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Prime Avoidance Property of *k*-th Powers of Piatetski–Shapiro Primes

par Helmut MAIER et Michael Th. RASSIAS

Résumé. Dans des articles antérieurs, les auteurs ont établi la propriété d'évitement des nombres premiers pour les puissances k-ièmes de nombres premiers et pour les nombres premiers dans les suites de Beatty. Dans cet article, les auteurs considèrent les puissances k-ièmes des nombres premiers de Piatetski–Shapiro.

Abstract. In previous papers the authors established the prime avoidance property of k-th powers of prime numbers and of prime numbers within Beatty sequences. In this paper the authors consider k-th powers of Piatetski–Shapiro primes.

1. Introduction

Let p_n denote the increasing sequence of prime numbers.

The fact that

$$\limsup_{n\to\infty} \frac{p_{n+1}-p_n}{\log p_n} = \infty$$

was first proved by Westzynthius [15]. Erdős [2] obtained that infinitely often one has:

$$p_{n+1}-p_n > C_1 \frac{\log p_n \ \log_2 p_n}{(\log_3 p_n)^2} \tag{1.1}$$

with appropriate $C_1 > 0$. Here $\log_k x := \log(\log_{k-1} x)$.

The order of magnitude of the lower bound (1.1) was slightly improved by Rankin [13] who showed that

$$p_{n+1}-p_n > C_2 \frac{\log p_n \ \log_2 p_n \ \log_4 p_n}{(\log_3 p_n)^2} = C_2 g_1(p_n)\,. \tag{1.2}$$

The proof of Rankin differs only slightly from that of Erdős. Rankin is using a better estimate for the number of smooth integers. The approach of Rankin and modifications of it have become known as the Erdős–Rankin method. In [2] and [13] these two authors for sufficiently large x construct a long interval of integers, which all have greatest common divisor greater than 1 with $\prod_{p<x} p$.

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The results (1.1) and (1.2) are closely related to the concept of prime-avoidance. The term "prime avoidance" first appears in the paper [5] of K. Ford, D. R. Heath-Brown and S. Konyagin, where they prove the existence of infinitely many "prime-avoiding" perfect k-th powers for any positive integer k. Their definition is as follows:

Definition 1.1. *Let*

$$g_1(m) := \frac{\log m \, \log_2 m \, \log_4 m}{(\log_3 m)^2} \, . \tag{1.3}$$

Then an integer m, for which $g_1(m) \geq m^{(0)}$ (with $m^{(0)} > 0$ being a fixed constant) is called prime-avoiding with constant $c_0 > 0$ if $m+u$ is composite for all integers u satisfying $|u| \leq c_0 g_1(m)$. We also say, m has the prime avoidance property with constant c_0.

For the sake of larger flexibility we also consider the following variations:

Definition 1.2. *Let $h(m)$ be defined for $m \geq m^{(1)}$ and let $h(m) \to \infty$ for $m \to \infty$. Let $c_1 > 0$ be a fixed constant.*

(i) *Then an integer $m \geq m^{(1)}$ (with $m^{(1)} > 0$ being a fixed constant) is called prime-avoiding with constant $c_1 > 0$ and function h if $h(m) \geq m^{(1)}$ and $m + u$ is composite for all integers u satisfying $|u| \leq c_1 h(m)$. We also say, that m has the prime avoidance property with constant c_1 and function h.*

(ii) *$m \geq m^{(1)}$ is called prime-avoiding to the right with function h and constant c_2 if $m + u$ is composite for all integers $u \in [0, c_2 h(m)]$. We say that m has the prime-avoidance property to the right.*

An analogous definition applies for "prime-avoiding to the left" and "prime avoidance property to the left".

(iii) *The terms "one-sided prime-avoidance property" is used, if prime-avoidance property to the right, to the left or both hold.*

The function g_1 in (1.3) has played a central role for a long time in the study of large gaps between consecutive primes. The connection between these questions can easily be seen as follows:

Let m be prime-avoiding with function h and constant $c_1 > 0$ and let p_n be the largest prime number $\leq m$ with p_{n+1} being the subsequent prime number. Then

$$p_{n+1} - p_n \geq 2c_1 h(m) \, .$$

Also the reverse (large gaps imply the existence of prime-avoiding numbers) can be seen easily.

The results (1.1) and (1.2) also imply prime-avoidance results for most integers of the interval (p_n, p_{n+1}).

A modified version of this method was applied by the authors of [5] on the problem of prime-avoiding k-th powers mentioned in the introduction.

In [10] the authors of the present paper extended this result by proving the existence of infinitely many prime-avoiding k-th powers of prime numbers. For the sake of simplicity they only treat one-sided prime avoidance. They prove the following:

There are infinitely many n, such that

$$p_{n+1} - p_n \geq C_3 g_1(n)$$

and the interval $[p_n, p_{n+1}]$ contains the k-th power of a prime number.

Their method of proof consists of a combination of the method of [5] with the matrix method of the first author [8]. The matrix $\mathcal{M}$ employed in this technique is of the form

$$\mathcal{M} := (a_{r,u} \ : \ u \in \mathcal{B}) \ ,$$

where $P(x) := \prod_{p<x} p$, D is a fixed positive integer, the *rows*

$$\mathcal{R}(r) := \{a_{r,u} \ : \ u \in \mathcal{B}\}$$

of the matrix are *translates* (in closer or wider sense) of the *base-row* $\mathcal{B}$.

The essential idea of the construction of the base-row $\mathcal{B}$ is the Erdős–Rankin method. Here however (as in all combinations with the matrix method) the base-row is partially coprime to $P(x)$. The columns

$$\mathcal{C}(u) := (a_{r,u})_{1 \leq r \leq P(x)^{D-1}} \ , \quad \text{with } (a_{r,u} \, , P(x)) = 1$$

are called *admissible columns.*

The number of prime numbers in these columns, which are arithmetic progressions (or in the case of [10], shifted powers of elements of arithmetic progressions), can be estimated using theorems on primes in arithmetic progressions.

A famous prize problem of Erdős, being open for more than 70 years, was to replace the function g_1 in (1.2) by a function of higher order of magnitude. This problem could finally be solved in the paper [4] by K. Ford, B. J. Green, S. Konyagin and T. Tao and independently in the paper [12] by J. Maynard. Later all five authors improved on this result in their joint paper [3]. They proved:

$$p_{n+1} - p_n \geq C_4 \, g_2(p_n)$$

infinitely often, where

$$g_2(m) := \frac{\log m \ \log_2 m \ \log_4 m}{\log_3 m} \ .$$

In the paper [10] the authors of the present paper combined the methods of the papers [3, 5, 8] to obtain the following theorem.

Theorem 1.1 of [10]**.** *There is a constant* $C_5 > 0$ *and infinitely many* n *such that*

$$p_{n+1} - p_n \geq C_5\, g_2(n)$$

and the interval $[p_n, p_{n+1}]$ *contains the* k*-th power of a prime.*

Using the Definition 1.2 this can also be phrased as follows:

There is a constant $C_5 > 0$ such that infinitely many k-th powers of primes are one-sided prime-avoidant with constant $\frac{1}{2}C_5$. In the paper [11] the authors investigated the prime-avoidance of the k-th powers of prime numbers with Beatty sequences.

Definition 1.3. *For two fixed real numbers* α *and* β*, the corresponding non-homogeneous Beatty sequence is the sequence of integers defined by*

$$\mathscr{B}_{\alpha,\beta} := (\lfloor \alpha n + \beta \rfloor)_{n=1}^{\infty}$$

($\lfloor u \rfloor$ *denotes the largest integer* $\leq u$*).*

Definition 1.4. *For an irrational number* γ *we define its type* τ *by the relation*

$$\tau := \sup\{\rho \in \mathbb{R} \;:\; \liminf\; n^{\rho} \|\gamma n\| = 0\}$$

(see [11]*).* $\|\cdot\|$ *denotes the distance to the nearest integer.* τ *is called of finite type, if* $\tau < \infty$*.*

In the paper [11] the authors prove:

Theorem 1.3 of [11]**.** *Let* $k \geq 2$ *be an integer. Let* α, β *be fixed real numbers with* α *being a positive irrational and of finite type. Then there is a constant* $C_6 > 0$*, depending only on* α *and* β*, such that for infinitely many* n *we have:*

$$p_{n+1} - p_n \geq C_6\, g_2(n)$$

and the interval $[p_n, p_{n+1}]$ *contains the* k*-th power of a prime* $\widetilde{p} \in \mathscr{B}_{\alpha,\beta}$*.*

In this paper we deal with the *Piatetski–Shapiro primes.* We prove:

Theorem 1.5. *Let* $c_4 \in (1, 18/17)$ *be fixed,* $k \in \mathbb{N}$*,* $k \geq 2$*. Then there is a constant* $C_6 > 0$*, depending only on* k *and* c_4*, such that for infinitely many* n *we have*

$$p_{n+1} - p_n \geq C_6 g_2(n)$$

and the interval $[p_n, p_{n+1}]$ *contains the* k*-th power of a prime* $\widetilde{p} = \lfloor l^{c_4} \rfloor$ *.*

Using Definition 1.2 we can phrase Theorem 1.5 as follows:

Theorem 1.6. *Let* $c_4 \in (1, 18/17)$ *be fixed,* $k \in \mathbb{N}$*,* $k \geq 2$*. Then there is a constant* $C_7 > 0$ *such that infinitely many* k*-th powers of primes of the form* $\widetilde{p} = \lfloor l^{c_4} \rfloor$ *have the one-sided prime avoidance property with constant* C_7 *and function* g_2*.*

One can achieve prime avoidance by a slight modification of the proof of Theorem 1.5.

2. Construction of the matrix $\mathcal{M}$

In several papers (e.g. [8, 9, 10, 11]) in which the matrix method was used, it was crucial for the estimate of prime numbers in arithmetic progressions mod q that q was a "good modulus".

The definitions and facts of these papers may be summarised as follows: We recall the definition:

Lemma 2.1. *There exists a constant $C_8 > 0$ such that for every $C > C_8$ there is a sequence of real numbers $(x_n(C))$, depending only on C, such that for every n the modulus $q_n := P(x_n(C))$ is good with respect to C in the following sense: One has $L(s,\chi) \neq 0$ for every character χ modulo q_n and every $s = \sigma + it$ satisfying*

$$\sigma > 1 - \frac{C}{\log[q(|t|+1)]}. \tag{2.1}$$

Lemma 2.2. *Let $C > 0$ and the sequence $(x_n) = (x_n(C))$ be given, such that (2.1) is satisfied. Then there exists a constant $D > 0$ depending only on C, such that*

$$\pi(x_n; q_n, a) \gg \frac{x_n}{\phi(q_n)\log x_n} \tag{2.2}$$

uniformly for $(a, q_n) = 1$ and $x_n \geq q^{D/2}$.

The constant D can be made arbitrarily large, if C is sufficiently large.

Proof. This is Lemma 2 of [8]. It is not due to Gallagher (as stated in [8]), but is a simple consequence of Theorem 7 of Gallagher [6]. □

In the sequel we fix constants C, D, such that D is sufficiently large and a sequence $(x_n) = (x_n(C))$, such that (2.1) and (2.2) are satisfied.

Lemma 2.3. *Let x be sufficiently large.*

$$y = C_9\, x\, \frac{\log x \log_3 x}{\log_2 x}\ , \tag{2.3}$$

$C_9 > 0$ being a sufficiently small constant. Let $C_{10} > 0$ be a fixed constant to be specified later. Then there is an integer m_0 satisfying

$$1 \leq m_0 < P(C_{10}x) \tag{2.4}$$

$$(m_0 + 1, P(C_{10}x)) = 1 \tag{2.5}$$

and an exceptional set V satisfying

$$\#V \ll x^{1/2+\epsilon}\ , \quad \epsilon > 0 \text{ arbitrarily small} \tag{2.6}$$

such that

$$(m_0 + 1 + rP(C_0x))^k + u - 1 \tag{2.7}$$

is composite for $2 \leq u \leq y$, unless $u \in V$.

Proof. By Lemma 3.10 of [10], there is an integer m_0 satisfying all the properties (2.3)–(2.7) with the possible exception of (2.5).

The additional property (2.5) follows from the construction of m_0 in [10]. In Definition 3.9 of [10], m_0 is defined by

$$1 \leq m_0 < P(C_{10}x) \ \ (C_{10} \text{ being named } C_0 \text{ in [10]})$$

and the congruences

$$\begin{aligned} &m_0 \equiv c_s \,(\mathrm{mod}\ s) \\ &m_0 \equiv d_p \,(\mathrm{mod}\ p) \\ (2.8)\quad &m_0 \equiv 0 \,(\mathrm{mod}\ q), \quad q \in (1, x], \ q \notin S \cup P \\ &m_0 \equiv e_u \,(\mathrm{mod}\ p_u), \\ &m_0 \equiv g_p \,(\mathrm{mod}\ p), \ \text{for all other primes } p \leq C_{10}x, \ g_p \text{ being arbitrary.} \end{aligned}$$

Since by the definition of $\mathcal{A}, \mathcal{B}$ we have

$$c_s \not\equiv \ -1 \,(\mathrm{mod}\ s)\ , \ d_p \not\equiv \ -1 \,(\mathrm{mod}\ p)$$

we have from the first four congruences in (2.8) that $m_0 + 1 \not\equiv 0$ for all $p \in [2, C_{10}x]$ and thus $(m_0 + 1, P(C_{10}x)) = 1$. □

The construction of V is described in Lemma 3.5 of [10]. We now construct the matrix $\mathcal{M}$. We start with the following:

Remark. In the proof of Lemma 3.10 in [10], prime numbers from the interval $(x, C_{10}x]$ are used to construct a number m_0 with $1 \leq m_0 < P(C_{10}x)$, such that

$$a_{r,u} = (m_0 + 1 + rP(x))^k + u - 1\,,$$

$2 \leq u \leq y$, is composite unless $u \in V$. For the investigation of primes in the columns of the matrix $\mathcal{M}$ the condition needed is that the moduli $P(C_{10}x)$ are good. This is transformed into the condition that $P(x)$ is good by renaming x into x/C_{10}.

Definition 2.4. *Let $n \in \mathbb{N}$ be sufficiently large and let $x = x_n = x_n(C)$. The existence has been established in Lemma 2.1. Let $q = q_n = P(x_n(C))$. Let y satisfy* (2.3)*, and m_0, V satisfy* (2.4)–(2.7) *with $C_{10} = 1$. We let*

$$\mathcal{M} := (a_{r,u})_{\substack{1 \leq r \leq P(x)^{D-1} \\ 1 \leq u \leq y}}\ ,$$

where $a_{r,u} := (m_0 + 1 + rP(x))^k + u - 1$

For $1 \leq r \leq P(x)^{D-1}$ we denote by

$$R(r) := (a_{r,u})_{1 \leq u \leq y}$$

the r-th row of $\mathcal{M}$ and for $1 \leq u \leq y$ we denote by

$$C(u) := (a_{r,u})_{1 \leq r \leq P(x)^{D-1}}$$

the u-th column of $\mathcal{M}$.

3. Piatetski–Shapiro primes in arithmetic progressions

Definition 3.1. *Let $c_6 > 1$. Let a and d be coprime integers, $w \geq 1$. Then we let*

$$\pi_{c_6}(w; d, a) = \#\{p \leq w \ : \ p \in \mathcal{P}^{(c_6)}, \ p \equiv a \mod d\}$$
$$\mathcal{P}^{(c_6)} = \{p \text{ prime} \ : \ p = \lfloor l^{c_6} \rfloor \text{ for some } l\} .$$

Lemma 3.2. *Let a and d be coprime integers, $d \geq 1$. For fixed $c_6 \in (1, 18/17)$ we have (with $\gamma = 1/c_6$):*

$$\pi_{c_6}(w; d, a) = \gamma w^{\gamma-1}\pi(w; d, a) + \gamma(1-\gamma)\int_2^w u^{\gamma-2}\pi(u; d, a)\mathrm{d}u + O\Big(w^{\frac{17}{39}+\frac{7\gamma}{13}+\epsilon}\Big),$$

with the constant being implied in O not depending on d or a.

Proof. This is Theorem 8 of [1]. □

We shall use Dirichlet-characters to evaluate $\pi(u; P(x), m_0 + 1)$ and use the familiar definition

$$\theta(u; g, r) = \sum_{\substack{p \leq u \\ p \equiv r \bmod g}} \log p.$$

Lemma 3.3. *Assume that $\chi \bmod r$ is induced by the primitive character χ^*. Then*

$$\sum_{p \leq u} |\chi(p)\log p - \chi^*(p)\log p| = O(\log^2(ru)).$$

Proof. $\chi(p) \neq \chi^*(p)$ implies $\chi(p) = 0$ and thus $p \mid \text{conductor}(x) \mid r$. Also

$$\sum_{p \mid r} 1 = O(\log r).$$

Thus

$$\sum_{p \leq u} |\chi(p)\log p - \chi^*(p)\log p| = O(\log u \cdot \log r) = O(\log(ur)^2).$$ □

Definition 3.4. *We let $l(r) := m_0 + 1 + rP(x)$,*

$$\mathcal{R}_1(\mathcal{M}) := \left\{r \ : \ 1 \leq r \leq P(x)^{D-1}, \ l(r) \in \mathcal{P}^{(c_6)}\right\},$$
$$\mathcal{R}_2(\mathcal{M}) := \left\{r \ : \ \begin{array}{l} 1 \leq r \leq P(x)^{D-1}, \ r \in \mathcal{R}_1(\mathcal{M}), \\ R(r) \text{ contains a prime number} \end{array}\right\}.$$

We observe that each row $R(r)$ with $r \in \mathcal{R}_1(\mathcal{M})$ has as its first element $a_{r,1}$, the k-th power of the prime $l(r) \in \mathcal{P}^{(c_6)}$. We now conclude the proof of Theorem 1.5 by showing that the set $\mathcal{R}_1(\mathcal{M}) \setminus \mathcal{R}_2(\mathcal{M})$ is non-empty.

Lemma 3.5. *We have*

$$\#\mathcal{R}_1(\mathcal{M}) = \frac{P(x)^{D\gamma}}{\phi(P(x))}\left(1 + O\left(e^{-c_7 D}\right)\right).$$

Proof. We apply Lemma 3.2 with $P(x)^D$ instead of w, $d = P(x)$, $a = m_0+1$ and obtain

$$\begin{aligned}(3.1)\quad \#\mathcal{R}_1(\mathcal{M}) &= \gamma P(x)^{D(\gamma-1)}\pi(P(x)^D; P(x), m_0+1)\\ &\quad + \gamma(1-\gamma)\int_2^{P(x)^D} u^{\gamma-2}\pi(u; P(x), m_0+1)\,\mathrm{d}u\\ &\quad + O\left(P(x)^{D\left(\frac{17}{39}+\frac{7\gamma}{13}+\epsilon\right)}\right).\end{aligned}$$

We now use Dirichlet-characters to evaluate $\pi(u; P(x), m_0+1)$. We use the familiar definition

$$\theta(u; g, r) = \sum_{\substack{p\le u\\ p\equiv r \bmod g}} \log p$$

and obtain

$$\begin{aligned}(3.2)\quad &\theta(u; P(x), m_0+1)\\ &= \frac{1}{\phi(P(x))}\sum_{p\le u}\log p + \frac{1}{\phi(P(x))}\sum_{\substack{\chi \bmod P(x)\\ \chi\ne\chi_0}}\sum_{p\le u}\chi(p)\log p\,.\end{aligned}$$

From Lemma 3.3 we obtain

$$\begin{aligned}&\left|\theta(u; P(x), m_0+1) - \frac{1}{\phi(P(x))}\sum_{p\le u}\log p\right|\\ &\quad\le \frac{1}{\phi(P(x))}\sum_{1<r\le P(x)}\ \sideset{}{^*}\sum_{\chi^* \bmod r}\left(\sum_{p\le u}\chi^*(p)\log p + O((\log P(x))^2)\right),\end{aligned}$$

where $\sideset{}{^*}\sum$ denotes the summation over primitive characters.

We first assume that $u > P(x)^{D/2}$ and apply Lemma 2.2 and obtain:

$$(3.3)\qquad \theta(u; P(x), m_0+1) = \frac{1}{\phi(P(x))}\left(\sum_{p\le u}\log p\right)\left(1 + O\left(e^{-c_7 D}\right)\right).$$

For $u \le P(x)^{D/2}$ we use the trivial estimate

$$\theta(u; P(x), m_0+1) = O(u\log^2 u).$$

Lemma 3.5 now follows from (3.1), (3.2) and (3.3). □

We now estimate $\mathcal{R}_2(\mathcal{M})$. We have

$$|\mathcal{R}_2(\mathcal{M})| \leq \sum_{v \in V} t(v)\,, \tag{3.4}$$

where

$$t(v) = \#\Big\{ r \,:\, l(r) \in \mathcal{P}^{(c_6)}, l(r)^k + v - 1 \text{ prime} \Big\}\,. \tag{3.5}$$

We majorize the set $\mathcal{P}^{(c_6)}$ by sieving the simpler set

$$\mathcal{N}^{(c_6)} = \{ n \in \mathbb{N} \,:\, n = \lfloor l^{c_6} \rfloor \text{ for some } l \}.$$

Definition 3.6. *Let*

$$s_c(x; h, i) = \#\Big\{ n \in \mathcal{N}^{(c_6)} \,:\, n \leq x, n \equiv i \bmod h \Big\}.$$

Lemma 3.7.

$$s_c(x; h, i) = \frac{x^{1/c}}{h} + O_c\Big(\Big(x^{1+5/c} (\log x)^3 \Big)^{1/7} \Big).$$

Proof. This is Proposition 1 of [14]. □

Let

$$Q(x) = \prod_{\substack{x < p \leq P(x) \\ p \text{ prime}}} p\,.$$

Let

$$n(r) = m_0 + 1 + rP(x)$$

$$N(v) := \#\left\{ r \,:\, \begin{array}{l} n(r) \in \mathcal{N}^{(c_6)},\ n(r) \leq 2P(x)^D, \\ (n(r), P(x)) = 1,\ (n(r)^k + v - 1, Q(x)) = 1 \end{array} \right\}. \tag{3.6}$$

By the sieve of Eratosthenes we have:

$$N(v) \leq \sum_{r\,:\, \substack{n(r) \in \mathcal{N}^{(c_6)}, \\ n(r) \leq 2P(x)^D}} \ \sum_{\tau_1 | (n(r), P(x))} \mu(\tau_1) \sum_{\tau_2 | (n(n(r)^k + v - 1), Q(x))} \mu(\tau_2). \tag{3.7}$$

We now apply a Fundamental Lemma in the theory of Combinatorial Sieves.

Lemma 3.8. *Let $\kappa > 0$ and $\Omega > 1$. There exist two sets of real numbers $\Lambda^+ = (\lambda_d^+)$ and $\Lambda^- = (\lambda_d^-)$ depending only on κ and Ω with the following properties:*

$$\lambda_1^{\pm} = 1\,, \tag{3.8}$$

$$|\lambda_d^{\pm}| < 1\,, \quad \text{if } 1 < d < \Omega\,, \tag{3.9}$$

$$\lambda_d^{\pm} = 0\,, \quad \text{if } d \geq \Omega\,, \text{ and for any integer } n > 1, \tag{3.10}$$

$$\sum_{d | n} \lambda_d^- \leq 0 \leq \sum_{d | n} \lambda_d^+\,. \tag{3.11}$$

Moreover, for any multiplicative function $g(d)$ with $0 \leq g(p) < 1$ and satisfying the dimension condition

$$\prod_{w \leq p < z} (1-g(p))^{-1} \leq \left(\frac{\log z}{\log w}\right)^{\kappa} \left(1 + \frac{K}{\log w}\right)$$

for all w, z such that $2 \leq w < z \leq y$ we have

$$\sum_{d \mid P(z)} \lambda_d^{\pm} g(d) = \left(1 + O\left(e^{-s}\left(1 + \frac{K}{\log z}\right)^{10}\right)\right) \prod_{p<z} (1-g(p)) \, ,$$

where $s = \frac{\log \Omega}{\log z}$.

Proof. This is Fundamental Lemma 6.3 in [7, p. 159]. □

For $x < \widetilde{p} \leq P(x)$ let $\rho(v, \widetilde{p})$ be the number of solutions (in s) of the congruence

$$s(s^k + v - 1) \equiv 0 \bmod \widetilde{p} \, .$$

We apply Lemma 3.8 with g given by

$$g(\widetilde{p}) = \begin{cases} 1/\widetilde{p}, & \text{for } 1 < \widetilde{p} \leq x \\ \rho(v, \widetilde{p})/\widetilde{p} & \text{for } x < \widetilde{p} \leq P(x). \end{cases}$$

The range of definition of g is extended to all integers by multiplicativity.

- λ_d^- is not needed.
- λ_d^+ is chosen satisfying (3.8)–(3.11), with $\Omega = P(x)^{D/100}$.

From (3.7) and Lemma 3.8, we obtain

$$N(v) \leq \sum_{\tau \mid P(x)} \sum_{\substack{n :\, n \in \mathcal{N}^{(c_6)},\, n \leq 2P(x)^D, \\ n \equiv m_0 + 1 \bmod P(x), \\ n(n^k + v - 1) \equiv 0 \bmod \tau}} 1 \tag{3.12}$$

Let $S(\tau, v)$ be the solution set of $s(s^k + v - 1) \equiv 0 \bmod \tau$. The system of congruences in the inner sum of (3.12) is equivalent to a union of congruence classes

$$n \equiv z(s) \bmod P(x), \quad s \in S(\tau, v).$$

Thus

$$N(v) \leq \sum_{\tau \mid P(x)} \lambda^+(\tau) \sum_{\substack{n :\, n \in \mathcal{N}^{c_6}, n \leq 2P(x)^D \\ n \equiv z(s) \bmod P(x)}} 1 \tag{3.13}$$

The inner sum in (3.13) can now be evaluated by the use of Lemma 3.7.

From (3.4), (3.5), (3.6), (3.7) and (3.13) it follows that

$$\#\mathcal{R}_2 = o(\#\mathcal{R}_1),$$

which finishes the proof of Theorem 1.3. □

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