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Kazuaki MURAKAMI

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Kida's formula for split prime $\mathbb{Z}_p$-extensions over imaginary quadratic fields

par Kazuaki MURAKAMI

Résumé. Soit p un nombre premier impair et k un corps quadratique imaginaire dans lequel p est décomposé en $\mathfrak{p}$ et $\mathfrak{p}^*$. Alors il existe une unique $\mathbb{Z}_p$-extension N_∞/k telle que l'idéal premier $\mathfrak{p}^*$ soit non ramifié. Pour une extension finie K/k, nous appelons $K_\infty = KN_\infty$ la $\mathbb{Z}_p$-extension de type *split prime* correspondant à $\mathfrak{p}$. Nous démontrons un analogue de la formule de Kida pour les $\mathbb{Z}_p$-extensions de type *split prime*. Comme application, nous appliquons cette formule aux modules d'Iwasawa $\mathfrak{p}$-ramifiés et déterminons les classes d'isomorphisme des modules d'Iwasawa non ramifiés associés aux $\mathbb{Z}_p$-extensions sur k.

Abstract. Let p be an odd prime number and k an imaginary quadratic field in which p splits into $\mathfrak{p}$ and $\mathfrak{p}^*$. Then there exists a uniquely defined $\mathbb{Z}_p$-extension N_∞/k such that the prime ideal $\mathfrak{p}^*$ does not ramify. For a finite extension K/k, we call $K_\infty = KN_\infty$ the split prime $\mathbb{Z}_p$-extension corresponding to $\mathfrak{p}$. We prove an analogue of Kida's formula for the split prime $\mathbb{Z}_p$-extensions. As an application, we apply this formula to $\mathfrak{p}$-ramified Iwasawa modules and determine the isomorphism classes of unramified Iwasawa modules associated to $\mathbb{Z}_p$-extensions over k.

1. Introduction

Let p be an odd prime number. For a number field k, there exists a $\mathbb{Z}_p$-extension k_∞/k, which is a Galois extension whose Galois group $\mathrm{Gal}(k_\infty/k)$ is topologically isomorphic to the additive group $\mathbb{Z}_p$ of p-adic integers. Let $L(k_\infty)$ be the maximal unramified pro-p abelian extension field of k_∞. We call $\mathrm{Gal}(L(k_\infty)/k_\infty)$ the unramified Iwasawa module associated to k_∞ and denote it by $X(k_\infty)$. The Galois group $\mathrm{Gal}(k_\infty/k)$ acts on $X(k_\infty)$ via the inner automorphism. Then it becomes a module over the completed group ring $\mathbb{Z}_p[\![\mathrm{Gal}(k_\infty/k)]\!]$. Iwasawa [11] proved that the unramified Iwasawa module is a finitely generated torsion $\mathbb{Z}_p[\![\mathrm{Gal}(k_\infty/k)]\!]$-module. By fixing a topological generator of $\mathrm{Gal}(k_\infty/k)$, we have an isomorphism between $\mathbb{Z}_p[\![\mathrm{Gal}(k_\infty/k)]\!]$ and $\mathbb{Z}_p[\![T]\!]$ the ring of power series over $\mathbb{Z}_p$ in one variable.

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From the structure theorem of $\mathbb{Z}_p[\![T]\!]$-modules ([23, Proposition 5.1.7]), the characteristic ideal of $X(k_\infty)$ is generated by a power series of the form $p^\mu f(T)$, where μ is a non-negative integer and $f(T)$ is a distinguished polynomial over $\mathbb{Z}_p$. We call μ and the degree of $f(T)$ μ- and λ-invariant of k_∞/k, respectively. The λ-invariant of k_∞/k is equal to the dimension of $X(k_\infty) \otimes_{\mathbb{Z}} \mathbb{Q}_p$ as a vector space over $\mathbb{Q}_p$ of p-adic numbers. These invariants have very important role in Iwasawa theory.

Let k_∞^c be the cyclotomic $\mathbb{Z}_p$-extension, namely, k_∞^c is the maximal p-extension of k in $\bigcup_{n=1}^\infty k(\zeta_{p^n})$, where ζ_{p^n} are primitive p^n-th roots of unity. Iwasawa [13] conjectured that μ-invariants of cyclotomic $\mathbb{Z}_p$-extensions vanish. Ferrero and Washington [5] proved this conjecture under the assumption that k is an abelian number field. For an odd prime p and a Galois extension L/K of a finite algebraic number field of degree p, Iwasawa [13] proved that $\mu(K_\infty^c/K) = 0$ if and only if $\mu(L_\infty^c/K) = 0$, where K_∞^c and L_∞^c are cyclotomic $\mathbb{Z}_p$-extensions, respectively. In the case of $p = 2$, the same result holds under the assumption that K is a totally imaginary field.

Concerning λ-invariants of cyclotomic $\mathbb{Z}_p$-extensions, Kida's formula [18] relates λ-invariants in p-extensions of CM-fields in terms of the degree and ramification index for an odd prime p. This formula is an analogue of the classical Riemann–Hurwitz formula. Iwasawa [14] also proved this formula under a general assumption including $p = 2$ using the theory of Galois cohomology for number fields whose degree are not necessarily finite over the field of rational numbers. Iwasawa's formula does not need the CM-field assumption.

In this paper, we assume that p is odd and that k is an imaginary quadratic field in which p splits into $\mathfrak{p}$ and $\mathfrak{p}^*$. We will give an analogue of Kida's formula for non-cyclotomic $\mathbb{Z}_p$-extensions. From class field theory, there exists a uniquely defined $\mathbb{Z}_p$-extension N_∞/k (respectively, N_∞^*/k) such that the prime ideal $\mathfrak{p}^*$ (respectively, $\mathfrak{p}$) does not ramify. Let F be an arbitrary finite extension field of k. We call

$$F_\infty = FN_\infty$$

the *split prime* $\mathbb{Z}_p$-extension of F corresponding to $\mathfrak{p}$. The split prime $\mathbb{Z}_p$-extension N_∞ over k has rather similar properties to the cyclotomic $\mathbb{Z}_p$-extension over $\mathbb{Q}$ of rational numbers. It is known that the Iwasawa μ-invariant of N_∞/k is zero. This fact was proved by Gillard [6] and Schneps [27] for $p \geq 5$, and Oukhaba–Viguié [24] for $p = 2, 3$. We denote by $M_\mathfrak{p}(F_\infty)$ the maximal pro-p abelian extension field of F_∞ unramified outside all prime ideals lying above $\mathfrak{p}$. Greenberg [9] proved that $X_\mathfrak{p}(F_\infty)$ is a finitely generated torsion $\mathbb{Z}_p[\![\mathrm{Gal}(F_\infty/F)]\!]$-module and that it has no non-trivial finite submodule. We call this module $\mathfrak{p}$-ramified Iwasawa module associated to F_∞. Under an isomorphism $\mathbb{Z}_p[\![\mathrm{Gal}(F_\infty/F)]\!] \cong \mathbb{Z}_p[\![T]\!]$ obtained via a fixed topological generator of $\mathrm{Gal}(F_\infty/F)$, we have a characteristic ideal of $X_{\mathfrak{p}}(F_\infty)$ whose generator is a power series of the form $p^\mu G(T)$, where $G(T)$ is a distinguished polynomial over $\mathbb{Z}_p$. We call μ above the μ-invariant of $X_{\mathfrak{p}}(F_\infty)$. For $p \geq 5$, Schneps [27] proved the vanishing of μ-invariant of $X_{\mathfrak{p}}(F_\infty)$ for $F = k$ whose class number is 1, and Gillard [6] proved the same result for arbitrary F abelian over k. Recently, it was proved by Oukhaba–Viguié [24] and Crisąn–Müller [3] that the μ-invariants of $X_{\mathfrak{p}}(F_\infty)$ is zero for $p = 2, 3$ under the assumption that F/k is a finite abelian extension. Choi–Kezuka–Li [2] also proved the same result in the case where F is the Hilbert class field of k for $p = 2$ under some technical condition on k. Oukhaba–Viguié, Crisąn–Müller and Choi–Kezuka–Li proved the results independently. Hence $X_{\mathfrak{p}}(F_\infty)$ is a free $\mathbb{Z}_p$-module of finite rank if we suppose that F/k is abelian.

In this paper, we relate the $\mathbb{Z}_p$-ranks of $\mathfrak{p}$-ramified Iwasawa modules in a p-extension of number fields that are abelian extensions over k in terms of the degree and ramification index. More precisely, we prove the following

Theorem 1.1. *Let p be an odd prime number and k an imaginary quadratic field in which p splits into $\mathfrak{p}$ and $\mathfrak{p}^*$. Let L/K be a Galois extension of degree p of abelian extensions over k. Let t be the number of primes of K_∞ which are not lying above $\mathfrak{p}$ and ramify in L_∞. Then we have*

$$\operatorname{rank}_{\mathbb{Z}_p}(X_{\mathfrak{p}}(L_\infty)) = \begin{cases} p \cdot \operatorname{rank}_{\mathbb{Z}_p}(X_{\mathfrak{p}}(K_\infty)) + (p-1)(t-1) & \text{if } K_\infty \cap L = K \\ \operatorname{rank}_{\mathbb{Z}_p}(X_{\mathfrak{p}}(K_\infty)) & \text{if } K_\infty \cap L \neq K, \end{cases}$$

where $\operatorname{rank}_{\mathbb{Z}_p}(*)$ *is the $\mathbb{Z}_p$-rank of $*$.*

We note that our proof of Theorem 1.1 is similar to that of Iwasawa [14], Jaulent [16], and Michel [19]. Wingberg [29] and Michel [19] proved an analogue of Kida's formula for the Selmer groups of elliptic curves with complex multiplication under several assumptions on the ramification of $\mathfrak{p}$ in L/K. Very recently, Kataoka [17, Theorem 4.5] also proved this theorem using Selmer complexes. More precisely, let k be a CM-field such that every p-adic prime splits in k/k^+, where k^+ is the maximal totally real subfield of k. Let k_∞/k be a $\mathbb{Z}_p$-extension such that no finite prime splits completely and Σ a set of primes of k lying above p that contains each pair of complex conjugate primes over p. His result is a Kida's formula for the Iwaswa module of p-abelian extensions unramified outside Σ and totally split at all other places. In the case where k is an imaginary quadratic field, $\Sigma = \{\mathfrak{p}\}$, and $k_\infty = N_\infty$, this Iwasawa module coincides with $X_{\mathfrak{p}}(N_\infty)$.

We apply Theorem 1.1 to Iwasawa modules attached to imaginary quadratic fields. Let $\mathfrak{D}_{\mathfrak{p}}$ (respectively, $\mathfrak{D}_{\mathfrak{p}^*}$) be the decomposition group of the prime $\mathfrak{p}$ (respectively, $\mathfrak{p}^*$) in $\widetilde{k}/k$, where $\widetilde{k}$ is the maximal multiple $\mathbb{Z}_p$-extension field of k. Minardi [20, Lemma 3.1] proved that the prime number

p is only finitely decomposed in $\widetilde{k}/\mathbb{Q}$. Hence the index $[\mathrm{Gal}(\widetilde{k}/k) : \mathfrak{D}_{\mathfrak{p}^*}]$ is finite. Let $M_{\mathfrak{p}}(\widetilde{k})$ be the maximal pro-p abelian extension field of $\widetilde{k}$ unramified outside all primes lying above $\mathfrak{p}$. We denote the Galois group $\mathrm{Gal}(M_{\mathfrak{p}}(\widetilde{k})/\widetilde{k})$ by $X_{\mathfrak{p}}(\widetilde{k})$. It is a finitely generated torsion $\mathbb{Z}_p[[\mathrm{Gal}(\widetilde{k}/k)]]$-module. The following theorem determines the structure of the Iwasawa module $X_{\mathfrak{p}}(\widetilde{k})$ as a $\mathbb{Z}_p[[\mathrm{Gal}(\widetilde{k}/N_\infty)]]$-module.

Theorem 1.2. *Let p be an odd prime number and k an imaginary quadratic field in which p splits into $\mathfrak{p}$ and $\mathfrak{p}^*$. Then $X_{\mathfrak{p}}(\widetilde{k})$ is a free $\mathbb{Z}_p[[\mathrm{Gal}(\widetilde{k}/N_\infty)]]$-module of rank $\mathrm{rank}_{\mathbb{Z}_p}(X_{\mathfrak{p}}(N_\infty)) + [\mathrm{Gal}(\widetilde{k}/k) : \mathfrak{D}_{\mathfrak{p}*}] - 1$.*

Let n_0 be the integer satisfying $p^{n_0} = [\mathrm{Gal}(\widetilde{k}/k) : \mathfrak{D}_{\mathfrak{p}}]$. Let k_∞^a be the anti-cyclotomic $\mathbb{Z}_p$-extension of k and $k_{n_0}^a$ the n_0-th layer of k_∞^a/k. In the previous paper, we studied the behavior of $\lambda(k_\infty/k)$ and $\mu(k_\infty/k)$ as k_∞ varies over all $\mathbb{Z}_p$-extension fields of an imaginary quadratic field k and gave an explicit upper bound of λ-invariants for all $\mathbb{Z}_p$-extensions under several assumptions ([21]). In particular, we proved that $\lambda(k_\infty/k) \leq p^{n_0}$ for all $\mathbb{Z}_p$-extension k_∞ which satisfies that $k_\infty \cap k_\infty^a \supset k_{n_0}^a$ ([21, Theorem 4.1]). Applying Theorem 1.1 and Theorem 1.2 to this case, we can determine the actual value of the λ-invariants and determine the isomorphism class of Iwasawa modules for such $\mathbb{Z}_p$-extensions. More precisely, we have the following

Theorem 1.3. *Let p be an odd prime number and k an imaginary quadratic field in which p splits into $\mathfrak{p}$ and $\mathfrak{p}^*$. Assume that k is p-rational, in other words, $L(k) \subset \widetilde{k}$, where $L(k)$ is the p-Hilbert class field of k. Assume also that $\mathfrak{D}_{\mathfrak{p}}$ is a normal subgroup of $\mathrm{Gal}(\widetilde{k}/\mathbb{Q})$. Let k_∞ be a $\mathbb{Z}_p$-extension satisfying $k_\infty \cap k_\infty^a \supset k_{n_0}^a$. Then $X(k_\infty)$ is a free $\mathbb{Z}_p$-module of rank p^{n_0} and is a cyclic $\mathbb{Z}_p[[\mathrm{Gal}(k_\infty/k)]]$-module. In other words, if we fix a topological generator γ in $\mathrm{Gal}(k_\infty/k)$, we have an isomorphism*

$$X(k_\infty) \cong \mathbb{Z}_p[[\mathrm{Gal}(k_\infty/k)]]/(\gamma^{p^{n_0}} - 1)$$

as a $\mathbb{Z}_p[[\mathrm{Gal}(k_\infty/k)]]$-module. In particular, $X(k_\infty^a)$ is a free $\mathbb{Z}_p$-module of rank p^{n_0} and is a cyclic $\mathbb{Z}_p[[\mathrm{Gal}(k_\infty^a/k)]]$-module.

The outline of this paper is as follows. In Section 2, we provide some basic definitions and notation. In Section 3, we prove Theorem 1.1 using Galois cohomology. In Section 4, we prove Theorems 1.2 and 1.3, and introduce some numerical examples which were computed using PARI/GP.

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2. Preliminaries

We assume that p is an odd prime number, and that k is an imaginary quadratic field in which p splits into $\mathfrak{p}$ and $\mathfrak{p}^*$. Let F be a finite extension field of k and S a finite set of primes of F. We put

$$T(F) = \{v \mid v \text{ is a prime ideal of } F \text{ lying above } \mathfrak{p}\},$$
$$F_S^\times = \{a \in F^\times \mid \mathrm{ord}_v(a) = 0 \text{ for all } v \text{ in } S\},$$
$$F_{T(F),\mathfrak{m}(n)}^\times = \{a \in F_{T(F)}^\times \mid a \equiv 1 \bmod v^n \text{ for all } v \text{ in } T(F)\},$$

where ord_v is the normalized additive valuation with respect to v and n is a non-negative integer. For $n = 0$, we have $F_{T(F),\mathfrak{m}(n)}^\times = F_{T(F)}^\times$.

For each prime ideal v of F, we denote by F_v and $U(F_v)$ the completion of F at v and the unit group of F_v, respectively. Let $\mathcal{O}_F$ be the ring of integers of F. For each non-negative integer n, we put

$$U^{(n)}(F_v) = \{a \in U(F_v) \mid \mathrm{ord}_v(a-1) \geq n\},$$
$$U_T^{(n)}(F) = \{a \in \mathcal{O}_F^\times \mid a \equiv 1 \bmod v^n \text{ for all } v \text{ in } T\},$$

where $\mathcal{O}_F^\times$ is the group of units in $\mathcal{O}_F$. For each prime v, we fix an embedding $i_v^F : F \to F_v$. For simplicity, we write $i_v = i_v^F$ and $T = T(F)$. By definition of $F_T^\times$, we have $i_v(F_T^\times) \subset U(F_v)$. We define a homomorphism

$$i_T^F : F_T^\times \longrightarrow \bigoplus_{v \in T} U(F_v)$$

by $i_T^F(a) = (i_v(a))_{v \in T}$, where $a \in F_T^\times$. Since $U^{(1)}(F_v)$ includes $i_v(U_T^{(1)}(F))$, the homomorphism i_T^F induces the following embedding

$$i_T^F|_{U_T^{(1)}(F)} : U_T^{(1)}(F) \longrightarrow \bigoplus_{v \in T} U^{(1)}(F_v).$$

Let $\overline{i_T^F(U_T^{(1)}(F))}$ be the closure of $i_T^F(U_T^{(1)}(F))$ in $\bigoplus_{v \in T} U^{(1)}(F_v)$ under the $\mathfrak{p}$-adic topology. Then there exists a non-negative integer $\delta_{F,\mathfrak{p}}$ such that

$$\mathrm{rank}_{\mathbb{Z}_p}\Big(\overline{i_T^F(U_T^{(1)}(F))}\Big) = r_2(F) - 1 - \delta_{F,\mathfrak{p}},$$

where $r_2(F)$ is the number of complex primes of F. Greenberg states that $\mathfrak{p}$-adic Leopoldt's conjecture holds if F/k is abelian (in the end of Section 4 in [9]). Namely we have the following

Proposition 2.1 ([9])**.** *Suppose that F/k is abelian. Then we have $\delta_{F,\mathfrak{p}} = 0$, in other words,* $\mathrm{rank}_{\mathbb{Z}_p}\Big(\overline{i_T^F(U_T^{(1)}(F))}\Big) = r_2(F) - 1$.

Proof. We note that $\delta_{F,\mathfrak{p}} = 0$ if and only if $R_{\mathfrak{p}}(F) \neq 0$, where $R_{\mathfrak{p}}(F)$ is the $\mathfrak{p}$-adic regulator of Leopoldt for F. We define $R_{\mathfrak{p}}(F)$ as follows: Let $\{\sigma_i \,|\, 1 \leq i \leq r_2(F)\}$ be the embeddings of F into $\overline{\mathbb{Q}}_p$ extending the embedding of k into $\mathbb{Q}_p$ given by $\mathfrak{p}$. Since F is totally imaginary, the $\mathbb{Z}$-rank of $U^{(1)}(F)$ is $r_2(F) - 1$. Let $\{\epsilon_i \,|\, 1 \leq i \leq r_2(F) - 1\}$ be a basis for $U^{(1)}(F)/U^{(1)}(F)_{tor}$, where $U^{(1)}(F)_{tor}$ is the torsion subgroup of $U^{(1)}(F)$. Let $\log_p$ be a p-adic logarithm such that $\log_p(p) = 0$. Then we define the $\mathfrak{p}$-adic regulator of Leopoldt by $\det(\log_p \sigma_i(\epsilon_j))_{i,j=1,\ldots,r_2-1}$. Since we suppose that F/k is abelian, $\{\log_p \epsilon_i \,|\, 1 \leq i \leq r_2(F) - 1\}$ are linearly independent by [1]. This implies that $R_{\mathfrak{p}}(F) \neq 0$. Thus we get the conclusion. □

For each non-negative integer n, let $i_T^{F,(n)}$ be a homomorphism

$$i_T^{F,(n)} : F_T^\times \longrightarrow \bigoplus_{v \in T} U(F_v)/U^{(n)}(F_v)$$

given by $i_T^{F,(n)}(a) = (i_v(a) \bmod U^{(n)}(F_v))_{v \in T}$, where a is an element of $F_T^\times$. We consider homomorphisms

$$\overline{i_T^F} : F_T^\times \otimes_{\mathbb{Z}} \mathbb{Z}_p \longrightarrow \bigoplus_{v \in T} U^{(1)}(F_v),$$

$$\overline{i_T^{F,(n)}} : F_T^\times \otimes_{\mathbb{Z}} \mathbb{Z}_p \longrightarrow \bigoplus_{v \in T} U^{(1)}(F_v)/U^{(n)}(F_v)$$

by

$$\overline{i_T^F}(a \otimes \alpha) = \left(\left(\frac{a}{\omega_{F_v}(a)}\right)^\alpha\right)_{v \in T} \quad \text{and}$$

$$\overline{i_T^{F,(n)}}(a \otimes \alpha) = \left(\left(\frac{a}{\omega_{F_v}(a)}\right)^\alpha \bmod U^{(n)}(F_v)\right)_{v \in T},$$

where ω_{F_v} is the Teichmüller character for F_v.

Remark 2.2. We may omit "F" from symbols i_T^F and $i_T^{F,(n)}$, if it can not cause confusion.

We use the following proposition to prove Theorem 1.1.

Proposition 2.3 ([7], Proposition 2.4.1)**.** *Let n be a positive integer. We have the following exact sequences of $\mathbb{Z}_p$-modules*

$$1 \longrightarrow F_{T,\mathfrak{m}(n)}^\times \otimes_{\mathbb{Z}} \mathbb{Z}_p \longrightarrow F_T^\times \otimes_{\mathbb{Z}} \mathbb{Z}_p \xrightarrow{\overline{i_T^{(n)}}} \bigoplus_{v \in T} U^{(1)}(F_v)/U^{(n)}(F_v) \longrightarrow 1$$

and

$$1 \longrightarrow \bigcap_{n=1}^{\infty} (F_{T,\mathfrak{m}(n)}^\times \otimes_{\mathbb{Z}} \mathbb{Z}_p) \longrightarrow F_T^\times \otimes_{\mathbb{Z}} \mathbb{Z}_p \xrightarrow{\overline{i_T}} \bigoplus_{v \in T} U^{(1)}(F_v) \longrightarrow 1.$$

Remark 2.4. We note that $\mathfrak{p}$-adic Leopoldt's conjecture holds for p and F if and only if $\ker(\overline{i_T}) \cap (\mathcal{O}_F^\times \otimes \mathbb{Z}_p) = 0$.

3. Kida's formula for split prime $\mathbb{Z}_p$-extensions

In this section, we prove Theorem 1.1 using Galois cohomological method which is similar to that of Iwasawa [14], Jaulent [16], and Michel [19]. Let G be a cyclic p-group. For a finitely generated $\mathbb{Z}_p[G]$-module M, we suppose that cohomology groups $\mathrm{H}^1(G, M)$ and $\mathrm{H}^2(G, M)$ are finite groups. We define

$$\chi(G, M) = \mathrm{ord}_p \left(\frac{\#\mathrm{H}^2(G, M)}{\#\mathrm{H}^1(G, M)} \right),$$

where ord_p is the normalized additive valuation on $\mathbb{Q}_p$ of p-adic numbers such that $\mathrm{ord}_p(p) = 1$. We also put $M_G = M/I_G M$, where I_G is the augmentation ideal of $\mathbb{Z}_p[G]$.

As in the previous section, we suppose that p splits in k into $\mathfrak{p}$ and $\mathfrak{p}^*$. Then there exists the split prime $\mathbb{Z}_p$-extension N_∞ over k corresponding to $\mathfrak{p}$. For a finite extension field F over k, let F_∞/F be the split prime $\mathbb{Z}_p$-extension corresponding to $\mathfrak{p}$. Then the $\mathfrak{p}$-ramified Iwasawa module $X_\mathfrak{p}(F_\infty)$ is a finitely generated torsion $\mathbb{Z}_p[\![\mathrm{Gal}(F_\infty/F)]\!]$-module and has no non-trivial finite submodule. Hence it is a free $\mathbb{Z}_p$-module.

Let L and K be abelian extension fields over k. In the following, we suppose that L/K is a Galois extension of degree p. Using the method of [20, Lemma 3.1], we can prove the following lemma. This is essential in Propositions 3.3 and 3.16.

Lemma 3.1. *Let F be a finite abelian extension field over k. Then all prime ideals of F are finitely decomposed F_∞/F.*

Proof. Since F/k is a finite extension, we have only to prove that all prime numbers are only finitely decomposed in $N_\infty/\mathbb{Q}$. The prime number p is finitely decomposed in $\widetilde{k}/\mathbb{Q}$ by [20, Lemma 3.1]. Let $\mathfrak{q}$ be a prime ideal of k which is not lying above p. For an algebraic number field E over k, we denote by $M_\mathfrak{p}(E)$ the maximal pro-p abelian extension field of E unramified outside all prime ideals lying above $\mathfrak{p}$. Since k is an imaginary quadratic field, $M_\mathfrak{p}(k)/N_\infty$ is a finite extension. Therefore we shall prove that $\mathfrak{q}$ is finitely decomposed in $M_\mathfrak{p}(k)/k$. We have an isomorphism

$$\mathrm{Gal}(M_\mathfrak{p}(k)/k) \cong \varprojlim_L \mathrm{Gal}(L/k)$$
$$(\sigma \in \mathrm{Gal}(M_\mathfrak{p}(k)/k) \leftrightarrow (\sigma|_L)_L \in \varprojlim_L \mathrm{Gal}(L/k)),$$

where L is running over all intermediate field extensions between $M_\mathfrak{p}(k)$ and k such that L/k is finite and the inverse limit is taken with respect to the restriction homomorphisms. Let $J(k)$ be the idéle group of k. From class field theory, global norm residue symbols induce a surjective homomorphism

$$(\,\cdot\,, M_{\mathfrak{p}}(k)/k) : J(k) \longrightarrow \mathrm{Gal}(M_{\mathfrak{p}}(k)/k).$$

The homomorphism $(\,\cdot\,, M_{\mathfrak{p}}(k)/k)$ induces an isomorphism

$$\mathrm{Gal}(M_{\mathfrak{p}}(k)/L(k)) \cong p\text{-part of} \left(\frac{k^{\times} \prod_{\mathfrak{q}\nmid\mathfrak{p}} U(k_{\mathfrak{q}}) \times U(k_{\mathfrak{p}})}{k^{\times} \prod_{\mathfrak{q}\nmid\mathfrak{p}} U(k_{\mathfrak{q}})} \right) \cong U^{(1)}(k_{\mathfrak{p}}),$$

where $\mathfrak{q}$ ranges over all primes of k not lying above p. Also we have an exact sequence

$$0 \longrightarrow U^{(1)}(k_{\mathfrak{p}}) \xrightarrow{\iota} \mathrm{Gal}(M_{\mathfrak{p}}(k)/k) \longrightarrow \mathrm{Gal}(L(k)/k) \longrightarrow 0,$$

where ι is an injective homomorphism induced from the isomorphism above. Let π be a prime element in $\mathfrak{q}$, in other words, $\pi \in \mathfrak{q}$ and $\pi \notin \mathfrak{q}^2$. Since $\pi^{p-1} \neq 1$ and $\pi^{p-1} \in U^{(1)}(k_{\mathfrak{p}})$, we see that $\overline{\langle \pi^{p-1} \rangle}$ has finite index in $U^{(1)}(k_{\mathfrak{p}})$, where $\overline{\langle * \rangle}$ is the topological closure of $\langle * \rangle$ in $U^{(1)}(k_{\mathfrak{p}})$. Let $\mathcal{D}_{\mathfrak{q}}$ be the decomposition group of the prime $\mathfrak{q}$ in $\mathrm{Gal}(M_{\mathfrak{p}}(k)/k)$. If we prove that $\iota(\overline{\langle \pi^{p-1} \rangle}) \subset \mathcal{D}_{\mathfrak{q}}$, we see that $\mathfrak{q}$ is finitely decomposed in $M_{\mathfrak{p}}(k)/k$. We shall prove this inclusion. We define $\alpha = (a_{\mathfrak{r}})_{\mathfrak{r}}$ and $\beta = (b_{\mathfrak{r}})_{\mathfrak{r}}$ in $J(k)$ by

$$a_{\mathfrak{r}} = \begin{cases} \pi^{p-1} & \text{if } \mathfrak{r} = \mathfrak{p}, \\ 1 & \text{if } \mathfrak{r} \neq \mathfrak{p}, \end{cases} \qquad \text{and} \qquad b_{\mathfrak{r}} = \begin{cases} \pi^{1-p} & \text{if } \mathfrak{r} = \mathfrak{q}, \\ 1 & \text{if } \mathfrak{r} \neq \mathfrak{q}. \end{cases}$$

Let $\gamma = (c_{\mathfrak{r}})_{\mathfrak{r}}$ an element of $J(k)$ satisfying $\alpha\beta^{-1} = \pi^{p-1}\gamma$. We note that $c_{\mathfrak{p}} = c_{\mathfrak{q}} = 1$ and $c_{\mathfrak{r}} = \pi^{1-p} \in U(k_{\mathfrak{r}})$ for $\mathfrak{r} \nmid \mathfrak{p}\mathfrak{q}$. Then we have

$$\begin{aligned} (\alpha\beta^{-1}, M_{\mathfrak{p}}(k)/k) &= ((\pi^{p-1}, L/k) \cdot (\gamma, L/k))_L \\ &= \left(\prod_{\mathfrak{r}\nmid\mathfrak{p}\mathfrak{q}} (\pi^{1-p}, L_{\mathfrak{r}}/k_{\mathfrak{r}}) \cdot (1, L_{\mathfrak{p}}/k_{\mathfrak{p}}) \cdot (1, L_{\mathfrak{q}}/k_{\mathfrak{q}}) \right)_L \\ &= (\mathrm{id}_L)_L, \end{aligned}$$

where $(*, L_{\mathfrak{r}}/k_{\mathfrak{r}})$ is the local residue symbol for a prime $\mathfrak{r}$. Here we note that $L_{\mathfrak{r}}/k_{\mathfrak{r}}$ is unramified for each prime τ with $\tau \nmid \mathfrak{p}\mathfrak{q}$. Hence we have $(\alpha, M_{\mathfrak{p}}(k)/k) = (\beta, M_{\mathfrak{p}}(k)/k)$. Since $(\beta, M_{\mathfrak{p}}(k)/k) = ((\pi^{1-p}, L_{\mathfrak{q}}/k_{\mathfrak{q}}))_L \in \mathcal{D}_{\mathfrak{q}}$, we get $\iota(\pi^{p-1}) = (\alpha, M_{\mathfrak{p}}(k)/k) \in \mathcal{D}_{\mathfrak{q}}$. Therefore we obtain $\iota(\overline{\langle \pi^{p-1} \rangle}) \subset \mathcal{D}_{\mathfrak{q}}$. □

In this section, we will investigate the relation between the $\mathbb{Z}_p$-rank of $X_{\mathfrak{p}}(K_\infty)$ and that of $X_{\mathfrak{p}}(L_\infty)$.

Lemma 3.2. *Let L and K be abelian extension fields over k. Suppose that $K_\infty \cap L \neq K$. Then we have* $\mathrm{rank}_{\mathbb{Z}_p}(X_{\mathfrak{p}}(L_\infty)) = \mathrm{rank}_{\mathbb{Z}_p}(X_{\mathfrak{p}}(K_\infty))$.

Proof. In the case of $K_\infty \cap L \neq K$, we have $L = K_1$. This implies that $L_\infty = K_\infty$. Thus, we arrive at the conclusion. □

We suppose that $K_\infty \cap L = K$. We put $G = \mathrm{Gal}(L_\infty/K_\infty)$. As we stated in introduction, the $\mathfrak{p}$-ramified Iwasawa module $X_\mathfrak{p}(L_\infty)$ is a free $\mathbb{Z}_p$-module. Using the method of Iwasawa [14, Section 9], we have the following relation between the $\mathbb{Z}_p$-rank of $X_\mathfrak{p}(L_\infty)$ and its Pontryagin dual.

Proposition 3.3. *Let L/K be a $\mathbb{Z}/p\mathbb{Z}$-extension of abelian extension fields over k which satisfies that $K_\infty \cap L = K$. Then we have*

$$\mathrm{rank}_{\mathbb{Z}_p}(X_\mathfrak{p}(L_\infty)) = p \cdot \mathrm{rank}_{\mathbb{Z}_p}(X_\mathfrak{p}(L_\infty)_G) + (p-1)\chi(G, X_\mathfrak{p}(L_\infty)^\vee),$$

where $X_\mathfrak{p}(L_\infty)^\vee$ is the Pontryagin dual

$$X_\mathfrak{p}(L_\infty)^\vee = \mathrm{Hom}_{\mathbb{Z}_p}(X_\mathfrak{p}(L_\infty), \mathbb{Q}_p/\mathbb{Z}_p).$$

Proof. We note that $X_\mathfrak{p}(L_\infty)$ is a torsion-free $\mathbb{Z}_p$-module of finite rank and that G is isomorphic to $\mathbb{Z}/p\mathbb{Z}$. The only indecomposable $\mathbb{Z}_p[G]$-modules having a free finite $\mathbb{Z}_p$-basis are (up to isomorphism) $\mathbb{Z}_p$, I_G, and $\mathbb{Z}_p[G]$ by [4, 10, 26]. Then there exist non-negative integers α, β, and γ such that

$$X_\mathfrak{p}(L_\infty) \cong \mathbb{Z}_p^{\oplus\alpha} \oplus I_G^{\oplus\beta} \oplus \mathbb{Z}_p[G]^{\oplus\gamma}.$$

Hence we get an isomorphism as $\mathbb{Z}_p[G]$-modules

$$X_\mathfrak{p}(L_\infty)^\vee \cong (\mathbb{Q}_p/\mathbb{Z}_p)^{\oplus\alpha} \oplus (I_G^\vee)^{\oplus\beta} \oplus (\mathbb{Q}_p/\mathbb{Z}_p[G])^{\oplus\gamma},$$

where $I_G^\vee$ is the Pontryagin dual of I_G. In the same way as [14, Section 9], we get $\alpha + \gamma = \mathrm{rank}_{\mathbb{Z}_p}(X_\mathfrak{p}(L_\infty)_G)$, $\beta - \alpha = \chi(G, X_\mathfrak{p}(L_\infty)^\vee)$. Hence we obtain

$$\mathrm{rank}_{\mathbb{Z}_p}(X_\mathfrak{p}(L_\infty)) = \alpha + (p-1)\beta + p\gamma = p(\alpha+\gamma) + (p-1)(\beta-\alpha).$$

Thus we have the conclusion. □

We will calculate $\mathrm{rank}_{\mathbb{Z}_p}(X_\mathfrak{p}(L_\infty)_G)$ and $\chi(G, X_\mathfrak{p}(L_\infty)^\vee)$. First we consider the $\mathbb{Z}_p$-rank of $X_\mathfrak{p}(L_\infty)_G$.

Proposition 3.4. *Let L/K be a $\mathbb{Z}/p\mathbb{Z}$-extension of abelian extension fields over k which satisfies that $K_\infty \cap L = K$. Then we have* $\mathrm{rank}_{\mathbb{Z}_p}(X_\mathfrak{p}(L_\infty)_G) = \mathrm{rank}_{\mathbb{Z}_p}(X_\mathfrak{p}(K_\infty))$.

Proof. Let $\mathrm{Ram}(L_\infty/K_\infty)$ be the set of prime ideals of K_∞ which ramify in L_∞/K_∞. We put

$$H = \{\mathfrak{P}_\infty^* \mid \mathfrak{P}_\infty^* \text{ is a prime ideal of } K_\infty \text{ lying above } \mathfrak{p}^*\} \cup \mathrm{Ram}(L_\infty/K_\infty) \backslash T(K_\infty),$$

where $T(K_\infty)$ is the set of prime ideals of K_∞ lying above $\mathfrak{p}$. By Lemma 3.1, $\mathfrak{p}^*$ is finitely decomposed in K_∞/k. Therefore H is a finite set.

Let M be the intermediate field of $M_{\mathfrak{p}}(L_\infty)$ abelian over K_∞. Let $\mathfrak{Q}$ be an element of H. We denote by $I_{\mathfrak{Q}}$ the inertia subgroup of $\mathrm{Gal}(M/K_\infty)$ for the prime ideal $\mathfrak{Q}$. Then we have an exact sequence

$$0 \longrightarrow \sum_{\mathfrak{Q}\in H} I_{\mathfrak{Q}} \longrightarrow \mathrm{Gal}(M/K_\infty) \longrightarrow \mathrm{Gal}(M_{\mathfrak{p}}(K_\infty)/K_\infty) \longrightarrow 0.$$

We note that the fixed field of M by $\sum_{\mathfrak{Q}\in H} I_{\mathfrak{Q}}$ coincides with $M_{\mathfrak{p}}(K_\infty)$ and that $X_{\mathfrak{p}}(L_\infty)_G \cong \mathrm{Gal}(M/K_\infty)$. By definition of H, we have $\mathrm{Gal}(M/L_\infty) \cap I_{\mathfrak{Q}} = 1$ for any prime $\mathfrak{Q}$ in H. Hence we get

$$I_{\mathfrak{Q}} \cong \begin{cases} \{0\} & \text{if } \mathfrak{Q} \text{ is unramified in } L_\infty, \\ \mathbb{Z}/p\mathbb{Z} & \text{otherwise} \end{cases}$$

for any $\mathfrak{Q}$ in H. Thus we have $\mathrm{rank}_{\mathbb{Z}_p}(I_{\mathfrak{Q}}) = 0$. From the exact sequence above, we obtain $\mathrm{rank}_{\mathbb{Z}_p}(X_{\mathfrak{p}}(L_\infty)_G) = \mathrm{rank}_{\mathbb{Z}_p}(X_{\mathfrak{p}}(K_\infty))$. □

Next we will calculate $\chi(G, X_{\mathfrak{p}}(L_\infty)^\vee)$. Let γ be a topological generator of $\mathrm{Gal}(L_\infty/L)$. We fix γ and put $\omega_n = \gamma^{p^n} - 1$ as an element of $\mathbb{Z}_p[\![\mathrm{Gal}(L_\infty/L)]\!]$ for each non-negative integer n. Let L_n be the intermediate field of the $\mathbb{Z}_p$-extension L_∞ such that L_n is the unique cyclic extension over L of degree p^n.

Lemma 3.5. *Assume the same notation above. Then we have an exact sequence*

$$0 \longrightarrow X_{\mathfrak{p}}(L_\infty)/\omega_n X_{\mathfrak{p}}(L_\infty) \longrightarrow \mathrm{Gal}(M_{\mathfrak{p}}(L_n)/L_n) \longrightarrow \mathrm{Gal}(L_\infty/L_n) \longrightarrow 0$$

as $\mathbb{Z}_p[\mathrm{Gal}(L_n/K_n)]$*-modules. Moreover, the quotient* $X_{\mathfrak{p}}(L_\infty)/\omega_n X_{\mathfrak{p}}(L_\infty)$ *is a* $\mathbb{Z}_p$*-module of finite order.*

Proof. Since L_n/k is abelian extension, $\mathfrak{p}$-adic Leopoldt's conjecture is valid for L_n by Proposition 2.1. This implies that the $\mathbb{Z}_p$-rank of $\mathrm{Gal}(M_{\mathfrak{p}}(L_n)/L_n)$ is 1. Hence $X_{\mathfrak{p}}(L_\infty)/\omega_n X_{\mathfrak{p}}(L_\infty)$ is a $\mathbb{Z}_p$-module of finite order. □

For each non-negative integer n, we denote by

$$\mathrm{Ver}^{(n)} : \mathrm{Gal}(M_{\mathfrak{p}}(L_n)/L_n) \longrightarrow \mathrm{Gal}(M_{\mathfrak{p}}(L_{n+1})/L_{n+1})$$

the transfer map from $\mathrm{Gal}(M_{\mathfrak{p}}(L_{n+1})/L_n)$ to $\mathrm{Gal}(M_{\mathfrak{p}}(L_{n+1})/L_{n+1})$. This homomorphism is defined as follows. We put $G_n = \mathrm{Gal}(M_{\mathfrak{p}}(L_{n+1})/L_n)$ and $H_n = \mathrm{Gal}(M_{\mathfrak{p}}(L_{n+1})/L_{n+1})$. Let R_n be a set of left representatives for the cosets of H_n in G_n satisfying $\mathrm{id}_{M_{\mathfrak{p}}(L_{n+1})} \in R_n$, where id_* denotes the identity map on $*$. Hence we have $G_n = R_n \cdot H_n$. We note that H_n is an abelian group. Let σ be an element of G_n. For each $\rho \in R_n$, there exist $\sigma_\rho \in H_n$ and $\rho' \in R_n$ such that $\sigma \circ \rho = \rho' \circ \sigma_\rho$. Then we define

$$\mathrm{Ver}^{(n)}(\sigma \bmod [G_n : G_n]) = \prod_{\rho\in R_n} \sigma_\rho,$$

where $[G_n : G_n]$ is the commutator subgroup of G_n. We note that transfer map is independent of the choice of the representatives.

We have the following commutative diagram of $\mathbb{Z}_p[\mathrm{Gal}(L_n/K_n)]$-modules as follows.

Proposition 3.6. *Let n be a non-negative integer. Then we have the following commutative diagram :*

$$\begin{array}{ccccccccc} 0 \longrightarrow & X_{\mathfrak{p}}(L_\infty)/\omega_n X_{\mathfrak{p}}(L_\infty) & \longrightarrow & \mathrm{Gal}(M_{\mathfrak{p}}(L_n)/L_n) & \longrightarrow & \mathrm{Gal}(L_\infty/L_n) & \longrightarrow 0 \\ & \downarrow{\scriptstyle \frac{\omega_{n+1}}{\omega_n}\times} & & \downarrow{\scriptstyle \mathrm{Ver}^{(n)}} & & \downarrow & \\ 0 \longrightarrow & X_{\mathfrak{p}}(L_\infty)/\omega_{n+1} X_{\mathfrak{p}}(L_\infty) & \longrightarrow & \mathrm{Gal}(M_{\mathfrak{p}}(L_{n+1})/L_{n+1}) & \longrightarrow & \mathrm{Gal}(L_\infty/L_{n+1}) & \longrightarrow 0, \end{array}$$

where the left vertical map is multiplication by ω_{n+1}/ω_n and the right vertical map is the p-th power map.

Proof. From Lemma 3.5, the rows are exact. Let $\widetilde{\gamma}$ be an extension of γ to $M_{\mathfrak{p}}(L_{n+1})$. Here, γ is the fixed topological generator of $\mathrm{Gal}(L_\infty/L)$ defined before Lemma 3.5. We set

$$R_n = \{\mathrm{id}_{M_{\mathfrak{p}}(L_{n+1})}, \widetilde{\gamma}^{p^n}, \widetilde{\gamma}^{2p^n}, \ldots, \widetilde{\gamma}^{(p-1)p^n}\},$$

which is a set of left representatives for the cosets of H_n in G_n. Note that $X_{\mathfrak{p}}(L_\infty)/\omega_n X_{\mathfrak{p}}(L_\infty) \cong \mathrm{Gal}(M_{\mathfrak{p}}(L_n)/L_\infty)$. Via this isomorphism, we identify these $\mathbb{Z}_p[\mathrm{Gal}(L_n/K_n)]$-modules.

First, we prove the commutativity of the left side of the diagram, i.e., we prove that $\mathrm{Ver}^{(n)}(\sigma) = \frac{\omega_{n+1}}{\omega_n}\sigma$ for any σ in $\mathrm{Gal}(M_{\mathfrak{p}}(L_n)/L_\infty)$. Suppose that $\sigma \in \mathrm{Gal}(M_{\mathfrak{p}}(L_n)/L_\infty)$. Let $\widetilde{\sigma}$ be an extension of σ to $M_{\mathfrak{p}}(L_{n+1})$. By the action of $\mathrm{Gal}(L_\infty/L_n)$ on $\mathrm{Gal}(M_{\mathfrak{p}}(L_{n+1})/L_\infty)$, we have

$$\gamma^{up^n} \cdot \widetilde{\sigma} = \widetilde{\gamma}^{up^n} \circ \widetilde{\sigma} \circ \widetilde{\gamma}^{-up^n}$$

for each u in $\{0, 1, 2, \cdots, p-1\}$. Here, $\widetilde{\gamma}^{up^n} = \mathrm{id}_{M_{\mathfrak{p}}(L_{n+1})}$ for $u = 0$. Since $M_{\mathfrak{p}}(L_n)/L_n$ is an abelian extension, we have $(\gamma^{up^n} \cdot \widetilde{\sigma})|_{M_{\mathfrak{p}}(L_n)} = \sigma$. Hence, there exists $\tau(u)$ in $\mathrm{Gal}(M_{\mathfrak{p}}(L_{n+1})/M_{\mathfrak{p}}(L_n))$ such that $\gamma^{up^n} \cdot \widetilde{\sigma} = \widetilde{\sigma} \circ \tau(u)$. Note that $\widetilde{\sigma} \circ \tau(u) \in H_n$. Thus we get

$$\widetilde{\sigma} \circ \widetilde{\gamma}^{-up^n} = \widetilde{\gamma}^{-up^n} \circ \widetilde{\sigma} \circ \tau(u).$$

This implies that

$$\mathrm{Ver}^{(n)}(\sigma) = \prod_{u=0}^{p-1} \widetilde{\sigma} \circ \tau(u) = \left(\prod_{u=0}^{p-1} \gamma^{up^n}\right) \cdot \sigma = \frac{\omega_{n+1}}{\omega_n}\sigma.$$

Next we prove the commutativity of the right side of the diagram, i.e., we prove that $\mathrm{Ver}^{(n)}(\sigma)|_{L_\infty} = (\sigma|_{L_\infty})^p$ for any σ in $\mathrm{Gal}(M_{\mathfrak{p}}(L_n)/L_n)$. Suppose that $\sigma \in \mathrm{Gal}(M_{\mathfrak{p}}(L_n)/L_n)$. Let $\widetilde{\sigma}$ be an extension of σ to $M_{\mathfrak{p}}(L_{n+1})$. We put

$\widetilde{\sigma}|_{L_\infty} = \gamma^{sp^n}$ for some p-adic integer s. For each integer u with $0 \le u \le p-1$, there exist $s(u)$ in $\{0, 1, 2, \cdots, p-1\}$ and σ_u in H_n such that

$$\widetilde{\sigma} \circ \widetilde{\gamma}^{up^n} = \widetilde{\gamma}^{s(u)p^n} \circ \sigma_u.$$

Here we note that $s(u) \equiv s + u \bmod p$. Thus we obtain

$$\mathrm{Ver}^{(n)}(\sigma)|_{L_\infty} = \left(\prod_{u=0}^{p-1} \sigma_u\right)|_{L_\infty} = \left(\prod_{u=0}^{p-1} \widetilde{\gamma}^{(s+u-s(u))p^n}\right)|_{L_\infty} = (\widetilde{\sigma}|_{L_\infty})^p.$$

Thus we get the conclusion. □

We put $G = \mathrm{Gal}(L_\infty/K_\infty)$. We consider $\mathbb{Z}_p[G]$-modules

$$\varinjlim_n X_\mathfrak{p}(L_\infty)/\omega_n X_\mathfrak{p}(L_\infty), \quad \varinjlim_n \mathrm{Gal}(M_\mathfrak{p}(L_n)/L_n), \quad \varinjlim_n \mathrm{Gal}(L_\infty/L_n),$$

where inductive limits are taken with respect to the same maps defined in Proposition 3.6. Taking direct limits is an exact functor in the category of modules, we have

Corollary 3.7. *We have the following exact sequence of $\mathbb{Z}_p[G]$-modules*

$$0 \longrightarrow \varinjlim_n X_\mathfrak{p}(L_\infty)/\omega_n X_\mathfrak{p}(L_\infty) \longrightarrow \varinjlim_n \mathrm{Gal}(M_\mathfrak{p}(L_n)/L_n) \longrightarrow \varinjlim_n \mathrm{Gal}(L_\infty/L_n) \longrightarrow 0.$$

Using the method of [12, Section 1] and [19, Section 3], we have the following proposition and lemma.

Proposition 3.8. *With the same notation as above, we have*

$$\chi(G, X_\mathfrak{p}(L_\infty)^\vee) = \chi\Big(G, \varinjlim_n X_\mathfrak{p}(L_\infty)/\omega_n X_\mathfrak{p}(L_\infty)\Big).$$

Proof. First we note that the Pontryagin dual $X_\mathfrak{p}(L_\infty)^\vee$ is pseudo-isomorphic to $\varinjlim_n X_\mathfrak{p}(L_\infty)/\omega_n X_\mathfrak{p}(L_\infty)$. Indeed, let $\alpha(X_\mathfrak{p}(L_\infty))$ be the adjoint of $X_\mathfrak{p}(L_\infty)$. In other words, we define

$$\alpha(X_\mathfrak{p}(L_\infty)) = \mathrm{Hom}_{\mathbb{Z}_p}\Big(\varinjlim_n X_\mathfrak{p}(L_\infty)/\omega_n X_\mathfrak{p}(L_\infty), \mathbb{Q}_p/\mathbb{Z}_p\Big).$$

Here $X_\mathfrak{p}(L_\infty)/\omega_n X_\mathfrak{p}(L_\infty)$ is a $\mathbb{Z}_p$-module of finite order by Lemma 3.5. We make $\alpha(X_\mathfrak{p}(L_\infty))$ into a $\mathbb{Z}_p[\![\mathrm{Gal}(L_\infty/L)]\!]$-module by defining the action $\gamma \cdot f(x) = f(\gamma \cdot x)$ for $f \in \alpha(X_\mathfrak{p}(L_\infty))$ and $x \in \varinjlim_n X_\mathfrak{p}(L_\infty)/\omega_n X_\mathfrak{p}(L_\infty)$. Since $\alpha(X_\mathfrak{p}(L_\infty))$ is pseudo-isomorphic to $X_\mathfrak{p}(L_\infty)$ (cf. [28, Corollary 15.31] and [12, p. 250]), the Pontryagin dual $X_\mathfrak{p}(L_\infty)^\vee$ is pseudo-isomorphic to $\varinjlim_n X_\mathfrak{p}(L_\infty)/\omega_n X_\mathfrak{p}(L_\infty)$. Thus, we obtain the conclusion. □

We note that L and K are abelian extension fields over k. Hence, G acts trivially on $\mathrm{Gal}(L_\infty/L_n)$ for all L_n. This leads to the following

Lemma 3.9. *With the same notation as above, we have*

$$\chi\Big(G, \varinjlim_n \mathrm{Gal}(L_\infty/L_n)\Big) = 1.$$

Proof. We have the following commutative diagram:

$$\begin{array}{ccc} \mathrm{Gal}(L_\infty/L_n) & \longrightarrow & \overline{\langle \gamma^{p^n} \rangle} \\ \downarrow & & \downarrow \\ \mathrm{Gal}(L_\infty/L_{n+1}) & \longrightarrow & \overline{\langle \gamma^{p^{n+1}} \rangle}, \end{array}$$

where the vertical maps are p-th power maps. We note that these maps are isomorphism. Thus we obtain $\varinjlim_n \mathrm{Gal}(L_\infty/L_n) \cong \mathbb{Z}_p$. Since G acts on $\varinjlim_n \mathrm{Gal}(L_\infty/L_n)$ trivially and G is a group of order p, we have

$$\mathrm{H}^2\Big(G, \varinjlim_n \mathrm{Gal}(L_\infty/L_n)\Big) \cong \mathrm{H}^2(G, \mathbb{Z}_p) \cong \mathbb{Z}/p\mathbb{Z} \text{ and } \mathrm{H}^1(G, \mathbb{Z}_p) = 0.$$

Thus, we obtain the conclusion. □

Combining Corollary 3.7 with the results above, we have the following

Proposition 3.10. *With the same notation as above, we have*

$$\chi(G, X_\mathfrak{p}(L_\infty)^\vee) = \chi\Big(G, \varinjlim_n \mathrm{Gal}(M_\mathfrak{p}(L_n)/L_n)\Big) - 1.$$

Proof. By Corollary 3.7, we get

$$\chi\Big(G, \varinjlim_n \mathrm{Gal}(M_\mathfrak{p}(L_n)/L_n)\Big) = \chi\Big(G, \varinjlim_n X_\mathfrak{p}(L_\infty)/\omega_n X_\mathfrak{p}(L_\infty)\Big) + \chi\Big(G, \varinjlim_n \mathrm{Gal}(L_\infty/L_n)\Big).$$

We obtain $\chi(G, X_\mathfrak{p}(L_\infty)^\vee) = \chi(G, \varinjlim_n \mathrm{Gal}(M_\mathfrak{p}(L_n)/L_n)) - 1$ using Proposition 3.8 and Lemma 3.9. Therefore, the conclusion follows. □

To calculate $\chi(G, \varinjlim_n \mathrm{Gal}(M_\mathfrak{p}(L_n)/L_n))$, we prepare some lemmas and propositions. Let F be an algebraic number field over k. Let $I(F)$ be the set of non-zero fractional ideals of F, or equivalently, the invertible $\mathcal{O}_F$-submodules of F. As in the previous section, we write T for $T(F)$ and fix an embedding $i_v^F : F^\times \to F_v^\times$ for $v \in T$. We denote by $I_T(F)$ the subgroup of $I(F)$ generated by non-zero fractional ideals prime to $\mathfrak{p}$. We denote by P_T^F the set of principal ideals of F generated by the elements of $F_T^\times$. Let n be a non-negative integer. We put

$$P^F_{T,\mathfrak{m}(n)} = \big\{\mathfrak{a} \in I_T(F) \;\big|\; \mathfrak{a} = (x) \text{ for some } x \in F^\times_{T,\mathfrak{m}(n)}\big\}.$$

Here, note that $P^F_{T,\mathfrak{m}(0)} = P^F_T$. We define homomorphisms

$$\psi^F_T : F^\times_T \longrightarrow I_T(F) \quad \text{and} \quad \overline{\psi^F_T} : F^\times_T \otimes_{\mathbb{Z}} \mathbb{Z}_p \longrightarrow I_T(F) \otimes_{\mathbb{Z}} \mathbb{Z}_p$$

by

$$\psi^F_T(a) = (a) \quad \text{and} \quad \overline{\psi^F_T}(a \otimes \alpha) = (a) \otimes \alpha$$

for $a \in F^\times_T$ and $\alpha \in \mathbb{Z}_p$. We note that $\overline{\psi^F_T}(F^\times_{T,\mathfrak{m}(n)} \otimes_{\mathbb{Z}} \mathbb{Z}_p) \subset P^F_{T,\mathfrak{m}(n)} \otimes_{\mathbb{Z}} \mathbb{Z}_p$.

Remark 3.11. The symbol "F" may be omitted from $I_T(F)$, $P^F_{T,\mathfrak{m}(n)}$, and ψ^F_T if there is no risk of confusion.

From the ideal-theoretic formulation of class field theory, we use the following

Lemma 3.12 ([15], Section 1)**.** *Let F/k be an algebraic extension. Then we have*

$$\mathrm{Gal}(M_\mathfrak{p}(F)/F) \cong (I_T(F) \otimes_{\mathbb{Z}} \mathbb{Z}_p)/\overline{\psi^F_T}(\ker(\overline{i_T})),$$

where $\overline{i_T}$ is the homomorphism defined in Section 2.

The following lemma will be used to prove Proposition 3.14.

Lemma 3.13. *Let F/k be a finite extension. Then we have*

$$\ker(\overline{\psi^F_T}) = \mathcal{O}^\times_F \otimes \mathbb{Z}_p.$$

In particular, if we assume that $\mathfrak{p}$-adic Leopoldt's conjecture holds for p and F, we have

$$\ker(\overline{\psi^F_T}) \cap \ker(\overline{i_T}) = 0.$$

Proof. By the definition of the homomorphism ψ^F_T, we have an exact sequence

$$0 \longrightarrow \mathcal{O}^\times_F \longrightarrow F^\times_T \longrightarrow P^F_T \longrightarrow 0,$$

where $P^F_T = \{(a) \in I_T(F) \mid a \in F^\times_T\}$. Hence we have $\ker(\overline{\psi^F_T}) = \mathcal{O}^\times_F \otimes \mathbb{Z}_p$. From Remark 2.4, we have $\ker(\overline{i_T}) \cap (\mathcal{O}^\times_F \otimes \mathbb{Z}_p) = 0$. Thus, the proof is complete. □

For each v in $T(n)$, we fix an embedding $i^{L_n}_v : L^\times_n \to (L_n)^\times_v$. The inclusion $L_n \subset L_{n+1}$ induces natural embeddings $I_{T(n)} \to I_{T(n+1)}$ and $\ker(\overline{i_{T(n)}}) \to \ker(\overline{i_{T(n+1)}})$ as $\mathbb{Z}_p[G]$-modules. Hence, we have $\mathbb{Z}_p[G]$-modules $\varinjlim_n (I_{T(n)} \otimes \mathbb{Z}_p)$ and $\varinjlim_n \ker(\overline{i_{T(n)}})$.

Proposition 3.14. *Assume the same notation as above. Then we have*

$$\chi\Big(G, \varinjlim_n \mathrm{Gal}(M_\mathfrak{p}(L_n)/L_n)\Big) = \chi\Big(G, \varinjlim_n (I_{T(n)} \otimes \mathbb{Z}_p)\Big) - \chi\Big(G, \varinjlim_n \ker(\overline{i_{T(n)}})\Big).$$

Proof. Note that the $\mathfrak{p}$-adic Leopoldt's conjecture holds for p and L_n. Hence, by Lemma 3.13, we have $\ker(\overline{\psi_{T(n)}}) \cap \ker(\overline{i_{T(n)}}) = 0$. Thus we obtain $\overline{\psi_{T(n)}}(\ker(\overline{i_{T(n)}})) \cong \ker(\overline{i_{T(n)}})$. Combining this isomorphism with Lemma 3.12, we obtain an exact sequence of $\mathbb{Z}_p[G]$-modules:

$$0 \longrightarrow \ker(\overline{i_{T(n)}}) \longrightarrow I_{T(n)} \otimes \mathbb{Z}_p \longrightarrow \mathrm{Gal}(M_{\mathfrak{p}}(L_n)/L_n) \longrightarrow 0.$$

Thus, we obtain an exact sequence of $\mathbb{Z}_p[G]$-modules:

$$0 \longrightarrow \varinjlim_n \ker(\overline{i_{T(n)}}) \longrightarrow \varinjlim_n (I_{T(n)} \otimes \mathbb{Z}_p) \longrightarrow \varinjlim_n \mathrm{Gal}(M_{\mathfrak{p}}(L_n)/L_n) \longrightarrow 0,$$

where the inductive limit of the right term is taken with respect to the transfer map $\mathrm{Ver}^{(n)}$ defined in Section 3. Therefore, we get

$$\chi(G, \varinjlim_n \mathrm{Gal}(M_{\mathfrak{p}}(L_n)/L_n)) = \chi(G, \varinjlim_n (I_{T(n)} \otimes \mathbb{Z}_p)) - \chi(G, \varinjlim_n \ker(\overline{i_{T(n)}})). \qquad \square$$

Let $\mathfrak{Q}$ be a prime ideal of L and $\mathfrak{Q}_\infty$ a prime ideal of L_∞ lying above $\mathfrak{Q}$. Then we define

$$\langle \mathfrak{Q}_\infty \rangle = \varinjlim_n \langle \mathfrak{Q}_n \rangle_{\mathbb{Z}},$$

where $\mathfrak{Q}_n = \mathfrak{Q}_\infty \cap L_n$ and the inductive limit is taken with respect to the natural embedding $I(L_n) \to I(L_{n+1})$ for $n \geq 0$.

Remark 3.15. The transition map only maps $\mathfrak{Q}_n$ to $\mathfrak{Q}_{n+1}$ if $\mathfrak{Q}$ is inert in L_{n+1}/L_n. This is the case for all n large enough.

By Proposition 3.14, to compute $\chi(G, X_{\mathfrak{p}}(L_\infty)^\vee)$, we only need to calculate $\chi(G, \varinjlim_n (I_{T(n)} \otimes \mathbb{Z}_p))$ and $\chi(G, \varinjlim_n \ker(\overline{i_{T(n)}}))$. The former can be calculated using the method described in [14, Section 1].

Proposition 3.16. *Let t be the number of primes of K_∞ that do not lie above $\mathfrak{p}$ and are ramified in L_∞. Then we have*

$$\chi(G, \varinjlim_n (I_{T(n)} \otimes \mathbb{Z}_p)) = t.$$

Proof. By Lemma 3.1, no prime ideal of L splits completely in L_∞/L. Let $\mathfrak{Q}$ be a prime ideal of L which is not lying above $\mathfrak{p}$. Then $\mathfrak{Q}$ is unramified in L_∞ because L_∞/L is the split prime $\mathbb{Z}_p$-extension corresponding to $\mathfrak{p}$. Let $\mathfrak{Q}_\infty$ be a prime ideal of L_∞ lying above $\mathfrak{Q}$. Then we have $\langle \mathfrak{Q}_\infty \rangle = \varinjlim_n \langle \mathfrak{Q}_n \rangle_{\mathbb{Z}} \cong \mathbb{Z}$, where $\mathfrak{Q}_n = \mathfrak{Q}_\infty \cap L_n$ and the inductive limit is taken with respect to the natural embedding $I(L_n) \to I(L_{n+1})$ for $n \geq 0$. Hence, we have $I_T(L_\infty) \cong \varinjlim_n I_{T(n)} \cong \bigoplus_{\mathfrak{Q}_\infty \nmid \mathfrak{p}} \langle \mathfrak{Q}_\infty \rangle_{\mathbb{Z}} \cong \bigoplus_{\mathfrak{Q}_\infty \nmid \mathfrak{p}} \mathbb{Z}$, where

$\mathfrak{Q}_\infty$ ranges over all prime ideals of L_∞ not lying above $\mathfrak{p}$. Thus, we obtain isomorphisms of $\mathbb{Z}_p[G]$-modules:

$$I_T(L_\infty)\otimes\mathbb{Z}_p\cong\varinjlim_n(I_{T(n)}\otimes\mathbb{Z}_p)\cong\bigoplus_{\mathfrak{Q}_\infty\nmid\mathfrak{p}}\langle\mathfrak{Q}_\infty\rangle_{\mathbb{Z}_p}\cong\bigoplus_{\mathfrak{Q}_\infty\nmid\mathfrak{p}}\mathbb{Z}_p.$$

Let $\mathfrak{q}$ be a prime ideal of K which is not lying above $\mathfrak{p}$ and $\mathfrak{q}_\infty$ a prime ideal of K_∞ lying above $\mathfrak{q}$. Since $\mathfrak{q}$ does not split completely in K_∞, $\mathfrak{q}_\infty$ is not inert in L_∞. Hence all prime ideals of K_∞ which are not lying above $\mathfrak{p}$ either ramify or split completely in L_∞. Then we have a direct summands of $\mathbb{Z}_p[G]$-modules:

$$\bigoplus_{\mathfrak{Q}_\infty\nmid\mathfrak{p}}\langle\mathfrak{Q}_\infty\rangle_{\mathbb{Z}_p}=\bigoplus_{\mathfrak{Q}_\infty\in A}\langle\mathfrak{Q}_\infty\rangle_{\mathbb{Z}_p}\oplus\bigoplus_{\mathfrak{Q}_\infty\in B}\langle\mathfrak{Q}_\infty\rangle_{\mathbb{Z}_p}\cong\bigoplus_{\mathfrak{Q}_\infty\in A}\mathbb{Z}_p\oplus\bigoplus_{\mathfrak{Q}_\infty\in B}\mathbb{Z}_p,$$

where we put

$$A=\left\{\mathfrak{Q}_\infty\;\middle|\;\begin{array}{c}\mathfrak{Q}_\infty\text{ is a prime ideal of }L_\infty\text{ which is not lying above }\mathfrak{p},\\ \mathfrak{Q}_\infty\cap K_\infty\text{ is ramified in }L_\infty\end{array}\right\},$$

$$B=\left\{\mathfrak{Q}_\infty\;\middle|\;\begin{array}{c}\mathfrak{Q}_\infty\text{ is a prime ideal of }L_\infty\text{ which is not lying above }\mathfrak{p},\\ \mathfrak{Q}_\infty\cap K_\infty\text{ splits completely in }L_\infty\end{array}\right\}.$$

For $i=1,2$, we obtain

$$\mathrm{H}^i\left(G,\bigoplus_{\mathfrak{Q}_\infty\nmid\mathfrak{p}}\langle\mathfrak{Q}_\infty\rangle_{\mathbb{Z}_p}\right)\cong\mathrm{H}^i\left(G,\bigoplus_{\mathfrak{Q}_\infty\in A}\langle\mathfrak{Q}_\infty\rangle_{\mathbb{Z}_p}\right)\oplus\mathrm{H}^i\left(G,\bigoplus_{\mathfrak{Q}_\infty\in B}\langle\mathfrak{Q}_\infty\rangle_{\mathbb{Z}_p}\right).$$

Since $\bigoplus_{\mathfrak{Q}_\infty\in B}\langle\mathfrak{Q}_\infty\rangle_{\mathbb{Z}_p}$ is an induced G-module, $\mathrm{H}^i(G,\bigoplus_{\mathfrak{Q}_\infty\in B}\langle\mathfrak{Q}_\infty\rangle_{\mathbb{Z}_p})=0$ for $i=1,2$. Note that G acts on $\langle\mathfrak{Q}_\infty\rangle_{\mathbb{Z}_p}$ ($\mathfrak{Q}_\infty\in A$) and $\bigoplus_{\mathfrak{Q}_\infty\in A}\langle\mathfrak{Q}_\infty\rangle_{\mathbb{Z}_p}$ trivially. Since G is a cyclic group, we have

$$\mathrm{H}^2\left(G,\bigoplus_{\mathfrak{Q}_\infty\in A}\langle\mathfrak{Q}_\infty\rangle_{\mathbb{Z}_p}\right)\cong\bigoplus_{\mathfrak{Q}_\infty\in A}\mathrm{H}^2(G,\langle\mathfrak{Q}_\infty\rangle_{\mathbb{Z}_p})\cong\bigoplus_{\mathfrak{Q}_\infty\in A}\mathbb{Z}_p/p\mathbb{Z}_p\cong(\mathbb{Z}/p\mathbb{Z})^{\oplus t}.$$

Here, we note that $t=\#A$. Since G is a cyclic group of order p, we have

$$\mathrm{H}^1\left(G,\bigoplus_{\mathfrak{Q}_\infty\in A}\langle\mathfrak{Q}_\infty\rangle_{\mathbb{Z}_p}\right)\cong\mathrm{Hom}\left(G,\bigoplus_{\mathfrak{Q}_\infty\in A}\langle\mathfrak{Q}_\infty\rangle_{\mathbb{Z}_p}\right)=0.$$

Therefore, we obtain $\chi(G,\varinjlim_n(I_{T(n)}\otimes\mathbb{Z}_p))=t$. □

For each positive integer n and prime w in $T(n)$, we select a symbol $\mathbf{e}_w$ and we let $\bigoplus_{w\in T(n)}\mathbb{Z}\mathbf{e}_w$ be the free $\mathbb{Z}$-module of rank $\#T(n)$, which has $\{\mathbf{e}_w\mid w\in T(n)\}$ as basis. From the isomorphism $G\cong\mathrm{Gal}(L_n/K_n)$, we let G act on $\bigoplus_{w\in T(n)}\mathbb{Z}\mathbf{e}_w$ by

$$\sigma\mathbf{e}_w=\mathbf{e}_{\sigma(w)}\quad\text{for }\sigma\in G.$$

We define homomorphisms

$$\mathrm{ord}^{(n)} : L_n^\times \longrightarrow \bigoplus_{w\in T(n)} \mathbb{Z}\mathbf{e}_w \quad \text{and} \quad \varphi_n : \bigoplus_{w\in T(n)} \mathbb{Z}\mathbf{e}_w \longrightarrow \bigoplus_{\tilde{w}\in T(n+1)} \mathbb{Z}\mathbf{e}_{\tilde{w}}$$

by

$$\mathrm{ord}^{(n)}(a) = (\mathrm{ord}_w(a)\mathbf{e}_w)_{w\in T(n)}$$
$$\text{and} \quad \varphi_n((a_w\mathbf{e}_w)_{w\in T(n)}) = (e(\tilde{w}/w)a_w\mathbf{e}_{\tilde{w}})_{\tilde{w}\in T(n+1)},$$

where $a \in L_n^\times$ and $(a_w\mathbf{e}_w)_{w\in T(n)} \in \bigoplus_{w\in T(n)} \mathbb{Z}\mathbf{e}_w$, $\tilde{w}$ is a prime ideal of L_{n+1} lying above w, and $e(\tilde{w}/w)$ is the ramification index of the extension $(L_{n+1})_{\tilde{w}}/(L_n)_w$. Note that $\mathrm{ord}^{(n)}$ and φ_n are homomorphisms of $\mathbb{Z}[G]$-modules.

To compute $\chi(G, \varinjlim_n \ker(\overline{i_{T(n)}}))$, we prepare the following two propositions.

Proposition 3.17. *With the same notation as above, we have*

$$\chi(G, \varinjlim_n (L_n)^\times_{T(n)} \otimes \mathbb{Z}_p) = 0,$$

where the inductive limit is taken with respect to the natural embedding $(L_n)^\times_{T(n)} \subset (L_{n+1})^\times_{T(n+1)}$.

Proof. We have an exact sequence of $\mathbb{Z}[G]$-modules:

$$0 \longrightarrow (L_n)^\times_{T(n)} \longrightarrow L_n^\times \xrightarrow{\mathrm{ord}^{(n)}} \bigoplus_{w\in T(n)} \mathbb{Z}\mathbf{e}_w \longrightarrow 1.$$

Using a homomorphism $\mathrm{ord}^{(n)} \otimes 1 : L_n^\times \otimes \mathbb{Z}_p \to \bigoplus_{w\in T(n)} \mathbb{Z}\mathbf{e}_w \otimes \mathbb{Z}_p$ and an isomorphism $\mathbb{Z}\mathbf{e}_w \otimes \mathbb{Z}_p \cong \mathbb{Z}_p\mathbf{e}_w$, we get an exact sequence of $\mathbb{Z}_p[G]$-modules:

$$0 \longrightarrow (L_n)^\times_{T(n)} \otimes \mathbb{Z}_p \longrightarrow L_n^\times \otimes \mathbb{Z}_p \xrightarrow{\mathrm{ord}^{(n)}\otimes 1} \bigoplus_{w\in T(n)} \mathbb{Z}_p\mathbf{e}_w \longrightarrow 1.$$

Then we have a commutative diagram:

$$\begin{array}{ccc} L_n^\times \otimes \mathbb{Z}_p & \xrightarrow{\mathrm{ord}^{(n)}\otimes 1} & \bigoplus_{w\in T(n)} \mathbb{Z}_p\mathbf{e}_w \\ \downarrow & & \downarrow{\scriptstyle \varphi_n\otimes 1} \\ L_{n+1}^\times \otimes \mathbb{Z}_p & \xrightarrow{\mathrm{ord}^{(n+1)}\otimes 1} & \bigoplus_{\tilde{w}\in T(n+1)} \mathbb{Z}_p\mathbf{e}_{\tilde{w}} \end{array}$$

where the left vertical map is the map induced by the natural embedding $L_n \subset L_{n+1}$. From the commutative diagram, we obtain an exact sequence of $\mathbb{Z}_p[G]$-modules:

$$0 \longrightarrow \varinjlim_n (L_n)^\times_{T(n)} \otimes \mathbb{Z}_p \longrightarrow \varinjlim_n L_n^\times \otimes \mathbb{Z}_p \xrightarrow{\mathrm{ord}^{(n)}\otimes 1} \varinjlim_n \bigoplus_{w\in T(n)} \mathbb{Z}_p\mathbf{e}_w \longrightarrow 1.$$

Hence, we get

$$\chi\left(G, \varinjlim_n (L_n)^\times_{T(n)} \otimes \mathbb{Z}_p\right) = \chi\left(G, \varinjlim_n L_n^\times \otimes \mathbb{Z}_p\right) - \chi\left(G, \varinjlim_n \bigoplus_{w\in T(n)} \mathbb{Z}_p \mathbf{e}_w\right).$$

First, we will show that $\chi(G, \varinjlim_n L_n^\times \otimes \mathbb{Z}_p) = 0$. We have

$$\mathrm{H}^i\left(G, \varinjlim_n L_n^\times \otimes \mathbb{Z}_p\right) = \varinjlim_n \mathrm{H}^i(G, L_n^\times \otimes \mathbb{Z}_p) \cong \varinjlim_n \mathrm{H}^i(G, L_n^\times) \otimes \mathbb{Z}_p.$$

By Hilbert's Theorem 90, we have $\mathrm{H}^1(G, L_n^\times) = 0$. This implies that $\mathrm{H}^1(G, \varinjlim_n L_n^\times \otimes \mathbb{Z}_p) = 0$. Since no finite prime of K splits completely in K_∞, we have $\mathrm{Br}(K_\infty) \otimes \mathbb{Z}_p = 0$ by [23, Theorem 8.1.14], where we denote by $\mathrm{Br}(*)$ the Brauer group of $*$. Then we obtain $\mathrm{H}^2(G, L_\infty^\times) \otimes \mathbb{Z}_p = 0$ from the exact sequence

$$0 \longrightarrow \mathrm{H}^2(G, L_\infty^\times) \longrightarrow \mathrm{Br}(K_\infty) \longrightarrow \mathrm{Br}(L_\infty).$$

Hence we get $\chi(G, \varinjlim_n L_n^\times \otimes \mathbb{Z}_p) = 0$.

Next we will show that $\chi(G, \varinjlim_n \bigoplus_{w\in T(n)} \mathbb{Z}_p \mathbf{e}_w) = 0$. Since K_∞/K is the split prime $\mathbb{Z}_p$-extension corresponding to $\mathfrak{p}$, there exists a positive integer n such that all prime ideals of K_n lying above $\mathfrak{p}$ are totally ramified in K_∞. By the same argument as in Proposition 3.16, we get

$$\mathrm{H}^i\left(G, \bigoplus_{w\in T(n),\, w\cap K_n \text{ splits}} \mathbb{Z}_p \mathbf{e}_w\right) = 0,$$

where w ranges over all primes in $T(n)$ such that $w \cap K_n$ splits completely in L_n. Hence we have

$$\mathrm{H}^i\left(G, \bigoplus_{w\in T(n)} \mathbb{Z}_p \mathbf{e}_w\right) \cong \mathrm{H}^i\left(G, \bigoplus_{w_1\in T(n)} \mathbb{Z}_p \mathbf{e}_{w_1}\right) \oplus \mathrm{H}^i\left(G, \bigoplus_{w_2\in T(n)} \mathbb{Z}_p \mathbf{e}_{w_2}\right)$$

for $i = 1, 2$, where w_1 and w_2 range over all primes in $T(n)$ such that $w_1 \cap K_n$ is ramified and $w_1 \cap K_n$ is inert in L_n, respectively. Since G acts trivially on $\bigoplus_{w_j\in T(n)} \mathbb{Z}_p \mathbf{e}_{w_j}$ $(j = 1, 2)$, we have

$$\mathrm{H}^i\left(G, \bigoplus_{w_j\in T(n)} \mathbb{Z}_p \mathbf{e}_{w_j}\right) \cong \begin{cases} \bigoplus_{w_j\in T(n)} \mathbb{Z}/p\mathbb{Z} & \text{if } i = 2, \\ 0 & \text{if } i = 1, \end{cases}$$

for $j = 1, 2$. Hence

$$\mathrm{H}^1\left(G, \varinjlim_n \bigoplus_{w\in T(n)} \mathbb{Z}_p \mathbf{e}_w\right) \cong \varinjlim_n \mathrm{H}^1\left(G, \bigoplus_{w\in T(n)} \mathbb{Z}_p \mathbf{e}_w\right) = 0 \quad \text{for } i = 1.$$

Furthermore, the homomorphism

$$\mathrm{H}^2\left(G, \bigoplus_{w_j\in T(n)} \mathbb{Z}_p\mathbf{e}_w\right) \longrightarrow \mathrm{H}^2\left(G, \bigoplus_{\tilde{w}_j\in T(n+1)} \mathbb{Z}_p\mathbf{e}_w\right)$$

is induced from the homomorphism

$$\varphi_n : \bigoplus_{w_j\in T(n)} \mathbb{Z}\mathbf{e}_w \longrightarrow \bigoplus_{\tilde{w}_j\in T(n+1)} \mathbb{Z}\mathbf{e}_{\tilde{w}},$$

which is the p-th power map. Therefore, we get

$$\mathrm{H}^2\left(G, \varinjlim_n \bigoplus_{w\in T(n)} \mathbb{Z}_p\mathbf{e}_w\right) = 0.$$

This completes the proof. □

Proposition 3.18. *With the same notation as above, we have*

$$\chi\left(G, \varinjlim_n \bigoplus_{w\in T(n)} U^{(1)}((L_n)_w)\right) = 0,$$

where the inductive limit is taken with respect to the natural inclusion $\bigoplus_{w\in T(n)} U^{(1)}((L_n)_w) \to \bigoplus_{\tilde{w}\in T(n+1)} U^{(1)}((L_{n+1})_{\tilde{w}})$.

Proof. We have an isomorphism of direct summands of $\mathbb{Z}_p[G]$-modules

$$\bigoplus_{w\in T(n)} U^{(1)}((L_n)_w) \cong \bigoplus_{w\in A(n)} U^{(1)}((L_n)_w) \oplus \bigoplus_{w\in B(n)} U^{(1)}((L_n)_w),$$

where we put

$$A(n) = \{w \in T(L_n) \mid w \cap K_n \text{ is unramified in } L_n\},$$
$$B(n) = \{w \in T(L_n) \mid w \cap K_n \text{ is ramified in } L_n\}.$$

If w is a prime ideal in $A(n)$, we have $\mathrm{H}^i(G, U^{(1)}((L_n)_w)) = 0$ for $i = 1, 2$. Hence we obtain $\mathrm{H}^i(G, \varinjlim_n \bigoplus_{w\in A(n)} U^{(1)}((L_n)_w)) = 0$ for $i = 1, 2$.

We will show that $\chi(G, \varinjlim_n \bigoplus_{w\in B(n)} U^{(1)}((L_n)_w)) = 0$. We suppose that $w \in B(n)$. Because K_∞/K and L_∞/L are split prime $\mathbb{Z}_p$-extensions, all prime ideals of K_n (respectively, L_n) above $\mathfrak{p}$ are totally ramified in K_∞ (respectively, L_∞) for sufficiently large integer n. Hence if $w \in B(n)$, then $\widetilde{w} \in B(n+1)$ for sufficiently large integer n. We put $v = w \cap K_n$. Then we have $\mathrm{H}^2(G, U^{(1)}((L_n)_w)) \cong \mathbb{Z}/p\mathbb{Z}$ since $(L_n)_w/(K_n)_v$ is a ramified extension. For each positive integer n, let $\mathrm{rec}_w^{(n)}$ be the reciprocity map

$$\mathrm{rec}_w^{(n)} : (K_n)_v^\times / N_{(L_n)_w/(K_n)_v}((L_n)_w) \longrightarrow \mathrm{Gal}((L_n)_w/(K_n)_v),$$

where $N_{(L_n)_w/(K_n)_v}$ is the norm map. Then we have the following commutative diagram:

$$\begin{array}{ccc}
(K_{n+1})_{\tilde{v}}^{\times}/N_{(L_{n+1})_{\tilde{w}}/(K_{n+1})_{\tilde{v}}}((L_{n+1})_{\tilde{w}}) & \xrightarrow{\mathrm{rec}_{\tilde{w}}^{(n+1)}} & \mathrm{Gal}((L_{n+1})_{\tilde{w}}/(K_{n+1})_{\tilde{v}}) \\
{\scriptstyle N_{\tilde{w}}^{(n+1)}}\downarrow & & \downarrow\cong \\
(K_n)_v^{\times}/N_{(L_n)_w/(K_n)_v}((L_n)_w) & \xrightarrow{\mathrm{rec}_w^{(n)}} & \mathrm{Gal}((L_n)_w/(K_n)_v),
\end{array}$$

where $N_{\tilde{w}}^{(n+1)}$ is the homomorphism induced from the norm map $(K_{n+1})_{\tilde{v}} \to (K_n)_v$ ($\widetilde{v} = \widetilde{w} \cap K_{n+1}$) and the right vertical map is the restriction map. We note that the right vertical map is an isomorphism since $\mathrm{Gal}((L_n)_w/(K_n)_v)$ and $\mathrm{Gal}((L_{n+1})_{\tilde{w}}/(K_{n+1})_{\tilde{v}})$ are groups of order p. Let a_n be an element of $U^{(1)}((K_n)_v)$. Then we have

$$\begin{aligned}
\mathrm{rec}_w^{(n)}(N_{\tilde{w}}^{(n+1)}(a_n \bmod N_{(L_{n+1})_{\tilde{w}}/(K_{n+1})_{\tilde{v}}}((L_{n+1})_{\tilde{w}})))& \\
&= \mathrm{rec}_w^{(n)}(a_n \bmod N_{(L_n)_w/(K_n)_v}((L_n)_w))^p \\
&= 1.
\end{aligned}$$

We have $a_n \bmod N_{(L_{n+1})_{\tilde{w}}/(K_{n+1})_{\tilde{v}}}((L_{n+1})_{\tilde{w}}) \in \ker(\mathrm{rec}_{\tilde{w}}^{(n+1)})$ from the commutativity of the diagram. Hence $a_n \in N_{(L_{n+1})_{\tilde{w}}/(K_{n+1})_{\tilde{v}}}((L_{n+1})_{\tilde{w}})$. This implies that $\mathrm{H}^2(G, \varinjlim_n U^{(1)}((L_n)_w)) = 0$. Thus we obtain

$$\mathrm{H}^2\left(G, \varinjlim_n \bigoplus_{w \in B(n)} U^{(1)}((L_n)_w)\right) = 0.$$

Next we will show that $\mathrm{H}^1(G, \varinjlim_n \bigoplus_{w\in B(n)} U^{(1)}((L_n)_w)) = 0$. We suppose that $w \in B(n)$. Then we have the following commutative diagram:

$$\begin{array}{ccc}
(L_n)_w^{\times} & \xrightarrow{\mathrm{ord}_w} & \mathbb{Z} \\
\downarrow & & \downarrow {\scriptstyle \times e((L_{n+1})_{\tilde{w}}/(L_n)_w)} \\
(L_{n+1})_{\tilde{w}}^{\times} & \xrightarrow{\mathrm{ord}_{\tilde{w}}} & \mathbb{Z},
\end{array}$$

where the right vertical map is the is multiplication by the ramification index $e((L_{n+1})_{\tilde{w}}/(L_n)_w)$. From $\mathrm{H}^1(G, U((L_n)_w)) \cong \mathrm{H}^1(G, U^{(1)}((L_n)_w))$ for each $w \in T(n)$, the diagram above induces

$$\begin{array}{ccc}
\mathbb{Z}/p\mathbb{Z} & \longrightarrow & \mathrm{H}^1(G, U^{(1)}((L_n)_w) \\
{\scriptstyle \times e((L_{n+1})_{\tilde{w}}/(L_n)_w)}\downarrow & & \downarrow \\
\mathbb{Z}/p\mathbb{Z} & \longrightarrow & \mathrm{H}^1(G, U^{(1)}((L_{n+1})_{\tilde{w}}),
\end{array}$$

where the left vertical map is the is multiplication by the ramification index $e((L_{n+1})_{\tilde{w}}/(L_n)_w)$. Here we note that $e((L_n)_w/(K_n)_v) = p$ since $w \in B(n)$. We have $e((L_{n+1})_{\tilde{w}}/(L_n)_w) = p$ for sufficiently large integer n. Therefore, the right vertical map is a zero map. Hence, we obtain

$$\mathrm{H}^1\left(G, \varinjlim_n \bigoplus_{w \in B(n)} U^{(1)}((L_n)_w)\right) = 0.$$

Thus, we reach the conclusion. □

Finally we can compute $\chi(G, \varinjlim_n \ker(\overline{i_{T(n)}}))$.

Proposition 3.19. *With the same notation as above, we have*

$$\chi\left(G, \varinjlim_n \ker(\overline{i_{T(n)}})\right) = 0.$$

Proof. By Proposition 2.3, the homomorphism $\overline{i_{T(n)}}$ is surjective. Thus we have an exact sequence

$$1 \longrightarrow \ker(\overline{i_{T(n)}}) \longrightarrow (L_n)^{\times}_{T(n)} \otimes \mathbb{Z}_p \xrightarrow{\overline{i_{T(n)}}} \bigoplus_{w \in T(n)} U^{(1)}((L_n)_w) \longrightarrow 1.$$

Using Propositions 3.17 and 3.18, we obtain

$$\begin{aligned} &\chi\left(G, \varinjlim_n \ker(\overline{i_{T(n)}})\right) \\ &\quad = \chi\left(G, \varinjlim_n (L_n)^{\times}_{T(n)} \otimes \mathbb{Z}_p\right) - \chi\left(G, \varinjlim_n \bigoplus_{w \in T(n)} U^{(1)}((L_n)_w)\right) \\ &\quad = 0. \end{aligned}$$ □

Now we can prove Theorem 1.1.

Proof of Theorem 1.1. By Propositions 3.10, 3.14, 3.16, 3.19, we obtain

$$\begin{aligned} \chi(G, X_{\mathfrak{p}}(L_\infty)^\vee) &= \chi\left(G, \varinjlim_n \mathrm{Gal}(M_{\mathfrak{p}}(L_n)/L_n)\right) - 1 \\ &= \chi\left(G, \varinjlim_n (I_{T(n)} \otimes \mathbb{Z}_p)\right) - \chi\left(G, \varinjlim_n \ker(\overline{i_{T(n)}})\right) - 1 \\ &= t - 1. \end{aligned}$$

Using Lemma 3.2 and Proposition 3.3, the proof is complete. □

4. Applications

In this section, we apply Theorem 1.1 to Iwasawa modules associated with imaginary quadratic fields. Let p be an odd prime number and k an imaginary quadratic field in which p splits into two distinct prime ideals $\mathfrak{p}$ and $\mathfrak{p}^*$. Let K be a $\mathbb{Z}_p$-extension or the $\mathbb{Z}_p^{\oplus 2}$-extension of k. We denote by $L(K)/K$ the maximal unramified pro-p abelian extension and put $X(K) = \mathrm{Gal}(L(K)/K)$. Since the Galois group $\mathrm{Gal}(K/k)$ acts naturally on $X(K)$, it becomes a $\mathbb{Z}_p[\![\mathrm{Gal}(K/k)]\!]$-module. It is known that $X(K)$ is a finitely generated torsion $\mathbb{Z}_p[\![\mathrm{Gal}(K/k)]\!]$-module ([8, 11]).

Since we have $\mathrm{Gal}(\widetilde{k}/k) \cong \mathbb{Z}_p^{\oplus 2}$, k has two independent $\mathbb{Z}_p$-extensions. For example, the cyclotomic $\mathbb{Z}_p$-extension k_∞^c and the anti-cyclotomic $\mathbb{Z}_p$-extension k_∞^a are disjoint over k and satisfy $\widetilde{k} = k_\infty^c k_\infty^a$. Let σ and τ be topological generators of $\mathrm{Gal}(\widetilde{k}/k_\infty^c)$ and $\mathrm{Gal}(\widetilde{k}/k_\infty^a)$, respectively. By the isomorphism

$$\mathrm{Gal}(\widetilde{k}/k) \cong \mathrm{Gal}(\widetilde{k}/k_\infty^c) \times \mathrm{Gal}(\widetilde{k}/k_\infty^a),$$

we fix an isomorphism

$$\mathbb{Z}_p[\![\mathrm{Gal}(\widetilde{k}/k)]\!] \cong \mathbb{Z}_p[\![S, T]\!] \qquad (\sigma \leftrightarrow 1+S,\ \tau \leftrightarrow 1+T). \tag{4.1}$$

We put $\Lambda = \mathbb{Z}_p[\![S, T]\!]$. Using this isomorphism, we regard $X(\widetilde{k})$ as a Λ-module.

4.1. Proof of Theorem 1.2.

In this subsection, we give a proof of Theorem 1.2. Let G be a profinite group. For any G-module M, we denote by M^G the subset of elements of M invariant under the action of G. We also denote by M_G the largest quotient module of M on which G acts trivially.

First we give a necessary and sufficient condition for the vanishing of the λ-invariant of $X_{\mathfrak{p}}(N_\infty)$.

Proposition 4.1. *Let p be an odd prime number and k an imaginary quadratic field in which p splits into $\mathfrak{p}$ and $\mathfrak{p}^*$. Then the followings are equivalent:*

(i) *k is p-rational, in other words, $L(k) \subset \widetilde{k}$,*
(ii) *$M_{\mathfrak{p}}(N_\infty) = N_\infty$.*

Proof. First we suppose that (i) holds. Since k is p-rational, we have $M_p(k) = \widetilde{k}$, where $M_p(k)$ the maximal unramified pro-p abelian extension field of k unramified outside all prime ideals lying above p. Hence we obtain $M_{\mathfrak{p}}(k) \subset M_p(k) = \widetilde{k}$. Furthermore, all prime ideals of N_∞ lying above $\mathfrak{p}^*$ are totally ramified in $\widetilde{k}/N_\infty$ because N_∞ coincides with the fixed field of $\widetilde{k}$ by the inertia subgroup of $\mathrm{Gal}(\widetilde{k}/k)$ for the prime ideal $\mathfrak{p}^*$ ([20, Lemma 3.2]). Thus we have $M_{\mathfrak{p}}(k) = N_\infty$. Since we have an isomorphism $X_{\mathfrak{p}}(N_\infty)_{\mathrm{Gal}(N_\infty/k)} \cong \mathrm{Gal}(M_{\mathfrak{p}}(k)/N_\infty)$, we get $X_{\mathfrak{p}}(N_\infty)_{\mathrm{Gal}(N_\infty/k)} =$

0. From Nakayama's lemma, we obtain $X_{\mathfrak{p}}(N_\infty) = 0$. This implies that $M_{\mathfrak{p}}(N_\infty) = N_\infty$.

Next we prove that (ii) implies (i). We suppose that (i) does not hold. This implies that $L(k)\widetilde{k} \neq \widetilde{k}$. Hence we obtain

$$N_\infty \subsetneq N_\infty L(k) \subset M_{\mathfrak{p}}(k).$$

This implies that $\mathrm{Gal}(M_{\mathfrak{p}}(k)/N_\infty) \neq 0$. Hence (ii) does not hold. Thus we get the conclusion. □

For each non-negative integer n, let k_n^c be the n-th layer of k_∞^c/k. We denote by $(k_n^c)_\infty$ the split prime $\mathbb{Z}_p$-extension of k_n^c corresponding to $\mathfrak{p}$. From Theorem 1.1, we have the following

Theorem 4.2. *Assume the same conditions in Proposition 4.1. Let $\mathfrak{D}_{\mathfrak{p}^*}$ is the decomposition group of the prime $\mathfrak{p}^*$ in $\mathrm{Gal}(\widetilde{k}/k)$. Then we have*

$$\mathrm{rank}_{\mathbb{Z}_p}(X_{\mathfrak{p}}((k_n^c)_\infty)) = p^n \cdot (\mathrm{rank}_{\mathbb{Z}_p}(X_{\mathfrak{p}}(N_\infty)) + [\mathrm{Gal}(\widetilde{k}/k) : \mathfrak{D}_{\mathfrak{p}*}] - 1) - ([\mathrm{Gal}(\widetilde{k}/k) : \mathfrak{D}_{\mathfrak{p}*}] - 1)$$

for any positive integer $n > 0$. Furthermore, we have

$$\mathrm{rank}_{\mathbb{Z}_p[\![\mathrm{Gal}(\tilde{k}/N_\infty)]\!]}(X_{\mathfrak{p}}(\widetilde{k})) = \mathrm{rank}_{\mathbb{Z}_p}(X_{\mathfrak{p}}(N_\infty)) + [\mathrm{Gal}(\widetilde{k}/k) : \mathfrak{D}_{\mathfrak{p}*}] - 1.$$

Proof. Let n be a positive integer. The extension $(k_n^c)_\infty/(k_{n-1}^c)_\infty$ is Galois with $[(k_n^c)_\infty : (k_{n-1}^c)_\infty] = p$. We note that the $\mathbb{Z}_p$-extension $\widetilde{k}/N_\infty$ is unramified outside all prime ideals lying above $\mathfrak{p}^*$ and that all prime ideals of N_∞ lying above $\mathfrak{p}^*$ are totally ramified in $\widetilde{k}/N_\infty$. By [22, Lemma 3.10], we have $[\mathrm{Gal}(\widetilde{k}/k) : \mathfrak{D}_{\mathfrak{p}*}] = [\mathrm{Gal}(N_\infty/k) : D_{\mathfrak{p}^*}]$, where $D_{\mathfrak{p}^*}$ is the decomposition group of $\mathfrak{p}^*$ in $\mathrm{Gal}(N_\infty/k)$. Hence the number of primes of $(k_{n-1}^c)_\infty$ which ramify in $(k_n^c)_\infty$ is equal to $[\mathrm{Gal}(\widetilde{k}/k) : \mathfrak{D}_{\mathfrak{p}*}]$. We put $n_0 = \mathrm{ord}_p([\mathrm{Gal}(\widetilde{k}/k) : \mathfrak{D}_{\mathfrak{p}*}])$. We apply Theorem 1.1 to $K = (k_{n-1}^c)_\infty$ and $L = (k_n^c)_\infty$. Then we obtain

$$\mathrm{rank}_{\mathbb{Z}_p}(X_{\mathfrak{p}}((k_n^c)_\infty) = p \cdot \mathrm{rank}_{\mathbb{Z}_p}(X_{\mathfrak{p}}(k_{n-1}^c)_\infty) + (p-1)(p^{n_0} - 1)$$

for positive integer n. It is easy to check that

$$\mathrm{rank}_{\mathbb{Z}_p}(X_{\mathfrak{p}}((k_n^c)_\infty)) = p^n \cdot (\mathrm{rank}_{\mathbb{Z}_p}(X_{\mathfrak{p}}(N_\infty)) + p^{n_0} - 1) - (p^{n_0} - 1).$$

For a non-negative integer n, let $L^{(n)}$ be the maximal abelian extension field over $(k_n^c)_\infty$ in $M_{\mathfrak{p}}(\widetilde{k})$. Let

$$\left\{\mathfrak{p}_{\infty,i}^{*(n)} \,\middle|\, 1 \leq i \leq p^{n_0}\right\}$$

be the set of prime ideals of $(k_n^c)_\infty$ lying above $\mathfrak{p}^*$, where n_0 is the integer satisfying $n_0 = \mathrm{ord}_p([\mathrm{Gal}(\widetilde{k}/k) : \mathfrak{D}_{\mathfrak{p}*}])$. We note that all prime ideals of k lying above p are totally ramified in the cyclotomic $\mathbb{Z}_p$-extension k_∞^c/k. Hence the number of primes of $(k_n^c)_\infty$ lying above $\mathfrak{p}^*$ is equal to p^{n_0}. We denote by $I_{\mathfrak{p}^{*(n)}_{\infty,i}}$ the inertia subgroup of $\mathrm{Gal}(L^{(n)}/(k_n^c)_\infty)$ for the prime ideal $\mathfrak{p}^{*(n)}_{\infty,i}$. Since $L^{(n)}/\widetilde{k}$ is an unramified extension outside all primes lying above $\mathfrak{p}$, we have $\mathrm{Gal}(L^{(n)}/\widetilde{k}) \cap I_{\mathfrak{p}^{*(n)}_{\infty,i}} = \{1\}$. Hence $I_{\mathfrak{p}^{*(n)}_{\infty,i}}$ is isomorphic to $\mathbb{Z}_p$ for each i. Then we have finitely generated $\mathbb{Z}_p$-module $\sum_{i=1}^{p^{n_0}} I_{\mathfrak{p}^{*(n)}_{\infty,i}}$. We note that the fixed field of $L^{(n)}$ by $\sum_{i=1}^{p^{n_0}} I_{\mathfrak{p}^{*(n)}_{\infty,i}}$ coincides with $M_{\mathfrak{p}}((k_n^c)_\infty)$. Then we have

$$p^n \cdot (\mathrm{rank}_{\mathbb{Z}_p}(X_{\mathfrak{p}}(N_\infty)) + p^{n_0} - 1) - p^{n_0} \le \mathrm{rank}_{\mathbb{Z}_p}\left(X_{\mathfrak{p}}(\widetilde{k})_{\mathrm{Gal}(\tilde{k}/(k_n^c)_\infty)}\right)$$

and

$$\begin{aligned}
&\mathrm{rank}_{\mathbb{Z}_p}\left(X_{\mathfrak{p}}(\widetilde{k})_{\mathrm{Gal}(\tilde{k}/(k_n^c)_\infty)}\right) \\
&= p^n \cdot (\mathrm{rank}_{\mathbb{Z}_p}(X_{\mathfrak{p}}(N_\infty)) + p^{n_0} - 1) - (p^{n_0} - 1) + \mathrm{rank}_{\mathbb{Z}_p}\left(\sum_{i=1}^{p^{n_0}} I_{\mathfrak{p}^{*(n)}_{\infty,i}}\right) - 1 \\
&\le p^n \cdot (\mathrm{rank}_{\mathbb{Z}_p}(X_{\mathfrak{p}}(N_\infty)) + p^{n_0} - 1).
\end{aligned}$$

From the structure theorem of $\mathbb{Z}_p[[\mathrm{Gal}(\widetilde{k}/N_\infty)]]$-modules, we obtain the latter part. □

Now we can prove Theorem 1.2.

Proof of Theorem 1.2. Theorem 4.2 gives a lower bound on the $\mathbb{Z}_p$-rank of the Galois coinvariant of $X_{\mathfrak{p}}(N_\infty)$:

$$\mathrm{rank}_{\mathbb{Z}_p}(X_{\mathfrak{p}}(N_\infty)) + p^{n_0} - 1 \le \mathrm{rank}_{\mathbb{Z}_p}\left(X_{\mathfrak{p}}(\widetilde{k})_{\mathrm{Gal}(\tilde{k}/N_\infty)}\right).$$

This inequality implies that the Galois coinvariant $X_{\mathfrak{p}}(\widetilde{k})_{\mathrm{Gal}(\tilde{k}/N_\infty)}$ has at least $\mathrm{rank}_{\mathbb{Z}_p}(X_{\mathfrak{p}}(N_\infty)) + p^{n_0} - 1$ generators. We use the same notation as in the proof of Theorem 4.2. For simplicity, we write L and $I_{\mathfrak{p}^*_{\infty,i}}$ for $L^{(0)}$ and $I_{\mathfrak{p}^{*(0)}_{\infty,i}}$, respectively. We note that $I_{\mathfrak{p}^*_{\infty,i}}$ is isomorphic to $\mathbb{Z}_p$ for each i and that the fixed field of L by $\sum_{i=1}^{p^{n_0}} I_{\mathfrak{p}^*_{\infty,i}}$ coincides with $M_{\mathfrak{p}}(N_\infty)$. Then we have an exact sequence

$$0 \longrightarrow \sum_{i=1}^{p^{n_0}} I_{\mathfrak{p}^*_{\infty,i}} \longrightarrow \mathrm{Gal}(L/N_\infty) \longrightarrow X_{\mathfrak{p}}(N_\infty) \longrightarrow 0.$$

Hence $\mathrm{Gal}(L/N_\infty)$ is finitely generated with at most $\mathrm{rank}_{\mathbb{Z}_p}(X_{\mathfrak{p}}(N_\infty)) + p^{n_0}$ elements as a $\mathbb{Z}_p$-module. This implies that $X_{\mathfrak{p}}(\widetilde{k})_{\mathrm{Gal}(\tilde{k}/N_\infty)}$ has at most $\mathrm{rank}_{\mathbb{Z}_p}(X_{\mathfrak{p}}(N_\infty)) + p^{n_0} - 1$ generators. Concerning the $\mathbb{Z}_p$-rank of

$\mathrm{Gal}(L/N_\infty)$, we have

$$\begin{aligned}\mathrm{rank}_{\mathbb{Z}_p}(\mathrm{Gal}(L/N_\infty)) &= \mathrm{rank}_{\mathbb{Z}_p}\left(\sum_{i=1}^{p^{n_0}} I_{\mathfrak{p}^*_{\infty,i}}\right) + \mathrm{rank}_{\mathbb{Z}_p}(X_{\mathfrak{p}}(N_\infty)) \\ &\le p^{n_0} + \mathrm{rank}_{\mathbb{Z}_p}(X_{\mathfrak{p}}(N_\infty)).\end{aligned}$$

Hence we get

$$\mathrm{rank}_{\mathbb{Z}_p}(\mathrm{Gal}(L/\widetilde{k})) = \mathrm{rank}_{\mathbb{Z}_p}\left(X_{\mathfrak{p}}(\widetilde{k})_{\mathrm{Gal}(\tilde{k}/N_\infty)}\right) \le p^{n_0} - 1 + \mathrm{rank}_{\mathbb{Z}_p}(X_{\mathfrak{p}}(N_\infty)).$$

This implies that $X_{\mathfrak{p}}(\widetilde{k})_{\mathrm{Gal}(\tilde{k}/N_\infty)}$ is a torsion-free $\mathbb{Z}_p$-module. Therefore we obtain

$$\mathrm{rank}_{\mathbb{Z}_p}\left(X_{\mathfrak{p}}(\widetilde{k})_{\mathrm{Gal}(\tilde{k}/N_\infty)}\right) = p^{n_0} - 1 + \mathrm{rank}_{\mathbb{Z}_p}(X_{\mathfrak{p}}(N_\infty)).$$

By [25], $X_{\mathfrak{p}}(\widetilde{k})$ has no nontrivial pseudo-null submodule. Hence we have $X_{\mathfrak{p}}(\widetilde{k})^{\mathrm{Gal}(\tilde{k}/N_\infty)} = 0$. This implies that $X_{\mathfrak{p}}(\widetilde{k})$ is a free $\mathbb{Z}_p[\![\mathrm{Gal}(\widetilde{k}/N_\infty)]\!]$-module of rank $p^{n_0} - 1 + \mathrm{rank}_{\mathbb{Z}_p}(X_{\mathfrak{p}}(N_\infty))$. □

4.2. Proof of Theorem 1.3. In this subsection, we prove Theorem 1.3. We put $\lambda^* = \lambda(k^c_\infty/k) - 1$. We use the following

Lemma 4.3 ([21, Lemma 3.3, Proposition 3.8])**.** *Suppose that $\lambda^* \ge 1$, where λ^* is the integer defined above. Then there exist power series $f(S,T) \in \Lambda$ and $g_i(S) \in \mathbb{Z}_p[\![S]\!]$ $(i = 0, \dots, \lambda^*-1)$ such that $f(S,T)$ is an annihilator of $X(\widetilde{k})$ of the form*

$$f(S,T) = T^{\lambda^*} + g_{\lambda^*-1}(S)T^{\lambda^*-1} + \cdots + g_1(S)T + g_0(S).$$

Furthermore, if we suppose that k is p-rational, we have a surjective homomorphism

$$\mathbb{Z}_p[\![S,T]\!]/(f(S,T)) \longrightarrow X(\widetilde{k})$$

as $\mathbb{Z}_p[\![S,T]\!]$-modules.

If k is p-rational and if $\mathfrak{D}_{\mathfrak{p}}$ is a normal subgroup of $\mathrm{Gal}(\widetilde{k}/\mathbb{Q})$, we can determine the ideal generated by $f(S,0) = g_0(S)$ in $\mathbb{Z}_p[\![S]\!]$.

Lemma 4.4 ([21, Proposition 3.5])**.** *Let k be p-rational. Assume that $\mathfrak{D}_{\mathfrak{p}}$ is a normal subgroup of $\mathrm{Gal}(\widetilde{k}/\mathbb{Q})$. Then there exists a power series $U(S) \in \mathbb{Z}_p[\![S]\!]^\times$ such that*

$$f(S,0) = \nu_{n_0}(S)U(S),$$

where we put $\nu_{n_0}(S) = \frac{(1+S)^{p^{n_0}}-1}{S}$.

Let $D_{\mathfrak{p}^*}$ be the decomposition group of $\mathfrak{p}^*$ in $\mathrm{Gal}(N^*_\infty/k)$. Let N_{n_0} be the n_0-th layer of N_∞/k. If we suppose that $\mathfrak{D}_{\mathfrak{p}}$ is a normal subgroup of $\mathrm{Gal}(\widetilde{k}/\mathbb{Q})$, then we have $N_{n_0} = k^a_{n_0}$ by [22, Lemma 3.8].

We prepare a proposition to prove Theorem 1.3.

Proposition 4.5. *Assume the same condition in Theorem 1.3. Let L be the maximal abelian extension field over N_∞ in $M_{\mathfrak{p}}(\widetilde{k})$. Then we have $L = \widetilde{N_{n_0}}$, where $\widetilde{N_{n_0}}$ is the composite of all $\mathbb{Z}_p$-extension fields over N_{n_0}.*

Proof. As in the proof of Theorem 1.2, we denote by $I_{\mathfrak{p}^*_{\infty,i}}$ the inertia subgroup of $\mathrm{Gal}(L/N_\infty)$ for the prime ideal $\mathfrak{p}^*_{\infty,i}$ of N_∞ lying above $\mathfrak{p}^*$. Then $I_{\mathfrak{p}^*_{\infty,i}}$ is isomorphic to $\mathbb{Z}_p$ for each i. By Proposition 4.1, we have $X_{\mathfrak{p}}(N_\infty) = 0$. Hence we get $\mathrm{Gal}(L/N_\infty) \cong \sum_1^{p^{n_0}} I_{\mathfrak{p}^*_{\infty,i}}$. For each i with $1 \le i \le p^{n_0}$, the decomposition group $D_{\mathfrak{p}^*}$ acts on $I_{\mathfrak{p}^*_{\infty,i}}$ because all prime ideals of N_{n_0} lying above $\mathfrak{p}^*$ do not split in N_∞. Since we have $\mathrm{Gal}(L/\widetilde{k}) \cap I_{\mathfrak{p}^*_{\infty,i}} = 1$, there exists an injective homomorphism $I_{\mathfrak{p}^*_{\infty,i}} \to \mathrm{Gal}(\widetilde{k}/N_\infty)$ as $\mathbb{Z}_p[\![\mathrm{Gal}(N_\infty/N_{n_0})]\!]$-modules. Then $D_{\mathfrak{p}^*}$ acts on $I_{\mathfrak{p}^*_{\infty,i}}$ trivially because $\widetilde{k}/N_{n_0}$ is an abelian extension. Hence $D_{\mathfrak{p}^*}$ acts on $\sum_{i=1}^{p^{n_0}} I_{\mathfrak{p}^*_{\infty,i}}$ trivially. This implies that L/N_{n_0} is an abelian extension. By [22, Lemma 3.10], we have $D_{\mathfrak{p}^*} = \mathrm{Gal}(N_\infty/N_{n_0})$. From the proof of Theorem 1.2, $\mathrm{Gal}(L/N_{n_0})$ is a free $\mathbb{Z}_p$-module of rank $p^{n_0}+1$. Since N_{n_0}/k is an abelian extension, Leopoldt's conjecture holds for p and N_{n_0} by Proposition 2.1. Hence we have $\mathrm{Gal}(\widetilde{N_{n_0}}/N_{n_0}) \cong \mathbb{Z}_p^{\oplus p^{n_0}+1}$. This implies that $L = \widetilde{N_{n_0}}$. □

Now we can prove Theorem 1.3.

Proof of Theorem 1.3. Let k_∞ be a $\mathbb{Z}_p$-extension satisfying $k_\infty \cap k^a_\infty \supset k^a_{n_0}$. Since $\widetilde{N_{n_0}}/k_\infty$ is an unramified extension, we have $\mathrm{rank}_{\mathbb{Z}_p}(X(k_\infty)) \ge p^{n_0}$. Using Lemma 4.3, we have a surjective homomorphism $\Lambda/(f(S,T)) \to X(\widetilde{k})$. By [22, Lemma 5.7], there exists a unit $u \in \mathbb{Z}_p^\times$ such that $\mathrm{Gal}(\widetilde{k}/k_\infty) = \overline{\langle \sigma^{up^m}\tau \rangle}$ with $m \ge n_0$. Then we have a surjective homomorphism

$$\mathbb{Z}_p[\![S]\!]/(f(S,(1+S)^{-up^m}-1)) \longrightarrow X(\widetilde{k})_{\mathrm{Gal}(\tilde{k}/k_\infty)}. \tag{4.2}$$

By definition of $f(S,T)$, we obtain

$$\begin{aligned}
&f(S,(1+S)^{-up^m}-1)\\
&= \{(1+S)^{-up^m}-1\}^{\lambda^*} + \sum_{i=0}^{\lambda^*-1} g_i(S)\{(1+S)^{-up^m}-1\}^i\\
&\equiv \{(1+S^{p^m})^{-u}-1\}^{\lambda^*} + \sum_{i=0}^{\lambda^*-1} g_i(S)\{(1+S^{p^m})^{-u}-1\}^i \quad \mathrm{mod}\ p
\end{aligned}$$

$$\equiv \left\{\sum_{j=1}^{\infty}\binom{-u}{j}S^{jp^m}\right\}^{\lambda^*} + \sum_{i=0}^{\lambda^*-1} g_i(S)\left\{\sum_{j=1}^{\infty}\binom{-u}{j}S^{jp^m}\right\}^{i} \mod p$$
$$\equiv g_0(S) \mod (p, S^{p^m}).$$

We note that $g_0(S) \equiv S^{p^{n_0}-1}U(S) \bmod p$ by Lemma 4.4. Then the module $\mathbb{Z}_p[\![S]\!]/(f(S,(1+S)^{-up^m}-1))$ is a finitely generated $\mathbb{Z}_p$-module. Hence we have $\mathrm{rank}_{\mathbb{Z}_p}(\mathbb{Z}_p[\![S]\!]/(f(S,(1+S)^{-up^m}-1)) = p^{n_0}-1$. Thus we obtain $\mathrm{rank}_{\mathbb{Z}_p}(X(\widetilde{k})_{\mathrm{Gal}(\tilde{k}/k_\infty)}) \le p^{n_0}-1$. Since $\widetilde{N_{n_0}}/\widetilde{k}$ is unramified, we see that $\mathrm{rank}_{\mathbb{Z}_p}(X(\widetilde{k})_{\mathrm{Gal}(\tilde{k}/k_\infty)}) \ge p^{n_0}-1$. Therefore the surjective homomorphism (4.2) is an isomorphism. It follows from the exact sequence

$$0 \longrightarrow X(\widetilde{k})_{\mathrm{Gal}(\tilde{k}/k_\infty)} \longrightarrow X(k_\infty) \longrightarrow \mathrm{Gal}(\widetilde{k}/k_\infty) \longrightarrow 0$$

that $\mathrm{rank}_{\mathbb{Z}_p}(X(k_\infty)) = \mathrm{rank}_{\mathbb{Z}_p}(X(\widetilde{k})_{\mathrm{Gal}(\tilde{k}/k_\infty)})+1 = p^{n_0}$. Hence $X(k_\infty)$ is a free $\mathbb{Z}_p$-module of rank p^{n_0}. This implies that $L(k_\infty) = \widetilde{N_{n_0}}$. Furthermore, $X(k_\infty)$ is a cyclic $\mathbb{Z}_p[\![\mathrm{Gal}(k_\infty/k)]\!]$-module. Indeed, we have an isomorphism

$$X(k_\infty)_{\mathrm{Gal}(k_\infty/k)} \cong \mathrm{Gal}(\widetilde{k}/k_\infty) \cong \mathbb{Z}_p$$

by the assumption that $L(k) \subset \widetilde{k}$. Since $\widetilde{N_{n_0}}/N_{n_0}$ is an abelian extension, $\mathrm{Gal}(k_\infty/k_{n_0})$ acts on $X(k_\infty)$ trivially. We have an isomorphism

$$X(k_\infty) \cong \mathbb{Z}_p[\![\mathrm{Gal}(k_\infty/k)]\!]/(\gamma^{p^{n_0}}-1)$$

as a $\mathbb{Z}_p[\![\mathrm{Gal}(k_\infty/k)]\!]$-module, where γ is the topological generator defined in the assumption of Theorem 1.3. □

4.3. Numerical examples. In this section, we introduce some numerical examples which were computed using PARI/GP. As in the previous section, suppose that s and n_0 are the integers satisfying

$$p^s = [L(k) \cap \widetilde{k} : k], \qquad p^{n_0} = [\mathrm{Gal}(\widetilde{k}/k) : \mathfrak{D}_{\mathfrak{p}*}].$$

We denote by $A(k)$ the p-Sylow subgroup of the ideal class group of k.

Example 4.6. Put $p = 3$ and put $k = \mathbb{Q}(\sqrt{-14})$. Then 3 splits completely in k. By PARI/GP, we have $A(k) = 0$. Hence we have $s = 0$. By Proposition 4.1, we have $X_{\mathfrak{p}}(N_\infty) = 0$. Using PARI/GP, we have $n_0 = 1$. Hence $\mathfrak{D}_{\mathfrak{p}}$ is not a normal subgroup of $\mathrm{Gal}(\widetilde{k}/\mathbb{Q})$. It follows from Theorem 1.2 that $X_{\mathfrak{p}}(\widetilde{k}) \cong \mathbb{Z}_p[\![\mathrm{Gal}(\widetilde{k}/N_\infty)]\!]^{\oplus 2}$.

Example 4.7. Put $p = 3$ and put $k = \mathbb{Q}(\sqrt{-743})$. Then 3 splits completely in k. We have $A(k) \cong \mathbb{Z}/3\mathbb{Z}$. We can check that $L(k) \subset \widetilde{k}$. Indeed, the class number of $\mathbb{Q}(\sqrt{2229})$ is 1. Using [20, Corollary of Proposition 6.B], we see $L(k) \subset \widetilde{k}$. Hence, we have $s = 0$ and $X_{\mathfrak{p}}(N_\infty) = 0$. Let $f(S,T)$ be the same power series defined in Lemma 4.3. By [22, Example 1], we have $n_0 = \mathrm{ord}_3(f(0,0)) = 1$. It follows from Theorem 1.2 that $X_{\mathfrak{p}}(\widetilde{k}) \cong \mathbb{Z}_p[\![\mathrm{Gal}(\widetilde{k}/N_\infty)]\!]^{\oplus 2}$. Since $\mathfrak{D}_{\mathfrak{p}}$ is a normal subgroup of $\mathrm{Gal}(\widetilde{k}/\mathbb{Q})$, we obtain $\lambda(k_\infty/k) = 3$ for all $\mathbb{Z}_p$-extension k_∞ satisfying $k_\infty \cap k_\infty^a \supset k_1^a$.

Example 4.8. Put $p = 7$ and put $k = \mathbb{Q}(\sqrt{-5207})$. Then 7 splits completely in k. By [21, Example (i)], we have $A(k) \cong \mathbb{Z}/7\mathbb{Z}$, $L(k) \subset \widetilde{k}$, and $s = n_0 = 1$. It follows from Theorem 1.2 that $X_{\mathfrak{p}}(\widetilde{k}) \cong \mathbb{Z}_p[\![\mathrm{Gal}(\widetilde{k}/N_\infty)]\!]^{\oplus 6}$. Since $\mathfrak{D}_{\mathfrak{p}}$ is a normal subgroup of $\mathrm{Gal}(\widetilde{k}/\mathbb{Q})$, we obtain $\lambda(k_\infty/k) = 7$ for all $\mathbb{Z}_p$-extension k_∞ satisfying $k_\infty \cap k_\infty^a \supset k_1^a$.

Example 4.9. Put $p = 3$ and put $k = \mathbb{Q}(\sqrt{-971})$. Then 3 splits completely in k. We have $A(k) \cong \mathbb{Z}/3\mathbb{Z}$. $\mathfrak{D}_{\mathfrak{p}}$ is not a normal subgroup of $\mathrm{Gal}(\widetilde{k}/\mathbb{Q})$. We can check that $L(k) \subset \widetilde{k}$. By the same method as Example 4.7, we get $s = 0$, $X_{\mathfrak{p}}(N_\infty) = 0$, and $n_0 = \mathrm{ord}_3(f(0,0)) = 5$. It follows from Theorem 1.2 that $X_{\mathfrak{p}}(\widetilde{k}) \cong \mathbb{Z}_p[\![\mathrm{Gal}(\widetilde{k}/N_\infty)]\!]^{\oplus 242}$.

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Kazuaki Murakami
Department of Childhood Education Faculty of Human Sciences,
Seinan Gakuin University,
6-2-92, Nishijin, Sawara-ku, Fukuoka 814-8511, Japan
E-mail: `kmurakami@seinan-gu.ac.jp`