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On the average size of 3-torsion in class groups of $C_2 \wr H$-extensions

par Jonas ISKANDER et Hari R. IYER

Résumé. Les heuristiques de Cohen, Lenstra, et Martinet conduisent à conjecturer que la taille moyenne de la p-torsion dans les groupes de classes des G-extensions d'un corps de nombres est finie. Dans un article de 2021, Lemke Oliver, Wang et Wood ont démontré cette conjecture dans le cas $p = 3$ pour les groupes de permutations G de la forme $C_2 \wr H$, pour une large famille de groupes de permutations H, incluant la plupart des groupes nilpotents. Cependant, leur théorème ne s'applique pas à certains groupes nilpotents d'intérêt, tels que $H = C_5$. Nous étendons leurs résultats afin de montrer que la taille moyenne de la 3-torsion dans les groupes de classes des $C_2 \wr H$-extensions est finie pour tout groupe nilpotent H.

Abstract. The Cohen–Lenstra–Martinet heuristics lead one to conjecture that the average size of the p-torsion in class groups of G-extensions of a number field is finite. In a 2021 paper, Lemke Oliver, Wang, and Wood proved this conjecture in the case of $p = 3$ for permutation groups G of the form $C_2 \wr H$ for a broad family of permutation groups H, including most nilpotent groups. However, their theorem does not apply for some nilpotent groups of interest, such as $H = C_5$. We extend their results to prove that the average size of 3-torsion in class groups of $C_2 \wr H$-extensions is finite for any nilpotent group H.

1. Introduction

For an extension of number fields K/k with degree n and Galois closure $\widetilde{K}$, let $\mathrm{Gal}(K/k)$ denote the Galois group of $\widetilde{K}/k$ acting as a transitive permutation group on the n-element set of k-linear embeddings $K \hookrightarrow \widetilde{K}$. Given a transitive permutation group $G \subseteq S_n$, a *G-extension* of k is a degree n extension K/k in $\bar{\mathbb{Q}}$ equipped with an isomorphism $\mathrm{Gal}(K/k) \xrightarrow{\sim} G$ as permutation groups; given a real number $X > 1$, we denote the set of all G-extensions K with $|\mathrm{Disc}(K)| \leq X$ by $E_k(G, X)$. Heuristics of Cohen, Lenstra, and Martinet [3, 4] suggest that the sizes $h_p(K)$ of the p-torsion subgroups of the class groups $\mathrm{Cl}(K)$ satisfy the following conjecture.

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Conjecture 1.1. *Let k be a number field, G a transitive permutation group, and p a prime such that $p \nmid |G|$. Then there exists a constant $c_{k,G,p} > 0$ such that*

$$\lim_{X\to\infty} \frac{1}{|E_k(G,X)|} \sum_{K\in E_k(G,X)} h_p(K) = c_{k,G,p}. \tag{1.1}$$

As explained in [11, p. 588], the result [12, Theorem 6.1] shows that the above is a consequence of the Cohen–Lenstra–Martinet heuristics. Davenport and Heilbronn [6] proved Conjecture 1.1 for the case where $p = 3$, $G = S_2$, and $k = \mathbb{Q}$ in 1971, and Datskovsky and Wright [5] generalized their results to arbitrary base fields k in 1988. In 2005, Bhargava [2] also proved (1.1) in the case where $p = 2$, $G = S_3$, and $k = \mathbb{Q}$.

Recent work of Lemke Oliver, Wang, and Wood [9] further generalizes [5] by proving Conjecture 1.1 for a much broader class of groups G. Specifically, they prove the conjecture for $p = 3$ and arbitrary k when G is a 2-group containing a transposition, as well as when $G = C_2 \wr H$ for H a transitive permutation group satisfying $|E_k(H,T)| \ll_{k,H,\epsilon} T^{\frac{1}{6}+\epsilon}$ for every $\epsilon > 0$ and $T \geq 1$.[1] In this paper, we further generalize their work by providing an optimization of an argument from [9] which yields the following result.

Theorem 1.2. *Let $H \subseteq S_n$ be a transitive permutation group, and set $G := C_2 \wr H$. Suppose that $E_k(H,\infty)$ is nonempty and that there exists a $\delta > 0$ such that*

$$|E_k(H,T)| \ll_{k,H} T^{\frac{1}{2}-\delta}$$

for all $T \geq 1$. Then there is a constant $c_{k,G,3}$ such that

$$\lim_{X\to\infty} \frac{1}{|E_k(G,X)|} \sum_{K\in E_k(G,X)} h_3(K) = c_{k,G,3}.$$

Explicitly, we have

$$c_{k,G,3} = \left(\sum_{F\in E_k(H,\infty)} \frac{h_3(F)\operatorname{Res}_{s=1}\zeta_F(s)}{\zeta_F(2)\operatorname{Disc}(F)^2} \cdot \left(1 + \frac{2^{r_1(F)}}{3^{r_1(F)+r_2(F)}}\right)\right) \times \left(\sum_{F\in E_k(H,\infty)} \frac{\operatorname{Res}_{s=1}\zeta_F(s)}{\zeta_F(2)\operatorname{Disc}(F)^2}\right)^{-1}, \tag{1.2}$$

where ζ_F denotes the Dedekind zeta function for F and $r_1(F)$ and $r_2(F)$ denote the numbers of real and complex embeddings of F, respectively.

[1] The result of Lemke Oliver et. al. also holds under the weaker condition that $\sum_{F\in E_k(H,T)} h_3(F) \ll_{k,H,\epsilon} X^{\frac{2}{3}+\epsilon}$ for all $\epsilon > 0$ and $T \geq 1$.

In light of the proof of the weak version of Malle's conjecture for nilpotent groups by Alberts [1, Corollary 1.8] (which was also proven in Klüners–Wang [8, Theorem 1.7] via a different method), we obtain the following corollary.

Corollary 1.3. *The conclusion of Theorem 1.2 holds when* $G = C_2 \wr H$ *for any transitive nilpotent permutation group* H*.*

It is possible to specify exactly what transitive nilpotent permutation groups H covered by this corollary are not covered by [9]. Denote by H_p any transitive permutation p-group containing a p-cycle. Then, by applying the classification proved in Corollary 2.4, the new nilpotent groups covered by our result are groups of the form H_5, $C_2 \times H_3$, $C_3 \times H_2$, and $C_5 \times H_2$, where these are written as direct products of transitive permutation groups.

Regarding non-nilpotent permutation groups, the smallest transitive permutation group $H \subseteq S_n$ for which the conclusion of Theorem 1.2 is not currently known is $H = S_3, n = 3$. Moreover, if we impose the condition that $3 \nmid |H|$, the smallest such group is $H = D_5, n = 5$. These groups could be interesting targets for future directions of study. In particular, the bound on $E_k(H, T)$ implied by the weak version of Malle's conjecture for these groups is not strong enough to apply Theorem 1.2, so a proof of Corollary 1.3 for $H \in \{S_3, D_5\}$ would require further innovations.

2. Proof of Results

Throughout the paper, given expressions e_1 and e_2, we write $e_1 \ll_{x_1,\dots,x_n} e_2$ to say that there exists a constant $C > 0$ depending only on the variables $x_1, \dots, x_n$ such that $e_1 \leq Ce_2$ in the domain on which e_1 and e_2 are defined. In addition, given an extension of number fields L/K, we write $\mathrm{Disc}(L/K)$ for the norm of the relative discriminant ideal of L/K, and given any number field L, we write $\mathrm{Disc}(L)$ for the *absolute value* of the discriminant of L.

The lemma below provides a modified version of Theorem 8.1 (2) from [9] that allows us to prove Conjecture 1.2 for a larger class of groups G.

Lemma 2.1. *Let* $H \subseteq S_n$ *be a transitive permutation group and set* $G = C_2 \wr H$*. If* $E_k(H, \infty)$ *is nonempty and there exists* $\delta > 0$ *such that*

$$\sum_{F \in E_k(H,T)} h_3(F) h_2(F)^{\frac{2}{3}} \ll_{k,H} T^{1-\delta}$$

for $T \geq 1$*, then we have that*

$$\lim_{X \to \infty} \frac{1}{E_k(G, X)} \sum_{K \in E_k(G,X)} h_3(K) = c_{k,G,3},$$

where $c_{k,G,3}$ *is the constant given in* (1.2)*.*

Proof. By [9, Theorem 8.1], for any G-extension K/k, there exists a unique index 2 subfield F_K, which is in fact an H-extension of k. Moreover, by [9, Corollary 3.2], for any number field F, $X \geq 1$, and $\epsilon > 0$, we have

$$\sum_{\substack{[K:F]=2\\ \mathrm{Disc}(K/F)\leq X}} h_3(K/F) \ll_{[F:\mathbb{Q}],\epsilon} \mathrm{Disc}(F)^{1+\epsilon} h_2(F)^{2/3} X.$$

Using the fact that $h_3(K) \leq h_3(K/F)h_3(F)$ and $\mathrm{Disc}(K/F) = \frac{\mathrm{Disc}(K)}{\mathrm{Disc}(F)^2}$ [10, Corollary III.2.10], we may then write

$$\sum_{\substack{[K:F]=2\\ \mathrm{Disc}(K)\leq X}} h_3(K) \leq \sum_{\substack{[K:F]=2\\ \mathrm{Disc}(K/F)\leq \frac{X}{\mathrm{Disc}(F)^2}}} h_3(F)h_3(K/F)$$
$$\ll_{[F:\mathbb{Q}],\epsilon} \frac{h_3(F)h_2(F)^{2/3}X}{\mathrm{Disc}(F)^{1-\epsilon}}.$$

Let $X, Y \geq 1$ and $0 < \epsilon < \delta$. We now wish to bound the quantity

$$\sum_{K\in E_k(G,X),\mathrm{Disc}(F_K)\geq Y} h_3(K).$$

Again by the equality $\mathrm{Disc}(K) = \mathrm{Disc}(F_K)^2\,\mathrm{Disc}(K/F_K)$, we know that if $\mathrm{Disc}(K) \leq X$, then $\mathrm{Disc}(F_K) \leq X^{1/2}$ and that K/F_K is a quadratic extension of the H-extension F_K. Hence, the above sum can be bounded above by summing $h_3(K)$ over all H-extensions F such that $Y \leq \mathrm{Disc}(F) \leq X^{1/2}$ and furthermore summing over all quadratic extensions K of such F (which certainly includes the sum over all the quadratic extension pairs of the form K/F_K with $\mathrm{Disc}(K) \leq X$ and $\mathrm{Disc}(F_K) \geq Y$), i.e.

$$\sum_{K\in E_k(G,X),\mathrm{Disc}(F_K)\geq Y} h_3(K) \leq \sum_{F\in E_k(H,X^{\frac{1}{2}})\setminus E_k(H,Y)} \sum_{\substack{[K:F]=2\\ \mathrm{Disc}(K)\leq X}} h_3(K).$$

Applying our previous bound for the inner sum yields the upper bound of

$$\sum_{K\in E_k(G,X),\mathrm{Disc}(F_K)\geq Y} h_3(K) \ll_{[k:\mathbb{Q}],G,\epsilon} \sum_{F\in E_k(H,X^{\frac{1}{2}})\setminus E_k(H,Y)} \frac{h_3(F)h_2(F)^{\frac{2}{3}}X}{\mathrm{Disc}(F)^{1-\epsilon}}.$$

Letting

$$A(T) := \sum_{F\in E_k(H,T)} h_3(F)h_2(F)^{\frac{2}{3}} \ll_{k,H} T^{1-\delta} \qquad \text{and} \qquad \phi(T) := \frac{X}{T^{1-\epsilon}},$$

we can then use Abel summation to write

$$\begin{aligned}\sum_{F\in E_k(H,X^{\frac{1}{2}})\setminus E_k(H,Y)} &\frac{h_3(F)h_2(F)^{\frac{2}{3}}X}{\mathrm{Disc}(F)^{1-\epsilon}}\\ &= A(X^{\frac{1}{2}})\phi(X^{\frac{1}{2}}) - A(Y)\phi(Y) - \int_Y^{X^{\frac{1}{2}}} A(T)\phi'(T)\mathrm{d}T\\ &\ll_{k,H} X^{\frac{1-\delta}{2}}\cdot X^{\frac{1+\epsilon}{2}} + \int_Y^{X^{\frac{1}{2}}} T^{1-\delta}\cdot\frac{X}{T^{2-\epsilon}}\mathrm{d}T\\ &= X^{1-\frac{\delta}{2}+\frac{\epsilon}{2}} + \left[\frac{X}{(\epsilon-\delta)T^{\delta-\epsilon}}\right]_{T=Y}^{X^{\frac{1}{2}}}\\ &\ll X^{1-\frac{\delta}{2}+\frac{\epsilon}{2}} + \frac{X}{Y^{\delta-\epsilon}}.\end{aligned}$$

Choosing $\epsilon = \delta/2$, we thus obtain

$$\sum_{K\in E_k(G,X),\mathrm{Disc}(F_K)\geq Y} h_3(K) \ll_{k,G} \frac{X}{Y^{\delta/2}} + X^{1-\frac{\delta}{4}}.$$

This is analogous to the first displayed equation of the proof of [9, Theorem 8.1(1)] with $h_3(K/F_K)$ replaced by $h_3(K)$, which merely introduces a factor of $h_3(F)$ in the middle expression of the aforementioned displayed equation.[2] Furthermore, the exponent of Y in the denominator here is $\delta/2$ instead of $\frac{1}{3n[k:\mathbb{Q}]}-\epsilon$ as written in [9], and we have an extra error term $X^{1-\frac{\delta}{4}}$ coming from the upper endpoint of the partial summation. However, just as in the proof of [9, Theorem 8.1(2)], our inequality yields the desired analog of the "tail estimate" [9, Theorem 5.1], since upon dividing by X (when averaging over K with bounded discriminant $\mathrm{Disc}(K) \leq X$) the expression on the right hand side vanishes in the large X and Y limits. To see why dividing by X suffices if we wish to average over K in this case, we note that $|E_k(G,X)|$ is asymptotically linear in X by [7, Theorem 5.8] under a mild growth condition on $|E_k(H,T)|$, which is certainly satisfied by our assumption $\sum_{F\in E_k(H,T)} h_3(F)h_2(F)^{\frac{2}{3}} \ll_{k,H} T^{1-\delta}$ which implies sublinear growth for $|E_k(H,T)|$.

Lastly, the rest of proof of [9, Theorem 8.1(2)] establishes a "soft analog" of [9, Theorem 6.5] (i.e. dealing with the sum over G-extensions K such that F_K has small discriminant $\mathrm{Disc}(F_K) \leq Y$) and applies to G-extensions for any transitive permutation group H, and is therefore unchanged in our

[2] The implied proof of [9, Theorem 8.1(2)] would also differ from the displayed equation in this way.

situation. These modifications allow us to proceed as in the proof of [9, Theorem 8.1(2)], and in this manner the theorem follows. The explicit formula for $c_{k,G,3}$ comes directly from [9, Section 6]. □

Proof of Theorem 1.2. Applying the substitution $h_6(F) = h_2(F)h_3(F)$ and the trivial bound $h(F) \ll_\epsilon \mathrm{Disc}(F)^{\frac{1}{2}+\epsilon}$ for any $\epsilon > 0$, we may write

$$\begin{aligned}\sum_{F\in E_k(H,T)} h_3(F)h_2(F)^{\frac{2}{3}} &= \sum_{F\in E_k(H,T)} h_3(F)^{\frac{1}{3}}h_6(F)^{\frac{2}{3}} \\ &\ll_\epsilon \sum_{F\in E_k(H,T)} \mathrm{Disc}(F)^{\frac{1}{2}+\epsilon}.\end{aligned}$$

From here, the bound $|E_k(H,T)| \ll_{k,H} T^{\frac{1}{2}-\delta}$ yields, for any $\epsilon > 0$,

$$\sum_{F\in E_k(H,T)} h_3(F)h_2(F)^{\frac{2}{3}} \ll_{\epsilon,k,H} T^{1-\delta+\epsilon}.$$

Choosing $\epsilon = \delta/2$ and applying Lemma 2.1 yields the theorem. □

To explore the implications of Theorem 1.2 for nilpotent groups, the following lemma of [8], which decomposes transitive nilpotent permutation groups into their p-Sylow subgroups, proves useful.

Lemma 2.2 ([8, Lemma 3.1])**.** *A transitive nilpotent permutation group $G \subseteq S_n$ is permutation isomorphic to the natural direct product of transitive permutation p-groups $G_p \subseteq S_{n_p}$,*

$$G \cong \prod_p G_p, \qquad \text{with} \qquad n = \prod_p n_p,$$

where the G_p are isomorphic to the p-Sylow subgroups of G and n_p is the maximal p-power dividing n.

Given a transitive permutation group $H \subseteq S_n$ and an element $h \in H$, let $\mathrm{orb}(h)$ denote the number of cycles (including fixed points) in the permutation corresponding to h, and let $a(H) := \min\{n - \mathrm{orb}(h) : 1 \neq h \in H\}$. Alberts showed that when H is nilpotent, the quantity $a(H)$ controls the growth rate of $E_k(H,T)$ as a function of T. More precisely, he proved that for all $\epsilon > 0$ and $T \geq 1$, we have $|E_k(H,T)| \ll_{k,\epsilon} T^{\frac{1}{a(H)}+\epsilon}$ [1, Corollary 1.8]. In light of this result, it is natural to ask whether nilpotent groups H achieving a particular value of $a(H)$ admit a simple classification. The following lemma is helpful for answering this question.

Lemma 2.3. *Let H be a nontrivial transitive nilpotent permutation group, and use Lemma 2.2 to write $H \cong \prod_p H_p$ for some transitive permutation p-groups $H_p \subseteq S_{n_p}$. Then we have*

$$\frac{a(H)}{n} = \min_{p|n} \frac{(p-1)m_p}{n_p},$$

where m_p is the smallest positive integer such that H_p contains a product of m_p p-cycles.

Proof. We will prove equality by showing that the left hand side is both less than or equal to and greater than or equal to the right hand side. To see that $a(H) \leq \min_{p|n}\left((p-1)\cdot m_p \cdot \frac{n}{n_p}\right)$, let $p \mid n$ be a prime, and choose an element $h_p \in H_p$ which is a product of m_p p-cycles. Then letting $H' := \prod_{q\neq p} H_q$, we see from the definition of a product of two permutation groups that $(h_p, 1) \in H_p \times H' \cong H$ decomposes as a product of $m_p \cdot |H'| = m_p \cdot \frac{n}{n_p}$ p-cycles. By the definition of $a(H)$, it follows that $a(H) \leq (p-1)\cdot m_p \cdot \frac{n}{n_p}$.

For the remaining inequality, adopt the notation $|g|$ for the order of a group element $g \in G$. Choose an element $h \in H$ such that $n - \mathrm{orb}(h) = a(H)$, let p be an arbitrary prime dividing $|h|$, and set $h_0 := h^{|h|/p}$. Then h_0 has order p. Consequently, if we set $H' := \prod_{q\neq p} H_q$ as before and let $(h_p, h') \in H_p \times H'$ be the element corresponding to h_0 under the isomorphism $H_p \times H' \cong H$, we see that $(h')^p = 1$, hence that $h' = 1$ because $p \nmid |H'|$. It follows in turn that h_p has order p and hence is a product of p-cycles. Letting $m \geq m_p$ denote the number of p-cycles in h_p, we conclude that h_0 is a product of $m \cdot |H'| = m \cdot \frac{n}{n_p}$ p-cycles, hence that

$$\begin{aligned} a(H) = n - \mathrm{orb}(h) \geq n - \mathrm{orb}(h_0) &= (p-1)\cdot m \cdot \frac{n}{n_p} \\ &\geq (p-1)\cdot m_p \cdot \frac{n}{n_p}. \end{aligned}$$

□

We can apply Lemma 2.3 to produce a classification of transitive nilpotent groups H for some small values of $a(H)$. For simplicity, we focus on the case where H is not a 2-group, as the case of 2-groups is already covered by Theorem 1.1 of [9].

Corollary 2.4. *Let H be a nontrivial transitive nilpotent permutation group, and set $a := a(H)$. Consider the finite set of triples*

$$T_a := \{(p, m, n') \in \mathbb{Z}^3_{\geq 1} : (p-1)\cdot m \cdot n' = a,\ p \text{ prime},\ p \nmid n'\}.$$

Then there exists a triple $(p, m, n') \in T_a$ such that $H \cong H_p \times H'$, where:

(i) *H_p is a transitive permutation p-group containing a product of m p-cycles; and*
(ii) *H' is a transitive nilpotent permutation group of order n'.*

In particular, suppose H is not a 2-group. Then:

(1) *We have $a(H) \geq 2$.*
(2) *If $a(H) = 2$, then H is a 3-group containing a 3-cycle.*
(3) *If $a(H) = 3$, then $H \cong C_3 \times H_2$ where H_2 is a 2-group containing a transposition.*

(4) *If* $a(H) = 4$*, then one of the following holds:*
   (a) $H = C_2 \times H_3$ *where* H_3 *is a* 3*-group containing a* 3*-cycle.*
   (b) H *is a* 3*-group containing a product of two* 3*-cycles.*
   (c) H *is a* 5*-group containing a* 5*-cycle.*

(5) *If* $a(H) = 5$*, then* $H = C_5 \times H_2$*, where* H_2 *is a* 2*-group containing a transposition.*

Proof. Adopting the notation from Lemma 2.3, we know that there exists a prime p such that $a(H) = (p-1) \cdot m_p \cdot \frac{n}{n_p}$. This in turn allows us to write $H \cong H_p \times H'$, where H_p is a transitive permutation p-group containing a product of m_p p-cycles and H' is a transitive permutation group of order $\frac{n}{n_p}$, proving the first part of the corollary. The rest of the corollary follows by noting that

$$T_1 = \{(2,1,1)\}, \qquad T_2 = \{(2,2,1),(3,1,1)\},$$
$$T_3 = \{(2,3,1),(2,1,3)\}, \qquad T_4 = \{(2,4,1),(3,1,2),(3,2,1),(5,1,1)\},$$
$$T_5 = \{(2,5,1),(2,1,5)\}.$$

The first entry in each set corresponds to a case in which H is a 2-group and hence is excluded. □

By applying Corollary 2.4 to the cases not covered directly by Theorem 1.2, we obtain Corollary 1.3.

Proof of Corollary 1.3. If H is a 2-group, then $G = C_2 \wr H$ is a transitive permutation 2-group containing a transposition, so the result follows by [9, Theorem 1.1] on the average size of 3-torsion in class groups of such extensions. Suppose that H is not a 2-group, in which case H necessarily satisfies $a(H) \geq 2$. If $a(H) \geq 3$, then since H is a transitive nilpotent permutation group, [1, Corollary 1.8] implies that for all number fields k, $\epsilon > 0$ and $T \geq 1$, we have $|E_k(H,T)| \ll_{k,\epsilon} T^{\frac{1}{a(H)}+\epsilon} \leq T^{\frac{1}{3}+\epsilon}$, which satisfies the hypothesis of Theorem 1.2 of this paper and therefore implies the result, i.e. the conclusion of Theorem 1.2. Lastly, if $a(H) = 2$, then Corollary 2.4 (i) implies that H is a 3-group. This case is covered by the remark after Theorem 8.1 in [9]. □

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