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New portions of $M \setminus L$ and a lower bound on the Hausdorff distance between L and M

par Clément RIEUTORD, Carlos Gustavo MOREIRA et Harold ERAZO

Résumé. Soient M et L les spectres de Markov et de Lagrange respectivement. Il est connu que L est contenu dans M, et Freiman démontra en 1968 que $M \setminus L \neq \emptyset$. En 2018, la première région de $M \setminus L$ fut découverte par C. Matheus et C. G. Moreira, réfutant ainsi une conjecture faite par Cusick en 1975. En 2022, les mêmes auteurs, avec L. Jeffreys, découvrirent une nouvelle région près de 3.938. Dans ce papier, nous étudions deux nouvelles régions de $M \setminus L$ au-dessus de $\sqrt{12}$, au voisinage de la valeur de Markov de deux mots périodiques non semi-symétriques de longueurs impaires, à savoir $\overline{212332111}$ et $\overline{123332112}$. Dans les deux cas, nous démontrons qu'il existe une lacune maximale de L et un ensemble de Gauss–Cantor à l'intérieur de cette lacune qui est contenu dans M. En outre, nous montrons que la dimension de Hausdorff locale à l'extrémité droite de ces lacunes est égale à 1. Après avoir étudié ces exemples, nous donnerons une minoration pour la distance de Hausdorff $d_H(M, L)$ entre M et L.

Abstract. Let M and L be the Markov and Lagrange spectra, respectively. It is known that L is contained in M and Freiman showed in 1968 that $M \setminus L \neq \emptyset$. In 2018 the first region of $M \setminus L$ above $\sqrt{12}$ was discovered by C. Matheus and C. G. Moreira, thus disproving a conjecture of Cusick of 1975. In 2022, the same authors together with L. Jeffreys discovered a new region near 3.938. In this paper, we will study two new regions of $M \setminus L$ above $\sqrt{12}$, in the vicinity of the Markov value of two periodic words of odd length that are non semisymmetric, which are $\overline{212332111}$ and $\overline{123332112}$. We will demonstrate that for both cases, there is a maximal gap of L and a Gauss–Cantor set inside this gap that is contained in M. Moreover we show that at the right endpoint of those gaps we have local Hausdorff dimension equal to 1.

After studying the mentioned examples, we will provide a lower bound for the value of $d_H(M, L)$ (the Hausdorff distance between M and L).

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Mots-clefs. Lagrange and Markov spectra, Hausdorff dimension, Hausdorff distance.

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1. Introduction

Definition 1.1. The best constant of Diophantine approximation $k(\alpha)$ for $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ is given by:

$$l(\alpha) = \limsup_{\substack{p,q\to\infty \\ p,q\in\mathbb{Z}^2}} \frac{1}{|q(q\alpha - p)|}.$$

The Lagrange spectrum is:

$$L = \{\, l(\alpha) \mid \alpha \in \mathbb{R} \setminus \mathbb{Q} \text{ and } l(\alpha) < \infty\}.$$

Definition 1.2. Let $q(x,y) = ax^2 + bxy + cy^2$ be a binary quadratic form with $(a,b,c) \in \mathbb{R}^3$ and $\Delta(q) = b^2 - 4ac$. We say q is indefinite if $\Delta(q) > 0$.

The Markov Spectrum is:

$$M = \left\{ \frac{\sqrt{\Delta(q)}}{\inf_{(x,y)\in\mathbb{Z}^2\setminus\{(0,0)\}} q(x,y)} \;\middle|\; q \text{ is an indefinite binary quadratic form} \right\}.$$

The Lagrange and Markov spectra were first studied by Markov around 1880 (see [14]). In particular he established that $L \cap (0,3) = M \cap (0,3)$ is a discrete set that accumulates at 3. M. Hall showed [8] in 1947 that $[c,\infty) \subset L$ for some positive constant $c > 3$. In [5] and [23], it was proved that in fact $[\sqrt{21},\infty) \subset L$ and finally in 1975, Freiman [6] determined the largest half-ray contained in L, namely $[\mu,\infty) \subset L$, where $\mu = 4.527829\ldots$ is an explicit quadratic surd. The set $[\mu,\infty)$ is known as *Hall's ray*.

It is well known that $L \subset M$ are closed subsets of $\mathbb{R}$ (see [3]). It took some time to decide if they were equal: only in 1968 Freiman [4] exhibited a countable subset of isolated points in $M \setminus L$ near 3.11; after that, Freiman proved in 1973 that $M \setminus L$ contains a point α_∞ near 3.29, and Flahive [7] showed in 1977 that α_∞ is the accumulation point of a countable subset of $M \setminus L$. These results led Cusick [2] to conjecture in 1975 that the Lagrange and Markov spectra coincide after $\sqrt{12}$, i.e. $(M \setminus L) \cap [\sqrt{12},\infty) = \varnothing$.

Only recently our knowledge of $M \setminus L$ changed significantly: in [15], Carlos Matheus and Carlos Gustavo Moreira showed that in fact there is a Gauss–Cantor set inside $M \setminus L$ that contains α_∞ (in particular $M \setminus L$ is uncountable).[1] As a by product of their work they showed that $\dim_{\mathrm{H}}(M \setminus L) > 0.353$, where $\dim_{\mathrm{H}}(M \setminus L)$ denotes the Hausdorff dimension of the set $M \setminus L$. Shortly after they showed that Cusick conjecture mentioned above is false: in [17] they found a Gauss–Cantor set near 3.7 of Hausdorff dimension larger than 1/2. Moreover they showed that in fact $\dim_{\mathrm{H}}(M \setminus L) < 0.987$

[1] In the PhD thesis by [24] in 1976, the author found that there is a Gauss–Cantor set near α_∞ contained in $M \setminus L$. Consequently, he proved that $M \setminus L$ is uncountable. However, it was not observed that this already implies that $M \setminus L$ has positive Hausdorff dimension, nor was there an attempt made to completely characterize the region. It is unfortunate that this work went essentially unnoticed; it is not even mentioned in the book of [3].

(in particular L and M have the same interior). The same authors together with M. Pollicott and P. Vytnova proved rigorously in [19] that

$$0.537152 < \dim_{\mathrm{H}}(M \setminus L) < 0.796445.$$

The upper bound is the current best upper bound on the Hausdorff dimension of $M \setminus L$, while in [9] the lower bound was improved to 0.593.

By a *region* of $M \setminus L$ we mean a non-empty intersection of M with a maximal gap of L. In [16] the region near 3.11 related to Freiman example of 1968 was completely characterized: again the authors found a Gauss–Cantor set contained on $M \setminus L$ (see the book [11] for more details).

In all the above examples, the regions of $M \setminus L$ turned out to be closed subsets of $\mathbb{R}$, more precisely, they consists of a Gauss–Cantor set together with an infinite set of isolated points. Motivated by this, T. Bousch asked whether the set $M \setminus L$ is a closed subset of $\mathbb{R}$. In a first attempt to answer negatively this question, in [12] Lima–Matheus–Moreira–Vieira gave some evidence towards the possibility that $3 \in L \cap \overline{(M \setminus L)}$, more precisely they exhibited a decreasing sequence of elements on M and proved that the first four of them belong to $M \setminus L$. In spite of the fact that $3 \in L \cap \overline{(M \setminus L)}$ remains an open question, the same authors managed to prove in [13] that $1 + \frac{3}{\sqrt{2}} \in L \cap \overline{(M \setminus L)}$, by constructing a decreasing sequence of elements of $M \setminus L$ that converge to $1 + \frac{3}{\sqrt{2}} \in L$.

More recently, in [9], one new region of $M \setminus L$ above $\sqrt{12}$ was discovered near 3.938. This region gives the current known maximal elements of $M \setminus L$. In fact, by means of computational search, four new regions of $M \setminus L$ above $\sqrt{12}$ were suggested in [9]. In this paper we study one of the regions they suggested, namely the one associated with 123332112. The other region we study is the one associated with 212332111, which was not noticed before.

We say that a finite word w is *semisymmetric*[2] if it is a palindrome or the concatenation of two palindromes, equivalently, if the orbits by the shift map of the bi-infinite sequences $\overline{w}$ and $\overline{w^T}$ are equal. For example 212332111 and 123332112 are non semi-symmetric words of odd length. As is explained in [17], this is one of the key notions behind the construction of all known regions of $M \setminus L$.

For more details about the spectra, we refer the reader to the survey [18], the classical textbook of Cusick–Flahive [3] or the book of Lima–Matheus–Moreira–Romaña [11] for a more modern point of view.

1.1. Perron's description of the Lagrange and Markov spectra. Perron managed to characterize the Lagrange and Markov spectra in terms of Dynamical Systems.

[2]The term was coined by [7], however this concept already appeared in the work of Berstein in [1, p. 42]

Definition 1.3. Let us call $\Sigma = (\mathbb{N}^*)^{\mathbb{Z}}$ and $\lambda_0 : \Sigma \to \mathbb{R}$ the function:

$$\forall\ (a_n)_{n\in\mathbb{Z}},\ \ \lambda_0((a_n)_{n\in\mathbb{Z}}) = [a_0, a_1, a_2, \dots] + [0, a_{-1}, a_{-2}, \dots],$$

with $[a_0, a_1, a_2, \dots] = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \dots}}$ the value of the continued fraction associated with the sequence $(a_n)_{n\geq 0}$. We also define the function λ_k, for $k \in \mathbb{Z}$ as follows: $\forall\ (a_n)_{n\in\mathbb{Z}}$, $\lambda_k((a_n)_{n\in\mathbb{Z}}) = \lambda_0((a_{n+k})_{n\in\mathbb{Z}})$.

Theorem 1.4 (Perron [22])**.** *We define the Langrage and Markov value of a sequence $S \in (\mathbb{N}^*)^{\mathbb{Z}}$ by:*

$$l(S) = \limsup_{k\to\infty} \lambda_k(S) \quad \textit{and} \quad m(S) = \sup_{k\in\mathbb{Z}} \lambda_k(S).$$

We have:

$$L = \left\{ l(S) \,\middle|\, S \in (\mathbb{N}^*)^{\mathbb{Z}} \textit{ and } l(S) < \infty \right\}$$

and

$$M = \left\{ m(S) \,\middle|\, S \in (\mathbb{N}^*)^{\mathbb{Z}} \textit{ and } m(S) < \infty \right\}.$$

For the future, a sequence marked with a "*" is a sequence where the index 0 is next to the left of the asterisk. For example, we have:

$$\lambda_0(\dots 1223\mathbf{3}^*211112\dots) = [\mathbf{3}, 2, 1, 1, 1, 1, 2, \dots] + [0, 3, 2, 2, 1, \dots].$$

1.2. New portions of $M \setminus L$. Our main result is the complete characterization of two new regions of $M \setminus L$. Moreover, we also give information about the local dimension at the right endpoint of the gaps of L where these regions are contained.

Theorem 1.5. *There are two non-empty open intervals A_1 and A_2 such that for $i \in \{1, 2\}$:*

$$A_i \cap L = \emptyset \textit{ and } \exists\ K_i \textit{ a Gauss–Cantor set such that } K_i \subset A_i \cap (M \setminus L).$$

Furthermore, we have:

(1) (a) $\inf A_1 = \lambda_0(\overline{21233^*2111}) \approx 3.6767$,
  (b) $|A_1| \approx 8.42651 \times 10^{-12}$,
  (c) $\sup A_1 \in L'$,
  (d) *For all* $\alpha > 0$,

$$\dim_{\mathrm{H}}\{M \cap (\sup A_1, \sup A_1 + \alpha)\} = \dim_{\mathrm{H}}\{L \cap (\sup A_1, \sup A_1 + \alpha)\} = 1,$$

(2) (a) $\inf A_2 = \lambda_0(\overline{12333^*2112}) \approx 3.72627$,
  (b) $|A_2| \approx 5.88429 \times 10^{-12}$,
  (c) $\sup A_2 \in L'$,
  (d) *For all* $\alpha > 0$,

$$\dim_{\mathrm{H}}\{M \cap (\sup A_2, \sup A_2 + \alpha)\} = \dim_{\mathrm{H}}\{L \cap (\sup A_2, \sup A_2 + \alpha)\} = 1,$$

where L' denotes the accumulation points of L.

For a detailed description of the regions of Theorem 1.5 see Theorem 4.17 and Theorem 5.13.

1.3. Hausdorff distance between M and L. Another interesting way of measure how different the spectra are is by computing the *Hausdorff distance* between them. Since $L \subset M$, we have that

$$d_H(L, M) = \sup_{m \in M \setminus L} d(L, m).$$

Theorem 1.6. *The Hausdorff distance between M and L is at least δ_0, where*

$$\begin{aligned}\delta_0 &= \frac{\begin{array}{l}272052036746460995 - 3973474319367040\sqrt{87} \\ \quad - 2762049221999040\sqrt{18229} + 353887557067187\sqrt{151905}\end{array}}{226488036203921280} \\ &\approx 9.1094243388 \times 10^{-8}.\end{aligned}$$

The reason why the bound is significant, is because in all regions discovered so far, including the ones described in Theorem 1.5, the one that is contained in the largest gap of L is the one given by the odd non semi-symmetric word 2112122 of length 7. This is precisely the region that contains the example discovered by Freiman in 1973 and that was later characterized in [15]. The intuition is that longer words should give regions of $M \setminus L$ that lie in smaller gaps. There are no odd non semi-symmetric words in the alphabet $\{1, 2\}$ of length less than 7. The shortest odd non semisymmetric word that contains a 3 and which is known to give a region of $M \setminus L$, was the region discovered in [17] which is associated to 1233222. The gap of L that contains this region is roughly of the size 7×10^{-11} (the right border was corrected in [10]).

There are no known examples of $M \setminus L$ associated with odd non semi-symmetric words that contain at least a 4. In fact there are no known examples above 3.945. For odd non semi-symmetric words in the alphabet $\{1, 2, 3, 4\}$ with at least one 3 or 4 with length less than 7, the method does not seem to work, because either the Markov value of the periodic word lies in Hall's ray or the property of local uniqueness is not know to hold (it could happen that the Markov value of these periodic words belongs to the interior of the spectra, so the local uniqueness could be impossible to establish).

Conjecture 2. The Hausdorff distance between M and L is precisely δ_0.

Most of this paper will consist in proving Theorem 1.5. In Section 2, we describe precisely the structure of $M \setminus L$ nearby $\lambda_0(\overline{21233^*2111})$. In Section 3, we describe the structure of $M \setminus L$ nearby $\lambda_0(\overline{12333^*2112})$. Finally, in Section 4, we give a proof of Theorem 1.6.

3. Preliminaries

In this paper, we will only work in the region $M \cap (-\infty, 4)$. Since $m(\dots 4 \dots) > 4$, we see that we must only deal with sequences $S \in \{1,2,3\}^{\mathbb{Z}}$.

Definition 3.1. For a finite word $w \in (\mathbb{N}^*)^{(\mathbb{Z})}$, we denote: $w^n = w \dots w$ (n times).

We denote the bi-infinite periodic sequence: $\dots w \dots w \dots = \overline{w}$.

Eventually, if S is a bi-infinite sequence with a periodic side w on the left (or on the right), we also write $S = \overline{w}\, S_n S_{n+1} \dots$.

Remark 3.2. We have for every finite word w:

$$m(\overline{w}) = l(\overline{w}).$$

And we write, for $w = w_1 \dots w_n$:

$$\lambda_0(\overline{w_1 \dots w_k^* \dots w_n}) = \lambda_0(\overline{w}\, w_1 \dots w_k^* \dots w_n\, \overline{w}).$$

Definition 3.3. For all sequences $S = (S_n)_{n \in \mathbb{Z}}$, we define the transpose of S by $S^T = (S_{-n})_{n \in \mathbb{Z}}$.

Clearly we have $\lambda_0(S) = \lambda_0(S^T)$ and therefore $m(S) = m(S^T)$ and $l(S) = l(S^T)$.

Remark 3.4. In order to simplify notation, if $w = w_{-p} \dots w_q \in (\mathbb{N}^*)^{p+q+1}$ is a finite word where $p, q \in \mathbb{N}$ and $S \in (\mathbb{N}^*)^{\mathbb{Z}}$ a bi-infinite sequence, we write $S = w_{-p} \dots w_0^* \dots w_q$ to denote

$$\forall\, k \in [-p, q] \cap \mathbb{Z}, \quad S_k = w_k.$$

Here, the asterisk on the finite word represents the position 0 on the sequence S.

Definition 3.5. Given $B = \{\beta_1, \dots, \beta_\ell\}$, $\ell \geq 2$, a finite alphabet of finite words $\beta_j \in (\mathbb{N}^*)^{r_j}$, $r_j \geq 1$ which is primitive (in the sense that β_i does not begin by β_j for all $i \neq j$) and a finite word $c_1 \dots c_r \in (\mathbb{N}^*)^r$, $r \geq 0$, then the set $K \subseteq [0,1]$ defined by

$$K = \{[0; c_1, \dots, c_r, \gamma_1, \gamma_2, \dots] \mid \gamma_i \in B\},$$

is a *Gauss–Cantor set* associated with B.

For the sequel, we need the following property on the function λ_0 (see [20, Lemma A.1] and [3, Chapter 1, Lemma 2]):

Proposition 3.6 (Properties of continuity of λ_0)**.**

(1) $\lambda_0(R^T\,.)$ *is bi-Lipschitz:*

$\forall\, R = R_1R_2\cdots \in (\mathbb{N}^*)^{\mathbb{N}^*}$, $\forall\, S \in (\mathbb{N}^*)^{\mathbb{N}}$, *we define the bi-infinite sequence $R^T S$ by $(R^T S)_n = S_n$ if $n \geq 0$ and $(R^T S)_n = R_{-n}$ if $n < 0$. There are $C_1, C_2 > 0$ such that $\forall\, S, S' \in \{1,2,3\}^{\mathbb{N}}$, $\forall\, R = R_1R_2\cdots \in (\mathbb{N}^*)^{\mathbb{N}^*}$ we have:*

$$\frac{C_1}{9^N} \leq \left|\lambda_0(R^T S) - \lambda_0(R^T S')\right| \leq \frac{C_2}{2^N}$$

with $N = \max\{n \mid \forall\, 0 \leq k \leq n,\ S_k = S'_k\}$.

(2) λ_0 *is Lipschitz:*

There is $C > 0$ such that $\forall\, S, S' \in \{1,2,3\}^{\mathbb{Z}}$ we have:

$$\left|\lambda_0(S) - \lambda_0(S')\right| \leq \frac{C}{2^N}$$

with $N = \max\{n \mid \forall\, |k| \leq n,\ S_k = S'_k\}$.

3.1. Boundaries of finite words. In order to estimate the λ_0 value of finite words that only contain $\{1,2,3\}$, we will use the following bounding rules:

Lemma 3.7. *We have for any $(a_n)_{n\in\mathbb{Z}} \in \{1,2,3\}^{\mathbb{Z}}$ that*

(1) $\lambda_0(\overline{13}\, a_{2n} \ldots a_0^* \ldots a_{2m}\overline{31}\,) \leq \lambda_0(\ldots a_{2n} \ldots a_0^* \ldots a_{2m} \ldots) \leq \lambda_0(\overline{31}\, a_{2n} \ldots a_0^* \ldots a_{2m}\overline{13}\,)$,

(2) $\lambda_0(\overline{13}\, a_{2n} \ldots a_0^* \ldots a_{2m+1}\overline{13}\,) \leq \lambda_0(\ldots a_{2n} \ldots a_0^* \ldots a_{2m+1} \ldots) \leq \lambda_0(\overline{31}\, a_{2n} \ldots a_0^* \ldots a_{2m+1}\overline{31}\,)$,

(3) $\lambda_0(\overline{31}\, a_{2n+1} \ldots a_0^* \ldots a_{2m}\overline{31}\,) \leq \lambda_0(\ldots a_{2n+1} \ldots a_0^* \ldots a_{2m} \ldots) \leq \lambda_0(\overline{13}\, a_{2n+1} \ldots a_0^* \ldots a_{2m}\overline{13}\,)$,

(4) $\lambda_0(\overline{31}\, a_{2n+1} \ldots a_0^* \ldots a_{2m+1}\overline{13}\,) \leq \lambda_0(\ldots a_{2n+1} \ldots a_0^* \ldots a_{2m+1} \ldots) \leq \lambda_0(\overline{13}\, a_{2n+1} \ldots a_0^* \ldots a_{2m+1}\overline{31}\,)$.

These boundaries are a little gross, because the word "13" is quickly going to be forbidden, but it is enough to construct an accurate tree of possibilities (Figures 4.1 and 5.1) and obtain the results we want.

3.2. Forbidden words. In the following pages, we refer a lot to forbidden words. In each section we fix a real number j_0.

Definition 3.8. A forbidden word is a finite word $w = a_{-n} \ldots a_0 \ldots a_m \in \{1,2,3\}^{(\mathbb{N})}$ that verifies

$$\inf\Big\{m(S)\ \Big|\ S \in \{1,2,3\}^{\mathbb{Z}}, w \text{ is subword of } S\Big\} > j_0 + \eta, \text{ for some } \eta > 0.$$

In other terms, w is forbidden if for any bi-infinite sequence $S \in \{1,2,3\}^{\mathbb{Z}}$ containing w as subword, the Markov value of S is at least $j_0 + \eta$ for some $\eta > 0$.

In practice we will forbid words according to the bounding rules of Lemma 3.7:

$$\lambda_0(\dots w \dots) \geq \min_{\varepsilon_l, \varepsilon_r \in \{\overline{13}, \overline{31}\}} \lambda_0(\varepsilon_l \, w \, \varepsilon_r)$$

Remark 3.9. Because $\lambda_0(S) = \lambda_0(S^T)$, if w is forbidden, then we also have that w^T is forbidden as well.

4. Portion of $M \setminus L$ in the vicinity of $\lambda_0(\overline{21233^*2111})$

In this section, we study the structure of M in the vicinity of $j_0 = \lambda_0(\overline{21233^*2111}) \approx 3.6766994172$.

Our aim is to determine the largest value j_1 for which $L \cap (j_0, j_1) = \emptyset$ is true, investigate the fractal structure of $(M \setminus L) \cap (j_0, j_1)$ and provide a description of the local structure of L above j_1. To accomplish this, we will establish two key properties of the function λ_0 on this region:

(1) Local uniqueness.
(2) Self-replication.

We will resume the process of local uniqueness and self-replication in a tree (Figure 4.1). We call for the future ω_1 the finite word 212332111. We also denote ω_1^* the word 21233*2111, where the asterisk represents the position zero.

4.1. Local uniqueness. In this part, we will show that if S verifies $|\lambda_0(S) - j_0| < \varepsilon$, with $\varepsilon \approx 10^{-6}$, then S must be written in the following form:

$$S_{-8} \dots S_8 = 2111 \, \omega_1^* \, 2123.$$

We will also determinate all forbidden words useful for the self-replication.

Lemma 4.1 (Forbidden words I)**.** *Let* $S \in \{1,2,3\}^{\mathbb{Z}}$. *Then*

(1) *If* $S = 13^*$, *then* $\lambda_0(S) > j_0 + 10^{-1}$.
(2) *If* $S = 23^*2$, *then* $\lambda_0(S) > j_0 + 10^{-2}$.
(3) *If* $S = 33^*23$, *then* $\lambda_0(S) > j_0 + 10^{-2}$.
(4) *If* $S = 233^*22$ *or* 333^*22, *then* $\lambda_0(S) > j_0 + 10^{-2}$.
(5) *If* $S = 333^*211$, *then* $\lambda_0(S) > j_0 + 10^{-3}$.
(6) *If* $S = 1233^*2112$, *then* $\lambda_0(S) > j_0 + 10^{-3}$.
(7) *If* $S = 111233^*21111$, *then* $\lambda_0(S) > j_0 + 10^{-4}$.

As a consequence, if S verifies $m(S) \leq j_0 + 10^{-4}$ then, $(S_n)_{n \in \mathbb{Z}}$ doesn't contain any of these subwords or their transpose.

Proof. By using the above boundaries (Lemma 3.7), it is simple computation. □

Corollary 4.2. *If* S *is a sequence such that* $m(S) \leq j_0 + 10^{-4}$, *then, the words* 322, 223 *and* 323 *are forbidden.*

Proof. If S contains 323, then since according to the Lemma 4.1, the words $13, 232$ and 3323 are forbidden, we can't extend the word 323 to the left without making a forbidden word appear.

If $S = \dots 322 \dots$, then according to the Lemma 4.1, since 13 and 232 are forbidden, the word 322 must extend to the left in such way: $S = \dots 3322 \dots$. However, since the words $13, 23322$ and 33322 are forbidden, we can't extend the word 3322 without having a contradiction. The same reasoning gives that 223 is forbidden. □

Lemma 4.3 (Local uniqueness I)**.** *Let $S \in \{1,2,3\}^{\mathbb{Z}}$ be such that $m(S) = \lambda_0(S) \le j_0 + 10^{-4}$. Then S or S^T must take on of these forms;*

(1) $S = 1^*$ *or* 2^* *and* $\lambda_0(S) < j_0 - 10^{-2}$*,*
(2) $S = 33^*3$ *and* $\lambda_0(S) < j_0 - 10^{-2}$*,*
(3) $S = 1233^*212$ *and* $\lambda_0(S) < j_0 - 10^{-3}$*,*
(4) $S = 2333^*212$ *or* 3333^*212 *and* $\lambda_0(S) < j_0 - 10^{-3}$*,*
(5) $S = 111233^*21112$ *and* $\lambda_0(S) < j_0 - 10^{-4}$*,*
(6) $S = \omega_1^*$*.*

As a consequence, if S verifies $|\lambda_0(S) - j_0| \le 10^{-4}$ and $m(S) \le j_0 + 10^{-4}$ then S must be of the form:

$$S = \omega_1^*.$$

Proof. Let S verifies the conditions of the lemma. If $S = 1^*, 2^*$ then, we have $\lambda_0(S) < j_0 - 10^{-2}$. Otherwise, $S = 3^*$ and since 13 and 232 are forbidden words, then:

$$S = 33^*2, 23^*3 \quad \text{or} \quad \left\{S = 33^*3 \text{ and } \lambda_0(S) < j_0 - 10^{-2}\right\}.$$

By symmetry, it is enough to study the case $S = 33^*2$. Since, 323 and 322 are forbidden, then $S = 33^*21$.

Again, 13 is forbidden so:

$$S = 333^*21 \quad \text{or} \quad S = 233^*21.$$

If $S = 333^*21$, using the forbidden words 13 and 333211, we must have:

$$S = 2333^*212 \text{ or } S = 3333^*212 \quad \text{and} \quad \lambda_0(S) < j_0 - 10^{-3}.$$

If $S = 233^*21$, using the forbidden words $323, 223$ and 13, we must have:

$$S = 1233^*211 \quad \text{or} \quad \left\{S = 1233^*212 \text{ and } \lambda_0(S) < j_0 - 10^{-3}\right\}.$$

If $S = 1233^*211$, since $13, 31$ and 12332112 are forbidden:

$$S = 11233^*2111 \quad \text{or} \quad S = 21233^*2111.$$

If $S = 11233^*2111$, since the words $13, 31, 21123321$ and 11123321111 are forbidden, then:

$$S = 111233^*21112 \quad \text{and} \quad \lambda_0(S) < j_0 - 10^{-4}.$$

□

Lemma 4.4 (Forbidden words II). *Let $S \in \{1,2,3\}^{\mathbb{Z}}$. Then*

(1) *If $S = \omega_1^*\,1$, then $\lambda_0(S) > j_0 + 10^{-3}$.*
(2) *If $S = \omega_1^*\,22$ or $\omega_1^*\,23$, then $\lambda_0(S) > j_0 + 10^{-5}$.*
(3) *If $S = 21\,\omega_1^*\,211$ or $21\,\omega_1^*\,212$, then $\lambda_0(S) > j_0 + 10^{-6}$.*
(4) *If $S = 11\,\omega_1^*\,211$, then $\lambda_0(S) > j_0 + 10^{-5}$.*
(5) *If $S = 12111\,\omega_1^*\,2123$ or $22111\,\omega_1^*\,2123$, then $\lambda_0(S) > j_0 + 10^{-7}$.*
(6) *If $S = 332111\,\omega_1^*\,212333$, then $\lambda_0(S) > j_0 + 10^{-8}$.*
(7) *If $S = 2332111\,\omega_1^*\,21233212$, then $\lambda_0(S) > j_0 + 8 \times 10^{-9}$.*
(8) *If $S = 112332111\,\omega_1^*\,2123321112$, then $\lambda_0(S) > j_0 + 3 \times 10^{-10}$.*

Corollary 4.5. *If $S \in \{1,2,3\}^{\mathbb{Z}}$ is such that $S = 21\,\omega_1^*\,21$, then $m(S) > j_0 + 10^{-6}$. Therefore, if $m(S) < j_0 + 10^{-6}$, then the word $21\,\omega_1\,21$ is forbidden.*

Proof. Let $S = 21\,\omega_1^*\,21$. Then according to the Lemma 4.4, if $S_7 \in \{1,2\}$, then $\lambda_0(S) > j_0 + 10^{-6}$. Otherwise, $S_7 = 3$ and $\lambda_7(S) > j_0 + 10^{-2}$. In both cases, we have $m(S) > j_0 + 10^{-6}$. □

Lemma 4.6 (Local uniqueness II). *Let $S \in \{1,2,3\}^{\mathbb{Z}}$ be such that*

$$S = 21233^*2111 = \omega_1^*, \quad \text{and} \quad \forall\, n \in \mathbb{Z},\ \lambda_n(S) \le j_0 + 3 \times 10^{-10},$$

then S must take one of these forms;

(1) *$S = 33\,\omega_1^*\,21$ and $\lambda_0(S) < j_0 - 10^{-4}$,*
(2) *$S = 12\,\omega_1^*\,21$ and $\lambda_0(S) < j_0 - 10^{-5}$,*
(3) *$S = 22\,\omega_1^*\,212$ and $\lambda_0(S) < j_0 - 10^{-6}$,*
(4) *$S = 222\,\omega_1^*\,211$, $211\,\omega_1^*\,212$ or $332\,\omega_1^*\,212$ and $\lambda_0(S) < j_0 - 10^{-6}$,*
(5) *$S = 1111\,\omega_1^*\,212$ and $\lambda_0(S) < j_0 - 10^{-6}$,*
(6) *$S = 1122\,\omega_1^*\,2111$, $1122\,\omega_1^*\,2112$, $2122\,\omega_1^*\,2111$, $2122\,\omega_1^*\,2112$, $2111\,\omega_1^*\,2121$, $2111\,\omega_1^*\,2122$, $2332\,\omega_1^*\,2111$, $2332\,\omega_1^*\,2112$, $3332\,\omega_1^*\,2111$, $3332\,\omega_1^*\,2112$ and $\lambda_0(S) < j_0 - 10^{-6}$,*
(7) *$S = 2111\,\omega_1^*\,2123$.*

As a consequence, if S verifies $|\lambda_0(S) - j_0| \le 10^{-6}$ and $m(S) \le j_0 + 10^{-6}$ then S must be of the form:

$$S = 2111\,\omega_1^*\,2123.$$

Proof. Assuming that $S = 21233^*2111 = \omega_1^*$ then since $\omega_1\,1$ and 13 are forbidden words, S must write itself in such way:

$$S = 1\,\omega_1^*\,2,\ 2\,\omega_1^*2 \quad \text{or} \quad 3\,\omega_1^*2.$$

(1) If $S = 3\,\omega_1^*2$, since the words $13, 232$, $\omega_1\,22$ and $\omega_1\,23$ are forbidden, then S must write itself in such way:

$$S = 33\,\omega_1^*21 \quad \text{and} \quad \lambda_0(S) < j_0 - 10^{-4}.$$

(2) If $S = 2\,\omega_1^*\,2$, then, using the forbidden words $\omega_1\,22$ and $\omega_1\,23$ we have three possibilities:
  (a) $S = 22\,\omega_1^*\,21$,
  (b) $S = 32\,\omega_1^*\,21$,
  (c) $S = 12\,\omega_1^*\,21$ and $\lambda_0(S) < j_0 - 10^{-5}$.

We analyze each case separately:
  (a) If $S = 22\,\omega_1^* 21$, then since 13 and 322 are forbidden we have:

$$S = 122\,\omega_1^*\,211 \text{ or }$$
$$\left\{S = 122\,\omega_1^*\,212, 222\,\omega_1^*\,212, 222\,\omega_1^*\,211 \text{ and } \lambda_0(S) < j_0 - 10^{-6}\right\}.$$

If $S = 122\,\omega_1^*\,211$, then since the words 13 and 31 are forbidden, we must have:

$$S = 1122\,\omega_1^*\,2111, 1122\,\omega_1^*\,2112, 2122\,\omega_1^*\,2111, 2122\,\omega_1^*\,2112$$
$$\text{and } \lambda_0(S) < j_0 - 10^{-5}.$$

  (b) If $S = 32\,\omega_1^*\,21$ then since, $13, 31$ and 232 are forbidden, we must have:

$$S = 332\,\omega_1^*\,211 \quad \text{or} \quad \left\{S = 332\,\omega_1^*\,212 \text{ and } \lambda_0(S) < j_0 - 10^{-4}\right\}.$$

If $S = 332\,\omega_1^*\,211$, then since $13, 31$ are forbidden, we must have:

$$S = 2332\,\omega_2^*\,2111, 2332\,\omega_2^*\,2112, 3332\,\omega_2^*\,2111, 3332\,\omega_2^*\,2112$$
$$\text{and} \quad \lambda_0(S) < j_0 - 10^{-6}.$$

(3) If $S = 1\,\omega_1^*\,2$, since 31, $\omega_1\,22$ and $\omega_1\,23$ are forbidden, so S can take 2 forms:

$$S = 11\,\omega_1^*\,21 \quad \text{or} \quad 21\,\omega_1^*\,21.$$

However, since the words $21\,\omega_1\,211, 21\,\omega_1\,212$ and 13 are forbidden, the word $21\,\omega_1\,21$ is not allowed.

Hence, if $S = 11\,\omega_1^*\,21$, then, since the words 13 and $11\,\omega_1\,211$ are forbidden, then :

$$S = 111\,\omega_1^*\,212 \quad \text{or} \quad \left\{S = 211\,\omega_1^*\,212 \text{ and } \lambda_0(S) < j_0 - 10^{-5}\right\}.$$

In the first case, since 13 is forbidden, then:
  (a) $S = 1111\,\omega_1^*\,2121, 1111\,\omega_1^*\,2122$ or $2111\,\omega_1^*\,2121$ and $\lambda_0(S) < j_0 - 10^{-5}$,
  (b) $S = 2111\,\omega_1^*\,2122$ or $1111\,\omega_1^*\,2123$ and $\lambda_0(S) < j_0 - 1.8 \times 10^{-6}$,
  (c) $S = 2111\,\omega_1^*\,2123$. □

Lemma 4.7 (Forbidden words III). *Let $S \in \{1,2,3\}^{\mathbb{Z}}$ be such that $m(S) < j_0 + 3 \times 10^{-10}$. Then*

(1) *If $S = 3\,\omega_1\,\omega_1^*\,\omega_1\,21$, then $\lambda_0(S) > j_0 + 1.22 \times 10^{-10}$.*
(2) *If $S = 12\,\omega_1\,\omega_1^*\,\omega_1\,21$, then $\lambda_0(S) > j_0 + 2.7 \times 10^{-11}$.*
(3) *If $S = 22\,\omega_1\,\omega_1^*\,\omega_1\,212$, then $\lambda_0(S) > j_0 + 10^{-10}$.*
(4) *If $S = 222\,\omega_1\,\omega_1^*\,\omega_1\,21$, then $\lambda_0(S) > j_0 + 2 \times 10^{-10}$.*
(5) *If $S = 211\,\omega_1\,\omega_1^*\,\omega_1\,212$, then $\lambda_0(S) > j_0 + 3.15 \times 10^{-11}$.*
(6) *If $S = 1111\,\omega_1\,\omega_1^*\,\omega_1\,212$, then $\lambda_0(S) > j_0 + 1.9 \times 10^{-12}$.*
(7) *If $S = 111\,\omega_1\,\omega_1^*\,\omega_1\,2121$, then $\lambda_0(S) > j_0 + 2 \times 10^{-11}$.*
(8) *If $S = 2111\,\omega_1\,\omega_1^*\,\omega_1\,2122$, then $\lambda_0(S) > j_0 + 2.3 \times 10^{-12}$.*

Proof. All computations are direct, except the item (4), where we have to use that $11\omega_1 211$ and 31 are forbidden to conclude that

$$\begin{aligned}\lambda_0(\dots 222\,\omega_1\,\omega_1^*\,\omega_1\,21\dots) &\geq \lambda_0(\dots 222\,\omega_1\,\omega_1^*\,\omega_1\,212\dots)\\ &> j_0 + 2 \times 10^{-10}.\end{aligned}$$ □

4.2. Sequence development Tree around $j_0 = \lambda_0(\overline{21233^*2111})$. The previous work, of finding forbidden words and small words (i.e. with value less than j_0), can be represented visually by a tree. Indeed, assume that we are looking around the possible extension of a finite word $w_i \in \{1,2,3\}^{(\mathbb{N})}$. In addition, suppose we have a set of forbidden words $F(w_i)$, which are all the forbidden words gathered by developing the sequence until the word w_i. Then, $\forall\,(s_l, s_r) \in \{1,2,3\}^2$ we have 4 possibilities:

(1) The word $s_l\,w_i\,s_r$ contains a forbidden word from $F(w_i)$. This case is immediately deleted.
(2) The word $s_l\,w_i\,s_r$ verifies $\min \lambda_0(\dots s_l\,w_i\,s_r\dots) > j_0$, according to the bounding rules from Lemma 3.7. Then, $s_l\,w_i\,s_r$ is added to the new list of forbidden words. The branch associated with $s_l\,w_i\,s_r$ is then colored in red and ended.
(3) The word $s_l\,w_i\,s_r$ verifies $\max \lambda_0(\dots s_l\,w_i\,s_r\dots) < j_0$ according to the bounding rules from Lemma 3.7. Then, the branch associated with $s_l\,w_i\,s_r$ is colored in blue and ended.
(4) The word $s_l\,w_i\,s_r$ verifies
$$\min \lambda_0(\dots s_l\,w_i\,s_r\dots) < j_0 < \max \lambda_0(\dots s_l\,w_i\,s_r\dots),$$
according to the bounding rules from Lemma 3.7. Then, the branch associated with $s_l\,w_i\,s_r$ is colored in black and must be extended in the next level.

Using these rules, we construct a finite tree from the development of the sequence $\overline{212332111}$.

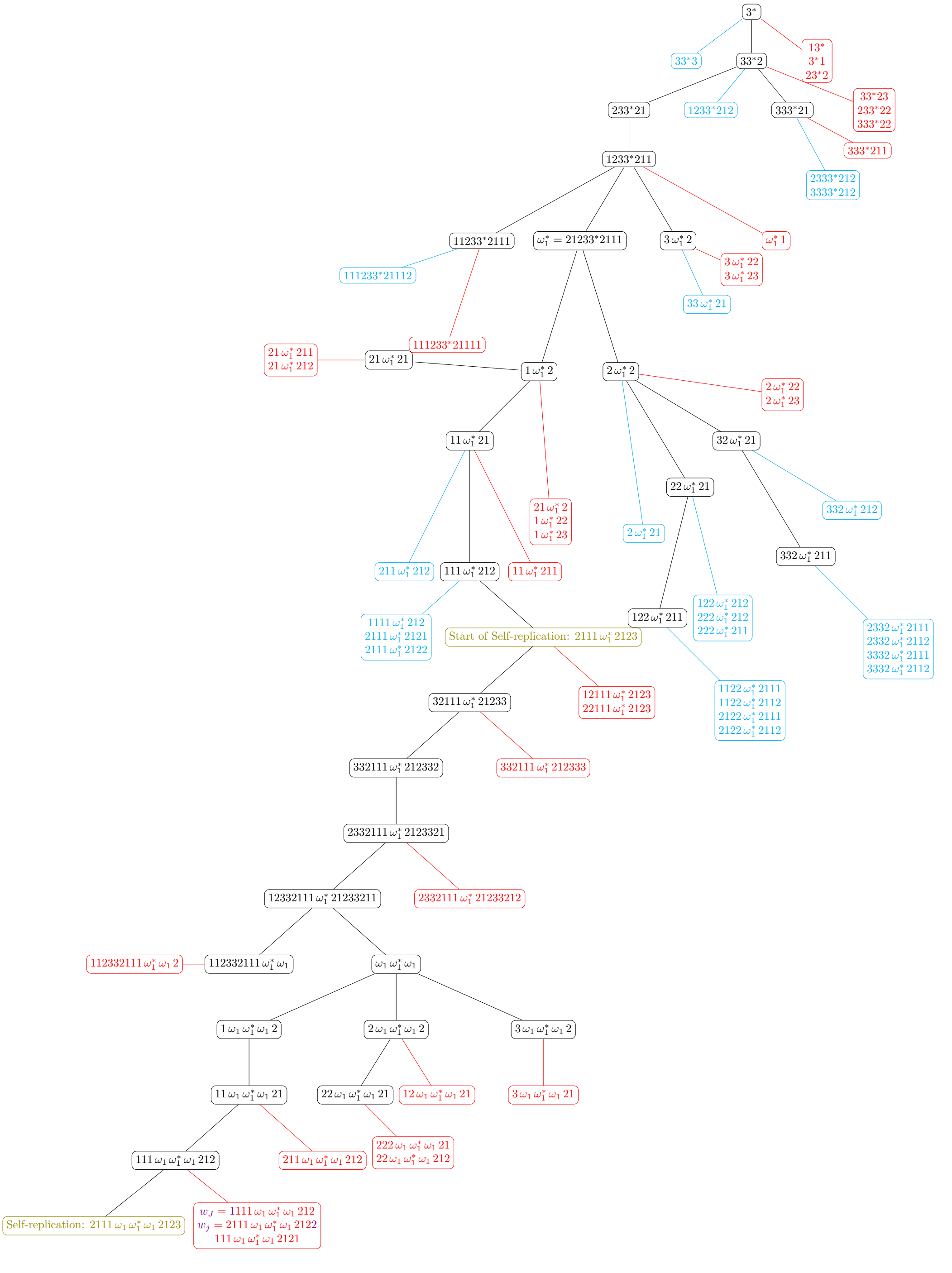

FIGURE 4.1. Sequence development Tree of $\overline{21233^*2111}$

4.3. Self-replication. We have gathered enough forbidden words to establish the self replication property of the function λ_0. Let us call the whole set of forbidden words F_{tot}. We have:

$$F = \left\{\begin{matrix} 13, 232, 323, 322, 333211, 12332112, 11123321111, \omega_1\,1, \omega_1\,23, \omega_1\,22, \\ 21\,\omega_1\,21, 11\,\omega_1\,211, 22111\,\omega_1\,2123, 12111\,\omega_1\,2123, 332111\,\omega_1\,212333, \\ 2332111\,\omega_1\,21233212, 112332111\,\omega_1\,\omega_1\,2, 3\,\omega_1\,\omega_1\,\omega_1\,21, \\ 12\,\omega_1\,\omega_1\,\omega_1\,21, 222\,\omega_1\,\omega_1\,\omega_1\,21, 22\,\omega_1\,\omega_1\,\omega_1\,212, 211\,\omega_1\,\omega_1\,\omega_1\,212, \\ 1111\,\omega_1\,\omega_1\,\omega_1\,212, 111\,\omega_1\,\omega_1\,\omega_1\,2121, 2111\,\omega_1\,\omega_1\,\omega_1\,2122 \end{matrix}\right\},$$

and

$$F_{tot} = F \cup F^T.$$

If we use all of them, we will have self replication on both sides and end by showing that j_0 is isolated in M. To obtain a more comprehensive description of $M \cap (j_0, j_1)$, we need to consider the forbidden words with the smallest λ_0 value.

4.3.1. *Focus on forbidden words with the smallest λ_0 value.* We now focus on words from F_{tot} with small λ_0 value to find the borders of L and M. Let us consider 2 following words:

(1) $w_j = 2111\,\omega_1\,\omega_1\,\omega_1\,2122 \in F_{tot}$.
(2) $w_J = 1111\,\omega_1\,\omega_1\,\omega_1\,212 \in F_{tot}$.

We also denote w_j^* and w_J^* respectively the words $2111\,\omega_1\,\omega_1^*\,\omega_1\,2122$ and $1111\,\omega_1\,\omega_1^*\,\omega_1\,212$ with the position zero marked with asterisk.

Theorem 4.8 (Self-replication)**.** *Let $S \in \{1,2,3\}^{\mathbb{Z}}$ such that:*

$$S_{n-8}\dots S_{n+8} = 2111\,\omega_1^* 2123 \quad \textit{for some } n \in \mathbb{Z}.$$

(1) *If the finite word $S_{n-17}\dots S_{n+17}$ doesn't contain any words of $F_{tot} \setminus \{w_j, w_j^T\}$, then $S_{n-17}\dots S_{n+16} = 2111\,\omega_1\,\omega_1^*\,\omega_1\,212$.*
(2) *If the finite word $S_{n-17}\dots S_{n+17}$ doesn't contain any word of F_{tot}, then $S_{n-17}\dots S_{n+17} = 2111\,\omega_1\,\omega_1^*\,\omega_1\,2123$.*

Proof. Let assume that the finite word $S_{n-17}\dots S_{n+17}$ doesn't contain any word of $F_{tot} \setminus \{w_j, w_j^T\}$, and that we have:

$$S_{-8}\dots S_8 = 2111\,\omega_1^*\,2123.$$

Forbidden words used $12111\,\omega_1\,2123, 22111\,\omega_1\,2123$:

$$S_{-9}\dots S_8 = 32111\,\omega_1^*\,2123.$$

Forbidden words used $13, 232$:

$$S_{-10}\dots S_8 = 332111\,\omega_1^*\,2123.$$

Forbidden words used $13, 333211$:

$$S_{-11}\dots S_8 = 2332111\,\omega_1^*\,2123.$$

Forbidden words used $223, 323$:

$$S_{-12} \dots S_8 = 12332111\, \omega_1^*\, 2123.$$

Forbidden words used $31, 232$:

$$S_{-12} \dots S_9 = 12332111\, \omega_1^*\, 21233.$$

Forbidden words used $31, 332111\, \omega_1\, 212333$:

$$S_{-12} \dots S_{10} = 12332111\, \omega_1^*\, 212332.$$

Forbidden words used $322, 323$:

$$S_{-12} \dots S_{11} = 12332111\, \omega_1^*\, 2123321.$$

Forbidden words used $13, 2332111\, \omega_1\, 21233212$:

$$S_{-12} \dots S_{12} = 12332111\, \omega_1^*\, 21233211.$$

Forbidden words used $13, 12332112$:

$$S_{-12} \dots S_{13} = 12332111\, \omega_1^*\, \omega_1.$$

Forbidden words used $13, \omega_1\, 1$:

$$S_{-12} \dots S_{14} = 12332111\, \omega_1^*\, \omega_1\, 2.$$

Forbidden words used $\omega_1\, 22, \omega_1\, 23$:

$$S_{-12} \dots S_{15} = 12332111\, \omega_1^*\, \omega_1\, 21.$$

Forbidden words used $11\, \omega_1\, 211, 13$:

$$S_{-12} \dots S_{16} = 12332111\, \omega_1^*\, \omega_1\, 212.$$

Forbidden words used $31, 112332111\, \omega_1\, \omega_1\, 2$:

$$S_{-13} \dots S_{16} = \omega_1\, \omega_1^*\, \omega_1\, 212.$$

Forbidden words used $3\, \omega_1\, \omega_1\, \omega_1\, 21, 12\, \omega_1\, \omega_1\, \omega_1\, 21, 22\, \omega_1\, \omega_1\, \omega_1\, 212, 322$:

$$S_{-14} \dots S_{16} = 1\, \omega_1\, \omega_1^*\, \omega_1\, 212.$$

Forbidden words used $31, 21\, \omega_1\, 21$:

$$S_{-15} \dots S_{16} = 11\, \omega_1\, \omega_1^*\, \omega_1\, 212.$$

Forbidden words used $31, 211\, \omega_1\, \omega_1\, \omega_1\, 212$:

$$S_{-16} \dots S_{16} = 111\, \omega_1\, \omega_1^*\, \omega_1\, 212.$$

Forbidden words used $31, 1111\, \omega_1\, \omega_1\, \omega_1\, 212 = w_J$:

$$S_{-17} \dots S_{16} = 2111\, \omega_1\, \omega_1^*\, \omega_1\, 212.$$

At this point we obtain the first result, since we only used words from the set $F_{tot} \setminus \{w_j\}$.

If w_j is forbidden in the sequence $S_{n-17}\dots S_{n+17}$, then using the forbidden words, $111\,\omega_1\,\omega_1\,\omega_1\,2121, w_j = 2111\,\omega_1\,\omega_1\,\omega_1\,2122$ we must have:

$$S_{-17}\dots S_{17} = 2111\,\omega_1\,\omega_1^*\,\omega_1\,2123,$$

which proves the second point of the theorem. □

Using the Theorem 4.8 repeatedly, we have the two following results.

Corollary 4.9. *Let $S \in \{1,2,3\}^{\mathbb{Z}}$ be such that: $S_{-8}\dots S_8 = 2111\,\omega_1^* 2123$.*

(1) *Then, if all the words from $F_{tot} \setminus \{w_j, w_j^T\}$ are forbidden in S, then we must have:*

$$S = \overline{\omega_1}\,\omega_1^*\,\omega_1\,212S_r \text{ with } S_r \in \{1,2,3\}^{\mathbb{N}}.$$

(2) *If all the words from F_{tot} are forbidden in S, then we must have:*

$$S = \overline{\omega_1} = \overline{21233^*2111}.$$

Lemma 4.10. *We have:*

(1)

$$\min\left\{ \lambda_0(S_l^T w_j^* S_r) \;\middle|\; \begin{array}{l} S_l, S_r \in \{1,2,3\}^{\mathbb{N}} \text{ with } \lambda_0(S_l^T w_j^* S_r) = m(S_l^T w_j^* S_r) \\ \text{and } S_l^T w_j^* S_r \text{ doesn't containany words} \\ \text{from } F_{tot} \setminus \{w_j, w_j^T\} \end{array} \right\}$$
$$= \lambda_0(\overline{\omega_1}\,\omega_1\,\omega_1^*\,\omega_1\,2122\,\overline{12}) \in M.$$

(2)

$$\min\left\{ \lambda_0(S_l^T\, w_J^*\, S_r) \;\middle|\; \begin{array}{l} S_r, S_l \in \{1,2,3\}^{\mathbb{N}}, \lambda_0(S_l^T\, w_J^*\, S_r) = m(S_l^T\, w_J^*\, S_r) \\ \text{and } S_l^T w_J^* S_r \text{ doesn't containany words} \\ \text{from } F_{tot} \setminus \{w_j, w_j^T, w_J, w_J^T\} \end{array} \right\}$$
$$= \lambda_0(\overline{21}\,1111\,\omega_1\,\omega_1^*\,\omega_1\,\omega_1\,2122\,1212\,\overline{333212}) \in M.$$

For the future, we call:

$$j = \lambda_0(\overline{\omega_1}\,\omega_1\,\omega_1^*\,\omega_1\,2122\,\overline{12}),\;\; J = \lambda_0(\overline{21}\,1111\,\omega_1\,\omega_1^*\,\omega_1\,\omega_1\,2122\,1212\,\overline{333212}).$$

We get numerically:

(1) $j \approx j_0 + 8.32039 \times 10^{-12}$.
(2) $J \approx j_0 + 8.42651 \times 10^{-12}$.

Remark 4.11. We will show later that the number j is the first element of the set $(M \setminus L) \cap (j_0, j_1)$ and that $J = j_1$ is the right border of the gap of L.

Proof of Lemma 4.10.

Proof of j. Let $S = S_l^T w_j^* S_r$ be such that S doesn't contain any words from $F_{tot} \setminus \{w_j, w_j^T\}$. Then, $S_{-8} \dots S_8 = 2111\,\omega_1^* 2123$ and according to the Corollary 4.9, we must have $S = \overline{\omega_1}\,\omega_1^*\,\omega_1\,2122 S_{18} \dots$.

Since we are minimizing the function λ_0, we must take $S_{18} = 1$ and because the word 13 is forbidden, we must continue with the periodic sequence $\overline{12}$. The sequence that minimises λ_0 is:

$$S = \overline{\omega_1}\,\omega_1\,\omega_1^*\,\omega_1\,2122\,\overline{12} = \overline{\omega_1}\,21233\,w_j^*\,\overline{12}.$$

Proof of J. Let $S = S_l^T w_J^* S_r$ such that S doesn't contain any words from $F_{tot} \setminus \{w_j, w_j^T, w_J, w_J^T\}$. Again, we have:

$$S = \dots S_{-18}\,1111\,\omega_1\,\omega_1^*\,\omega_1\,212 S_{17} \dots$$

In order to minimise λ_0, since 13 is forbidden, the left extension must be:

$$\dots S_{18} = \overline{21}$$

So:

$$S = \overline{21}\,1111\,\omega_1\,\omega_1^*\,\omega_1\,212 S_{17} \dots$$

Again, in order to minimise λ_0, we must take $S_{17} = 3$ and therefore we have:

$$S_1 \dots S_{17} = 2111\,\omega_1\,2123$$

Using the second point of the Theorem 4.8, we must have:

$$S = \overline{21}\,1111\,\omega_1\,\omega_1^*\,\omega_1\,\omega_1\,212 S_{26} \dots$$

Since the word $111\,\omega_1\,\omega_1\,\omega_1\,2121$ is forbidden (Lemma 4.7 (7)) and that we minimise $\lambda_0(S)$, we must have $S_{26} = 2$ and

$$S = \overline{21}\,1111\,\omega_1\,\omega_1^*\,\omega_1\,\omega_1\,2122 S_{27} \dots$$

At this stage, we have a competition between the value of $\lambda_0(S)$ and $\lambda_9(S)$. Indeed, we have to extend S in such a way that $\lambda_9(S)$ is small enough so we can have $\lambda_9(S) \le \lambda_0(S)$ and that in addition, $\lambda_0(S)$ is minimised. However, we know that:

$$\lambda_9(\overline{21}\,1111\,\omega_1\,\omega_1^*\,\omega_1\,\omega_1\,2122\,\overline{12}) = \lambda_0(\overline{21}\,1111\,\omega_1\,\omega_1\,\omega_1^*\,\omega_1\,2122\,\overline{12}) < j.$$

So we define, for $n \ge 0$:

$$L_n^{(9)} = \max\left\{ \lambda_9(\overline{21}\,1111\,\omega_1\,\omega_1^*\,\omega_1\,\omega_1\,2122[12]_n T) \;\middle|\; \begin{array}{l} T \text{ doesn't contain} \\ \text{any words from} \\ F_{tot} \setminus \{w_j, w_j^T, w_J, w_J^T\} \end{array} \right\},$$

$$l_n^{(0)} = \min\left\{ \lambda_0(\overline{21}\,1111\,\omega_1\,\omega_1^*\,\omega_1\,\omega_1\,2122[12]_n T) \;\middle|\; \begin{array}{l} T \text{ doesn't contain} \\ \text{any words from} \\ F_{tot} \setminus \{w_j, w_j^T, w_J, w_J^T\} \end{array} \right\},$$

where $[12]_n$ represents the n^{th} first terms of the sequence $(\overline{12})_{k\geq 0}$.

Let us determinate the exact value of $l_{2n}^{(0)}$. We minimise the λ_0 value of sequence of following form $S = \overline{21}\,1111\,\omega_1\,\omega_1^*\,\omega_1\,\omega_1\,2122(12)^n S_{27+2n}\ldots$, so we must have:

$$S = \overline{21}\,1111\,\omega_1\,\omega_1^*\,\omega_1\,\omega_1\,2122\,(12)^n 3S_{28+2n}\ldots$$

But since the words 31 and 232 are forbidden, we must have:

$$S = \overline{21}\,1111\,\omega_1\,\omega_1^*\,\omega_1\,\omega_1\,2122\,(12)^n 33S_{29+2n}\ldots$$

Since the word 31 is forbidden, to minimise λ_0, we must have:

$$S = \overline{21}\,1111\,\omega_1\,\omega_1^*\,\omega_1\,\omega_1\,2122\,(12)^n 333S_{30+2n}\ldots$$

Since, the word 31 is forbidden, and that we are minimising λ_0, we must have:

$$S = \overline{21}\,1111\,\omega_1\,\omega_1^*\,\omega_1\,\omega_1\,2122\,(12)^n 3332S_{31+2n}\ldots$$

Since, the words 323 and 322 are forbidden, we must have:

$$S = \overline{21}\,1111\,\omega_1\,\omega_1^*\,\omega_1\,\omega_1\,2122\,(12)^n 33321S_{32+2n}\ldots$$

Since the word 333211 is forbidden, we must have:

$$S = \overline{21}\,1111\,\omega_1\,\omega_1^*\,\omega_1\,\omega_1\,2122\,(12)^n 333212S_{33+2n}\ldots$$

Then, we repeat the process by induction, since $2n+33$ and $2n+27$ have the same parity. So finally, we have:

$$S = \overline{21}\,1111\,\omega_1\,\omega_1^*\,\omega_1\,\omega_1\,2122\,(12)^n\,\overline{333212}.$$

And:

$$l_{2n}^{(0)} = \lambda_0\big(\overline{21}\,1111\,\omega_1\,\omega_1^*\,\omega_1\,\omega_1\,2122\,(12)^n\,\overline{333212}\big).$$

Now we determine the exact value of $l_{2n+1}^{(0)}$. If we minimise the λ_0 value of the sequence with the form $S = \overline{21}\,1111\,\omega_1\,\omega_1^*\,\omega_1\,\omega_1\,2122\,(12)^n\,1\,S_{2n+28}\ldots$, then we must have:

$$S = \overline{21}\,1111\,\omega_1\,\omega_1^*\,\omega_1\,\omega_1\,2122\,(12)^n 1\,1\ldots$$

And then, using the fact that the words $13, 31$ are forbidden, we must continue with the periodic sequence $\overline{12}$, so we have:

$$l_{2n+1}^{(0)} = \lambda_0\big(\overline{21}\,1111\,\omega_1\,\omega_1^*\,\omega_1\,\omega_1\,2122\,(12)^n 1\,\overline{12}\big).$$

Since 0 is even and 9 is odd, we have that $\forall\, n \in \mathbb{N}$:

$$L_{2n}^{(9)} = \lambda_9\big(\overline{21}\,1111\,\omega_1\,\omega_1^*\,\omega_1\,\omega_1\,2122\,(12)^n\,\overline{333212}\big)$$

$$L_{2n+1}^{(9)} = \lambda_0\big(\overline{21}\,1111\,\omega_1\,\omega_1^*\,\omega_1\,\omega_1\,2122\,(12)^n\,1\,\overline{12}\big).$$

So finally, we have that:

$$J = \min_{n\in\mathbb{N}}\Big\{\max\Big(l_n^{(0)}, L_n^{(9)}\Big)\Big\}.$$

The sequence $(L_n^{(9)})_{n\geq 0}$ is decreasing to $L^{(9)}$ and the sequence $(l_n^{(0)})_{n\geq 0}$ is increasing to $l^{(0)}$ with:

$$L^{(9)} < j < l^{(0)}.$$

If we define $N = \min\{n \in \mathbb{N} \mid l_n^{(0)} \geq L_n^{(9)}\}$, then we have:

$$J = l_N^{(0)}.$$

We get computationally that $N = 4$ and therefore:

$$J = l_4^{(0)} = \lambda_0(\overline{21}\,1111\,\omega_1\,\omega_1^*\,\omega_1\,\omega_1\,2122\,1212\,\overline{333212}) \approx j_0 + 8.42651 \times 10^{-12}. \qquad \square$$

Proposition 4.12. *Let $S \in \{1,2,3\}^{\mathbb{Z}}$ be a sequence such that $\lambda_0(S) = m(S) = \sup_{n\in\mathbb{Z}} \lambda_n(S)$.*

According to the definition of j and J in Lemma 4.10, we have:

(1) *If $\lambda_0(S) \in M \cap (J, J+\alpha)$, for $\alpha > 0$ small enough, then at least one of the words w_j, w_j^T and w_J, w_J^T must appear in S.*
(2) *If $\lambda_0(S) \in M \cap (j_0, J)$, no forbidden words from $F_{tot} \setminus \{w_j, w_j^T\}$ are allowed to appear in S.*
(3) *If $\lambda_0(S) \in M \cap (j_0, j)$, no forbidden words from F_{tot} are allowed to appear in S.*

Proof. Let $m \in M$ such that $m > j_0$ and $S \in \{1,2,3\}^{\mathbb{Z}}$ a sequence such that:

$$m = \lambda_0(S) = \sup_{n\in\mathbb{Z}} \lambda_n(S).$$

If $m \in (J, J+\alpha)$, with $\alpha < 2 \times 10^{-11}$, then because of Lemma 4.1, Lemma 4.3, Lemma 4.4, Lemma 4.6 and 4.7, necessarily one of w_j, w_j^T and w_J, w_J^T must appear in the sequence S.

Let assume that $m \in (j_0, J)$. By contradiction, suppose that there exists $w_f \in F_{tot} \setminus \{w_j, w_j^T\}$ such that $S = S_l^T w_f S_r$. If $w_f \in F_{tot} \setminus \{w_j, w_j^T, w_J, w_J^T\}$, then according to Lemmas 4.1, 4.4 and 4.7, we must have $\lambda_N(S) > j_0 + 2 \times 10^{-11}$ for $N \in \mathbb{Z}$, which is impossible. *So S doesn't contain any word from $F_{tot} \setminus \{w_j, w_j^T, w_J, w_J^T\}$.*

If S contains w_J, then we have $S = S_l^T w_J S_r$. By definition of J (cf. Lemma 4.10), we must have $\lambda_N(S) \geq J$ for $N \in \mathbb{Z}$ and therefore, $m \geq J$, which is impossible. The same is true with w_J^T. *So S doesn't contain any subwords from $F_{tot} \setminus \{w_j, w_j^T\}$.*

Let assume that $m \in (j_0, j)$. Then according to above, S doesn't contain any words from $F_{tot} \setminus \{w_j, w_j^T\}$. If $S = S_l^T\, w_j\, S_r$ then by definition of j (cf. Lemma 4.10), we must have $\lambda_N(S) \geq j$ and therefore: $m(S) \geq j$. Again, the same is true for w_j^T. Hence, the sequence S doesn't contain any words from F_{tot}. $\square$

The main consequence of all of this is the following Theorem:

Theorem 4.13. *We have:*
(1) $M \cap (j_0, j) = \emptyset$.
(2) $L \cap (j_0, J) = \emptyset$.

Proof. Let assume by contradiction that there exist $m \in M \cap (j_0, j)$. Then we can find a sequence $S \in \{1,2,3\}^{\mathbb{Z}}$ such that $m = \sup_{n\in\mathbb{Z}} \lambda_n(S) = m(S) = \lambda_0(S)$. Then, we have $j_0 < \lambda_0(S) < j < j_0 + 3 \times 10^{-10}$.

So according to the Lemma 4.6, we must have: $S_{-8} \dots S_8 = 2111\,\omega_1^* 2123$.

In addition, according to the Proposition 4.12 the words from the set F_{tot} are forbidden in S. Then, because of the Corollary 4.9, the sequence S must extend in such way:

$$S = \overline{\omega_1},$$

and so $m(S) = j_0$, a contradiction.

Let assume by contradiction that there exists $l \in L \cap (j_0, J)$. We use the fact that the Markov values of periodic sequences are dense in L (see [3, Theorem 2, Chapter 3]). Therefore, $\exists\ (l_n)_{n\in\mathbb{N}} \in L^{\mathbb{N}}$ such that:

$$\forall\, n \in \mathbb{N},\ \exists\, \sigma^{(n)} \in \mathbb{N}^{(\mathbb{N})}, \quad l_n = m\Big(\overline{\sigma^{(n)}}\Big).$$

And:

$$\lim_{n\to\infty} l_n = l.$$

Let us write $\forall\, n \in \mathbb{N}, S^{(n)} = \overline{\sigma^{(n)}}$. We can assume (even if it means taking $n \geq n_0$ with n_0 big enough), $\forall\, n \in \mathbb{N}$, $S^{(n)} \in \{1,2,3\}^{\mathbb{Z}}$, $l_n < J$. So we have $\lambda_0(S^{(n)}) = l_n < J$. Using the Lemma 4.3 and 4.6, we have that $S_{-8}^{(n)} \dots S_8^{(n)} = 2111\,\omega_1\,2123$ (or its transpose). Since $l_n = \lambda_0(S^{(n)}) \in M \cap (j_0, J)$, according to the Proposition 4.12, no forbidden words from $F_{tot} \setminus \{w_j, w_j^T\}$ are allowed to appear in $S^{(n)}$. Therefore, using the Corollary 4.9, we have:

$$\forall\, n \in \mathbb{N},\ \ S^{(n)} = \overline{\omega_1}\,\omega_1^*\,\omega_1\,212\,S_r^{(n)} \ \text{ with } \ S_r^{(n)} \in \{1,2,3\}^{\mathbb{N}}.$$

Hence, $\forall\, n \in \mathbb{N}$, $\overline{\sigma^{(n)}} = \overline{\omega_1}\,\omega_1^*\,\omega_1\,212\,S_r^{(n)}$ and necessarily $\forall\, n \in \mathbb{N}$, $\overline{\sigma^{(n)}} = S^{(n)} = \overline{\omega_1}$. So, for all $n \in \mathbb{N}$, we have $l_n = m(\overline{\omega_1}) = j_0$ and $l = j_0$, which is impossible. □

4.4. The description of $(M \setminus L) \cap (j_0, j_1)$. Now, let us characterize the set $(M \setminus L) \cap (j_0, J)$.

Lemma 4.14. *We have:*
(1) $\lambda_0(\overline{\omega_1}\,\omega_1^*\,\omega_1\,21223\dots) > \lambda_0(\overline{\omega_1}\,\omega_1^*\,\omega_1\,21222\dots) > J$,
(2) $\lambda_0(\overline{\omega_1}\,\omega_1^*\,\omega_1\,212211\dots) > J$,
(3) $\lambda_0(\overline{\omega_1}\,\omega_1^*\,\omega_1\,2122123\dots) > \lambda_0(\overline{\omega_1}\,\omega_1^*\,\omega_1\,2122122\dots) > J$,
(4) $\lambda_0(\overline{\omega_1}\,\omega_1^*\,\omega_1\,21221211\dots) > J$.

Corollary 4.15. *Let $m \in M \cap (j, J)$ and $S \in \{1,2,3\}^{\mathbb{Z}}$ be a sequence such that $m = m(S) = \lambda_0(S)$. Then we have, up to transposition:*

$$S = \overline{\omega_1}\, \omega_1^*\, \omega_1\, 21221212\, S_r,$$

satisfying

(1) $1212 S_r$ *doesn't contain any words from*

$$F_{tot} \setminus \{w_j, w_j^T\} \cup \{2111\, \omega_1\, 2123\},$$

(2) *If $\exists\, n \in \mathbb{Z}$ such that $S_{n-8} \dots S_{n+8} = 3212\, \omega_1^T\, 1112$, then we have $S = \dots S_{n-16}\, 212\, \omega_1^T\, \overline{\omega_1^T}$.*

Proof. Let $m(S) = \lambda_0(S) \in (j_0, J)$. According to the Lemma 4.3 and 4.6, we have:

$$S_{-8} \dots S_8 = 2111\, \omega_1^*\, 2123.$$

Since $m(S) < J$, the sequence S doesn't contain any words from $F_{tot} \setminus \{w_j, w_j^T\}$. Then, because of the Corollary 4.9, we have:

$$S = \overline{\omega_1}\, \omega_1^*\, \omega_1\, 212 \dots$$

If $S = \overline{\omega_1}\, \omega_1^*\, \omega_1\, 2121 \dots$, then according to Lemma 4.7(7), $\lambda_0(S) > j_0 + 2 \times 10^{-11} > J$, which is impossible. If $S = \overline{\omega_1}\, \omega_1^*\, \omega_1\, 2123 \dots$, then $\lambda_0(S) < \lambda_0(\overline{\omega_1}\, \omega_1^*\, \omega_1\, 2122\overline{12}) = j$ which again, is impossible.

So $S = \overline{\omega_1}\, \omega_1^*\, \omega_1\, 2122 \dots$. However, since we have $\lambda_0(S) < J$, using the forbidden word 13 and the Lemma 4.14, we must have:

$$S = \overline{\omega_1}\, \omega_1^*\, \omega_1\, 21221212\, S_r.$$

- If $1212 S_r$ contains any words from $F_{tot} \setminus \{w_j, w_j^T\}$, we have obviously $m(S) > J$.
- If $1212 S_r$ contains the subword $2111\, \omega_1\, 2123$, then there exists an index $n \geq 22$ such that

  $$S_{n-8} \dots S_n \dots S_{n+8} = 2111\, \omega_1\, 2123.$$

  By applying self-replication to the left, as stated in Theorem 4.8, we obtain

  $$\dots S_n \dots S_{n+17} = \dots \omega_1 \omega_1 \omega_1.$$

  However, this leads to a contradiction: the subword 1212 appears in $(S_k)_{k \leq 22}$ but does not belong to $\overline{\omega_1}$.
- If S contains $3212\, \omega_1^T\, 1112$, we have $n \in \mathbb{Z}$ such that $S_{n-8} \dots S_{n+8} = 3212\, \omega_1^T\, 2111$ then because of self-replication (Theorem 4.8), applied with ω_1^T, we must have $S_{n-16} \dots S_{n+4} \dots = 212\, \omega_1^T\, \overline{\omega_1^T}$. □

We can now make a global description of the set $(M \setminus L) \cap (j, J)$.

First, let us call $\widetilde{F}$ the set of the forbidden words (from F_{tot}) that doesn't contain the subword $2111\, \omega_1\, 2123$. We call $\widetilde{F}^T$ the set of forbidden words (from F_{tot}^T) that doesn't contain the subword $3212\,\omega_1^T\,1112$. We have, according to Lemma 4.1, 4.4 and 4.7:

$$\widetilde{F} = \left\{ \begin{matrix} 13, 232, 323, 322, 333211, 12332112, 11123321111, \\ \omega_1\,1, \omega_1\,22, \omega_1\,23, 21\,\omega_1\,21, 11\,\omega_1\,211 \end{matrix} \right\}.$$

We can improve the set of forbidden words:

Lemma 4.16. *Let $S \in \{1,2,3\}^{\mathbb{Z}}$.*

(1) *If $S = \dots 3321111\dots$, then, $m(S) > j_0 + 10^{-4}$.*
(2) *If $S = \dots 332112\dots$, then, $m(S) > j_0 + 10^{-4}$.*

Proof. Let $S \in \{1,2,3\} \in \mathbb{Z}$ with $S = \dots 3321111\dots$. Then using the forbidden words 31 and 333211, we have, according to Lemma 4.1:

$$m(S) > j_0 + 10^{-3} \text{ or } S = \dots 23321111\dots$$

In the last case, using the forbidden words 323 and 322, according to the Lemma 4.1 and 4.4 we have:

$$m(S) > j_0 + 10^{-4} \text{ or } S = \dots 123321111\dots$$

In the last case, using the forbidden words 31 and $\omega_1\,1$, according to the Lemma 4.1 and 4.4 we have:

$$m(S) > j_0 + 10^{-3} \text{ or } S = \dots 1123321111\dots$$

In the last case, using the forbidden words 31, $(12332112)^T$ and 11123321111, according to the Lemma 4.1 we have:

$$m(S) > j_0 + 10^{-4}.$$

Hence, in every case we have: $m(S) > j_0 + 10^{-4}$. Using the same reasoning with the forbidden words $13, 333211, 322, 323$ and 12332112 we see that: If $S = \dots 332112$, then $m(S) > j_0 + 10^{-4}$. □

The new set of forbidden words is:

$$\widetilde{F_1} = \left\{ \begin{matrix} 13, 232, 323, 322, 333211, 332112, 3321111, \\ \omega_1 22, \omega_1 23, 21\omega_1 2, 1\omega_1 211 \end{matrix} \right\}.$$

We finally add the self-replicating word $2111\,\omega_1\,2123$ and its transpose $3212\,\omega_1^T\,1112$, so we get:

$$\widetilde{F_2} = \widetilde{F_1} \cup \widetilde{F_1}^T \cup \{2111\,\omega_1\,2123, 3212\,\omega_1^T\,1112\}.$$

Theorem 4.17. *We have:*

$$(M \setminus L) \cap (j, J) = C \cup D \cup X,$$

where

$$X = \left\{ \begin{matrix} \lambda_0(\overline{\omega_1}\omega_1^*\omega_1 21221212212\omega_1^T\overline{\omega_1^T}), \lambda_0(\overline{\omega_1}\omega_1^*\omega_1 2122121212212\omega_1^T\overline{\omega_1^T}), \\ \lambda_0(\overline{\omega_1}\omega_1^*\omega_1 212212121212212\omega_1^T\overline{\omega_1^T}) \end{matrix} \right\},$$

$$D = \left\{ \lambda_0(\overline{\omega_1}\, \omega_1^*\, \omega_1\, 21221212\, s\, 21212212\, \omega_1^T\, \overline{\omega_1^T}) \;\middle|\; \begin{array}{l} 212\, s\, 212 \in \{1,2,3\}^{(\mathbb{N})} \\ \textit{doesn't contain any} \\ \textit{words from } \widetilde{F_2} \textit{ and} \\ [0,s] \leq [0,s^T] \end{array} \right\},$$

and

$$C = \left\{ \lambda_0(\overline{\omega_1}\omega_1^*\omega_1 21221212 S) \;\middle|\; S \in \{1,2,3\}^{\mathbb{N}} \textit{ doesn't contain words from } \widetilde{F_2} \right\}.$$

Proof. Let $S \in \{1,2,3\}^{\mathbb{Z}}$ be a sequence such that $m(S) = \lambda_0(S) \in M \cap (j, J)$. Then according to the Corollary 4.15, up to transposition we have $S = \overline{\omega_1}\, \omega_1^*\, \omega_1\, 21221212\, S_r$, with $S_r \in \{1,2,3\}^{\mathbb{N}}$ not containing any words from $F_{tot} \setminus \{w_j, w_j^T\} \cup \{2111\, \omega_1\, 2123\}$.

If S doesn't contain the word $3212\, \omega_1^T\, 1112$, then the set of forbidden words can be simplified and the sequence S doesn't contain any word from $\widetilde{F_2}$. So $m(S) \in C$.

Otherwise, let $N = \min\{n \in \mathbb{N} / 3212\, \omega_1^T\, 1112 \text{ appears in} (S_k)_{k \geq n}\}$. Then, we have $S_N \dots S_{N+16} = 3212\, \omega_1^T 1112$ and because of Corollary 4.15, we must have:

$$S_{N-8} \dots S_N \dots = 212\, \omega_1^T\, \overline{\omega_1^T}.$$

By definition of N we must have $S_{N-9} \in \{1,2\}$ and using the Lemma 4.14, we must have:

$$S_{N-13} \dots = 21212212\, \omega_1^T\, \overline{\omega_1^T}.$$

Then we have:

(1) $S = \overline{\omega_1}\, \omega_1^*\, \omega_1\, 21221212212\, \omega_1^T\, \overline{\omega_1^T},$ $(N = 30)$,
(2) $S = \overline{\omega_1}\, \omega_1^*\, \omega_1\, 2122121212212\, \omega_1^T\, \overline{\omega_1^T},$ $(N = 32)$,
(3) $S = \overline{\omega_1}\, \omega_1^*\, \omega_1\, 212212121212212\, \omega_1^T\, \overline{\omega_1^T},$ $(N = 34)$,
(4) $S = \overline{\omega_1}\, \omega_1^*\, \omega_1\, 21221212\, s\, 21212212\, \omega_1^T\, \overline{\omega_1^T},$ $(N \geq 35)$.

In the last case, we also need to add the conditions:

$$\lambda_0\Big(\overline{\omega_1}\, \omega_1^*\, \omega_1\, 21221212\, s\, 21212212\, \omega_1^T\, \overline{\omega_1^T}\Big) \geq \lambda_0\Big(\overline{\omega_1}\, 21221212\, s\, 21212212\, \omega_1^T\, (\omega_1^T)^*\, \overline{\omega_1^T}\Big),$$

which is equivalent to $[0,s] \leq [0,s^T]$, and also that the sequence $212\, s\, 212$ doesn't contain any words from the simplified set $\widetilde{F_2}$. So $m(S) \in D$ and $(M \setminus L) \cap (j, J) \subset C \cup D \cup X$.

Now we prove the reverse inclusion. We start by assuming $m \in C$. Then $m = \lambda_0(S)$ with $S = \overline{\omega_1}\, \omega_1^*\, \omega_1\, 21221212\, S_r$ and $1212 S_r$ not containing any words from $\widetilde{F_1} \cup \widetilde{F_1}^T \cup \{2111\, \omega_1\, 2123, 3212\, \omega_1^T\, 1112\}$. Clearly, we have $m \in (j, J)$. We want to show that $m = \sup_{n \in \mathbb{Z}} \lambda_n(S)$ and therefore $m \in M \cap (j, J)$.

Firstly, we have, $\forall\, k \in \mathbb{N}^*$, $\lambda_{-9k}(S) = \lambda_0(\dots \omega_1\, \omega_1\, \omega_1^*\, \omega_1\, \omega_1\, \dots) < j_0 + 10^{-14} < j$.

We also have $\lambda_9(S) = \lambda_0(\dots 2111\, \omega_1^*\, 2122 \dots) < j_0 - 10^{-6}$ according to the Lemma 4.6, so:

$$\sup_{n \leq 20} \lambda_n(S) \leq \lambda_0(S).$$

If we assume by contradiction that $\exists\, n \geq 21$ such that $\lambda_n(S) > \lambda_0(S) > j$, since S doesn't contain any words from $\widetilde{F_2}$ and that $\lambda_n(S) > j_0$, using Lemmas 4.3 and 4.6 (ignoring the cases where $\lambda_n(S) < j_0$), we see that S must take one of these forms:

$$S_{n-8} \dots S_{n+8} = 2111\, \omega_1^*\, 2123 \quad \text{or} \quad 3212\, (\omega_1^*)^T\, 1112,$$

which is impossible by definition of S. Then, $\forall\, n \geq 21, \lambda_n(S) \leq \lambda_0(S)$ and therefore $\lambda_0(S) = m(S)$. So $C \subset (M \setminus L) \cap (j, J)$. The same reasoning gives $D \subset (M \setminus L) \cap (j, J)$. □

Lemma 4.18. *The set D previously defined is a set of isolated points of $M \setminus L$.*

Proof. Let $m \in D$ and s be a finite sequence such that

$$m = \lambda_0(\overline{\omega_1}\, \omega_1^*\, \omega_1\, 21221212\, s\, 21212212\, \omega_1^T\, \overline{\omega_1^T}),$$

with s such that $212\, s\, 212 \in \{1, 2, 3\}^{(\mathbb{N})}$ doesn't contain any words from $\widetilde{F}_1 \cup \widetilde{F}_1^T \cup \{2111\, \omega_1\, 2123, 3212\, \omega_1^T\, 1112\}$.

Let assume there is a sequence $(m_n)_{n \in \mathbb{N}} \in M^{\mathbb{N}}$ such that:

$$m = \lim_{n \to \infty} m_n.$$

We can assume that we have in addition $\forall\, n \in \mathbb{N}$, $m_n \in M \cap (j, J)$. Hence, according to the Corollary 4.15, $\forall\, n \in \mathbb{N}$, $\exists\, S^{(n)}$ a sequence such that:

$$m_n = \lambda_0(\overline{\omega_1}\, \omega_1^*\, \omega_1\, 21221212\, S^{(n)}).$$

Using the Proposition 3.6, we can find a integer n_0 such that $\forall\, n \geq n_0$, we have:

$$S^{(n)} = s\, 21212212\, \omega_1^T \omega_1^T \omega_1^T\, \widetilde{S}^{(n)}.$$

So the sequence $S^{(n)}$ contains the words $3212\, \omega_1^T 1112$ and according to the Corollary 4.15, we must have:

$$\forall\, n \geq n_0, \quad S^{(n)} = s\, 21212212\, \overline{\omega_1^T}.$$

Therefore $\forall\, n \geq n_0,\ m_n = m$. Hence, m is an isolated point in $(M \setminus L) \cap (j, J)$. □

Corollary 4.19. *We have :*

$$\max(M \setminus L) \cap (j, J) = \lambda_0(\overline{\omega_1}\, \omega_1^*\, \omega_1\, 212212\, \overline{123332}) \approx j + 4.4064196 \times 10^{-14}.$$

Thus:

$$(M \setminus L) \cap (j_0, J) \subset (j, j + 4.4064196 \times 10^{-14}).$$

Proof. This is a computation, we want to solve the following problem:

$$\max\left\{ \lambda_0(\overline{\omega_1}\,\omega_1^*\,\omega_1\,21221212\,S_r) \;\middle|\; \begin{array}{l} S_r \text{ doesn't contain forbidden words} \\ \text{from } \widetilde{F_1} \cup \widetilde{F_1}^T \cup \{2111\,\omega_1\,2123\} \end{array} \right\}.$$

Let $S = \overline{\omega_1}\,\omega_1^*\,\omega_1\,21221212\,S_{22}\dots$ be a sequence that maximise λ_0 and that doesn't contain forbidden words from $\widetilde{F_1} \cup \widetilde{F_1}^T \cup \{2111\,\omega_1\,2123\}$. We have:

$$S = \overline{\omega_1}\,\omega_1^*\,\omega_1\,21221212\,3S_{23}\dots$$

Then using the forbidden words $31, 232, 323, 322$ and 333211, we must have:

$$S = \overline{\omega_1}\,\omega_1^*\,\omega_1\,21221212\,333212S_{28}\dots$$

Using the same argument repeatedly, since both 22 and 28 are even, we have:

$$S = \overline{\omega_1}\,\omega_1^*\,\omega_1\,212212\,\overline{123332}.$$ □

4.5. The local border of L.

Theorem 4.20. *$J \in L' \subset L$ and therefore (j_0, J) is the largest gap to the right of j_0 and we found that $j_1 = J$.*

Remark 4.21. With the notation of Theorem 1.5, this shows that $A_1 = (j_0, j_1)$.

Proof. By the proof of Lemma 4.10 we have

(4.1)

$$J = \min\left\{ \lambda_0(S_l^T\,1111\,\omega_1\omega_1^*\omega_1\omega_1\,2122\,1212\,S_r) \;\middle|\; \begin{array}{l} (S_r, S_l) \in \{1,2,3\}^{\mathbb{N}} \text{ and} \\ S_l^T w_J S_r \text{ doesn't contain} \\ \text{any words from} \\ F_{tot} \setminus \{w_j, w_j^T, w_J, w_J^T\} \end{array} \right\}$$

$$= \lambda_0(\overline{21}\,1111\,\omega_1\,\omega_1^*\,\omega_1\,\omega_1\,2122\,1212\,\overline{333212}) \approx j_0 + 8.42651 \times 10^{-12}.$$

Let us call, for all $n \geq 2$ big enough:

$$S^{(n)} = \overline{(21)^n\,1111\,\omega_1\,\omega_1^*\,\omega_1\,\omega_1\,2122\,1212\,(333212)^n}.$$

We have:

(1) $\lambda_{-9}(S^{(n)}) = \lambda_0(\dots 1111\,\omega_1^*\,212\dots) < j_0 - 10^{-6}$ according to Lemma 4.6 (5).
(2) $\lambda_0(S^{(n)}) \geq J$ by the above characterization of J.

(3)

$\lambda_9(S^{(n)})$
$= \lambda_0(\dots S_{-27-2n}(21)^n 1111\omega_1\omega_1\omega_1^*\omega_1 21221212(333212)^n S_{22+6n}\dots)$
$\leq \lambda_0(\overline{21}\,1111\,\omega_1\,\omega_1\,\omega_1^*\,\omega_1\,2122\,1212\,(333212)^n\,3\dots)$
$\leq \lambda_0(\overline{21}\,1111\,\omega_1\,\omega_1\,\omega_1^*\,\omega_1\,2122\,1212\,\overline{333212}) + \varepsilon(n)$.

By direct computation we have:

$$\lambda_0(\overline{21}\,1111\,\omega_1\,\omega_1\,\omega_1^*\,\omega_1\,2122\,1212\,\overline{333212}) < J.$$

Therefore, for n big enough, we have

$$\varepsilon(n) \leq J - \lambda_0(\overline{21}\,1111\,\omega_1\,\omega_1\,\omega_1^*\,\omega_1\,2122\,1212\,\overline{333212})$$

and so $\lambda_9(S^{(n)}) < J$.

(4) $\lambda_{18}(S^{(n)}) = \lambda_0(\dots 2111\,\omega_1^*\,2122\dots) < j_0 - 10^{-6}$ according to Lemma 4.6 (6).

(5) Lastly, $\forall\ 0 \leq k < n$:

$$\max_{i\in\{-1,0,1\}} \lambda_{6k+32+i}(S) = \lambda_{6k+31}(S) = \lambda_0(\dots 2123^*33212\dots) < j_0 - 10^{-3},$$

according to Lemma 4.3 (4).

So we have:

$$\begin{aligned} l_n &= m\Big(\overline{(21)^n\,1111\,\omega_1\,\omega_1^*\,\omega_1\,\omega_1\,2122\,1212\,(333212)^n}\Big)\\ &= l\Big(\overline{(21)^n\,1111\,\omega_1\,\omega_1^*\,\omega_1\,\omega_1\,2122\,1212\,(333212)^n}\Big)\\ &= \lambda_0\Big(\overline{(21)^n\,1111\,\omega_1\,\omega_1^*\,\omega_1\,\omega_1\,2122\,1212\,(333212)^n}\Big). \end{aligned}$$

Therefore, $\forall\ n \geq 2$, we have $l_n \in L$, and we see that:

$$\lim_{n\to\infty} l_n = J.$$

Since L is closed, we have $J \in L$. □

Theorem 4.22. *We have,* $\forall\ \alpha > 0$, $HD(M \cap (j_1, j_1 + \alpha)) = 1$.

Proof. Let $\alpha > 0$. Using the continuity of the function λ_0, there is a $n_0 \in \mathbb{N}$ such that $\forall\ n > n_0$, $\forall\ S_l, S_r \in \{1,2\}^{\mathbb{N}}$:

$$\lambda_0\Big(S_l^T\,(21)^n\,1111\,\omega_1\,\omega_1^*\,\omega_1\,\omega_1\,2122\,1212(333212)^n\,S_r\Big) \in [j_1, j_1 + \alpha).$$

In addition, we have:

$$\begin{aligned} &\lambda_0\Big(S_l^T\,(21)^n\,1111\,\omega_1\,\omega_1^*\,\omega_1\,\omega_1\,2122\,1212(333212)^n\,S_r\Big)\\ &\qquad = m\Big(S_l^T\,(21)^n\,1111\,\omega_1\,\omega_1^*\,\omega_1\,\omega_1\,2122\,1212(333212)^n\,S_r\Big) \end{aligned}$$

Indeed, if we call $S = S_l^T\,(21)^n\,1111\,\omega_1\,\omega_1^*\,\omega_1\,\omega_1\,2122\,1212(333212)^n\,S_r$, we have:

(1) $\lambda_{-9}(S) = \lambda_0(\dots 1111\,\omega_1^*\,212\dots) < j_0 - 10^{-6}$ according to Lemma 4.6 (5).
(2) $\lambda_0(S) \geq J$ from the characterization (4.1).
(3)

$$\begin{aligned}&\lambda_9(S)\\ &= \lambda_0(\dots S_{-27-2n}(21)^n 1111\omega_1\omega_1\omega_1^*\omega_1 21221212(333212)^n S_{22+6n}\dots)\\ &\leq \lambda_0(\overline{21}\,1111\,\omega_1\,\omega_1\,\omega_1^*\,\omega_1\,2122\,1212\,(333212)^n\,3\dots)\\ &\leq \lambda_0(\overline{21}\,1111\,\omega_1\,\omega_1\,\omega_1^*\,\omega_1\,2122\,1212\,\overline{333212}) + \varepsilon(n)\end{aligned}$$

By direct computation as before we obtain:

$$\lambda_9(S) \leq \lambda_0(\overline{21}\,1111\,\omega_1\,\omega_1\,\omega_1^*\,\omega_1\,2122\,1212\,\overline{333212}) + \varepsilon(n) < J.$$

(4) $\lambda_{18}(S) = \lambda_0(\dots 2111\,\omega_1^*\,2122\dots) < j_0 - 10^{-6}$ according to Lemma 4.6 (6).
(5) Lastly, $\forall\, 0 \leq k < n$:

$$\max_{i\in\{-1,0,1\}} \lambda_{6k+32+i}(S) = \lambda_{6k+31}(S) = \lambda_0(\dots 2123^*33212\dots) < j_0 - 10^{-3}.$$

So $\lambda_0(S) \in M$ and

$$X = \left\{\lambda_0\left(S_l^T(21)^n 1111\omega_1\omega_1^*\omega_1\omega_1 21221212(333212)^n S_r\right) \,\middle|\, (S_l, S_r) \in \{1,2\}^{\mathbb{N}}\right\}$$

is contained in $M \cap [j_1, j_1 + \alpha)$. We define two dynamical Cantor sets:

$$A_n = \left\{[3,2,1,1,1,(\omega_1)^2,2,1,2,2,(1,2)^2,(3,3,3,2,1,2)^n,S_r] \,\middle|\, S_r \in \{1,2\}^{\mathbb{N}}\right\},$$
$$B_n = \left\{[0,3,2,1,2,\omega_1^T,1,1,1,1,(1,2)^n,S_l] \,\middle|\, S_l \in \{1,2\}^{\mathbb{N}}\right\}.$$

So $X = A_n + B_n$, where A_n, B_n are two sets diffeomorphic to:

$$C(2) := \{[0,a_1,a_2,\dots] \mid \forall\, n \in \mathbb{N}\; a_n \in \{1,2\}\} \text{ with } HD(C(2)) > 0.5,$$

where $C(2)$ is a regular Cantor set of class $\mathcal{C}^2$ non-essentially affine (see [20, Proposition 1]. Thus we have:

$$HD(A_n) = HD(B_n) > 0.5$$

So according to the Moreira's dimension formula [21]:

$$HD(X) = \min\{1, HD(A_n) + HD(B_n)\} = 1,$$

and therefore:

$$1 \geq HD(M \cap (j_1, j_1 + \alpha)) \geq HD(X) = 1. \qquad \square$$

Corollary 4.23. *We have:*

$$\forall\, \alpha > 0,\; HD(L \cap (j_1, j_1 + \alpha)) = 1.$$

Proof. It is a consequence from:

$$HD(M \setminus L) < 1.$$

Indeed:

$$M \cap (j_1, j_1 + \alpha) = L \cap (j_1, j_1 + \alpha) \sqcup (M \setminus L) \cap (j_1, j_1 + \alpha),$$

and:

$$\begin{aligned} 1 &= HD(M \cap (j_1, j_1 + \alpha)) \\ &= \max\{HD(L \cap (j_1, j_1 + \alpha)), HD((M \setminus L) \cap (j_1, j_1 + \alpha))\}. \end{aligned}$$

So necessarily, we must have:

$$HD(L \cap (j_1, j_1 + \alpha)) = 1. \qquad \square$$

5. Portion of $M \setminus L$ in the vicinity of $\lambda_0(\overline{12333^*2112})$

For this section, we denote $\omega_2 = 123332112$ and $\omega_2^* = 12333^*2112$, with a asterisk when we refer to it with the position zero. We recall that for this section $j_0 = \lambda_0(\overline{\omega_2})$.

5.1. Local uniqueness. We call $\mathbb{Z}^- = \rrbracket -\infty, 0 \rrbracket$.

Lemma 5.1 (Forbidden words)**.** *Let $S \in \{1,2,3\}^{\mathbb{Z}}$ be a bi-infinite sequence. Then*

(1) *If $S = 13^*$, then $\lambda_0(S) > j_0 + 10^{-1}$.*
(2) *If $S = 23^*2$, then $\lambda_0(S) > j_0 + 10^{-2}$.*
(3) *If $S = 33^*23$, then $\lambda_0(S) > j_0 + 10^{-2}$.*
(4) *If $S = 233^*22$ or 333^*22, then $\lambda_0(S) > j_0 + 10^{-3}$.*
(5) *If $S = 1\,\omega_2^*\,12$, then $\lambda_0(S) > j_0 + 10^{-5}$.*
(6) *If $S = 32\,\omega_2^*\,12$, then $\lambda_0(S) > j_0 + 10^{-5}$.*
(7) *If $S = 12\,\omega_2^*\,121, 12\,\omega_2^*\,122, 22\,\omega_2^*\,121, 22\,\omega_2^*\,122$, then $\lambda_0(S) > j_0 + 10^{-6}$.*
(8) *If $S = 122\,\omega_2^*\,12, 222\,\omega_2^*\,12$, then $\lambda_0(S) > j_0 + 10^{-6}$.*
(9) *If $S = 3322\,\omega_2^*\,123, 322\,\omega_2^*\,1233$, then $\lambda_0(S) > j_0 + 10^{-7}$.*
(10) *If $S = 12112\,\omega_2^*\,1233, 22112\,\omega_2^*\,1233, 2112\,\omega_2^*\,12332$, then $\lambda_0(S) > j_0 + 10^{-8}$.*
(11) *If $S = 332112\,\omega_2^*\,123333, 2332112\,\omega_2^*\,1233321, 3332112\,\omega_2^*\,12333212, 23332112\,\omega_2^*\,123332111, 33332112\,\omega_2^*\,123332111$ then $\lambda_0(S) > j_0 + 10^{-10}$.*
(12) *If $S = 33332112\,\omega_2^*\,\omega_2$, then $\lambda_0(S) > j_0 + 10^{-10}$.*
(13) *If $S = \omega_2\,\omega_2^*\,\omega_2\,2, \omega_2\,\omega_2^*\,\omega_2\,3$, then $\lambda_0(S) > j_0 + 3 \times 10^{-10}$.*
(14) *If $S = 2\,\omega_2\,\omega_2^*\,\omega_2\,11$, then $\lambda_0(S) > j_0 + 4 \times 10^{-11}$.*
(15) *If $S = 212\,\omega_2\,\omega_2^*\,\omega_2\,123$, then $\lambda_0(S) > j_0 + 1.82 \times 10^{-12}$.*
(16) *If $S = 1112\,\omega_2\,\omega_2^*\,\omega_2\,123$, then $\lambda_0(S) > j_0 + 1.09 \times 10^{-13}$.*

Corollary 5.2. *Let $S \in \{1,2,3\}^{\mathbb{Z}}$ such that $m(S) \leq j_0 + 10^{-4}$. Then, the words $322, 223$ and 323 are forbidden.*

Proof. If S contains 323, then since according to the Lemma 5.1, the words $13, 232$ and 3323 are forbidden, we can't extend the word 323 to the left without making a forbidden word appear.

If $S = \dots 322 \dots$, then according to the Lemma 5.1, since 13 and 232 are forbidden, the word 322 must extend to the left in such way: $S = \dots 3322 \dots$. However, since the words $13, 23322$ and 33322 are forbidden, we can't extend the word 3322 without having a contradiction. The same reasoning gives that 223 is forbidden. ☐

Lemma 5.3 (Local uniqueness)**.** *Let $S \in \{1,2,3\}^{\mathbb{Z}}$ be such that $m(S) = \lambda_0(S) \leq j_0 + 10^{-7}$ then S or S^T must take one of these forms;*

(1) $S = 1^*, 2^*$ *or* 33^*3 *and* $\lambda_0(S) < j_0 - 10^{-2}$,
(2) $S = 233^*21$ *or* 333^*212 *and* $\lambda_0(S) < j_0 - 10^{-3}$,
(3) $S = 2333^*2111, 3333^*2111$ *or* 3333^*2112 *and* $\lambda_0(S) < j_0 - 10^{-4}$,
(4) $S = \omega_2^*\, 2$ *or* $\omega_2^*\, 3$ *and* $\lambda_0(S) < j_0 - 10^{-3}$,
(5) $S = 1\,\omega_2^*\, 11, 2\,\omega_2^*\, 11, 212\,\omega_2^*\, 123$ *and* $\lambda_0(S) < j_0 - 10^{-6}$,
(6) $S = 1112\,\omega_2^*\, 123$ *and* $\lambda_0(S) < j_0 - 10^{-7}$,
(7) $S = 2112\,\omega_2^*\, 1233$.

As a consequence, if S verifies $|\lambda_0(S) - j_0| \leq 10^{-7}$ then S must be of the form:

$$S = 2112\,\omega_2^*\, 1233.$$

Proof. Let $S \in \{1,2,3\}^{\mathbb{Z}}$ such that $m(S) = \lambda_0(S) \leq j_0 + 10^{-7}$. Then, according to Lemma 5.1, all words until item 9 are forbidden. If $S = 1^*$ or 2^*, then using the bounding rules, $\lambda_0(S) < j_0 - 10^{-2}$.
If $S = 3^*$, then, using the forbidden words $13, 232, 31$, we must have:

$$S = 23^*3, 33^*2 \quad \text{or} \quad \Big\{S = 33^*3 \text{ and } \lambda_0(S) < j_0 - 10^{-2}\Big\}.$$

By symmetry (taking S^T instead of S), we only need to develop the sequence from the sub-word 33^*2. If $S = 33^*2$, then, using the forbidden words $13, 323$ and 322, we must have:

$$S = 333^*21 \quad \text{or} \quad \Big\{S = 233^*21 \text{ and } \lambda_0(S) < j_0 - 10^{-3}\Big\}.$$

If $S = 333^*21$, then, using the forbidden words 13, we must have:

$$S = 2333^*211, 3333^*211 \quad \text{or} \quad \Big\{S = 333^*212 \text{ and } \lambda_0(S) < j_0 - 10^{-3}\Big\}.$$

If $S = 3333^*211$, then, using the forbidden words 13, we must have:

$$S = 3333^*2111, 3333^*2112 \quad \text{and} \quad \lambda_0(S) < j_0 - 10^{-3}.$$

If $S = 2333^*211$, then, using the forbidden words $13, 323, 223$, we must have:

$$S = \omega_2^* \quad \text{or} \quad \Big\{S = 2333^*2111 \text{ and } \lambda_0(S) < j_0 - 10^{-4}\Big\}.$$

If $S = \omega_2^*$, then, using the forbidden words 31, we must have:

$$S = 1\,\omega_2^*\,1, 2\,\omega_2^*\,1 \quad \text{or} \quad \Big\{S = \omega_2^*\,2, \omega_2^*\,3 \text{ and } \lambda_0(S) < j_0 - 10^{-3}\Big\}.$$

If $S = 1\,\omega_2^*\,1$, then, using the forbidden words $31, 13, 1\,\omega_2\,12$, we must have:

$$S = 1\,\omega_2^*\,11, \quad \text{and} \quad \lambda_0(S) < j_0 - 10^{-6}.$$

If $S = 2\,\omega_2^*\,1$, then, using the forbidden words $13, 32\,\omega_2\,12$, we must have:

$$S = 12\,\omega_2^*\,12, 22\,\omega_2^*\,12 \quad \text{or} \quad \Big\{S = 2\,\omega_2^*\,11 \text{ and } \lambda_0(S) < j_0 - 10^{-6}\Big\}.$$

If $S = 22\,\omega_2^*\,12$, then, using the forbidden words $122\,\omega_2\,12, 222\,\omega_2\,12$ and 322 (see Lemma 5.4), S can no longer be extended. So the word $22\,\omega_2\,12$ is forbidden.
If $S = 12\,\omega_2^*\,12$, then, using the forbidden words $13, 12\,\omega_2\,122, 12\,\omega_2\,121$, we must have:

$$S = 112\,\omega_2^*\,123 \quad \text{or} \quad \Big\{S = 212\,\omega_2^*\,123 \text{ and } \lambda_0(S) < j_0 - 10^{-6}\Big\}.$$

If $S = 112\,\omega_2^*\,123$, then, using the forbidden words $13, 232$, we must have:

$$S = 2112\,\omega_2^*\,1233 \quad \text{or} \quad \Big\{S = 1112\,\omega_2^*\,123 \text{ and } \lambda_0(S) < j_0 - 10^{-7}\Big\},$$

which proves the property of local-uniqueness. □

5.2. Sequence development Tree around $j_0 = \lambda_0(\overline{12333^*2112})$. Again, we can summarize the development of the sequence around the vicinity of j_0 with a tree of possibilities in Figure 5.1 following the same rules explained in Subsection 4.2.

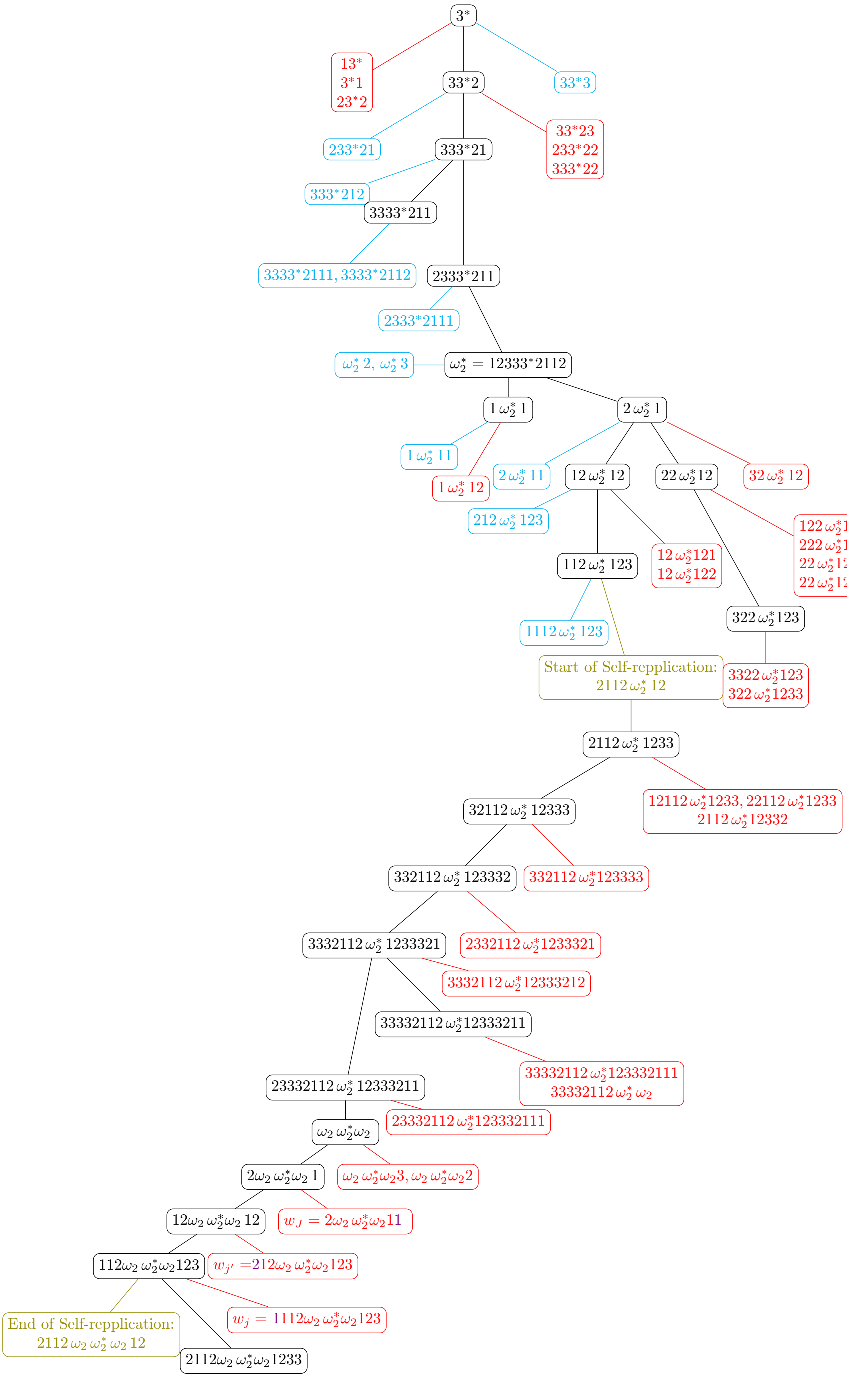

Figure 5.1. Sequence development Tree of $\overline{12333^*2112}$

5.3. Self-replication.

5.3.1. *Simplification of forbidden words.*

Lemma 5.4. *Let $S \in \{1,2,3\}^{\mathbb{Z}}$ be a bi-infinite sequence. Then*

1. *If $S = 3^*23$, then $m(S) > j_0 + 10^{-2}$.*
2. *If $S = 3^*22$, then $m(S) > j_0 + 10^{-3}$.*
3. *If $S = 22\,\omega_2^*\,12$, then $m(S) > j_0 + 10^{-7}$.*
4. *If $S = 33332112\,\omega_2^*\,12333211$, then $m(S) > j_0 + 9 \times 10^{-11}$.*

Proof. If $S = 3^*23$ then using the forbidden words $13, 232$ and 3323, we see, according to Lemma 5.1 (1) and (2) that we must have:

$$m(S) > j_0 + 10^{-2}.$$

If $S = 3^*22$, then using the forbidden words 13 and 232 (otherwise, we also get $m(S) > j_0 + 10^{-2}$), we must have $S = 33^*22$, then according to Lemma 5.1 (4), we must have:

$$m(S) > j_0 + 10^{-2}.$$

If $S = 22\,\omega_2^*\,12$, then according to Lemma 5.1 (7) and (8), if $S_{-7} \in \{1,2\}$ or $S_7 \in \{1,2\}$, then $\lambda_0(S) > j_0 + 10^{-6}$. Otherwise, $S = 322\,\omega_2^*\,123$ and using the forbidden words $3322\,\omega_2\,123, 232, 13, 31$ (otherwise, we also get $m(S) > j_0 + 10^{-7}$), we must have:

$$\inf_{S_g, S_d \in \{1,2,3\}} \sup_{n \in \mathbb{Z}} \lambda_n(\dots S_g\,322\,\omega_2^*\,123\,S_d \dots) > j_0 + 10^{-7}.$$

If $S = 33332112\,\omega_2^*\,12333211$, then using the forbidden words $33332112\,\omega_2\,12332111$, $33332112\,\omega_2\,\omega_2$, 13, according to Lemma 5.1, we must have:

$$\inf_{S_g, S_d \in \{1,2,3\}} \sup_{n \in \mathbb{Z}} \lambda_n(\dots S_g\,33332112\,\omega_2^*\,12333211\,S_d \dots) > j_0 + 10^{-10}.$$

If $S = 33332112\,\omega_2^*\,12333211$, then we have:

$$\inf_{S_g, S_d \in \{1,2,3\}^3} \sup_{n \in \mathbb{Z}} \lambda_n(\dots S_g\,33332112\,\omega_2^*\,12333211\,S_d \dots)$$
$$\geq \lambda_0(\overline{31}\,123\,33332112\,\omega_2^*\,12333211\,212\,\overline{13}),$$

with:

$$\lambda_0(\overline{31}\,123\,33332112\,\omega_2^*\,12333211\,212\,\overline{13}) \geq j_0 + 9 \times 10^{-11}. \qquad \square$$

Using the "new" forbidden words, we now call F_{tot} the set of every forbidden word (and its transposes), in the developing sequence of $\overline{123332112}$.

We have:

$$F = \left\{ \begin{array}{l} 13, 232, 322, 323, 1\,\omega_2\,12, 22\,\omega_2\,12, 32\,\omega_2\,12, 12\,\omega_2\,121, 12\,\omega_2\,122, \\ 22112\,\omega_2\,1233, 2112\,\omega_2\,12332, 12112\,\omega_2\,1233, 332112\,\omega_2\,123333, \\ 2332112\,\omega_2\,1233321, 3332112\,\omega_2\,12333212, 33332112\,\omega_2\,12333211, \\ 23332112\,\omega_2\,123332111,\ \omega_2\,\omega_2\,\omega_2\,2,\ \omega_2\,\omega_2\,\omega_2\,3, 2\,\omega_2\,\omega_2\,\omega_2\,11, \\ 212\,\omega_2\,\omega_2\,\omega_2\,123, 1112\,\omega_2\,\omega_2\,\omega_2\,123 \end{array} \right\},$$

and:

$$F_{tot} = F \cup F^T.$$

Let us call:

(1) $w_j = \mathbf{1}112\,\omega_2\,\omega_2\,\omega_2\,123$,
(2) $w_{j'} = \mathbf{2}12\,\omega_2\,\omega_2\,\omega_2\,123$,
(3) $w_J = 2\,\omega_2\,\omega_2\,\omega_2\,1\mathbf{1}$.

Theorem 5.5 (Self-replication)**.** *Let $S \in \{1,2,3\}^{\mathbb{Z}}$ such that:*

$$S_{n-8} \dots S_{n+6} = 2112\,\omega_2^* 12 \quad \textit{for } n \in \mathbb{Z}.$$

(1) *If the finite word $S_{n-15} \dots S_{n+15}$ doesn't contain any word of $F_{tot} \setminus \{w_j, w_j^T, w_{j'}, w_{j'}^T\}$, then $S_{n-15} \dots S_{n+15} = 12\,\omega_2\,\omega_2^*\,\omega_2\,12$.*
(2) *If the sequence $S_{n-17} \dots S_{n+15}$ doesn't contain any word of F_{tot}, then $S_{n-17} \dots S_{n+15} = 2112\,\omega_2\,\omega_2^*\,\omega_2\,12$.*

Proof. If $S = 2112\,\omega_2^*\,12$, then, using the forbidden words $12\,\omega_2\,121$, $12\,\omega_2\,122$, we must have:

$$S = 2112\,\omega_2^*\,123.$$

If $S = 2112\,\omega_2^*\,123$, then, using the forbidden words $31, 232$, we must have:

$$S = 2112\,\omega_2^*\,1233.$$

If $S = 2112\,\omega_2^*\,1233$, then, using the forbidden words 31, $22112\,\omega_2\,1233$, $2112\,\omega_2\,12332$, $12112\,\omega_2\,1233$, we must have:

$$S = 32112\,\omega_2^*\,12333.$$

If $S = 32112\,\omega_2^*\,12333$, then, using the forbidden words 13, 31, 232, $332112\,\omega_2\,123333$, we must have:

$$S = 332112\,\omega_2^*\,123332.$$

If $S = 332112\,\omega_2^*\,123332$, then, using the forbidden words 13, 323, 322, $2332112\,\omega_2\,1233321$, we must have:

$$S = 3332112\,\omega_2^*\,1233321.$$

If $S = 3332112\,\omega_2^*\,1233321$, then, using the forbidden words 13, $3332112\,\omega_2\,12333212$, $33332112\,\omega_2\,12333211$ we must have:

$$S = 23332112\,\omega_2^*\,12333211.$$

If $S = 23332112\,\omega_2^*\,12333211$, then, using the forbidden words 13, 3233, 223333, $23332112\,\omega_2\,123332111$, we must have:

$$S = \omega_2\,\omega_2^*\,\omega_2.$$

If $S = \omega_2\,\omega_2^*\,\omega_2$, then, using the forbidden words 31, $1\,\omega_2\,12$, $\omega_2\,\omega_2\,\omega_2\,2$, $\omega_2\,\omega_2\,\omega_2\,3$, we must have:

$$S = 2\,\omega_2\,\omega_2^*\,\omega_2\,1.$$

If $S = 2\,\omega_2\,\omega_2^*\,\omega_2\,1$, then, using the forbidden words 13, $22\,\omega_2\,12$, $32\,\omega_2\,12$, $2\,\omega_2\,\omega_2\,\omega_2\,11$, we must have:

$$S = 12\,\omega_2\,\omega_2^*\,\omega_2\,12,$$

which proves the first point. Again, using the forbidden words, 31, $w_{j'} = 212\,\omega_2\,\omega_2\,\omega_2\,123$, $12\,\omega_2\,121$ and $12\,\omega_2\,122$ we must have:

$$S = 112\,\omega_2\,\omega_2^*\,\omega_2\,12.$$

If $S = 112\,\omega_2\,\omega_2^*\,\omega_2\,12$, then, using the forbidden words 31, $w_j = 1112\,\omega_2\,\omega_2\,\omega_2\,123$, we must have:

$$S = 2112\,\omega_2\,\omega_2^*\,\omega_2\,12,$$

which concludes the proof. □

Using the Lemma 5.5 repeatedly, we have two following results.

Corollary 5.6. *Let $S \in \{1,2,3\}^{\mathbb{Z}}$ such that: $S_{-8}\dots S_6 = 2112\,\omega_2^*12$. Then*

(1) *If all the words from $F_{tot} \setminus \{w_j, w_{j'}, w_j^T, w_{j'}^T\}$ are forbidden in S, then we must have:*

$$S = S_l^T\,12\,\omega_2\,\omega_2^*\,\overline{\omega_2} \quad \textit{with } S_l \in \{1,2,3\}^{\mathbb{N}}.$$

(2) *If all the words from F_{tot} are forbidden in S, then we must have:*

$$S = \overline{\omega_2} = \overline{12333^*2112}.$$

Lemma 5.7. *We have:*

(1)

$$J = \min\left\{\lambda_0(S_l^T\,w_J^*\,S_r)\;\middle|\; \begin{array}{l} S_r, S_l \in \{1,2,3\}^{\mathbb{N}} \textit{ and } S_l^T\,w_J S_r \textit{ doesn't contain} \\ \textit{any words from } F_{tot} \setminus \{w_j, w_{j'}, w_J, w_j^T, w_{j'}^T, w_J^T\} \end{array}\right\}$$
$$= \lambda_0(\overline{12}\,\omega_2\,\omega_2\,\omega_2^*\,\omega_2\,11\,\overline{12}) \approx j_0 + 5.88429645 \times 10^{-11}.$$

(2)

$$j' = \min\left\{\lambda_0(S_l^T\,w_{j'}^*S_r)\;\middle|\; \begin{array}{l} S_r, S_l \in \{1,2,3\}^{\mathbb{N}} \textit{ and } S_l^T\,w_{j'}S_r \textit{ doesn't contain} \\ \textit{any words from } F_{tot} \setminus \{w_j, w_{j'}, w_j^T, w_{j'}^T\} \end{array}\right\}$$
$$= \lambda_0(\overline{21}\,12332121\,12333212\,\omega_2\,\omega_2^*\,\overline{\omega_2}) \approx j_0 + 2.2055806 \times 10^{-12}.$$

(3)

$$j = \min\left\{ \lambda_0(S_l^T w_j^* S_r) \;\middle|\; \begin{array}{l} S_r, S_l \in \{1,2,3\}^{\mathbb{N}} \textit{ and } S_l^T w_j S_r \textit{ doesn't} \\ \textit{contain any words from } F_{tot} \setminus \{w_j, w_j^T\} \end{array} \right\}$$
$$= \lambda_0(\overline{21}\,1112\,\omega_2\,\omega_2^*\,\overline{\omega_2}) \approx j_0 + 4.77646040 \times 10^{-13}.$$

In particular:

$$j < j' < J.$$

Remark 5.8. We also have:

(1) $j = \lambda_0(\overline{21}\,1112\,\omega_2\,\omega_2^*\,\overline{\omega_2}) = m(\overline{21}\,1112\,\omega_2\,\omega_2^*\,\overline{\omega_2})$.
(2) $j' = \lambda_0(\overline{21}\,12332121\,12333212\,\omega_2\,\omega_2^*\,\overline{\omega_2})$
$= m(\overline{21}\,12332121\,12333212\,\omega_2\,\omega_2^*\,\overline{\omega_2})$.
(3) $J = \lambda_0(\overline{12}\,\omega_2\,\omega_2\,\omega_2^*\,\omega_2\,11\,\overline{12}) = m(\overline{12}\,\omega_2\,\omega_2\,\omega_2^*\,\omega_2\,11\,\overline{12})$.

So:

$$(j, j', J) \in M^3.$$

It is relevant to understand the geometric role of these constants. We will prove later that j is the first element in M coming next after j_0 and is the left border of the region of $M \setminus L$ to the right of j_0, i.e. $\min(M \setminus L) \cap (j_0, J) = j$. It is also the first element of M in the vicinity of j_0, containing a forbidden word, which is w_j. The point j' is the first element of M containing another forbidden word, which is $w_{j'}$. Finally, J represents the right border of the region of $M \setminus L$ to the right of j_0, and is also the first element of M containing the forbidden word w_J.

We will prove later that J is a point in L' since above it, sequences $S \in \{1,2,3\}^{\mathbb{Z}}$ such that $m(S) = \lambda_0(S)$, can be different from $\overline{\omega_2}$ in both directions (left and right), whereas below J and above j_0, it can only be different to $\overline{\omega_2}$ to the left direction.

Proof of Lemma 5.7.

Proof of J. Let $S = S_l^T w_J S_r$ such that S doesn't contain any words from $F_{tot} \setminus \{w_j, w_j^T, w_{j'}, w_{j'}^T, w_J, w_J^T\}$. Again, we have:

$$S = \dots S_{-15}\,2\,\omega_2\,\omega_2^*\,\omega_2\,11\,S_{16}\dots$$

In order to minimise λ_0, since 13 is forbidden, the right extension must be:

$$S_{16}\dots = \overline{12}.$$

So:

$$S = \dots S_{-15}\,2\,\omega_2\,\omega_2^*\,\omega_2\,11\,\overline{12}.$$

Using the forbidden words $22\,\omega_2\,12$ and $32\,\omega_2\,12$, we must have:

$$S = \dots S_{-16}\,12\,\omega_2\,\omega_2^*\,\omega_2\,11\,\overline{12}.$$

Since we are minimising $\lambda_0(S)$, we must have:

$$S = \ldots S_{-17}\, 112\, \omega_2\, \omega_2^*\, \omega_2\, 11\, \overline{12}.$$

Using the forbidden words 31, we must have:

$$S = \ldots S_{-18}\, 2112\, \omega_2\, \omega_2^*\, \omega_2\, 11\, \overline{12}.$$

Then, we can apply the proof of the first point of Theorem 5.5 to $(S_{n-9})_{n\in\mathbb{Z}}$, so we get:

$$S = \ldots S_{-25}\, 12\, \omega_2\, \omega_2\, \omega_2^*\, \omega_2\, 11\, \overline{12}.$$

Since we are minimising $\lambda_0(S)$, then we must have:

$$S = \overline{12}\, \omega_2\, \omega_2\, \omega_2^*\, \omega_2\, 11\, \overline{12}.$$

Therefore:

$$J = \lambda_0(\overline{12}\, \omega_2\, \omega_2\, \omega_2^*\, \omega_2\, 11\, \overline{12}) \approx j_0 + 5.88429645 \times 10^{-11}.$$

Proof of j'. Let $S = S_l^T w_{j'} S_r$ such that S doesn't contain any words from $F_{tot} \setminus \{w_j, w_j^T, w_{j'}, w_{j'}^T\}$. Again, we have:

$$S = \ldots S_{-17}\, 212\, \omega_2\, \omega_2^*\, \omega_2\, 12\, S_{16} \ldots$$

Then, using Corollary 5.6, we must have:

$$S = \ldots S_{-17}\, 212\, \omega_2\, \omega_2^*\, \overline{\omega_2}.$$

Then, since we are minimising $\lambda_0(S)$, we must have:

$$S = \ldots S_{-18}\, 3212\, \omega_2\, \omega_2^*\, \overline{\omega_2}.$$

Then, using the forbidden words 13 and 232, we must have:

$$S = \ldots S_{-19}\, 33212\, \omega_2\, \omega_2^*\, \overline{\omega_2}.$$

Then, since we are minimising $\lambda_0(S)$, we must have:

$$S = \ldots S_{-20}\, 333212\, \omega_2\, \omega_2^*\, \overline{\omega_2}.$$

Then, using the forbidden words 13 and by minimising $\lambda_0(S)$, we must have:

$$S = \ldots S_{-21}\, 2333212\, \omega_2\, \omega_2^*\, \overline{\omega_2}.$$

Then, using the forbidden words 323 and 322, we must have:

$$S = \ldots S_{-22}\, 12333212\, \omega_2\, \omega_2^*\, \overline{\omega_2}.$$

Then, by minimising $\lambda_0(S)$ we can add 2121, so we get:

$$S = \ldots S_{-26}\, 2121\, 12333212\, \omega_2\, \omega_2^*\, \overline{\omega_2} = \ldots S_{-26}\, 21\, \omega_2^T\, 2\, 123332112\, \omega_2^*\, \overline{\omega_2}.$$

Then, since $221\, \omega_2^T\, 21$ and $121\, \omega_2^T\, 21$ are forbidden, we must have:

$$S = \ldots S_{-27}\, 32121\, 12333212\, \omega_2\, \omega_2^*\, \overline{\omega_2}.$$

Then, using the forbidden words 13 and 232, we must have:

$$S = \ldots S_{-28}\, 332121\, 12333212\, \omega_2\, \omega_2^*\, \overline{\omega_2}.$$

Then, using the forbidden words 13 and minimising $\lambda_0(S)$, we must have:

$$S = \dots S_{-29}\, 2332121\, 12333212\, \omega_2\, \omega_2^*\, \overline{\omega_2}.$$

Then, using the forbidden words 323 and 322, we must have:

$$S = \dots S_{-30}\, 12332121\, 12333212\, \omega_2\, \omega_2^*\, \overline{\omega_2}.$$

Finally, we minimise with the periodic sequence, $\overline{21}$, so we get:

$$S = \overline{21}\, 12332121\, 12333212\, \omega_2\, \omega_2^*\, \overline{\omega_2}.$$

Therefore:

$$j' = \lambda_0(\overline{21}\, 12332121\, 12333212\, \omega_2\, \omega_2^*\, \overline{\omega_2}) \approx j_0 + 2.2055806 \times 10^{-12}.$$

Proof of j. Let $S = S_l^T w_j S_r$ such that S doesn't contain any words from $F_{tot} \setminus \{w_j, w_j^T\}$. Again, we have:

$$S = \dots S_{-18}\, 1112\, \omega_2\, \omega_2^*\, \omega_2\, 123\, S_{17} \dots$$

Then, using Corollary 5.6, we must have:

$$S = \dots S_{-18}\, 1112\, \omega_2\, \omega_2^*\, \overline{\omega_2}.$$

Then, we minimise $\lambda_0(S)$ by adding the sequence $\overline{21}$ to the left:

$$S = \overline{21}\, 1112\, \omega_2\, \omega_2^*\, \overline{\omega_2}.$$

Therefore:

$$j = \lambda_0(\overline{21}\, 1112\, \omega_2\, \omega_2^*\, \overline{\omega_2}) \approx j_0 + 4.77646040 \times 10^{-13}.$$ □

Proposition 5.9. *Let $S \in \{1,2,3\}^{\mathbb{Z}}$ be a sequence such that $\lambda_0(S) = m(S) = \sup_{n\in\mathbb{Z}} \lambda_n(S)$. According to Lemma 5.1 and 5.7 we have:*

(1) *If $\lambda_0(S) \in M \cap (J, J+\alpha)$, for $\alpha > 0$ small enough, then, at least one of the subwords $w_j, w_j^T, w_{j'}, w_{j'}^T, w_J, w_J^T$ must appear in S.*
(2) *If $\lambda_0(S) \in M \cap (j_0, J)$, no forbidden words from $F_{tot} \setminus \{w_j, w_{j'}, w_j^T, w_{j'}^T\}$ are allowed to appear in S.*
(3) *If $\lambda_0(S) \in M \cap (j_0, j')$, no forbidden words from $F_{tot} \setminus \{w_j, w_j^T\}$ are allowed to appear in S.*
(4) *If $\lambda_0(S) \in M \cap (j_0, j)$, no forbidden words from F_{tot} are allowed to appear in S.*

Proof. Let $m \in M$ such that $m > j_0$ and $S \in \{1,2,3\}^{\mathbb{Z}}$ a sequence such that:

$$m = \lambda_0(S) = \sup_{n\in\mathbb{Z}} \lambda_n(S).$$

If $J + \alpha > m > j_0$, with $\alpha < 9 \times 10^{-11}$, then, according to the Lemma 5.1, Lemma 5.3 and Corollary 5.6, one of the words $w_j, w_j^T, w_{j'}, w_{j'}^T, w_J$ and w_J^T must be in the sequence S.

Let assume that $m \in (j_0, J)$. By contradiction, suppose that there exists $w_f \in F_{tot} \setminus \{w_j, w_j^T, w_{j'}, w_{j'}^T\}$ such that $S = S_l^T w_f S_r$. If $w_f \in F_{tot} \setminus \{w_j, w_j^T, w_{j'}, w_{j'}^T, w_J, w_J^T\}$, then according to Lemma 5.1 and 5.4, we must have $\lambda_N(S) > j_0 + 9 \times 10^{-11}$ for some $N \in \mathbb{Z}$, which is impossible. *So S doesn't contain any word from $F_{tot} \setminus \{w_j, w_j^T, w_{j'}, w_{j'}^T, w_J, w_J^T\}$.* If S contains w_J, then we have $S = S_l^T w_J S_r$. By definition of J, we must have $\lambda_N(S) \geq J$ for some $N \in \mathbb{Z}$ and therefore, $m \geq J$, which is impossible. The same is true with w_J^T. *So S doesn't contain any subwords from $F_{tot} \setminus \{w_j, w_j^T, w_{j'}, w_{j'}^T\}$.*

Let assume that $m \in (j_0, j')$. Then according to above, S doesn't contain any words from $F_{tot} \setminus \{w_j, w_j^T, w_{j'}, w_{j'}^T\}$. If $S = S_l^T w_{j'} S_r$ then by definition of j', we must have $\lambda_N(S) \geq j'$ and therefore: $m(S) \geq j'$. Again, the same is true for $w_{j'}^T$. Hence, the sequence S doesn't contain any words from $F_{tot} \setminus \{w_j, w_j^T\}$.

Finally, let assume that $m \in (j_0, j)$. Then according to above, S doesn't contain any words from $F_{tot} \setminus \{w_j, w_j^T\}$. If $S = S_l^T w_j S_r$, then by definition of j, we must have $\lambda_N(S) \geq j$ and therefore: $m(S) \geq j$. Again, the same is true for w_j^T. Hence, the sequence S doesn't contain any words from F_{tot}. □

The main consequence that follows from all of this:

Theorem 5.10. *We have:*

(1) $M \cap (j_0, j) = \emptyset$.
(2) $L \cap (j_0, J) = \emptyset$.

Proof. Let assume by contradiction that there exist $m \in M \cap (j_0, j)$. Then we can find a sequence S such that $m = \sup_{n \in \mathbb{Z}} \lambda_n(S) = m(S) = \lambda_0(S)$. Then, we have $j_0 < \lambda_0(S) < j < j_0 + 10^{-12}$.

So according to the Lemma 5.3, we must have: $S_{-8} \dots S_8 = 2112\, \omega_2^*\, 1233$. In addition, according to the Proposition 5.9 the words from the set F_{tot} are forbidden in S. Then, because of the Corollary 5.6, the sequence S must extend in such way:

$$S = \overline{\omega_2},$$

and so $m(S) = j_0$, a contradiction.

Let assume by contradiction that there exists $l \in L \cap (j_0, J)$. We use the fact that the Markov values of periodic sequences are dense in L (see [3, Theorem 2, Chapter 3]). Therefore, $\exists\, (l_n)_{n \in \mathbb{N}} \in L^{\mathbb{N}}$ such that:

$$\forall\, n \in \mathbb{N},\ \exists\, \sigma^{(n)} \in \mathbb{N}^{(\mathbb{N})}, \quad l_n = m\Big(\overline{\sigma^{(n)}}\Big) \text{ and } \lim_{n \to \infty} l_n = l.$$

Let us write $\forall\, n \in \mathbb{N}, S^{(n)} = \overline{\sigma^{(n)}}$. We can assume (even if it means taking $n \geq n_0$ with n_0 big enough), $\forall\, n \in \mathbb{N}, S^{(n)} \in \{1, 2, 3\}^{\mathbb{Z}}$ and that $l_n < J$. So we have $\lambda_0(S^{(n)}) = l_n < J$. Using Lemma 5.3, we have that $S_{-8}^{(n)} \dots S_8^{(n)} = 2112\, \omega_2^*\, 1233$ (or its transpose). Since $l_n = \lambda_0(S^{(n)}) \in M \cap (j_0, J)$, according to the Proposition 5.9, no forbidden words from $F_{tot} \setminus \{w_j, w_j^T, w_{j'}, w_{j'}^T\}$ are allowed to appear in $S^{(n)}$. Therefore, using the Corollary 5.6, we have:

$$\forall\, n \in \mathbb{N}, \quad S^{(n)} = S_l^{(n)} 12\, \omega_2\, \omega_2^*\, \overline{\omega_2} \quad \text{with} \quad S_l^{(n)} \in \{1,2,3\}^{\mathbb{Z}^-}.$$

Hence $\forall\, n \in \mathbb{N}$, $\overline{\sigma^{(n)}} = S_l^{(n)} 12\, \omega_2\, \omega_2^*\, \overline{\omega_2}$ and necessarily $\overline{\sigma^{(n)}} = S^{(n)} = \overline{\omega_2}$ since $\overline{\sigma^{(n)}}$ is periodic. So for all n we have $l_n = m(\overline{\omega_2}) = j_0$ and $l = j_0$, which is impossible. □

5.4. A portion of $M \setminus L$. Now, we can characterize the set $(M \setminus L) \cap (j_0, J)$.

Corollary 5.11. *Let $m \in M \cap (j, J)$ and $S \in \{1,2,3\}^{\mathbb{Z}}$ be a sequence such that $m = m(S) = \lambda_0(S)$. Then we have, up to transposition:*

$$S = S_l\, t\, 12\, \omega_2\, \omega_2^*\, \overline{\omega_2},$$

with $S_l \in \{1,2,3\}^{\mathbb{Z}^-}$ and $t \in \{2, 11\}$ such that:

1. *The sequence S cannot contain any words from $F_{tot} \setminus \{w_j, w_j^T, w_{j'}, w_{j'}^T\}$.*
2. *$S_l\, t\, 12$ cannot contain the word $2112\, \omega_2\, 12$.*
3. *If $\exists\, n \in \mathbb{Z}^-$ such that $S_{n-6}^T \dots S_{n+8}^T = 21\, \omega_2^T\, 2112$, then we have $S = \overline{\omega_2^T}\, 21\, S_{n+16} \dots$.*

Proof. Let $m(S) = \lambda_0(S) \in (j_0, J)$. According to the Lemma 5.3, we have:

$$S_{-8} \dots S_6 = 2112\, \omega_2^*\, 12.$$

Since $m(S) < J$, the sequence S doesn't contain any words from $F_{tot} \setminus \{w_j, w_{j'}, w_j^T, w_{j'}^T\}$. Then, because of the Corollary 5.6, we have:

$$S = \dots S_{-16} 12\, \omega_2\, \omega_2^*\, \overline{\omega_2}.$$

Since the word 31 is forbidden, we have $S_{-16} \in \{1, 2\}$.

If $S_{-16} = 1$, then using the forbidden word 31, we have $S_{-17} \in \{1, 2\}$. If $S_{-17} = 2$, then because of Theorem 5.5, applied to $(S_{n-9})_{n \in \mathbb{Z}}$ we must have:

$$S = \dots 12\, \omega_2\, \omega_2\, \omega_2^* \overline{\omega_2},$$

and $\lambda_0(S) < j_0 + 10^{-19} < j$, which is impossible. So $S_{-17} = 1$ (and so $t = 11$).

Therefore $S = S_l\, t\, 12\, \omega_2\, \omega_2^*\, \overline{\omega_2}$ with $S_l \in \{1,2,3\}^{\mathbb{Z}^-}$ and $t \in \{2, 11\}$. If $S_l\, t\, 12$ contains the word $2112\, \omega_2\, 12$, then because of self-replication to the right, we have no uniqueness in the writing of S. So $S_l\, t 12$ does not contain the word $2112\, \omega_2\, 12$. If $\exists\, n \in \mathbb{Z}^-$ such that:

$$S_{n-6} \dots S_{n+8} = 21\, \omega_2^T\, 2112.$$

Then, since the words from $F_{tot} \setminus \{w_j, w_j^T, w_{j'}, w_{j'}^T\}$ are forbidden, using self replication applied to the sequence S to the left, we have:

$$S = \overline{\omega_2^T}\, 21\, S_{n+16} \dots \qquad \square$$

Before making a global description of the set $(M \setminus L) \cap (j_0, J)$, we can simplify a little bit the set of forbidden words.

Lemma 5.12. *Let us call:*

$$F_0 = \left\{ \begin{matrix} 13, 232, 322, 323, 1\omega_2 12, 22\omega_2 12, 32\omega_2 12, \\ 12\omega_2 121, 12\omega_2 122, 2112\omega_2 12 \end{matrix} \right\}.$$

Then, if $S \in \{1,2,3\}^{\mathbb{Z}}$ is a sequence such that the set F_0 is forbidden in S, then the words $\omega_2\, 121$ and $\omega_2\, 122$ are forbidden as well.

Proof. Indeed, if $S = \dots \omega_2\, 121 \dots$, then using the forbidden words 31 and $1\,\omega_2\, 12$, then we just have:

$$S = \dots 2\,\omega_2\, 121 \dots$$

But we can no longer extend S to the left since the words $32\,\omega_2\, 12, 22\,\omega_2\, 12$ and $12\,\omega_2\, 121$ are forbidden. The same is true for the word $\omega_2\, 122$. $\square$

We can now make a global description of the set $(M \setminus L) \cap (j_0, J)$.

Theorem 5.13. *Let us call F the following set:*

$$F = \{13, 232, 322, 323, 1\omega_2 12, 22\omega_2 12, 32\omega_2 12, \omega_2 121, \omega_2 122, 2112\omega_2 12\},$$

and:

$$\widetilde{F} = F \cup F^T.$$

We have:

$$(M \setminus L) \cap (j_0, J) = C \cup D \cup X,$$

with:

$$C = \left\{ \lambda_0(S_l\, t\, 12\, \omega_2\, \omega_2^*\, \overline{\omega_2}) \;\middle|\; \begin{array}{l} (S_l, t) \in \{1,2,3\}^{\mathbb{N}} \times \{11, 2\} \text{ and} \\ S\, t\, 12 \text{ doesn't contain any words from } \widetilde{F} \end{array} \right\},$$

$$D = \left\{ \lambda_0(\overline{\omega_2^T}\, 21\, s\, 12\, \omega_2\, \omega_2^*\, \overline{\omega_2}) \;\middle|\; \begin{array}{l} s \in \{1,2,3\}^N, N \geq 1 \text{ such that} \\ (s_1 s_2, s_{N-1} s_N) \in \{11, 2\}^2, \\ [0, s^T] \leq [0, s], \\ \text{and } 1\, \omega_2^T\, \omega_2^T\, 21\, s\, 12\, \omega_2\, \omega_2\, 1 \\ \text{doesn't contain any words from } \widetilde{F} \end{array} \right\},$$

and:

$$X = \left\{ \lambda_0(\overline{\omega_2^T}\, 2\, \omega_2\, \omega_2^*\, \overline{\omega_2}), \lambda_0(\overline{\omega_2^T}\, 212\, \omega_2\, \omega_2^*\, \overline{\omega_2}), \lambda_0(\overline{\omega_2^T}\, 21112\, \omega_2\, \omega_2^*\, \overline{\omega_2}) \right\}.$$

Proof. Let $m \in (M \setminus L) \cap (j_0, J)$ and $S \in \{1,2,3\}^{\mathbb{Z}}$ a sequence such that $\lambda_0(S) = m$. Then according to Corollary 5.11 (1) and (2), we have:

$$S = S_l\, t\, 12\, \omega_2\, \omega_2^*\, \overline{\omega_2},$$

with $t \in \{2, 11\}$ and S such that the sequence S cannot contain any words from $F_{tot} \setminus \{w_j, w_j^T, w_{j'}, w_{j'}^T\} \cup \{2112\, \omega_2\, 12\}$. If in addition, the sequence S doesn't contain the word $21\, \omega_2^T\, 2112$, then we can simplify the set of forbidden words, since every word from F_{tot} containing the subword $2112\, \omega_2\, 12$ can no longer appear. Using the previous Lemma 5.12, the new set of forbidden words of $S_l\, t\, 12$ is exactly $\widetilde{F}$ and therefore, $m = \lambda_0(S) \in C$.

If the sequence S contains the word $21\, \omega_2^T\, 2112$, then, let us define $N = \max\{n \in \mathbb{Z} \mid S_{n-6} \dots S_{n+8} = 21\, \omega_2^T\, 2112\}$. By definition of N:

$$S = \dots 21\, \omega_2^T\, 2112\, S_{N+9} \dots$$

Then according to Corollary 5.11 (3), we have:

$$S = \overline{\omega_2^T}\, 21\, S_{N+16} \dots$$

And by definition of N, we must have $S_{N+16}S_{N+17} = 11$ or $S_{N+16} = 2$. Now we have two writings of the sequence S. So S can be of the following forms:

(1) $S = \overline{\omega_2^T}\, 2\, \omega_2\, \omega_2^*\, \overline{\omega_2}$, $(N = -33)$,
(2) $S = \overline{\omega_2^T}\, 212\, \omega_2\, \omega_2^*\, \overline{\omega_2}$, $(N = -35)$,
(3) $S = \overline{\omega_2^T}\, 21112\, \omega_2\, \omega_2^*\, \overline{\omega_2}$, $(N = -37)$,
(4) $S = \overline{\omega_2^T}\, 21\, s\, 12\, \omega_2\, \omega_2^*\, \overline{\omega_2}$, $(N \le -37)$.

The first three cases correspond to the set X. In the last case, s must verify:

$$\begin{aligned}\lambda_0(\overline{\omega_2^T}\, 21\, s\, 12\, \omega_2\, \omega_2^*\, \overline{\omega_2}) &\ge \lambda_{-31-N}(\overline{\omega_2^T}\, 21\, s\, 12\, \omega_2\, \omega_2^*\, \overline{\omega_2}) \\ &= \lambda_0(\overline{\omega_2^T}\, (\omega_2^T)^*\, \omega_2^T\, 21\, s\, 12\, \overline{\omega_2}) \\ &= \lambda_0(\overline{\omega_2^T}\, 21\, s^T\, 12\, \omega_2\, \omega_2^*\, \overline{\omega_2}),\end{aligned}$$

which is equivalent to:

$$[0, s^T] \le [0, s].$$

By hypothesis on N, the finite sequence $1\, \omega_2^T\, \omega_2^T\, 21\, s\, 12\, \omega_2\, \omega_2\, 1$ doesn't contain the word $21\, \omega_2^T\, 2112$, so like above, we must have that the finite sequence $1\, \omega_2^T\, \omega_2^T\, 21\, s\, 12\, \omega_2\, \omega_2\, 1$ doesn't contain any words from $\widetilde{F}$. In conclusion, we have:

$$(M \setminus L) \cap (j_0, J) \subset C \cup D \cup X.$$

Now, we show the reverse inclusion. Let $S_l \in \{1,2,3\}^{\mathbb{Z}^-}$ and $t \in \{2, 11\}$ such that $S_l\, t\, 12$ doesn't contain any words from $\widetilde{F}$. Let us show that $\lambda_0(S_l\, t\, 12\, \omega_2\, \omega_2^*\, \overline{\omega_2}) = m(S_l\, t\, 12\, \omega_2\, \omega_2^*\, \overline{\omega_2})$. Since we also have

$$\lambda_0(S_l\, t\, 12\, \omega_2\, \omega_2^*\, \overline{\omega_2}) \in (j_0, J),$$

it will follow that $\lambda_0(S_l\, t\, 12\, \omega_2\, \omega_2^*\, \overline{\omega_2}) \in (M \setminus L) \cap (j_0, J)$.

We call $S = S_l\, t\, 12\, \omega_2\, \omega_2^*\, \overline{\omega_2}$. If $\lambda_k(S) = m(S)$ for some $k \in \mathbb{Z}$, then since forbidden words from $\widetilde{F} \setminus \{2112\, \omega_2\, 12, 21\, \omega_2^T\, 2112\}$ can not appear in S, we have that either $S = 2112\, \omega_2^*\, 1233$, $S = 3321\, (\omega_2^*)^T\, 2112$ or S has one of the forms of Lemma 5.3. In other terms, if $S_k = 3$, the possible extensions (up to transposition) of S around S_k are given by the blue words of the tree in Figure 5.1 (so they satisfy $\lambda_k(S) < j_0$) or $S = 2112\, \omega_2^*\, 1233$ or $S = 3321\, (\omega_2^*)^T\, 2112$. By hypothesis, the last two subwords do not appear in $S_l\, t\, 12$, so $k \geq -9$ and by the shape of S we have that $k = 9r$ for some integer $r \geq -1$. For $r \geq 1$, is easy to see that $\lambda_{9r}(S) \leq j_0 + 10^{-19} < j$ and $\lambda_{-9}(S) < j_0 - 10^{-7}$ according to Lemma 5.3 (5) and (6). Finally, by the definition of j, we have that $\lambda_0(S) \geq j$, so the Markov value is necessarily being attained at the zero position of S. So in conclusion, $\lambda_0(S) = m(S)$ as we wanted.

We verify by direct computation that $X \subset (M \setminus L) \cap (j_0, J)$. If $m \in D$, then the condition $[0, s^T] \leq [0, s]$ ensures that

$$m = \lambda_0(\overline{\omega_2^T}\, 21\, s\, 12\, \omega_2\, \omega_2^*\, \overline{\omega_2}) = m(\overline{\omega_2^T}\, 21\, s\, 12\, \omega_2\, \omega_2^*\, \overline{\omega_2}). \qquad \square$$

Lemma 5.14. *The set D previously defined is a set of isolated points of $M \setminus L$.*

Proof. Let $m \in D$ and s be a finite sequence such that

$$m = \lambda_0(\overline{\omega_2^T}\, 21\, s\, 12\, \omega_2\, \omega_2^*\, \overline{\omega_2}).$$

Assume there is a sequence $(m_n)_{n\in\mathbb{N}} \in M^{\mathbb{N}}$ such that:

$$m = \lim_{n\to\infty} m_n.$$

We can assume that we have in addition $\forall\, n \in \mathbb{N}$, $m_n \in M \cap (j, J)$. Hence, according to the Corollary 5.11, $\forall\, n \in \mathbb{N}$, $\exists\, S_l^{(n)} \in \{1, 2, 3\}^{\mathbb{Z}^-}$ a sequence such that:

$$m_n = \lambda_0(S_l^{(n)}\, t\, 12\, \omega_2\, \omega_2^*\, \overline{\omega_2}).$$

Using the Proposition 3.6, we can find a integer n_0 such that $\forall\, n \geq n_0$, we have:

$$S_l^{(n)} = \widetilde{S_l}^{(n)}\, 21\, \omega_2^T\, \omega_2^T\, 21\, s.$$

So the sequence $S_l^{(n)}$ contains the words $21\, \omega_2^T\, 2112$ and according to the Corollary 5.11, we must have:

$$\forall\, n \geq n_0,\ \ S_l^{(n)} = \overline{\omega_2^T}\, 21\, s.$$

Therefore, $\forall\, n \geq n_0$, $m_n = m$. Hence, m is an isolated point in $(M \setminus L) \cap (j, J)$. $\square$

Lemma 5.15. *We have:*

$$(M \setminus L) \cap (j_0, J) \subset \big[\lambda_0(\overline{21}\, 1112\, \omega_2\, \omega_2^*\, \overline{\omega_2}), \lambda_0(\overline{21}\, 212\, \omega_2\, \omega_2^*\, \overline{\omega_2})\big].$$

Proof. Since we have $M \cap (j_0, j) = \emptyset$ and $L \cap (j_0, J) = \emptyset$, then, we have:

$$(M \setminus L) \cap (j_0, J) \subset [j, J).$$

Then, since we have $\max \lambda_0(\dots 1112\, \omega_2\, \omega_2^*\, \overline{\omega_2}) \le \min \lambda_0(\dots 212\, \omega_2\, \omega_2^*\, \overline{\omega_2})$, then,

$$\max (M \setminus L) \cap (j_0, J) = \max \lambda_0(\dots 212\, \omega_2\, \omega_2^*\, \overline{\omega_2}).$$

If S is a sequence such that

$$\max (M \setminus L) \cap (j_0, J) = \lambda_0(S),$$

then:

$$S = \dots S_{-17}\, 212\, \omega_2\, \omega_2^*\, \overline{\omega_2}.$$

Since we are maximising the function λ_0, we must add the sequence $\overline{21}$ to the left, so we get:

$$S = \overline{21}\, 212\, \omega_2\, \omega_2^*\, \overline{\omega_2}.$$

By computation, we have:

$$\lambda_0(\overline{21}\, 212\, \omega_2\, \omega_2^*\, \overline{\omega_2}) - \lambda_0(\overline{21}\, 1112\, \omega_2\, \omega_2^*\, \overline{\omega_2}) \approx 2.409522 \times 10^{-12}. \qquad \square$$

5.5. The local border of L.

Theorem 5.16. *$J \in L' \subset L$ and therefore, (j_0, J) is the largest gap of L near of j_0 and we found that $A_2 = (j_0, J)$*

Proof. We have, for all $n \ge 2$

$$\begin{aligned} l_n &= m\Big(\overline{(12)^n\, \omega_2\, \omega_2\, \omega_2^*\, \omega_2\, 11\, (12)^n}\Big) = l\Big(\overline{(12)^n\, \omega_2\, \omega_2\, \omega_2^*\, \omega_2\, 11\, (12)^n}\Big) \\ &= \lambda_0\Big(\overline{(12)^n\, \omega_2\, \omega_2\, \omega_2^*\, \omega_2\, 11\, (12)^n}\Big). \end{aligned}$$

So:

$$\lim_{n \to \infty} l_n = \lambda_0(\overline{12}\, \omega_2\, \omega_2\, \omega_2^*\, \omega_2\, 11\, \overline{12}).$$

In addition, we have $\forall\ n \in \mathbb{N}, l_n \in L$. So $J \in L'$ and since the Lagrange spectrum is closed, we finally get that $J \in L$. $\square$

Theorem 5.17. *We have, $\forall\ \alpha > 0$, $HD(M \cap (j_1, j_1 + \alpha)) = HD(L \cap (j_1, j_1 + \alpha)) = 1$.*

Proof. Let $\alpha > 0$. Using the continuity of the function λ_0 according to Proposition 3.6, there exists $n \in \mathbb{N}$ such that $\forall\ (S_l, S_r) \in \{1,2\}^{\mathbb{Z}^-} \times \{1,2\}^{\mathbb{N}}$, we have:

$$\lambda_0(S) = \lambda_0(S_l\, (12)^n\, \omega_2\, \omega_2\, \omega_2^*\, \omega_2\, 11\, (12)^n\, S_r) \in [J, J + \alpha).$$

In addition, we have;

(1) $\lambda_9(S) = \lambda_0(\dots 2\, \omega_2^*\, 11 \dots) < j_0 - 10^{-6}$ according to Lemma 5.3 (5).
(2) $\lambda_{-9}(S) \le \lambda_0(\dots 212\, \omega_2\, \omega_2^*\, \omega_2\, \omega_2 \dots) < j_0 + 3 \times 10^{-12} < J$.
(3) $\lambda_{-18}(S) = \lambda_0(\dots 212\, \omega_2^*\, 123 \dots) < j_0 - 10^{-6}$ according to Lemma 5.3 (5).

So we get $\lambda_0(S) = m(S)$ and finally,

$$X = \left\{ \lambda_0(S_l\,(12)^n\,\omega_2\,\omega_2\,\omega_2^*\,\omega_2\,11\,(12)^n\,S_r) \,\middle|\, (S_l, S_r) \in \{1,2\}^{\mathbb{Z}^-} \times \{1,2\}^{\mathbb{N}} \right\}$$

is contained in $M \cap [J, J+\alpha)$. We define dynamical Cantor sets:

$$A = \left\{ [3,2,1,1,2,\omega_2,1,1,(1,2)^n, S_r] \,\middle|\, S_r \in \{1,2\}^{\mathbb{N}} \right\},$$
$$B = \left\{ [0,3,3,2,1,\omega_2^T,\omega_2^T,(2,1)^n, S_l^T] \,\middle|\, S_l \in \{1,2\}^{\mathbb{Z}^-} \right\}.$$

We have $X = A_n + B_n$ and moreover A_n, B_n are diffeomorphic to:

$$C(2) := \{[0, a_1, a_2, \dots] \mid \forall\, n \in \mathbb{N}\; a_n \in \{1,2\}\} \text{ with } HD(C(2)) > 0.5,$$

where $C(2)$ is a regular Cantor set of class $\mathcal{C}^2$, non-essentially affine (see [20, Proposition 1]). We have:

$$HD(A) = HD(B) > 0.5.$$

So according to the Moreira's dimension formula [21]:

$$HD(X) = \min\{1, HD(A_n) + HD(B_n)\} = 1,$$

and thus:

$$1 \geq HD(M \cap (j_1, j_1 + \alpha)) \geq HD(X) = 1.$$

Finally, since $HD((M \setminus L)) < 1$, the same argument of Corollary 4.23 shows that $HD(L \cap (j_1, j_1+\alpha)) = 1$ for any $\alpha > 0$. □

6. Lower bound on the Hausdorff distance between M and L

In this section, we focus on a portion of the set $M \setminus L$, above the value $b_\infty = \lambda_0(\overline{2212^*112}) \approx 3.2930442439$.

We also call the word $w = 2212112$ and $w^* = 2212^*112$. According to C. G. Moreira and C. Matheus article [15], we have a good understanding of the spectra around b_∞.

Theorem 6.1. *The largest interval containing only points of $M \setminus L$ near b_∞ is (b_∞, B_∞) where*

$$B_\infty = \lambda_0(\overline{211211212221}\,www^*w\,\overline{112122212112})$$

and

$$B_\infty - b_\infty \approx 2.374867 \times 10^{-7}.$$

They also completely characterized the portion of $M \setminus L$ contained in (b_∞, B_∞), with the following theorem:

Theorem 6.2. *Let $m \in (M \setminus L) \cap (b_\infty, B_\infty)$. Then $m = m(B) = \lambda_0(B)$ for a sequence $B \in \{1,2\}^{\mathbb{Z}}$ such that:*

(1) $B_{-10} \dots B_0 B_1 \dots B_7 \dots = ww^*\overline{w}$.

(2) *The sequence $(B_n)_{n \le -11}$ has the following properties:*
   (a) *It does not contain the following words:* 21212*,* 121211*,* 112121*,* 212111*,* 111212*,* $2w21$*,* $12w^T2$*,* $12w22$*.*
   (b) *If $\exists\ n \le -21$, such that $B_{n-4}\dots B_{n+5} = 2w^T21$, then: $\dots B_{n-7}\dots B_{n+10} = \overline{w^T}\, w^T$.*

Giving two sets $A, B \subset \mathbb{R}$, we denote $d(A,B) = \inf_{x\in A, y\in B}|x-y|$. Recall that since $L \subset M$, we have that $d_H(L,M) = \sup_{m\in M\setminus L} d(L,m)$ where $d_H(L,M)$ denotes the Hausdorff distance between L and M. In particular, the lower bound of Theorem 1.6 follows from the following result.

Theorem 6.3. *We have*

$$\begin{aligned}&\sup_{m\in(M\setminus L)\cap(b_\infty,B_\infty)} d(L,m)\\ &\quad= \lambda_0(\overline{211211212221}\,www^*w\,\overline{112122212112}) - \lambda_0(\overline{221211}\,ww^*\,\overline{w})\\ &\quad\approx 9.1094243388\times 10^{-8}\end{aligned}$$

Proof. Letting $\delta_0 = \sup_{m\in(M\setminus L)\cap(b_\infty,B_\infty)} d(L,m)$, it can be written in the form:

$$\delta_0 = \frac{B_\infty - b_\infty}{2} - \inf_{m\in M}\left|m - \frac{B_\infty + b_\infty}{2}\right|.$$

So we only need to focus on the following quantity:

$$\begin{aligned}\varepsilon_0 &= \inf\left\{\left|m - \frac{b_\infty + B_\infty}{2}\right| \;\middle|\; m\in M\right\}\\ &= \inf\left\{\left|m - \frac{b_\infty + B_\infty}{2}\right| \;\middle|\; m\in (M\setminus L)\cap(b_\infty,B_\infty)\right\}.\end{aligned}$$

This calculation requires very few steps, thanks to the nice properties of the function λ_0.

Lemma 6.4. *We have:*
(1) $\lambda_0(\dots 2\,ww^*\overline{w}) \le \lambda_0(\overline{12}\,2\,ww^*\overline{w}) \le \frac{b_\infty+B_\infty}{2} - 9\times 10^{-8}$.
(2) $\lambda_0(\dots 1\,ww^*\overline{w}) \ge \lambda_0(\overline{21}\,1\,ww^*\overline{w}) \ge \frac{b_\infty+B_\infty}{2} + 2\times 10^{-8}$.

Because of the Lemma 6.4, to find ε_0, we only need to calculate the following values:

$$m_1 = \min\{\lambda_0(S1\,ww^*\overline{w}) \mid S1 \text{ verifies the properties of Theorem 6.2}\},$$

$$m_2 = \max\{\lambda_0(S2\,ww^*\overline{w}) \mid S2 \text{ verifies the properties of Theorem 6.2}\}.$$

Thus we must have $\varepsilon_0 = \min\{m_1 - \frac{b_\infty+B_\infty}{2}, \frac{b_\infty+B_\infty}{2} - m_2\}$.

Let S be a sequence such that $S = \dots S_{-12}\,1\,ww^*\overline{w}$.

Since we are minimizing $\lambda_0(S)$, we must develop S with:

$$S = \dots S_{-16}\,21211\,ww^*\overline{w}.$$

Since the word 121211 is forbidden, we must have:

$$S = \ldots S_{-17}\,221211\,ww^*\overline{w}.$$

Since the word $2w21$ is forbidden, we must have:

$$S = \ldots S_{-18}\,1221211\,ww^*\overline{w}.$$

Then we use repeatedly the same argument, since -18 and -12 are both even and finally, we get $S = \overline{221211}\,ww^*\overline{w}$, and:

$$m_1 = m(\overline{221211}\,\overline{w}) = \lambda_0(\overline{221211}\,ww^*\,\overline{w}) \approx \frac{b_\infty + B_\infty}{2} + 2.764903258 \times 10^{-8}.$$

Remark 6.5. The point m_1 is accumulated on the right by points of M.

The first point of Lemma 6.4 ensures that:

$$\varepsilon_0 = m_1 - \frac{b_\infty + B_\infty}{2}$$

Therefore δ_0 is equal to

$$B_\infty - m_1 = \lambda_0(\overline{211211212221}\,www^*w\,\overline{112122212112}) - \lambda_0(\overline{221211}\,ww^*\,\overline{w}).$$

Since

$$\lambda_0(\overline{211211212221}\,www^*w\,\overline{112122212112}) = \frac{780369102362985 + 454284412153\sqrt{151905}}{290742023368320},$$

$$\lambda_0(\overline{221211}\,ww^*\,\overline{w}) = \frac{82\sqrt{87} + 57\sqrt{18229} + 6931}{4674},$$

the exact formula of δ_0 can be written as follows

$$\delta_0 = \frac{\begin{array}{l}272052036746460995 - 3973474319367040\sqrt{87}\\ \quad - 2762049221999040\sqrt{18229} + 353887557067187\sqrt{151905}\end{array}}{226488036203921280}$$
$$\approx 9.1094243388 \times 10^{-8}. \qquad \square$$

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