mathbf{x} \in C_k^s$  such that*

$$(1.2) \quad \mathbf{c} \cdot \mathbf{x} = 0, \quad x_i \neq x_j \quad (i \neq j).$$

**Corollary 1.5.** *If  $1 < c < 12/11$ , and  $\mathbb{N}$  is partitioned into finitely many colour classes, then  $\text{PS}_c$  contains a monochromatic Schur triple.*

Coincidentally, Piatetski-Shapiro [30] also obtained the threshold  $12/11$  when proving a version of the prime number theorem for  $\text{PS}_c$ .

We thank Sean Prendiville for drawing our attention to the following more general problem, which he attributes to Frantzikinakis, and which inspired the present work. Note that the case  $c = 1$  is solved by Theorem 1.1, and that the case  $c = 2$  is Problem A.

**Problem B.** Prove or disprove that the conclusion of Corollary 1.5 holds for any  $c \in [1, 2]$ .

More generally, Bergelson asked whether  $\text{PS}_c$  is an IP set, see [15, Question 4].

**1.2. Relative sparsity of 3AP-free sets.** In a very influential paper, Roth [35] showed that if  $N$  is large and  $A \subseteq [N]$  with  $|A| \gg N$  then  $A$  contains non-trivial three-term arithmetic progressions, i.e. non-trivial solutions to  $x + y = 2z$ . More precisely, Roth proved that if  $A \subseteq [N]$  does not contain any non-trivial three-term arithmetic progressions, then  $|A| \ll N/\log \log N$ .

Recently, there have been strong quantitative refinements of Roth’s theorem by Kelley–Meka [23] and Bloom–Sisask [4]. To state them, we say that  $A \subset \mathbb{N}$  is *3AP-free* if  $x + y = 2z$  has no non-trivial solutions with  $x, y, z \in A$ . Remarkably, the bound has the same shape as the best known lower bound for the largest 3AP-free set, which comes from a convexity construction going back to Behrend [2].

**Theorem 1.6** (Kelley–Meka [23] and Bloom–Sisask [4]). *If  $N \geq 2$  and  $A \subset [N]$  is 3AP-free, then*

$$|A| \leq \exp(-c \log^{1/9} N)N,$$

*for some constant  $c > 0$ .*

Turning to the non-linear setting, consider the following density analogue of Problem A.

**Problem C.** Prove or disprove that if  $A$  is a 3AP-free subset of the squares then  $|A \cap [N]| = o(\sqrt{N})$ .

Equivalently, show that  $x^2 + y^2 = 2z^2$  is density regular, meaning that any  $A \subseteq [N]$  which lacks non-trivial solutions to this equation must satisfy  $|A| = o(N)$ .

For diagonal quadratic equations in five or more variables, such as

$$x_1^2 + \cdots + x_4^2 = 5x^2,$$

quantitative density regularity was characterised by Browning and Prendiville [6].

**Theorem 1.7** (Browning–Prendiville [6]). *Let  $s \geq 5$  and  $c_1, \dots, c_s$  be non-zero integers with  $c_1 + \cdots + c_s = 0$ . Let  $N \in \mathbb{N}$  be large, and suppose  $A \subseteq \text{PS}_2(N)$  is such that there does not exist  $\mathbf{x} \in A^s$  satisfying (1.2). Then, for any  $\varepsilon > 0$ ,*

$$|A| \ll_{\mathbf{c}, \varepsilon} \frac{|\text{PS}_2(N)|}{(\log \log \log N)^{(s-2-\varepsilon)/2}}.$$

Browning and Prendiville’s techniques were subsequently generalised to higher perfect powers and to primes in [10], and generalised further in [7, 8, 9]. By adapting these methods, we solve a version of Problem C for Piatetski-Shapiro numbers, with a quantitative density bound.

**Theorem 1.8.** *Let  $\tau(3) = 9$ , and set  $\tau(s) = 7$  for all  $s \geq 4$ . Let  $s \geq 3$  be an integer, and let  $1 < c < c^\dagger(s)$ . Let  $c_1, \dots, c_s$  be non-zero integers with  $c_1 + \cdots + c_s = 0$ . Let  $N \in \mathbb{N}$  be large. If  $A \subseteq \text{PS}_c(N)$  is such that there does not exist  $\mathbf{x} \in A^s$  satisfying (1.2), then*

$$|A| \leq \frac{|\text{PS}_c(N)|}{\exp(C^{-1}(\log \log N)^{1/\tau(s)})},$$

for some constant  $C > 1$  which depends only on  $c$  and  $\mathbf{c}$ .

**Remark 1.9.** Our density saving is substantially greater than the saving of  $(\log \log \log N)^{O(1)}$  that Browning and Prendiville obtained in the case of squares. This is partly because the  $W$ -trick is not required in our setting. We make further gains by strengthening the Fourier-analytic transference principle [6, Proposition 2.8] in Section 4. We do so by incorporating recent developments [12, 24].

In contrast with Theorem 1.7, we only need to assume that  $s \geq 3$ . With our constraint on  $c$ , we are able to obtain more precise estimates for exponential sums. This leads to a sharp restriction estimate at a lower exponent — see Section 1.3 — which ultimately reduces the number of variables that we require.

Note that the natural analogue of Problem C for Piatetski-Shapiro sequences with parameter  $c \in [1, 2)$  has already been settled. A result of Saito and Yoshida [36, Corollary 5] — which uses Szemerédi’s theorem [38] to find progressions in graphs of “slightly curved sequences” — implies the following.

**Theorem 1.10** (Saito–Yoshida [36]). *Fix  $c \in [1, 2)$  and an integer  $k \geq 3$ . Let  $N \in \mathbb{N}$  be large, and suppose  $A \subseteq \text{PS}_c(N)$  with*

$$|A| \gg |\text{PS}_c(N)|.$$

*Then  $A$  contains a non-trivial  $k$ -term arithmetic progression.*

One can deduce the following application using the further observation that if  $c_1 + \dots + c_s = 0$  then, for any  $\lambda \in \mathbb{R}$  and  $\mu_1 = \dots = \mu_s \in \mathbb{R}$ ,

$$\mathbf{c} \cdot \mathbf{x} = 0 \Rightarrow \mathbf{c} \cdot (\lambda \mathbf{x} + \boldsymbol{\mu}) = 0 \quad (\mathbf{x} \in \mathbb{R}^s).$$

**Corollary 1.11.** *Let  $s \geq 3$  be an integer, and let  $1 < c < 2$ . Let  $c_1, \dots, c_s$  be non-zero integers with  $c_1 + \dots + c_s = 0$ . Let  $N \in \mathbb{N}$ , and suppose  $A \subseteq \text{PS}_c(N)$  is such that there does not exist  $\mathbf{x} \in A^s$  satisfying (1.2). Then  $|A| = o(|\text{PS}_c(N)|)$  as  $N \rightarrow \infty$ .*

Compared with these wonderful findings, Theorem 1.8 has the advantage of providing a quantitative density bound.

One can also consider these problems for Piatetski-Shapiro primes, i.e. primes that are Piatetski-Shapiro numbers. Mirek [27] achieved this in the case of 3AP-free sets, but required that  $c < 72/71$ . We state Mirek’s result below, writing  $\mathcal{P}$  for the set of primes.

**Theorem 1.12** (Mirek [27]). *Let  $1 < c < 72/71$ . If  $N$  is large and  $A$  is a 3AP-free subset of  $\text{PS}_c(N) \cap \mathcal{P}$ , then  $|A| = o(|\text{PS}_c(N) \cap \mathcal{P}|)$ .*

One should think twice before looking for monochromatic Schur triples of Piatetski-Shapiro primes, however, as there are no Schur triples of odd numbers. Using our methods and the ingredients [27, Theorem 1.8 and Lemma 1.10], one can show that if  $1 < c < 72/71$  and  $\text{PS}_c \cap \mathcal{P} = C_1 \cup \dots \cup C_r$  then there exist  $k \in [r]$  and  $x, y, z \in C_k$  such that

$$x + y = z + 1.$$

See [9, 26, 40] for some similar results and open problems. We leave it to the interested reader to explore further generalisations.

**1.3. Methods.** We use the Fourier-analytic transference principle, as pioneered by Green [19]. We now explain briefly how this works. For a more detailed discussion of the transference machinery, see [31].

We introduce a weight function  $\nu$  supported on  $\text{PS}_c(N)$  such that  $\|\nu\|_1 \sim N$  as  $N \rightarrow \infty$ . The idea is to compare the  $\nu$ -weighted count of solutions with

the  $1_{[N]}$ -weighted count, essentially by combining some estimates in Fourier space. The latter count was estimated by Frankl, Graham and Rödl [13]. The key ingredients for the density result, Theorem 1.8, are as follows.

- (1) Fourier decay: For some  $\chi \in (0, 1)$ , we have

$$\|\widehat{\nu} - \widehat{1}_{[N]}\|_\infty \ll_\chi N^{1-\chi}.$$

- (2) Restriction estimate: For some  $t \in (s-1, s)$ , we have

$$\sup_{|\psi| \leq \nu} \int_{\mathbb{T}} |\widehat{\psi}(\alpha)|^t d\alpha \ll_t N^{t-1}.$$

- (3) Trivial count:

$$\sum_{\mathbf{n} \in \mathcal{K}} \prod_{i \leq s} \nu(n_i) = o(N^{s-1}) \quad (N \rightarrow \infty),$$

where

$$\mathcal{K} = \{\mathbf{n} \in \mathbb{Z}^s : \mathbf{c} \cdot \mathbf{n} = 0, \quad n_i = n_j \text{ for some } i \neq j\}.$$

There is one other ingredient, *density transfer*, which roughly asserts that

$$\sum_{n \in A} 1_{[N]}(n) \gg N \quad \implies \quad \sum_{n \in A} \nu(n) \gg N,$$

but this is easy to confirm. The Ramsey-theoretic result, Theorem 1.4, requires further combinatorial machinery, which we import from [11].

We gauge the strength of our results by the value of  $c^\dagger(s)$ . This depends on the values of  $\chi$  and  $t$  relating to Fourier decay and restriction, so we have attempted to maximise  $\chi$  and minimise  $t$ . Although we are able to obtain power-saving Fourier decay for all  $1 < c < 2$  in Proposition 2.1, we do need to assume that  $c < c^\dagger(s)$  in order to also obtain the restriction estimate (3.3), which is a limitation when  $s \leq 4$ .

**Organisation.** We establish a Fourier decay estimate in Section 2, and a restriction estimate in Section 3. In Section 4, we strengthen the transference principle provided by Browning in Prendiville in [6, Proposition 2.8], using recent developments. Finally, in Section 5, we prove Theorems 1.4 and 1.8.

**Notation.** For  $x \in \mathbb{R}$ , we write  $e(x) = e^{2\pi i x}$ . We put  $\mathbb{T} = [0, 1]$ . For  $f \in \ell^1(\mathbb{Z})$ , we define the Fourier transform by

$$\widehat{f}(\alpha) = \sum_{n \in \mathbb{Z}} f(n)e(n\alpha).$$

We employ the Vinogradov and Bachmann–Landau asymptotic notations, as we now describe. For complex-valued functions  $f$  and  $g$ , we write  $f \ll g$  or  $f = O(g)$  if there exists a constant  $C$  such that  $|f| \leq C|g|$  pointwise. We indicate the dependence of the implicit constant  $C$  on some

parameters  $\lambda_1, \dots, \lambda_t$  using subscripts, for example  $f \ll_{\lambda_1, \dots, \lambda_t} g$  or  $f = O_{\lambda_1, \dots, \lambda_t}(g)$ . We write  $f \asymp g$  if  $f \ll g \ll f$ . We write  $f = o(g)$  if  $f/g \rightarrow 0$  and  $f \sim g$  if  $f/g \rightarrow 1$ , in some specified limit.

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## 2. Fourier decay

The purpose of this section is to prove the following Fourier decay estimate for our weight function

$$\nu(n) = \nu_c(n) := \phi'(n)^{-1} 1_{\text{PS}_c(N)}(n),$$

where  $\phi(n) = n^{1/c}$ .

**Proposition 2.1** (Fourier decay). *Let  $c \in (1, 2)$  and  $\varepsilon > 0$ . Then*

$$\|\widehat{\nu} - \widehat{1}_{[N]}\|_{\infty} \ll_{c, \varepsilon} N^{\varepsilon + 6/5 - 2/(5c)}.$$

The broad strategy used to understand the Fourier coefficients of  $\nu$  is given in [21], where it is attributed to Vaughan.

**Lemma 2.2.** *For all  $c > 1$  and  $n \in \mathbb{N}$ ,*

$$1_{\text{PS}_c}(n) = \lfloor -\phi(n) \rfloor - \lfloor -\phi(n+1) \rfloor.$$

*Proof.* Since  $c > 1$ , the mean value theorem implies that the function

$$n \mapsto \lfloor -\phi(n) \rfloor - \lfloor -\phi(n+1) \rfloor$$

has image  $\{0, 1\}$ . Observe that  $n \in \mathbb{N}$  is mapped to 1 by this function if and only if there exists  $m \in \mathbb{N}$  such that

$$\phi(n) \leq m < \phi(n+1).$$

Applying  $\phi^{-1}$ , this is equivalent to

$$n \leq m^c < n+1,$$

which holds if and only if  $n = \lfloor m^c \rfloor$ . □

An application of this lemma reveals that

$$(2.1) \quad \widehat{\nu}(\alpha) = \sum_{n \leq N} \phi'(n)^{-1} e(n\alpha) (\lfloor -\phi(n) \rfloor - \lfloor -\phi(n+1) \rfloor).$$

To further modify this expression, we deploy the saw-tooth function

$$\psi(x) := x - \lfloor x \rfloor - 1/2.$$

For each  $n \in \mathbb{N}$ , observe that

$$\lfloor -\phi(n) \rfloor - \lfloor -\phi(n+1) \rfloor = \phi(n+1) - \phi(n) + \psi(-\phi(n+1)) - \psi(-\phi(n)).$$

By Taylor's theorem with Lagrange remainder, there exists  $\xi_n \in [0, 1]$  such that

$$\phi(n+1) - \phi(n) = \phi'(n) + \phi''(n + \xi_n)/2.$$

Thus,

$$\begin{aligned} \sum_{n \leq N} \phi'(n)^{-1} e(n\alpha) (\phi(n+1) - \phi(n)) \\ &= \sum_{n \leq N} e(n\alpha) \left( 1 + \frac{(1-c)(n + \xi_n)^{\frac{1}{c}-2}}{2cn^{\frac{1}{c}-1}} \right) \\ &= \widehat{1}_{[N]}(\alpha) + O(\log N). \end{aligned}$$

Inserting this into (2.1), we find that

$$\widehat{\nu}(\alpha) - \widehat{1}_{[N]}(\alpha) = \sum_{n \leq N} \frac{\psi(-\phi(n+1)) - \psi(-\phi(n))}{\phi'(n)} e(n\alpha) + O(\log N).$$

To make further progress, we split the sum on the right-hand side into dyadic ranges:

$$(2.2) \quad \widehat{\nu}(\alpha) - \widehat{1}_{[N]}(\alpha) = \sum_{1 \leq k \leq \frac{\log N}{\log 2} + 1} S(N/2^k) + O(\log N),$$

where

$$S(P) = S(P; \alpha) = \sum_{P < n \leq 2P} \frac{\psi(-\phi(n+1)) - \psi(-\phi(n))}{\phi'(n)} e(n\alpha).$$

To prove Proposition 2.1, we require estimates for the exponential sums  $S(P)$ . As noted in [21, Section 2], we have the following standard approximation for  $\psi$  by trigonometric sums.

**Lemma 2.3.** *For any integer  $M \geq 2$ ,*

$$\psi(t) - \sum_{0 < |m| \leq M} \frac{e(-mt)}{2\pi im} \ll \min\{1, (M\|t\|)^{-1}\} = \sum_{m \in \mathbb{Z}} b_m e(mt),$$

for some

$$b_m \ll \min \left\{ \frac{\log M}{M}, \frac{1}{|m|}, \frac{M}{m^2} \right\}.$$

We apply this lemma with  $M \in [2, P]$  equal to a power of  $P$  to be specified later. This gives

$$S(P) - S_0 \ll S_1 + S_2,$$

where

$$S_0 = \sum_{0 < |m| \leq M} \frac{1}{2\pi i m} \sum_{P < n \leq 2P} \frac{e(n\alpha + m\phi(n+1)) - e(n\alpha + m\phi(n))}{\phi'(n)},$$

$$S_1 = \sum_{P < n \leq 2P} \frac{\min\{1, (M\|\phi(n)\|)^{-1}\}}{\phi'(n)},$$

$$S_2 = \sum_{P < n \leq 2P} \frac{\min\{1, (M\|\phi(n+1)\|)^{-1}\}}{\phi'(n)}.$$

We begin by bounding  $S_0$ . Using partial summation, we find that

$$\begin{aligned} \sum_{P < n \leq 2P} \frac{e(n\alpha + m\phi(n+1)) - e(n\alpha + m\phi(n))}{\phi'(n)} \\ = U_m(2P)\varphi_m(2P) - \int_P^{2P} U_m(x)\varphi'_m(x) dx, \end{aligned}$$

where

$$\varphi_m(x) = \frac{e(m(\phi(x+1) - \phi(x))) - 1}{\phi'(x)}$$

and

$$U_m(x) = \sum_{P < n \leq x} e(n\alpha + m\phi(n)).$$

We compute that  $\varphi_m(x) \ll m$  and  $\varphi'_m(x) \ll m/x$ , whence

$$S_0 \ll \sum_{0 < |m| < M} \sup_{x \in (P, 2P]} |U_m(x)|.$$

Notice that, for  $P < x \leq 2P$ , the function  $F(x) = \alpha x + m\phi(x)$  satisfies  $|F''(x)| \asymp |m|P^{1/c-2}$ . Invoking van der Corput's lemma [27, Lemma 3.1], we deduce that

$$U_m(x) \ll |m|^{1/2} P^{1/(2c)} + P^{1-1/(2c)} |m|^{-1/2},$$

and so

$$S_0 \ll M^{3/2} P^{1/(2c)} + M^{1/2} P^{1-1/(2c)}.$$

We now consider the sum  $S_1$ . The key ingredient is the following estimate observed by Heath-Brown [21, Lemma 1].

**Lemma 2.4.** *Let  $m \in \mathbb{Z}$  and  $1/2 < \gamma < 1$ . Then, whenever we have  $1 \leq P \leq P_1 \leq 2P$ ,*

$$\sum_{P < n \leq P_1} e(mn^\gamma) \ll \min\{P, |m|^{-1} P^{1-\gamma} + (|m|P^\gamma)^{1/2}\}.$$

**Remark 2.5.** This can be refined in some ranges using the theory of exponent pairs [18]. However, such a refinement does not seem to affect the final outcome of Proposition 2.1.

To simplify our notation, let  $\gamma = 1/c$ , so that  $\phi(n) = n^\gamma$ . By the triangle inequality,

$$\phi'(P)S_1 \leq \sum_{m \in \mathbb{Z}} |b_m| \left| \sum_{P < n \leq 2P} e(m\phi(n)) \right|.$$

For further convenience, we write

$$T_m = T_m(P) = \sum_{P < n \leq 2P} e(m\phi(n)).$$

Our goal is to establish bounds for

$$\sum_{m \in \mathbb{Z}} |b_m T_m|.$$

We accomplish this by examining the contributions from various ranges for  $m \in \mathbb{Z}$ .

We begin with “small”  $|m|$  for which the bound  $b_m \ll (\log M)/M$  from Lemma 2.3 suffices. The contribution from  $m = 0$  is immediate:

$$|b_0 T_0| = |b_0|P \ll \frac{P \log M}{M}.$$

Notice that, for any  $m, P \geq 1$ , we have

$$(mP^\gamma)^{1/2} \leq m^{-1}P^{1-\gamma}$$

if and only if

$$m \leq P^{2/3-\gamma}.$$

Let  $L = \lfloor (2P)^{2/3-\gamma} \rfloor$ . By Lemma 2.4,

$$\begin{aligned} \sum_{0 < |m| \leq L} |b_m T_m| &\ll \frac{\log M}{M} P^{1-\gamma} \sum_{0 < |m| \leq L} \frac{1}{|m|} \\ &\ll \frac{(\log M) \log L}{M} P^{1-\gamma} \ll_\varepsilon \frac{P^{1-\gamma+\varepsilon}}{M}. \end{aligned}$$

A similar computation reveals that

$$\sum_{L < |m| \leq M} |b_m T_m| \ll \frac{\log M}{M} P^{\gamma/2} \sum_{L < |m| \leq M} |m|^{1/2} \ll M^{1/2} P^{\gamma/2} \log M.$$

For  $|m| > M$ , we use the bounds  $|b_m| \ll M/m^2$  and  $|T_m|^2 \ll |m|P^\gamma$  from Lemmas 2.3 and 2.4, respectively, to deduce that

$$\sum_{|m| > M} |b_m T_m| \ll MP^{\gamma/2} \sum_{|m| > M} |m|^{-3/2} \ll M^{1/2} P^{\gamma/2}.$$

By combining all of the above observations, we see that

$$\sum_{m \in \mathbb{Z}} |b_m T_m| \ll (M^{-1}P + M^{1/2}P^{\gamma/2}) \log M.$$

Since  $\phi'(P) = \gamma P^{\gamma-1}$ , we therefore have

$$S_1 \ll (M^{-1}P^{2-\gamma} + M^{1/2}P^{1-\gamma/2}) \log M.$$

Using an almost identical argument, we can also show that

$$S_2 \ll (M^{-1}P^{2-\gamma} + M^{1/2}P^{1-\gamma/2}) \log M.$$

We have therefore demonstrated that

$$S(P) \ll M^{3/2}P^{\gamma/2} + (M^{-1}P^{2-\gamma} + M^{1/2}P^{1-\gamma/2}) \log M.$$

Recall that  $c < 2$ , whence  $\gamma > 1/2$ . Choosing  $M = P^{4/5-3\gamma/5}$  gives

$$S(P) \ll_{\varepsilon} P^{\varepsilon+6/5-2\gamma/5},$$

for any  $\varepsilon > 0$ . Inserting this estimate into (2.2) finishes the proof of Proposition 2.1.

Before moving on to the next section, we record the following observations. By Proposition 2.1, if  $\varepsilon > 0$  then

$$\|\nu\|_1 = \widehat{\nu}(0) = N + O(N^{\varepsilon+6/5-2/(5c)}).$$

In particular, since  $c < 2$ ,

$$(2.3) \quad \|\widehat{\nu}\|_{\infty} = \|\nu\|_1 \sim N \quad (N \rightarrow \infty).$$

### 3. Fourier restriction

In this section, we use Bourgain’s epsilon-removal process [5] to establish two restriction estimates for our weight function  $\nu = \nu_c$ .

**Proposition 3.1** (Fourier restriction I). *Let  $1 < c < 2$ . Suppose we have*

$$(3.1) \quad \|\widehat{\nu} - \widehat{1}_{[N]}\|_{\infty} \ll_{c,\chi} N^{1-\chi},$$

for some  $\chi \in (0, 1)$ . Define  $t_0 > 2$  by

$$(3.2) \quad \chi = 2 \frac{1 - 1/c}{t_0 - 2},$$

and let  $t > t_0$ . Let  $\psi : \mathbb{Z} \rightarrow \mathbb{C}$  with  $|\psi| \leq \nu$ . Then

$$(3.3) \quad \int_{\mathbb{T}} |\widehat{\psi}(\alpha)|^t d\alpha \ll_{c,\chi,t} N^{t-1}.$$

*Proof.* A naïve approach would be to consider  $t = 2$  and use orthogonality to obtain

$$(3.4) \quad \int_{\mathbb{T}} |\widehat{\psi}(\alpha)|^2 d\alpha \leq \sum_{n \leq N} \nu(n)^2 \ll N^{2-1/c}.$$

This only saves  $1/c$  in the exponent. We need to save an additional exponent of  $1 - 1/c$  and to do this, we use an epsilon removal process. This method is usually used to remove an arbitrarily small power  $\varepsilon$ , but it can in fact achieve more, as in [11, Appendix E].

Note from (2.3) that  $\|\widehat{\psi}\|_\infty \ll N$ . For  $0 < \delta \ll 1$ , define the large spectra by

$$\mathcal{R}_\delta = \{\alpha \in \mathbb{T} : |\widehat{\psi}(\alpha)| > \delta N\}.$$

We claim that it suffices to prove that

$$(3.5) \quad \text{meas}(\mathcal{R}_\delta) \ll_{\varepsilon, t_0} \frac{1}{\delta^{t_0 + \varepsilon} N}$$

for all sufficiently small  $\varepsilon > 0$ . To see this, let  $C > 0$  be such that  $\sum_{n \leq N} \nu(n) \leq CN$  for all  $N \geq 1$ . Then

$$\int_{\mathbb{T}} |\widehat{\psi}(\alpha)|^t d\alpha \leq \sum_{n=0}^{\infty} (2^{-n} CN)^t \text{meas}(\mathcal{R}_{2^{-n}C}) \ll N^{t-1} \sum_{n=0}^{\infty} 2^{(t_0 - t + \varepsilon)n}.$$

Then we simply take  $\varepsilon < t - t_0$ .

Now, by (3.4),

$$\delta^2 N^2 \text{meas}(\mathcal{R}_\delta) \leq \int_{\mathbb{T}} |\widehat{\psi}(\alpha)|^2 d\alpha \ll N^{2-1/c},$$

and therefore

$$\text{meas}(\mathcal{R}_\delta) \ll \delta^{-2} N^{-1/c}.$$

So (3.5) holds if

$$\delta^{-2} N^{-1/c} \leq \delta^{-t_0 - \varepsilon} N^{-1},$$

which is equivalent to

$$\delta \leq N^{-\eta},$$

where

$$(3.6) \quad \eta = \frac{1 - 1/c}{t_0 - 2 + \varepsilon}.$$

We may therefore suppose that

$$(3.7) \quad N^{-\eta} < \delta \ll 1.$$

Let  $\theta_1, \dots, \theta_R \in \mathcal{R}_\delta$  be such that  $\|\theta_i - \theta_j\| \geq 1/N$  whenever  $i \neq j$ . Let  $c_r$  and  $a_n$  be such that

$$|\widehat{\psi}(\theta_r)| = c_r \widehat{\psi}(\theta_r), \quad \psi(n) = a_n \nu(n),$$

noting that  $|c_r|, |a_n| \leq 1$ . Cauchy's inequality gives

$$\begin{aligned} \delta^2 N^2 R^2 &\leq \left( \sum_{r=1}^R |\widehat{\psi}(\theta_r)| \right)^2 = \left( \sum_{r=1}^R c_r \sum_n a_n \nu(n) e(n\theta_r) \right)^2 \\ &\ll N \sum_n \nu(n) \left| \sum_{r=1}^R c_r e(n\theta_r) \right|^2 \leq N \sum_{1 \leq r, r' \leq R} |\widehat{\nu}(\theta_r - \theta_{r'})|. \end{aligned}$$

Consequently

$$\delta^2 N R^2 \ll \sum_{1 \leq r, r' \leq R} |\widehat{\nu}(\theta_r - \theta_{r'})|$$

and, by applying Hölder’s inequality with  $\kappa > 2$  a constant to be chosen later,

$$\delta^{2\kappa} N^\kappa R^2 \ll \sum_{1 \leq r, r' \leq R} |\widehat{\nu}(\theta_r - \theta_{r'})|^\kappa.$$

It follows from (3.1) and standard upper bounds for  $\widehat{1}_{[N]}$  that

$$\widehat{\nu}(\theta) \ll \frac{N}{1 + N\|\theta\|} + N^{1-\chi} \quad (\theta \in \mathbb{T}).$$

Now

$$\delta^{2\kappa} N^\kappa R^2 \ll N^\kappa \sum_{1 \leq r, r' \leq R} F(\theta_r - \theta_{r'}) + N^{\kappa(1-\chi)} R^2,$$

where  $F(\theta) = (1 + N\|\theta\|)^{-\kappa}$ . Recalling (3.2) and (3.6), we see that  $\eta < \chi/2$ . Combining this with (3.7), we find that  $N^{\kappa(1-\chi)} R^2 = o(\delta^{2\kappa} N^\kappa R^2)$ . Therefore

$$\delta^{2\kappa} R^2 \ll \sum_{1 \leq r, r' \leq R} F(\theta_r - \theta_{r'}).$$

Then, by the argument in [5, Section 4], but with  $N^2$  replaced by  $N$ , the exponent  $\gamma/2$  replaced by  $\kappa$  in the definition of  $F$ , and with  $Q = 1$ , we obtain

$$\delta^{2\kappa} R^2 \ll 1.$$

Hence  $R \ll \delta^{-\kappa}$ . Recalling that  $\theta_1, \dots, \theta_R$  are  $1/N$ -spaced points in  $\mathcal{R}_\delta$ , and supposing that this set is as large as possible, it follows that

$$\text{meas}(\mathcal{R}_\delta) \ll R/N \ll \frac{1}{\delta^\kappa N}.$$

Choosing  $\kappa = t_0 + \varepsilon$  then gives (3.5) and completes the proof.  $\square$

Our second restriction estimate is similar, but we bootstrap a fourth moment estimate — a bound on the additive energy — instead of a second moment estimate. Specifically, we invoke [1, Equation 47], which is essentially [34, Theorem 2]. This tells us that

$$\#\{(x_1, \dots, x_4) \in [X]^4 : |x_1^c + x_2^c - x_3^c - x_4^c| \leq 2\} \ll_{c, \varepsilon} X^{2+\varepsilon} + X^{4-c+\varepsilon},$$

for any  $c > 1$  and any  $\varepsilon > 0$ . Consequently, if  $1 < c < 2$  and  $\varepsilon > 0$ , then

$$\begin{aligned} E_c(N) &:= \#\{(n_1, \dots, n_4) \in \text{PS}_c(N)^4 : n_1 + n_2 = n_3 + n_4\} \\ (3.8) \quad &\ll_{c, \varepsilon} N^{\frac{4}{c}-1+\varepsilon}. \end{aligned}$$

**Proposition 3.2** (Fourier restriction II). *Let  $1 < c < 2$ . Let  $t > 4$ , and suppose we have (3.1) for some  $\chi \in (0, 1)$ . Let  $\psi : \mathbb{Z} \rightarrow \mathbb{C}$  with  $|\psi| \leq \nu$ . Then we have (3.3).*

*Proof.* We follow the proof of Proposition 3.1, *mutatis mutandis*. We replace  $t_0$  by 4. For any  $\varepsilon > 0$ , we see from (3.8) that

$$\delta^4 N^4 \operatorname{meas}(\mathcal{R}_\delta) \leq \int_{\mathbb{T}} |\widehat{\psi}(\alpha)|^4 d\alpha \leq \|\nu\|_\infty^4 E_c(N) \ll_\varepsilon N^{3+\varepsilon},$$

so

$$\operatorname{meas}(\mathcal{R}_\delta) \ll_\varepsilon \delta^{-4} N^{\varepsilon-1}.$$

Essentially the same proof then carries through.  $\square$

#### 4. An update on the transference principle

In this section, we refine the general transference machinery given by Browning and Prendiville in [6, Proposition 2.8]. Kościuszko mentioned this possibility in [24]. To elucidate the quantitative aspects of this result, we begin by recalling the key concept of pseudorandomness.

**Definition 4.1.** Let  $\mathbf{c} \in \mathbb{Z}^s$ , and let  $\nu : \mathbb{Z} \rightarrow [0, \infty)$  be supported on  $[N]$ . We say that  $\nu$  is  $\mathbf{c}$ -pseudorandom if, for any  $\delta > 0$ , there exists  $c(\delta) > 0$  such that if  $0 \leq f \leq \nu$  then

$$\|f\|_1 \geq \delta \|\nu\|_1 \implies \sum_{\mathbf{c} \cdot \mathbf{x} = 0} \prod_{i \leq s} f(x_i) \geq c(\delta) \sum_{\mathbf{c} \cdot \mathbf{x} = 0} \prod_{i \leq s} \nu(x_i).$$

For the pseudorandomness of  $1_{[N]}$ , Browning and Prendiville infer a quantitative rate  $c(\delta)$  from earlier work of Bloom [3]. We now update this rate by incorporating recent advances due to Kościuszko [24] and Filmus, Hatami, Hosseini, and Kelman [12]. The latter generalises the aforementioned breakthrough of Kelley and Meka [23].

**Lemma 4.2.** Let  $\tau(3) = 9$ , and set  $\tau(s) = 7$  for all  $s \geq 4$ . Let  $s \geq 3$  and let  $c_1, \dots, c_s$  be non-zero integers summing to zero. Then  $1_{[N]}$  is  $\mathbf{c}$ -pseudorandom with quantitative rate

$$c(\delta) = \exp(-C \log^{\tau(s)}(2/\delta)),$$

where  $C$  is a large, positive constant depending on  $\mathbf{c}$  alone.

*Proof.* Throughout this proof, the letter  $C$  will denote a sufficiently large positive constant which depends on  $\mathbf{c}$  alone. The value of  $C$  may change from line to line.

We claim that it suffices to show that for all  $N \in \mathbb{N}$  and all  $0 < \delta \leq 1$ , if  $A \subseteq [N]$  satisfies  $|A| \geq \delta N$ , then

$$(4.1) \quad |\{\mathbf{x} \in A^s : \mathbf{c} \cdot \mathbf{x} = 0\}| \geq N^{s-1} \exp(-C \log^{\tau(s)}(2/\delta)).$$

To see this, let  $f : [N] \rightarrow [0, 1]$  with  $\|f\|_1 \geq \delta N$ , where  $N \in \mathbb{N}$  and  $0 < \delta \leq 1$ , and let  $A = \{n \in [N] : f(n) \geq \delta/2\}$ . By averaging, we have  $|A| \geq \delta N/2$ . Since  $f \geq (\delta/2)1_A$ , we find that

$$\sum_{\mathbf{c} \cdot \mathbf{x} = 0} \prod_{i \leq s} f(x_i) \geq (\delta/2)^s N^{s-1} \exp(-C \log^{\tau(s)}(2/\delta)).$$

After adjusting the value of  $C$  slightly, we infer the required bound

$$(4.2) \quad \sum_{\mathbf{c} \cdot \mathbf{x} = 0} \prod_{i \leq s} f(x_i) \geq \exp(-C \log^{\tau(s)}(2/\delta)) \sum_{\mathbf{c} \cdot \mathbf{x} = 0} \prod_{i \leq s} 1_{[N]}(x_i).$$

It therefore remains to verify (4.1). For  $s \geq 4$ , this follows immediately from [24, Theorem 3].

For  $s = 3$ , we need to perform some additional manoeuvres to extract (4.1) from [12]. We begin with [12, Theorem 1.5(ii)], which tells us that, for any prime  $p > \|\mathbf{c}\|_1$ , if  $B \subseteq \mathbb{Z}/p\mathbb{Z}$  does not admit a non-trivial solution to  $\mathbf{c} \cdot \mathbf{x} \equiv 0 \pmod{p}$  with  $\mathbf{x} \in B^s$ , then

$$|B|/p \leq \exp(-C^{-1} \log^{1/9} p).$$

To ensure that the conditions of the cited theorem hold, one uses the parametrisation  $((c_1 + c_2)x + c_2z, (c_1 + c_2)y - c_1z, c_1x + c_2y)$  for solutions to  $\mathbf{c} \cdot \mathbf{x} = 0$  over  $\mathbb{Z}/p\mathbb{Z}$ . This parametrisation “captures” solutions to  $\mathbf{c} \cdot \mathbf{x} = 0$ , in the sense that every solution is counted exactly  $p$  many times. The “underlying graph” is then a complete graph of order 3, which is “2-degenerate” — see [12, Example 1.7] and the paragraph preceding it.

If we choose  $\|\mathbf{c}\|_1 N < p \leq 2\|\mathbf{c}\|_1 N$ , then  $\mathbf{x} \in [N]^s$  satisfies  $\mathbf{c} \cdot \mathbf{x} = 0$  if and only if  $\mathbf{c} \cdot \mathbf{x} \equiv 0 \pmod{p}$ . Hence, by embedding  $[N]$  into  $\mathbb{Z}/p\mathbb{Z}$  in the usual way, the cited theorem implies — upon rearranging — that there exists a positive integer  $k \leq \exp(C \log^9(2/\delta))$  such that every  $S \subseteq [k]$  with  $|S| \geq \delta k/2$  contains a non-trivial solution to  $\mathbf{c} \cdot \mathbf{x} = 0$ . Since the entries of  $\mathbf{c}$  sum to zero, an averaging argument of Varnavides [39] — with the calculation as in [19, Proof of Lemma 6.8] — reveals that if  $A \subseteq \mathbb{Z}/p\mathbb{Z}$  and  $|A| \geq \eta p \geq 1$  then

$$|\{\mathbf{x} \in A^s : \mathbf{c} \cdot \mathbf{x} = 0\}| \geq \frac{\eta p^2}{2k^2} \geq N^2 \exp(-C \log^9(2/\eta)).$$

This verifies (4.1), upon choosing  $\eta = \delta/(2\|\mathbf{c}\|_1)$ . □

This lemma leads to the following refinement of [6, Proposition 2.8].

**Theorem 4.3.** *Let  $s \geq 3$  and let  $\tau(s)$  be as in the statement of Lemma 4.2. Let  $c_1, \dots, c_s$  be non-zero integers summing to zero. Let  $\mathcal{K}$  be a finite union of  $k$  many proper subspaces of the hyperplane defined by  $\mathbf{c} \cdot \mathbf{x} = 0$ . Let  $\nu : \mathbb{Z} \rightarrow [0, \infty)$  be supported on  $[N]$ , with the following properties:*

(i) (Fourier decay) For some  $\theta \in (0, 1]$ ,

$$\|\widehat{\nu} - \widehat{1}_{[N]}\|_\infty \leq \theta N.$$

(ii) (Fourier restriction) For some  $t \in [s - 1, s)$ ,

$$\sup_{|\psi| \leq \nu} \int_{\mathbb{T}} |\widehat{\psi}(\alpha)|^t d\alpha \ll_t \|\nu\|_1^t N^{-1}.$$

(iii) (*K-trivial saving*) For some  $\eta > 0$ ,

$$\sum_{\mathbf{x} \in \mathcal{K}} \prod_{i \leq s} \nu(x_i) \ll_{s,k} \|\nu\|_1^s N^{-1-\eta}.$$

Let  $\mathcal{A} \subseteq \text{supp}(\nu)$  be such that if  $\mathbf{x} \in \mathcal{A}^s$  and  $\mathbf{c} \cdot \mathbf{x} = 0$ , then  $\mathbf{x} \in \mathcal{K}$ . Then, for some sufficiently large constant  $C = C(\mathbf{c}, t, k, \eta) > 1$ ,

$$\sum_{n \in \mathcal{A}} \nu(n) \leq \frac{N}{\exp(C^{-1} \min\{\log \log(2/\theta), \log N\}^{1/\tau(s)})}.$$

*Proof.* After inserting the bounds given in Lemma 4.2 into [6, (2.4) and (2.5)], the theorem follows with the same proof as [6, Proposition 2.8].  $\square$

## 5. Applying transference

In this final section, we use our Fourier decay and restriction estimates to prove Theorems 1.4 and 1.8.

**5.1. Bounding the weighted number of trivial solutions.** Let  $c_1, \dots, c_s$  be non-zero integers, where  $s \geq 3$ . Let  $1 < c < 2$ , and put  $\gamma = 1/c$ .

**Lemma 5.1.** *Let*

$$\mathcal{K} = \{\mathbf{n} \in \mathbb{Z}^s : \mathbf{c} \cdot \mathbf{n} = 0, \quad n_i = n_j \text{ for some } i \neq j\}.$$

Then, for some  $\eta > 0$ ,

$$\sum_{\mathbf{n} \in \mathcal{K}} \prod_{i \leq s} \nu(n_i) \ll_{c,\eta} N^{s-1-\eta}.$$

*Proof.* Put

$$e_1 = c_1, \dots, e_{s-2} = c_{s-2}, \quad e_{s-1} = c_{s-1} + c_s.$$

By symmetry, it suffices to prove that

$$(5.1) \quad \sum_{\mathbf{n} \in K} \prod_{i \leq s} \nu(n_i) \ll N^{s-1-\eta},$$

where

$$\begin{aligned} K &= \{\mathbf{n} \in \mathbb{Z}^s : \mathbf{c} \cdot \mathbf{n} = 0, \quad n_{s-1} = n_s\} \\ &= \{(n_1, n_2, \dots, n_{s-1}, n_{s-1}) \in \mathbb{Z}^s : e_1 n_1 + \dots + e_{s-1} n_{s-1} = 0\}. \end{aligned}$$

Recall that  $\|\widehat{\nu}\|_\infty \ll N$  and  $\|\nu\|_\infty \ll N^{1-\gamma}$ . By orthogonality, Hölder's inequality, and periodicity,

$$\begin{aligned} \sum_{\mathbf{n} \in K} \prod_{i \leq s} \nu(n_i) &\ll N^{1-\gamma} \sum_{\mathbf{n} \in K} \prod_{i \leq s-1} \nu(n_i) \\ &= N^{1-\gamma} \int_{\mathbb{T}} \widehat{\nu}(e_1 \alpha) \cdots \widehat{\nu}(e_{s-2} \alpha) \widehat{\nu}(e_{s-1} \alpha) \, d\alpha \\ &\ll N^{s-2-\gamma} \int_{\mathbb{T}} |\widehat{\nu}(\alpha)|^2 \, d\alpha \ll N^{s-2\gamma} = N^{s-1-\eta}, \end{aligned}$$

where  $\eta = 2\gamma - 1 > 0$ . This confirms (5.1), thereby completing the proof of Lemma 5.1.  $\square$

**5.2. Proof of Theorem 1.4.** We establish the restriction estimate (3.3) by dividing the analysis into two cases. If  $s \leq 4$ , then we have  $1 < c < c^*(s)$ . Our definition (1.1) of  $c^*(s)$  then guarantees that there exist  $\chi, t_0, t \in \mathbb{R}$  satisfying (3.2) and

$$s - 1 < t_0 < t < s, \quad 0 < \chi < \frac{2}{5c} - \frac{1}{5}.$$

Hence, Proposition 2.1 provides us with the Fourier decay estimate (3.1). Then, by Proposition 3.1, we have the restriction estimate (3.3). If instead  $s \geq 5$  then, by Propositions 2.1 and 3.2, we again have (3.3) for some  $t \in (s - 1, s)$ . For any  $s \geq 3$ , it follows from (3.1) that

$$\left\| \frac{\widehat{\nu}}{\|\nu\|_1} - \frac{\widehat{1}_{[N]}}{\|1_{[N]}\|_1} \right\|_\infty \leq \frac{1}{M},$$

for some  $M \asymp N^\chi$ .

For  $F_1, \dots, F_s : \mathbb{Z} \rightarrow \mathbb{C}$  compactly supported, define

$$T(F_1, \dots, F_s) = \sum_{\mathbf{c} \cdot \mathbf{n} = 0} F_1(n_1) \cdots F_s(n_s).$$

We abbreviate

$$T(F) = T(F, \dots, F).$$

For  $i \in [r]$ , let

$$f_i(n) = \begin{cases} \phi'(n)^{-1}, & \text{if } n \in C_i \\ 0, & \text{otherwise,} \end{cases}$$

so that  $\sum_{i \leq r} f_i = \nu$ . By the modelling lemma [11, Proposition 14.1], there exist  $g_1, \dots, g_r : [N] \rightarrow [0, \infty)$  such that

$$g_1 + \cdots + g_r = (1 + M^{-1/2})1_{[N]}$$

and

$$\left\| \frac{\widehat{f}_i}{\|\nu\|_1} - \frac{\widehat{g}_i}{N} \right\|_\infty \ll (\log N)^{-1/(t+2)}.$$

The generalised von Neumann lemma [11, Lemma C.3] now gives

$$(5.2) \quad T(f_i/\|\nu\|_1) - T(g_i/N) \ll N^{-1}(\log N)^{-(s-t)/(t+2)} \quad (1 \leq i \leq r).$$

To complete the proof of Theorem 1.4, we use the following “functional FGR” lemma [11, Lemma 15.2].

**Lemma 5.2.** *Let  $s \geq 3$  be a positive integer. Let  $c_1, \dots, c_s \in \mathbb{Z} \setminus \{0\}$  with  $\sum_{i \in I} c_i = 0$  for some non-empty  $I \subseteq [s]$ . For any  $r \in \mathbb{N}$  there exists  $N_0 = N_0(\mathbf{c}, r) \in \mathbb{N}$  and  $\eta_0 = \eta_0(\mathbf{c}, r) \in (0, 1)$  such that the following is true. For*

any positive integer  $N \geq N_0$  and any functions  $F_1, \dots, F_s : [N] \rightarrow [0, \infty)$  with  $F_1 + \dots + F_s \geq 1_{[N]}$ , we have

$$\sum_{i=1}^r T(F_i, \dots, F_i) \geq \eta_0 N^{s-1}.$$

An application of this lemma shows that there exists  $k$  such that

$$T(g_k) \gg N^{s-1}.$$

Since  $\|\nu\|_1 \asymp N$ , for this value of  $k$  it follows from (5.2) that

$$T(f_k) \gg N^{s-1}.$$

By Lemma 5.1, this shows that there exists  $\mathbf{x} \in C_k^s$  with (1.2).

**5.3. Proof of Theorem 1.8.** Let  $A \subseteq \text{PS}_c(N)$  be as in the statement of Theorem 1.8. We compute that

$$\sum_{n \in A} \nu(n) \geq \sum_{x \leq |A|} cx^{c-1} \geq \int_0^{|A|} cx^{c-1} dx = |A|^c.$$

With the same constants  $\chi > 0$  and  $t \in (s-1, s)$  from the previous subsection, we again have the Fourier decay estimate (3.1) and the restriction estimate (3.3). Thus, by Lemma 5.1, all of the sufficient conditions in Theorem 4.3 are met, delivering the claim.

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