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The genus of division algebras over discrete valued fields

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Résumé. Étant donné un corps muni d'un ensemble de valuations discrètes V, nous montrons comment le genre d'une algèbre à division sur ce corps est lié aux genres des algèbres résiduelles par rapport à V et à des propriétés de ramification. Nous démontrons la trivialité du genre pour les algèbres de quaternions sur une large classe de corps, y compris les corps locaux multidimensionnels et les corps de fonctions des courbes définies sur un corps local multidimensionnel ou sur un corps réel clos. En outre, nous considérons le cas du corps de fonctions d'une courbe ayant un point rationnel sur un corps global et établissons un lien entre le problème de genre sur ce corps et la 2-torsion du groupe de Tate–Shafarevich de sa jacobienne. En particulier, nous montrons que sur les corps de fonctions des courbes elliptiques, les méthodes développées donnent de meilleurs bornes pour le genre et expliquons comment elles peuvent être calculées directement en utilisant des informations arithmétiques, en donnant des exemples.

Abstract. Given a field with a set of discrete valuations V, we show how the genus of a division algebra over the field is related to the genus of the residue algebras at various valuations in V and the ramification data. When the division algebra is a quaternion, we show the triviality of genus over many fields which include higher local fields, function fields of curves over higher local fields and function fields of curves over real closed fields. We also consider function fields of curves over global fields with a rational point and show how the genus problem is related to the 2-torsion of the Tate–Shafarevich group of its Jacobian. As a special case, we show how the methods developed yield better bounds on the size of the genus over function fields of elliptic curves and demonstrate how they can be computed directly using arithmetic data of the elliptic curve with a number of examples.

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Mots-clefs. Brauer groups, genus of division algebras, quaternions, elliptic curves, Tate–Shafarevich group, discrete valued fields, higher local fields, semi-global fields.

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1. Introduction

Given a finite dimensional division algebra D over K, one can ask how much of information is carried by the collection of maximal subfields that D contains. Let $[D]$ denote the class of D in $\mathrm{Br}(K)$. Then the *genus* of $[D]$, denoted $\mathrm{gen}([D])$ is the collection of all classes in $\mathrm{Br}(K)$ whose underlying division algebras share the same maximal subfields as D. One can ask if this collection is finite and if so what is its size. This has been the studied extensively over the past few years (see [6], [7], [8], [18], [34]) and plays an important role in the analysis of weak commensurability of Zariski-dense subgroups of the corresponding algebraic group ([32, Remark 5.4], [33, §6]).

Note that if $[D]$ is of exponent $n \geq 3$ in $\mathrm{Br}(K)$, for every i coprime to n, the underlying division algebra of $D^{\otimes i}$ has the same collection of maximal subfields as D. In particular, the $\mathrm{gen}([D])$ contains classes other than $[D]$ unless $[D]$ is of exponent 2. In the case of finitely generated fields of characteristic coprime to the exponent of $[D]$, it shown in [34] that $\mathrm{gen}([D])$ is always finite and explicit upper bounds on the size of the genus are given for many cases in [8]. When D is a quaternion algebra or in general an exponent 2 division algebra, a natural question is to know if $\mathrm{gen}([D])$ is always trivial i.e. is a singleton set. In other words, we may ask if quaternion algebras are completely determined by the collection of maximal subfields contained in them. Although the answer is no in general, the non-triviality of the genus for quaternions is known to happen over very large fields of infinite transcendence degree ([13, §2]). So one may wonder if it is trivial for reasonably nice fields. The answer to this question is unknown although this is known to be true for some fields. Examples include global fields ([6, §3.6]) and more generally for *transparent fields* ([13, §6]). Several other examples can be found in [18]. Moreover it is shown in [6, Theorem 3.5] that when $\mathrm{char}\, K \neq 2$, the property of genus being trivial for exponent 2 algebras is stable under purely transcendental extensions.

An important tool used in the literature so far for genus computations is the *unramified Brauer group* of K with respect to a set of discrete valuations V, denoted by $\mathrm{Br}(K)_V$. For example, it is shown in [6, Theorem 2.2] that if V satisfies certain conditions and if the n-torsion component ${}_n\mathrm{Br}(K)_V$ is finite then for any division algebra D of exponent and degree n, its genus is upper bounded by $|{}_n\mathrm{Br}(K)_V| \cdot \phi(n)^r$ where ϕ is the Euler's Totient function and r is the number of ramification places of D. While this bound is extremely useful in showing finiteness of genus and in bounding its size, it does not give any explicit description of the elements in the genus. Moreover, often times the bound given by ${}_n\mathrm{Br}(K)_V$ is loose and sometimes ${}_n\mathrm{Br}(K)_V$ is not finite for some fields that arise naturally in arithmetic geometry. For example, take $K = \mathbb{Q}((t))$ with V being the t-adic valuation. Then by [36, Chapter XXII, Theorem 2]) we have $\mathrm{Br}(\mathbb{Q}) \simeq \mathrm{Br}(\mathbb{Q}((t)))_V$ and hence ${}_n\mathrm{Br}(K)_V$ is not finite. Another example is when $K = k((t))(C)$ and V is the set of discrete valuations arising from codimension one points of some regular proper model of C over $k[[t]]$. Then $\mathrm{Br}(k) \subseteq \mathrm{Br}(K)_V$ and therefore ${}_n\mathrm{Br}(K)_V$ is not finite if ${}_n\mathrm{Br}(k)$ is not finite (this happens for example when $k = \mathbb{Q}$).

In this paper, we define the *separable genus* of $[D]$, denoted $\mathrm{gen}^s([D])$ to be collection of all classes of division algebras in $\mathrm{Br}(K)$ whose underlying division algebras share the same *separable* maximal subfields as D. For simplicity, instead of using the term "separable genus", we simply refer to it as the "genus". We include separability assumption in our definition of genus to be consistent with the definition of genus of the corresponding algebraic group $SL_{1,D}$ (see [6, Remark 5.1]) and also to be consistent with the notion defined in [18]. Note that when $\mathrm{char}\, K$ is zero or is coprime to the degree n of D (which is assumed for the fields considered in [6]), every maximal subfield of D is separable over K and the two notions coincide i.e., $\mathrm{gen}^s([D]) = \mathrm{gen}([D])$. We study the genus of tame division algebras over discrete valued fields that are not necessarily finitely generated. We also do not assume that ${}_n\mathrm{Br}(K)_V$ is finite. We show how the genus over these fields is related to the genus of residue algebras and the ramification data. In many cases, we give explicit description of the elements in the genus and show that we get better bounds on its size than known before. Roughly speaking, the idea behind our arguments is to describe the genus of the division algebra locally in terms of the genus of its "unramified component" and the "ramified component". The genus of the unramified component is directly related to the genus of the associated residue algebra which can be computed either directly or through an induction process. Once we have local information on the genus, we use some kind of local-global information to compute the genus globally.

We use the above techniques to give a short proof of the *Stability theorem* ([6, Theorem 3.5]) for purely transcendental extensions in Section 5. We also give an explicit description for the elements in the genus of division algebras over $k(x)$ where k is a number field and show that the bounds on the size of the genus thus obtained are tighter than known before. Next in Section 6, we analyze the genus problem for tame algebras over complete discrete valued fields and show how it is related to the genus of residue algebras. In this section, we also consider a different notion of genus which we call as the *splitting genus*, denoted by gen^s_{spl}. By definition $\mathrm{gen}^s_{spl}([D])$ is the collection of classes of division algebras sharing the same separable finite dimensional splitting fields with D[1]. Based on a decomposition lemma, we describe elements in $\mathrm{gen}^s_{spl}([D])$ in terms of the genus of its ramified and unramified components. Finally, in Section 7, we apply the above techniques for the

[1]A similar notion without separability assumption is considered in [19]

special case of quaternion algebras over a field K with a set of discrete valuations V. For $v \in V$, let K_v and $\overline{K_v}$ denote respectively the completion and residue field of K with respect to v. Assume for every $v \in V$, that $\overline{K_v}$ satisfies the property that the genus of any quaternion algebra over $\overline{K_v}$ is trivial. Then we show that for a quaternion Q over K that is tame with respect to V, any element in $\mathrm{gen}^s([Q])$ is Brauer equivalent to $Q \otimes_K Q'$ for some $[Q'] \in {}_2\text{Ш}^{\mathrm{Br}}(K,V)$. Here

$$
{}_2\text{Ш}^{\mathrm{Br}}(K,V) := \ker\left({}_2\mathrm{Br}(K) \longrightarrow \prod_{v\in V} {}_2\mathrm{Br}(K_v)\right)
$$

is the 2-torsion component of the kernel of the local-global map on the Brauer group with respect to V. Hence we get that $|\mathrm{gen}^s([Q])| \leq |{}_2\text{Ш}^{\mathrm{Br}}(K,V)|$. Comparing with the bound $|\mathrm{gen}^s([Q])| \leq |{}_2\mathrm{Br}(K)_V|$ given in [6, Theorem 2.2], we see that we get a better upper bound on the genus since the residue maps factor through the local-global maps and therefore ${}_2\text{Ш}^{\mathrm{Br}}(K,V) \subseteq {}_2\mathrm{Br}(K)_V$ (see (4.3)). Continuing with our examples before, for $K = \mathbb{Q}(\!(t)\!)$ with V being the obvious t-adic valuation, we get ${}_2\text{Ш}^{\mathrm{Br}}(K,V) = 0$ trivially and for $K = \mathbb{Q}(\!(t)\!)(C)$ with V the set of discrete valuations arising from codimension one points of some regular proper model of C over $k[\![t]\!]$, ${}_2\text{Ш}^{\mathrm{Br}}(K,V)$ is also trivial by [10, Theorem 4.2 and Theorem 4.3 (ii)]. Hence $\mathrm{gen}^s([Q])$ is trivial for any quaternion over K for the above fields although ${}_2\mathrm{Br}(K)_V$ is not even finite as shown before. We give a number of examples of fields over which the genus of any quaternion is trivial using the above techniques. These include higher local fields, function fields of curves over higher local fields (these are semi-global fields and have been extensively studied in over the recent years) and function fields of curves over real closed fields.

In the special case when K is the function field of a curve C over a global field with a rational point, we show that the size of genus of any quaternion algebra over K is bounded by size of the 2-torsion subgroup of the Tate–Shafarevich group of the Jacobian, ${}_2\text{Ш}(J_C)$. In particular, when $C = E$ is an elliptic curve we have $|\mathrm{gen}^s([Q])| \leq |{}_2\text{Ш}(E)|$ and therefore is trivial whenever ${}_2\text{Ш}(E)$ is trivial. In the cases where ${}_2\text{Ш}(E)$ is not trivial, we demonstrate how the bounds on the genus can be improved by directly using the arithmetic properties of E.

2. Notation

Although the term genus was originally coined for division algebras in [6], in this paper, we denote by $\mathrm{gen}^s(\alpha)$ (resp. $\mathrm{gen}^s_{spl}(\alpha)$) for $\alpha \in \mathrm{Br}(K)$, the Brauer classes whose underlying division algebras have the same separable maximal subfields (resp. same finite dimensional separable splitting fields) as the underlying division algebra of α. This is a well defined notion and makes it convenient to talk about the genus of a Brauer class.

For a division algebra D over K, $[D]$ denotes its Brauer class in $\mathrm{Br}(K)$. We write $D \sim E$ if $[D] = [E]$ in $\mathrm{Br}(K)$. If D is a valued division algebra, its residue algebra is denoted by $\overline{D}$. If K has a discrete valuation v, D_v denotes $D \otimes_K K_v$ and $\overline{D_v}$ denotes the residue algebra of the underlying division algebra of D_v. If K contains a primitive n-th root of unity ω, the symbol algebra generated by i, j over K with the relations $i^n = a, j^n = b, ij = \omega ji$ is denoted by $(a, b)_\omega$.

The n-torsion part of a group G is denoted by ${}_nG$. For $g \in G$, $\mathrm{ord}(g)$ denotes the order of g. The symbol (m, n) denotes the greatest common divisor of m and n. The separable closure of K is denoted by K^{sep}. All the valuations considered in this paper are discrete.

3. Main Results

For a division algebra D over a field k, let $\widetilde{D} := D \otimes_k k(x)$.

Theorem 3.1 (Genus over purely transcendental extension of number fields, Section 5)**.** *Let k be a number field containing a primitive n-th root of unity ω. For any $[D] \in {}_n\mathrm{Br}(k(x))$, write $[D] \simeq [\widetilde{C} \otimes R_1 \otimes R_2 \otimes \cdots \otimes R_r]$ where $[C] \in {}_n\mathrm{Br}(k)$, each $[R_i]$ is ramified at exactly one point in $\mathbb{P}^1_0 - \infty$ and nowhere else and $[R_i] \sim \otimes_j (f_{ij}, g_{ij})_\omega$ where for every j either f_{ij} or g_{ij} is a non-constant monic polynomial in $k[x]$(this is possible by Lemma 5.2). Then*

$$\mathrm{gen}^s([D]) \subseteq \left\{ [\widetilde{C'} \otimes R_1^{i_1} \otimes R_2^{i_2} \otimes \cdots \otimes R_r^{i_r}] \middle| \begin{matrix} [C'] \in \mathrm{gen}^s([C]) \text{ and} \\ (i_l, \mathrm{ord}([R_l])) = 1 \ \ \forall\, 1 \le l \le r \end{matrix} \right\}$$

In particular,

$$|\mathrm{gen}^s([D])| \le |\mathrm{gen}^s([C])| \phi(n)^r$$

where ϕ is the Euler's Totient function.

Now assume that k is a complete discrete valued field with residue $\overline{k}$. Let $\mathrm{SBr}(k)$ and $\mathrm{IBr}(k)$ denote respectively the subgroup of inertially split (tame) and inertial (unramified) classes in $\mathrm{Br}(k)$(Section 4.1).

Theorem 3.2 (Theorem 6.6)**.** *Let $[D] \in \mathrm{SBr}(k)$ be of prime index p. Then*

$$\mathrm{gen}^s([D]) \begin{cases} \subseteq \{[C] \in \mathrm{IBr}(k) | [\overline{C}] \in \mathrm{gen}^s([\overline{D}])\} & \text{if } [D] \text{ is unramified} \\ = \{[D^{\otimes i}] | 1 \le i \le p-1\} & \text{otherwise} \end{cases}$$

In particular,

$$|\mathrm{gen}^s([D])| \begin{cases} \le |\mathrm{gen}^s(\overline{D})| & \text{if } [D] \text{ is unramified} \\ = p - 1 & \text{otherwise} \end{cases}$$

If $\overline{k}$ is perfect, then $\subseteq$ and $\leq$ in the above expressions are equalities. Moreover, all the above statements also hold if $\mathrm{gen}^s([D])$ *is replaced with* $\mathrm{gen}^s_{spl}([D])$.

As an easy consequence, we get

Corollary 3.3 (Corollary 6.7). *Let $\overline{k}$ satisfy the property that the genus (resp. splitting genus) is trivial for any quaternion algebra over $\overline{k}$. Then the genus (resp. splitting genus) of any tame quaternion algebra over k is trivial.*

For a division algebra D, let e_D denote the ramification index of D. Recall the definition of gen^s_{spl} from Section 1.

Theorem 3.4 (Genus decomposition for gen^s_{spl}, Theorem 6.12). *Let I be unramified and N be NSR (Section 4.1) division algebras over k. Then*

$$\mathrm{gen}^s_{spl}([I \otimes_k N]) \subseteq \{[I' \otimes_k N'] \,|\, [I'] \in \mathrm{gen}^s([I]) \text{ and } [N'] \in \mathrm{gen}^s([N])\}$$

Corollary 3.5 (Corollary 6.15). *Let $[D] \in \mathrm{SBr}(k)$. Write $D \sim I \otimes_k N$ where I is inertial and N is NSR. Then any element in $\mathrm{gen}^s_{spl}([D])$ is of the form $[I' \otimes_k N^{\otimes j}]$ for some $(j, e_D) = 1$ where $[I'] \in \mathrm{gen}^s([I])$. The algebra I' is the (unique) inertial lift of some division algebra whose class lies in $\mathrm{gen}^s([\bar{I}])$. In particular*

$$|\mathrm{gen}^s_{spl}([D])| \leq |\mathrm{gen}^s([\bar{I}])| \cdot \phi(e_D)$$

where ϕ denotes the Euler's Totient function. If moreover $\overline{k}$ is perfect, then I' above is the (unique) inertial lift of some division algebra whose class lies in $\mathrm{gen}^s_{spl}([\bar{I}])$ and

$$|\mathrm{gen}^s_{spl}([D])| \leq |\mathrm{gen}^s_{spl}([\bar{I}])| \cdot \phi(e_D)$$

Now let K is an arbitrary field of char $\neq 2$ with a set of discrete valuations V. We denote the completion of K with respect to v and its residue field respectively by K_v and $\overline{K_v}$. Recall from Section 1 that ${}_2\text{Ш}^{\mathrm{Br}}(K, V)$ denotes the 2-torsion of the kernel of the local-global map on the Brauer group with respect to V.

For a division algebra D over K, we say that D (or its class in $\mathrm{Br}(K)$) is tame with respect to V if $D_v := D \otimes_K K_v$ is tame for every $v \in V$. We denote that set of tame elements of $\mathrm{Br}(K)$ with respect to V by $\mathrm{SBr}(K, V)$.

Theorem 3.6. *(Theorem 7.3) Let $[Q] \in \mathrm{SBr}(K, V)$ be the class of a quaternion division algebra over K. Suppose for every $v \in V$, the genus of any quaternion division algebra over $\overline{K_v}$ is trivial. Then*

$$\mathrm{gen}^s([Q]) \subseteq [Q] + {}_2\text{Ш}^{\mathrm{Br}}(K, V)$$

In particular,

$$|\mathrm{gen}^s([Q])| \leq |{}_2\text{Ш}^{\mathrm{Br}}(K, V)|$$

Corollary 3.7 (Corollary 7.5)**.** *If K is one of the following fields, the genus of any quaternion division algebra over K is trivial*

1. *Higher local fields where the final residue field has characteristic $\neq 2$*
2. *Iterated Laurent series $k((t_1))((t_2))\ldots((t_n))$ (resp. their finite extensions) where $\operatorname{char} k \neq 2$ and every quaternion division algebra over k (resp. every finite extension of k) has trivial genus*
3. *Function fields of one variable over fields in* (1)
4. *Function fields of one variable any real closed field*
5. *Function fields of one variable over fields in* (2) *where for any curve C over k (where k is as in* (2)*), every quaternion algebra over every finite extension of k and over $k(C)$ has trivial genus (for example, $k((t_1))((t_2))\ldots((t_n))(C)$ where one can take k to be any real closed field by* (4)*)*

For an abelian variety A over a global field k, let Ш(A) denote the Tate–Shafarevich group.

Theorem 3.8 (Section 7.1)**.** *Let C be a smooth projective geometrically integral curve over a global field k with a rational point. Let Q be a quaternion algebra over $k(C)$ that is tame with respect to every (dyadic) discrete valuation arising from a codimension one point of a regular projective model of C. Then*

$$|\mathrm{gen}^s([Q])| \leq |{}_2\text{Ш}(J_C)|$$

where J_C is the Jacobian of C. In particular, when $C = E$ is an elliptic curve, we have

$$|\mathrm{gen}^s([Q])| \leq |{}_2\text{Ш}(E)|$$

In particular, we get $\mathrm{gen}^s([Q])$ *is trivial whenever* ${}_2$Ш(E) *is trivial.*

4. Preliminaries

4.1. Division algebras over complete discrete valued fields. Let D be a finite dimensional division algebra over a complete discrete valued field k with residue field $\bar{k}$. Since k is complete, the valuation on k extends uniquely to a valuation v on D ([41, Corolary 2.2]). Associated to v, one can define the residue algebra ([41, §2]

$$\overline{D} = V_D/M_D$$

where

$$V_D = \{a \in D^* | v(a) \geq 0\} \cup 0$$
$$M_D = \{a \in D^* | v(a) > 0\} \cup 0$$

One can easily see that $\overline{D}$ is a division algebra over its center $Z(\overline{D})$ (which may or may not be equal to the residue field $\bar{k}$). Let Γ_D and Γ_k denote respectively the value group of D and k. Note that, the valuation v is discrete and we have $\Gamma_k \subseteq \Gamma_D \subseteq \frac{1}{\sqrt{[D:k]}}\Gamma_k$ ([41, (2.7)]). Therefore, Γ_D/Γ_k is cyclic and the number $e_D := |\Gamma_D : \Gamma_k|$ is called the *ramification index* of D. Moreover, we have the following *fundamental equality* ([41, (2.10)], [26, p. 359])

$$[D : k] = [\overline{D} : \overline{k}][\Gamma_D : \Gamma_k] \tag{4.1}$$

We say that D is *unramified or inertial* if $Z(\overline{D}) = \overline{k}$ and $[\overline{D} : \overline{k}] = [D : k]$. An interesting fact similar to the case of fields is that, given any division algebra $\widetilde{D}$ over $\overline{k}$, there is a unique unramified (also called inertial) division algebra D over k with the property that $\overline{D} \simeq \widetilde{D}$. This is called the *inertial lift* of $\widetilde{D}$ over k ([17, Theorem 2.8 (a)]). Note that if D is unramified, the ramification index $e_D = 1$ by the above equality. When e_D is larger that 1, D is said to be *ramified*. One special case of ramified division algebras is that of *nicely semiramified* (abbreviated as NSR) algebras. A division algebra N over k is said to be NSR if it contains a maximal subfield that is unramified as well as a maximal subfield that is totally ramified k ([17, §4], [28, Theorem 2.4]). In this case, $\overline{N}$ is a field and

$$[\overline{N} : \overline{k}] = [\Gamma_N : \Gamma_k] = \sqrt{[N : k]}$$

See [17, §4] for more details.

We say that D is *inertially split* if D is split by an unramified (inertial) extension of k. By [17, Lemma 6.2] and [40, Remark 3.2 (a)], this is equivalent to D being *tame* as defined in [17, Definition §6]. When $\overline{k}$ is perfect or if char $\overline{k}$ is coprime to the degree of D, every D over k is tame ([36, Theorem 1, Exercise 3, Chapter XII]). Note that an unramified division algebra D is automatically tame since it contains the inertial lift of a separable maximal subfield of $\overline{D}$ ([17, Theorem 2.9]). Following [41] we define

$$\mathrm{IBr}(k) = \{[D] | D \text{ is a finite dimensional unramified division algebra over } k\}$$
$$\mathrm{SBr}(k) = \{[D] | D \text{ is a finite dimensional tame division algebra over } k\}$$

We observe the following:

Proposition 4.1. *Let D be finite dimensional tame division algebra over k such that $\overline{D}$ is a field. Then D is cyclic. In particular, any NSR division algebra over k is cyclic.*

Proof. By [17, Lemma 5.1], $\overline{D}/\overline{k}$ is a cyclic field extension and

$$[\overline{D} : \overline{k}] = [\Gamma_D : \Gamma_k] = \sqrt{[D : k]}$$

by (4.1). Now D contains the inertial lift L of $\overline{D}$ by [17, Theorem 2.9]. Note that L is a maximal subfield and is cyclic([36, Chapter III, §5, Theorem 3])).

Therefore D is cyclic. The last statement follows from the previous statement and fact that any NSR division algebras is tame (since it contains an unramified maximal subfield) and the residue algebra is a field. $\square$

Theorem 4.2 ([17, Lemma 5.14, Theorem 5.15])**.** *Given any* $[D] \in \mathrm{SBr}(k)$, *there is a (non-canonical) decomposition in* $\mathrm{Br}(k)$ *given by*

$$D \sim I \otimes_k N$$

where I *is unramified and* N *is NSR. Moreover* $Z(\overline{D}) = \overline{N}$, $\overline{D} \simeq \overline{I} \otimes_{\overline{k}} \overline{N}$, $e_D = e_N$

Since non-tame division algebras pose certain difficulties that we are not able to handle at the moment, we will consider only tame algebras in this paper.

4.2. The unramified Brauer group. Let K be a field equipped with a discrete valuation v. Let K_v, $\mathcal{O}_v$ and $\overline{K_v}$ denote respectively the completion of K at v, its valuation ring and the residue field. For every n such that n is coprime to char $\overline{K_v}$ or when $\overline{K_v}$ is perfect, one can define the *residue homomorphism* ([35, Chapter 10])

$$\begin{aligned} \rho_v^n : {}_n\mathrm{Br}(K_v) &\longrightarrow H^1(\overline{K_v}, \mathbb{Z}/n\mathbb{Z}) \\ [D] &\longrightarrow \chi_D \end{aligned}$$

We recall that the image of χ_D is Γ_D/Γ_{K_v} ([17, Theorem 5.6 (b)]) and the fixed field of $\ker(\chi_D)$ is the center of $\overline{D}$ ([41, Theorem 3.5]). The kernel of ρ_v^n can be identified with ${}_n\mathrm{Br}(\mathcal{O}_v)$ and is the n-torsion component of the *unramified Brauer group of* K *at* v. A non-trivial class $[D] \in \mathrm{Br}(K_v)$ is unramified at v if and only if D is unramified in the sense of Section 4.1 ([41, (3.9)]). Therefore we have ${}_n\mathrm{IBr}(K_v) = \ker(\rho_n^v)$ and there is isomorphism([41, (3.7)])

$$\begin{aligned} \mathrm{IBr}(K_v) &\simeq \mathrm{Br}(\overline{K_v}) \\ D &\longmapsto \overline{D} \end{aligned} \tag{4.2}$$

Now let K be equipped with a set of discrete valuations V. Assuming char $\overline{K_v}$ is coprime to n or $\overline{K_v}$ is perfect for every $v \in V$, we have the residue map relative to V given by

$$\rho_V^n : {}_n\mathrm{Br}(K) \longrightarrow \prod_{v \in V} {}_n\mathrm{Br}(K_v) \longrightarrow \prod_{v \in V} H^1(\overline{K_v}, \mathbb{Z}/n\mathbb{Z})$$

where the first map is the local-global map on the Brauer group with respect to V whose kernel is denoted by ${}_n\text{Ш}^{\mathrm{Br}}(K, V)$. We say that the *local-global principle holds on* ${}_n\mathrm{Br}(K)$ *with respect to* V if ${}_n\text{Ш}^{\mathrm{Br}}(K, V)$ is trivial. The n-torsion of the *unramified Brauer group of* K *with respect to* V is defined to be

$${}_n\mathrm{Br}(K)_V := Ker\ \rho_V^n$$

We note that

$$ {}_n\text{Ш}^{\text{Br}}(K, V) \subseteq {}_n\text{Br}(K)_V \tag{4.3} $$

Let X be a regular integral noetherian scheme with function field $k(X)$ and let V be the valuations corresponding to the codimension 1 points $X^{(1)}$ of X. Let $k(x)$ denote the residue field at $x \in X^{(1)}$. Then we have an exact sequence ([31, Theorem 6.8.3])

$$ 0 \longrightarrow \text{Br}(X) \longrightarrow \text{Br}(k(X)) \xrightarrow{\rho_V} \bigoplus_{x \in X^{(1)}} H^1(k(x), \mathbb{Q}/\mathbb{Z}) $$

where we exclude the p-primary components in all of the above groups if $k(x)$ is imperfect of characteristic p for some x ([31, Theorem 6.8.3], [4, Remark 6.4]).

4.3. Higher local fields. We will quickly state a few facts on higher local fields that we use in this paper. We will use the notations from ([27, Definition 2.1]).

Given a field F, the *complete discrete valuation dimension* of F, denoted $\text{cdv.}\dim(F)$ is defined to be

$$ \text{cdv.}\dim(F) := \begin{cases} 0 & \text{if } F \text{ is not compete discrete valued field} \\ \text{cdv.}\dim(\overline{F}) + 1 & \text{otherwise} \end{cases} $$

where $\overline{F}$ is the residue field.

Set $F^{(0)} := F$. For every i, if $F^{(i)}$ is a complete discrete valued field, one can define $F^{(i+1)}$ to be the residue field of $F^{(i)}$. A field F is said to be an *n-dimensional local field* for some $n \geq 0$ if $(F) = n$ and the final residue field $F^{(n)}$ is finite.

By [27, Proposition 2.15], every finite extension of an n-dimensional local field is an n-dimensional local field. The classification of n-dimensional local fields can be found in [43, §1.1], [27, §2.1] (although we will not be needing it in this paper).

4.4. An important lemma. We will repeatedly use a lemma at many places in genus computations. For a division algebra D over K equipped with a discrete valuation v, let D_v denote $D \otimes_K K_v$ and $\overline{D_v}$ denote the residue algebra of the underlying division algebra of D_v.

Lemma 4.3. *Let D be a division algebra over a field K with discrete valuation v. Assume that D is tame with respect to v. If $[D'] \in \text{gen}^s([D])$, then $[\overline{D'_v}] \in \text{gen}^s([\overline{D_v}])$.*

Proof. By [34, Corollary 2.4], the underlying division algebras of D_v and D'_v have the same maximal separable subfields. We may assume that D_v (and hence D'_v) are division algebras. Since D_v (and hence D'_v) is tame, the center $Z(\overline{D_v})$ is separable over $\overline{K_v}$ ([41, Theorem 3.4 (iii)]). Moreover, by [6, Lemma 2.3 and Remark 2.6], $Z(\overline{D_v}) \simeq Z(\overline{D'_v})$ over $\overline{K_v}$. Let $\widetilde{L} \subset \overline{D_v}$ be a separable maximal subfield over $Z(\overline{D_v})$ (hence separable over $\overline{K_v}$) and let L be the inertial lift of $\widetilde{L}$ over K_v. Then by [17, Theorem 2.9], we conclude that $L \subset D_v$ is a separable subfield. By the fundamental equality (4.1) and [17, Lemma 5.1 (iii)], L is a separable maximal subfield of D_v. By hypothesis, L is also a separable maximal subfield of D'_v. Hence, $\widetilde{L} \subset \overline{D'_v}$ is a separable maximal subfield. Since the above argument is symmetric with respect to D_v and D'_v, we conclude that $[\overline{D'_v}] \in \mathrm{gen}^s([\overline{D_v}])$. □

5. Genus over purely transcendental extensions

Let k be a field with n invertible. Recall the Faddeev's split exact sequence ([14, Corollary 6.9.3]):

$$0 \longrightarrow {}_n\mathrm{Br}(k) \longrightarrow {}_n\mathrm{Br}(k(x)) \xrightarrow{\rho:=\oplus\rho_P} \bigoplus_{P\in\mathbb{P}^1_0-\infty} H^1(k(P),\mathbb{Z}/n\mathbb{Z}) \longrightarrow 0 \tag{5.1}$$

where the third arrow coincides with the residue map in Galois cohomology with respect to the discrete valuation corresponding to the points $P \in \mathbb{P}^1_0 - \infty$. We say that an element in $\mathrm{Br}(k(x))$ is ramified or unramified at P, if it is so for the discrete valuation corresponding to P.

The exact sequence (5.1) splits. Let

$$\theta : \bigoplus_{P\in\mathbb{P}^1_0-\infty} H^1(k(P),\mathbb{Z}/n\mathbb{Z}) \longrightarrow {}_n\mathrm{Br}(k(x))$$

be a splitting. So any class $[D] \in {}_n\mathrm{Br}(k)$ can be written as

$$[D] \sim [\widetilde{C} \otimes_{k(x)} R] \tag{5.2}$$

where $[\widetilde{C}] \in {}_n\mathrm{Br}(k(x))$ is the restriction of a unique $[C] \in {}_n\mathrm{Br}(k)$ and $[R]$ is an element in the image under θ. Write $[R] = [R_1 \otimes R_2 \otimes \cdots \otimes R_r]$ where r is the size of the ramification locus of D i.e., number of points $P \in \mathbb{P}^1_0 - \infty$ at which the residue map is non-zero and $[R_i]$ are the image of the ramification components under θ.

For any division algebra D over $k(x)$ and $P \in \mathbb{P}^1_0 - \infty$, let $D_P := D \otimes_{k(x)} k(x)_P$ where $k(x)_P$ is the completion of $k(x)$ with respect to the discrete valuation corresponding to P and let $\overline{D_P}$ denote the residue algebra of the division algebra in the class $[D_P]$. Similar notations, α_P and $\overline{\alpha_P}$ are used for any $\alpha \in \mathrm{Br}(k(x))$.

The decomposition of D given in (5.2) into unramified and ramified components easily yields a short proof of the *Stability Theorem*.

Theorem 5.1 (Stability Theorem, [6, Theorem 3.5])**.** *Let k be a field of characteristic $\neq 2$ with the property that the genus of any element in ${}_2\mathrm{Br}(k)$ is trivial. Then the same property holds for ${}_2\mathrm{Br}(k(x))$.*

Proof. Let $[D], [E] \in {}_2\mathrm{Br}(k(x))$ be such that $[E] \in \mathrm{gen}^s([D])$. Write

$$[D] \sim [\widetilde{C} \otimes R_1 \otimes R_2 \otimes \cdots \otimes R_r]$$

where $[\widetilde{C}] \in {}_2\mathrm{Br}(k(x))$ is the restriction of a unique $[C] \in {}_2\mathrm{Br}(k)$ and each $[R_i]$ is ramified at exactly $P_i \in \mathbb{P}^1_0 - \infty$ and nowhere else. By [6, Lemma 2.5], $\rho([E]) = \rho([D])$. Therefore,

$$[E] \sim [\widetilde{C'} \otimes R_1 \otimes R_2 \otimes \cdots \otimes R_r]$$

where $[\widetilde{C'}] \in {}_2\mathrm{Br}(k(x))$ is the restriction of a unique $[C'] \in {}_n\mathrm{Br}(k)$. We need to show $[C'] \sim [C]$. Let $P = \mathbb{P}^1_0(k) - \infty$ be a k-rational point where each $[R_i]$ is unramified. Note that D and E are unramified at P and recall that the map (4.2) is a homomorphism. Therefore residue algebras at P are given by

$$[\overline{E_P}] \sim [C' \otimes \overline{R_{1P}} \otimes \overline{R_{1P}} \otimes \cdots \otimes \overline{R_{1P}}]$$
$$[\overline{D_P}] \sim [C \otimes \overline{R_{1P}} \otimes \overline{R_{1P}} \otimes \cdots \otimes \overline{R_{1P}}]$$

in ${}_2\mathrm{Br}(k)$. Here $[\overline{\widetilde{C}_P}] = [C]$ because the composite

$$\mathrm{Br}(k) \longrightarrow \mathrm{Br}(k(x)) \longrightarrow \mathrm{Br}(k(x)_P) \longrightarrow \mathrm{Br}(\overline{k(x)_P})$$

is identity.

Since $[E] \in \mathrm{gen}^s([D])$, by Lemma 4.3, we get

$$[\overline{E_P}] \in \mathrm{gen}^s([\overline{D_P}])$$

By hypothesis the genus of any element in ${}_2\mathrm{Br}(k)$ is trivial. Therefore, $[\overline{E_P}] \sim [\overline{D_P}]$ and we conclude that $[C'] \sim [C]$. □

It is tempting to know if the above theorem can be generalized to compute a formula for the genus of elements in ${}_n\mathrm{Br}(k(x))$ for $n \geq 3$. We will now derive a formula for genus of arbitrary elements in ${}_n\mathrm{Br}(k(x))$ for $n \geq 3$ when k is a number field containing primitive n-th root of unity and show that resulting the bound on the size of the genus is sharp. The idea is to relate the genus of a given Brauer class to the genus of its unramified component and the ramification data.

Suppose k is any field containing a primitive n-th root of unity ω. By Merkurjev–Suslin theorem (originally due to S. Bloch for $k(x)$), the n-torsion of the Brauer group of $k(x)$ is generated by symbol algebras, so each element in $\mathrm{Br}(k(x))$ is Brauer equivalent to $\otimes_j (f_j, g_j)_\omega$ where $f_j, g_j \in k(x)$. By manipulation of symbol algebras ([1, Chapter VII]), we can assume that $f_j, g_j \in k[x]$ and that

$$(f_j, g_j)_\omega \sim (\otimes_l (a_l, b_l)_\omega) \otimes (\otimes_m (f_m, g_m)_\omega)$$

where $a_l, b_l \in k^*$ and for each m, either $f_m$or g_m is a non-constant monic polynomial. This observation together with the split exact sequence (5.1) yields:

Lemma 5.2. *For any $[D] \in {}_n\mathrm{Br}(k(x))$, we have $D \sim \widetilde{C} \otimes R_1 \otimes R_2 \otimes \cdots \otimes R_r$ where $[\widetilde{C}] \in {}_n\mathrm{Br}(k(x))$ is the restriction of a unique $[C] \in {}_n\mathrm{Br}(k)$ and each $[R_i]$ is ramified at exactly $P_i \in \mathbb{P}^1_0 - \infty$ and nowhere else. Moreover $[R_i] \sim \otimes_j (f_{ij}, g_{ij})_\omega$ where for every j either f_{ij}or g_{ij} is a non-constant monic polynomial in $k[x]$.*

For the rest of this section, let k denote a number field.

Theorem 5.3. *Let $n \geq 3$ and k be a number field containing a primitive n-th root of unity. Let $D \sim \widetilde{C} \otimes R_1 \otimes R_2 \otimes \cdots \otimes R_r$ as in Lemma 5.2. Then*

$$\mathrm{gen}^s([D]) \subseteq \left\{ [\widetilde{C'} \otimes R_1^{i_1} \otimes R_2^{i_2} \otimes \cdots \otimes R_r^{i_r}] \middle| \begin{matrix} [C'] \in \mathrm{gen}^s([C]) \text{ and} \\ (i_l, \mathrm{ord}([R_l])) = 1 \;\; \forall\, 1 \leq l \leq r \end{matrix} \right\}$$

In particular,

$$|\mathrm{gen}^s([D])| \leq |\mathrm{gen}^s(C)| \phi(n)^r$$

where $\phi(n)$ is the Euler's Totient function.

Remark 5.4. One can view $\widetilde{C}$ and $R_1 \otimes R_2 \otimes \cdots \otimes R_r$ in Lemma 5.2 respectively as unramified and ramified components of D (note that this decomposition is not unique). Then Theorem 5.3 expresses the genus of D in terms of the genus of its unramified component and the ramification data. Note that the genus of the unramified component $\widetilde{C}$ is expressed in terms of genus of the residue algebra C with respect to the valuation corresponding to any rational point.

In order to prove the theorem, we need another lemma. Note that for a class $[D] \in \mathrm{Br}(k(x))$ that is unramified at $P \in \mathbb{P}^1_0(k) - \infty$, the residue class $[\overline{D_P}]$ is in $\mathrm{Br}(k)$. For a place v of k, let $[\overline{D_P}]_v$ denote the completion of $[\overline{D_P}]$ at the place v of k. Let $PF(k)$ denote the set of finite places of k.

Lemma 5.5. *In the notation of Theorem 5.3, given any $v \in PF(k)$, there exists a k-rational point $P_v \in \mathbb{P}^1_0(k) - \infty$ such that for every i, $[R_i]$ is unramified at P_v and $[\overline{(R_i)_{P_v}}]_v$ is trivial in $\mathrm{Br}(k_v)$.*

Proof. By Lemma 5.2, $[R_i] \sim \otimes_j (f_{ij}, g_{ij})_\omega$ where for every j either f_{ij} or g_{ij} is a non-constant monic polynomial. Without loss of generality assume that g_{ij} is a non-constant monic polynomial for every i and j. Outside a finite set S of points in $\mathbb{P}^1_0(k) - \infty$, the extensions $k(x)(\sqrt[n]{f_{ij}})/k(x)$ and $k(x)(\sqrt[n]{g_{ij}})/k(x)$ are unramified for every i, j. Let $\pi_v \in k$ be a uniformizer in $\mathcal{O}_v$. Let $m \in \mathbb{Z}$ be smallest valuation (with respect to v) of the set of coefficients in g_{ij} for all i, j. Let N be a non-negative integer such that $n|N$ and $N \geq -m+1$. Set $P_N := \frac{1}{\pi_v^N} \in \mathbb{P}^1_0(k) - \infty - S$. Then note that

$$g_{ij}(P_N) = \pi^{-\deg(g_{ij})N} \cdot b_{ij}$$

where $b_{ij} \in (1 + \pi_v^{N+m}\mathcal{O}_v)$. Therefore,

$$\overline{((f_{ij}, g_{ij})_\omega)_{P_N}} \sim (a_{ij}, b_{ij})_\omega$$

for some $a_{ij} \in k^*$. Since k_v is a local field, there are only finitely many extensions of degree at most n ([21, Chapter II, §5, Proposition 14]). Let M be the supremum of the conductors of these extensions. So by choosing $N \geq M - m$ together with the above conditions on N, we deduce that $b_{ij} \in Norm(k_v(\sqrt[n]{a_{ij}})/k_v))$ and hence $(a_{ij}, b_{ij})_\omega \otimes_k k_v$ are split for all i, j. Setting $P_v = P_N$ proves the lemma. □

We are now ready to prove Theorem 5.3.

Proof of Theorem 5.3. Let $[E] \in \text{gen}^s([D])$. We now make the following observation. By [6, Lemma 2.5], $[E]$ is unramified at a valuation if and only if $[D]$ is. Moreover, for any point P, $\rho_P([E]) = \rho_P([D^{\otimes i_P}])$ for some i_P coprime to $\text{ord}([D])$. Recall that each $[R_i]$ is ramified exactly at P_i and nowhere else, so by Faddeev's exact sequence (5.1),

$$\rho([E]) = \rho([R_1^{i_1} \otimes R_2^{i_2} \otimes \cdots \otimes R_r^{i_r}])$$

Therefore we get

$$[E] \sim [\widetilde{C'} \otimes R_1^{i_1} \otimes R_2^{i_2} \otimes \cdots \otimes R_r^{i_r}]$$

where $(i_l, \text{ord}(\rho[R_l]) = 1) \;\; \forall\ 1 \leq l \leq r$ and $C' \in \text{Br}(k)$. So to prove the theorem it remains show that $[C'] \in \text{gen}^s([C])$.

Let $v \in PF(k)$. By Lemma 5.5, there exists a k-rational point $P_v \in \mathbb{P}^1_0(k) - \infty$ such that for every i, $[R_i]$ is unramified at P_v and $[\overline{R_{iP_v}}]_v$ is split over k_v. Therefore in ${}_n\text{Br}(k_v)$, we have

$$[\overline{D_{P_v}}]_v \sim [\overline{\widetilde{C}_{P_v}}]_v = [C]_v$$

$$[\overline{E_{P_v}}]_v \sim [\overline{\widetilde{C'}_{P_v}}]_v = [C']_v$$

By hypothesis, $[E] \in \text{gen}^s([D])$. Therefore by Lemma 4.3 and [34, Corollary 2.4], we conclude that $[C']_v \in \text{gen}^s([C]_v)$ for every $v \in PF(k)$. Since ${}_n\text{Br}(k_v)$ is cyclic and period equals index for Brauer classes over local fields, we deduce that for every $v \in PF(k)$, we have $[C']_v \sim [C^{\otimes j_v}]_v$ for some j_v such that $(j_v, n) = 1$. Note that since k contains n-th root of unity, and $n \geq 3$, k has no real embeddings. Now by [30, §18.4, Corollary b], it is easy to see that $[C'] \in \text{gen}^s([C])$. □

Theorem 5.3 can be viewed as a generalization of "genus stability" for higher dimensional division algebras which was queried in [34, p. 284].

Remark 5.6. The behaviour of genus of division algebras of degree ≥ 3 is studied in [6, Theorem 3.3]. However, we note that their results do not yield any finite upper bound for the genus over a number field k since the size of $\mathrm{gen}^s(\Delta) \cap {}_n\mathrm{Br}(k)$ is not bounded for $[\Delta] \in \mathrm{Br}(k)$. See ([7, §1]).

Yet another bound in [8, Corollary 8.4] is given by

$$\mathrm{gen}^s([D]) \leq \phi(n)^r \cdot n^{|S|}$$

where r is the number of ramification places of D in V (where V and S are given in [8, §8.2]). One can easily check that our bound given in Theorem 5.3 is better than the bound above.

Remark 5.7. The bound given in Theorem 5.3 is also sharp. Let $k := \mathbb{Q}(\omega)$ where ω is a primitive n-th root of unity for $n \geq 3$. Let $f \in k[x]$ be a non-constant monic polynomial and $a \in k^*$ such that $D = (f, a)_\omega$, is a division algebra ramified at exactly one point in $\mathbb{P}^1_0 - \infty$. Then Theorem 5.3 yields:

$$\mathrm{gen}^s([D]) \subseteq \{[D^{\otimes i}] | (i, n) = 1\}$$

But the above inclusion is equality since $[D^{\otimes i}]$ and $[D]$ generate the same subgroup of the Brauer group whenever $(i, n) = 1$.

The proof of Theorem 5.3 makes use of results from number fields and local fields. So it cannot be directly used to prove a similar statement for purely transcendental extensions of arbitrary fields. There are no results so far in the literature to compute the genus of division algebras over such fields of degree ≥ 3 in terms of its ramified and unramified components. It would be interesting to know if this is true in general.

Question 5.8. Does Theorem 5.3 hold if k is replaced by an arbitrary field?

6. Genus over complete discrete valued fields

In this section, we first give a formula for the genus of division algebras of prime degree over k. We then describe the splitting genus of division algebras of arbitrary degree over a complete discrete valued field in terms of the genus of its ramified and unramified components.

Throughout this section k denotes a complete discrete valued field with residue field $\bar{k}$. Recall from Section 4.1 that $\mathrm{IBr}(k)$ and $\mathrm{SBr}(k)$ denote the subgroup of inertial and tame classes in $\mathrm{Br}(k)$. For a division algebra D over k, let $\overline{D}$ denote the residue division algebra. If $\alpha \in \mathrm{Br}(k)$ denotes the class of D, we set $e_\alpha := e_D = |\Gamma_D : \Gamma_k|$.

6.1. The genus of division algebras of prime degree. First we study the behaviour of Brauer classes under totally ramified extensions of k.

Lemma 6.1. *Suppose D is an unramified division algebra over k that is split by a totally ramified extension, then $D \cong k$.*

Proof. This is basically [25, Theorem 1] together with the fact that discrete valuations are defectless ([26, p. 359]). □

Lemma 6.2. *Let N be a tame division algebra split by a totally ramified extension*[2]*. Then* $\operatorname{ord}([N]) = e_{[N]}$.

Proof. Since $e_{[N]} | \operatorname{ord}([N])$ ([17, Corollary 6.10]), it suffices to show that $[N^{\otimes e_{[N]}}]$ is trivial. Note that $[N^{\otimes e_{[N]}}]$ is unramified by ([17, Theorem 5.6 (b)]). By hypothesis, $[N^{\otimes e_{[N]}}]$ is split by a totally ramified extension of k and therefore is trivial by Lemma 6.1. □

Proposition 6.3. *Let N be a tame division algebra split by a totally ramified extension. Then*

$$\operatorname{gen}^s_{spl}([N]) = \operatorname{gen}^s([N]) = \{[N^{\otimes i}] | (i, e_{[N]}) = 1\}$$

Proof. By Lemma 6.2, if $(i, e_{[N]}) = 1$, $[N^{\otimes i}]$ and $[N]$ have same splitting fields (and hence same subfields) since they generate the same subgroup of the Brauer group. Moreover, $\operatorname{gen}^s_{spl}([N]) \subseteq \operatorname{gen}^s([N])$. So it suffices to show that if $[M] \in \operatorname{gen}^s([N])$ then $[M] = [N^{\otimes i}]$ for some i such that $(i, e_{[N]}) = 1$. Let $M \in \operatorname{gen}^s([N])$. Then by [6, Lemma 2.5], we have $[M \otimes_k N^{(\otimes i)op}]$ is unramified for some i where $(i, e_{[N]}) = 1$. By [28, Theorem 2.4], N is a NSR division algebra and therefore, N is contains a totally ramified separable maximal subfield L by Appendix B. Since $[M] \in \operatorname{gen}^s([N])$, $[M \otimes_k N^{(\otimes i)op}]$ is split by L. Now we use Lemma 6.1 to conclude that $[M] = [N^{\otimes i}]$. □

Lemma 6.4. *Suppose D is a tame division algebra over k. If* $[D'] \in \operatorname{gen}^s([D])$ *(resp.*$[D'] \in \operatorname{gen}^s_{spl}([D])$*), then* $[\overline{D}] \in \operatorname{gen}^s([\overline{D'}])$ *(resp.* $[\overline{D}] \in \operatorname{gen}^s_{spl}([\overline{D'}])$*).*

Proof. The fact that $[D'] \in \operatorname{gen}^s([D])$ implies $[\overline{D}] \in \operatorname{gen}^s([\overline{D}])$ follows from Lemma 4.3. Now suppose $[D'] \in \operatorname{gen}^s_{spl}([D])$. First note that $Z(\overline{D}) = Z(\overline{D'})$ ([6, Lemma 2.3 and Remark 2.6]). Let $\overline{L} \subset \overline{D}$ be a separable finite dimensional splitting field over $Z(\overline{D})$ with inertial lift L. Then by [17, Corollary 3.5], L splits D and hence also splits D' by hypothesis. Therefore again by [17, Corollary 3.5], $\overline{L}$ splits $\overline{D'}$. Since the argument is symmetric with respect ot D and D' we conclude that $[\overline{D}] \in \operatorname{gen}^s_{spl}([\overline{D'}])$. □

Proposition 6.5. *Let I be an unramified division algebra over k. Then the isomorphism in* (4.2)

$$\phi : \operatorname{IBr}(k) \longrightarrow \operatorname{Br}(\bar{k})$$
$$[D] \longmapsto [\overline{D}]$$

[2]This is equivalent to saying that N is NSR by [28, Theorem 2.4])

induces an injective maps of sets

$$\operatorname{gen}^s([I]) \longrightarrow \operatorname{gen}^s([\bar{I}])$$
$$\operatorname{gen}^s_{spl}([I]) \longrightarrow \operatorname{gen}^s_{spl}([\bar{I}])$$

If moreover, $\bar{k}$ is perfect, then the above maps are bijective

Proof. The first claim follows from Lemma 6.4. Now assume that $\bar{k}$ is perfect or $\operatorname{char}\bar{k}$ is coprime to the degree of I. It remains to show that if $[\bar{J}] \in \operatorname{gen}^s([\bar{I}])$ (resp. $[\bar{J}] \in \operatorname{gen}^s_{spl}([\bar{I}])$), then $[J] \in \operatorname{gen}^s([I])$ (resp. $[J] \in \operatorname{gen}^s_{spl}([I])$) where J is the unique inertial lift of $\bar{J}$ (Section 4.1).

Let $L \subset I$ be a separable subfield. Clearly, by the assumption on $\bar{k}$, $\bar{L} \subseteq \bar{I}$ is a separable subfield and therefore is also a separable subfield of $\bar{J}$. By [17, Theorem 2.9], we conclude that $L \subset J$ is a separable subfield. Since the above arguments are symmetric with respect to I and J, we have $J \in \operatorname{gen}^s([I])$.

The proof of $[J] \in \operatorname{gen}^s_{spl}([I])$ follows by similar argument as above by replacing the phrase "separable subfield" with "separable finite dimensional splitting field" together with Lemma 6.13 and [17, Corollary 3.5]. We leave the details to the reader. □

We will now describe the genus of division algebras of prime degree in terms of its residue algebra and ramification.

Theorem 6.6. *Let $[D] \in \operatorname{SBr}(k)$ be of prime index p. Then*

$$\operatorname{gen}^s([D]) \begin{cases} \subseteq \{[C] \in \operatorname{IBr}(k) | [\overline{C}] \in \operatorname{gen}^s([\overline{D}])\} & \text{if } [D] \text{ is unramified} \\ = \{[D^{\otimes i}] | 1 \leq i \leq p-1\} & \text{otherwise} \end{cases}$$

In particular,

$$|\operatorname{gen}^s([D])| \begin{cases} \leq |\operatorname{gen}^s(\overline{D})| & \text{if } [D] \text{ is unramified} \\ = p-1 & \text{otherwise} \end{cases}$$

If $\bar{k}$ is perfect, then $\subseteq$ and $\leq$ in the above expressions are equalities. Moreover, all the above statements also hold if $\operatorname{gen}^s([D])$ is replaced with $\operatorname{gen}^s_{spl}([D])$.

Proof. By the fundamental equality (4.1), we have that $e_D = [\Gamma_D : \Gamma_k]$ is either 1 or p since there are no tame and totally ramified division algebras over complete discrete valued fields ([40, Remark 3.2 (a)]). If $e_{[D]} = 1$, $[D]$ is unramified. Therefore, $\operatorname{gen}^s([D]) \subseteq \{C | \overline{C} \in \operatorname{gen}^s(\overline{D})\}$ and $\operatorname{gen}^s_{spl}([D]) \subseteq \{C | \overline{C} \in \operatorname{gen}^s_{spl}(\overline{D})\}$) (with $\subseteq$ replaced with $=$ if $\bar{k}$ is perfect) by Proposition 6.5. If $e_{[D]} = p$, then D contains an element a such that whose valuation generates Γ_D/Γ_k implying that D contains a totally ramified maximal subfield. Therefore by Proposition 6.3, $\operatorname{gen}^s_{spl}([D]) = \operatorname{gen}^s([D]) = \{[D^{\otimes i}] | 1 \leq i \leq p-1\}$. □

The case $p = 2$ yields:

Corollary 6.7. *Let $\overline{k}$ satisfy the property that the genus (resp. splitting genus) is trivial for any quaternion algebra over $\overline{k}$. Then the genus (resp. splitting genus) of any tame quaternion algebra over k is trivial.*

Remark 6.8. Quaternion algebras sharing same separable maximal subfields are said to be *totally separably linked* in [5]. It is shown in [5, Corollay 6.2, Corollary 6.9] that if $\overline{k}$ is perfect of characteristic 2, then the genus of any quaternion algebra over $k = \overline{k}((t))$ is trivial. This is a special case of Corollary 6.7, since in this case ${}_2\mathrm{Br}(\overline{k})$ is trivial ([11, Chapter I, Theorem 1.3.7]) and any quaternion algebra over k is tame ([36, Theorem 1, Exercise 3, Chapter XII]).

One can recursively use Corollary 6.6 to compute genus of degree p division algebras over fields of the form $k((t_1))((t_2))\dots((t_n))$.

Example 6.9. Let $K = k((t_1))(t_2))\dots((t_n))$ where k is a field of characteristic $\neq p$. Let D be a tame division algebra over K be of index p. Set $D^{(0)} := D$ and let $D^{(k)}$ denote the residue algebra of $D^{(k-1)}$, so that $D^{(n)}$ is a division algebra over k. Then by recursive application of Corollary 6.6, we get

$$|\mathrm{gen}^s([D])| \leq \begin{cases} p-1 & \text{if } D^{(n)} \text{ is a field} \\ |\mathrm{gen}^s([D]^{(n)})| \text{otherwise} \end{cases}$$

Example 6.10. In the above example, taking $p = 2$ and using the fact that genus of any tame quaternion over global field is trivial ([6, §3.6]), we see that the genus of any quaternion algebra over $K = \mathbb{Q}((t_1))((t_2))\dots((t_n))$ is trivial.

6.2. Genus decomposition for gen^s_{spl}. We will now derive a formula for $\mathrm{gen}^s_{spl}([D])$ for any $[D] \in \mathrm{SBr}(k)$. Recall from Section 4.1 that we have decomposition

$$D \sim I \otimes_k N$$

in $\mathrm{SBr}(k)$ where I is inertial and N is NSR over k and $e_D = e_N$.

Lemma 6.11. *In the above decomposition, if $[D] \in {}_n\mathrm{SBr}(K)$, so are $[I]$ and $[N]$*

Proof. It suffices to show that $[N^{\otimes n}]$ is trivial. Now $e_D|(ord[D])$ ([17, Corollary 6.10]) and therefore $e_D|n$. By Lemma 6.2 $e_D = e_N = \mathrm{ord}([N])$ and the result follows. □

Theorem 6.12. *Let I be unramified and N be NSR division algebras over k. Then*

$$\mathrm{gen}^s_{spl}([I \otimes_k N]) \subseteq \{[I' \otimes_k N'] \mid [I'] \in \mathrm{gen}^s([I]) \text{ and } [N'] \in \mathrm{gen}^s([N])\}$$

Proof. Let $D_1 \sim I_1 \otimes_k N_1$ and $D_2 \sim I_2' \otimes_k N_2$ where $[D_2] \in \mathrm{gen}^s_{spl}([D_1])$. Since I_1 and I_2' are unramified, for $i = 1, 2$, we have $\chi_{D_i} = \chi_{N_i}$ (see Section 4.2). As D_1 and D_2 have same splitting fields (and hence same subfields), by [6, Lemma 2.5], we have $\ker(\chi_{D_1}) = \ker(\chi_{D_2})$. Therefore $e_{D_1} = e_{D_2}$ and $\chi_{N_2} = \chi_{N_1^{\otimes j}}$ for some j where $(j, e_{N_1}) = 1$ (here we use the fact that $e_{N_i} = e_{D_i}$). Hence $N_2 \sim N_1^{\otimes j} \otimes_k C$ where $[C] \in \mathrm{IBr}(k)$. By replacing I_2' with $I_2 := I_2' \otimes_k C$, we get $D_2 \sim I_2 \otimes_k N_1^{\otimes j}$. Hence by Proposition 6.3, we have $D_2 \sim I_2 \otimes N_2$ where $[N_2] \in \mathrm{gen}^s([N_1])$. It remains to show that $[I_2] \in \mathrm{gen}^s([I_1])$.

Since N_1 is NSR, it contains a totally ramified maximal subfield. By Appendix B, N_1 contains a totally ramified separable maximal subfield L/k. Therefore L is also a maximal subfield of N_2. So we have $D_i \otimes_k L \sim I_i \otimes_k L$. Now let F/k be a finite separable maximal subfield of I_1. Note that since I_1 is unramified, $[\Gamma_F : \Gamma_k] = 1$ (although $\overline{F}/\overline{k}$ need not be separable). By the proof of [12, Lemma 2.5.8], F and L are linearly disjoint (as subfields of a fixed algebraic closure of k). Therefore $FL \simeq F \otimes_k L$ is separable and splits D_1. Therefore it also splits D_2. Therefore

$$D_2 \otimes_k FL \sim I_2 \otimes_k FL \sim (I_2 \otimes_k F) \otimes_F FL$$

is split. But FL/F is totally ramified. So $I_2 \otimes_k F$ is split by Lemma 6.1. In particular, $\deg(I_2) \leq \deg(I_1)$. Reversing the role of I_1 and I_2 we get $\deg(I_1) \leq \deg(I_2)$. Therefore $\deg(I_1) = \deg(I_2)$ and F is a maximal subfield of I_2. Since the argument is symmetric with respect to I_1 and I_2, we get $I_2 \in \mathrm{gen}^s([I_1])$. □

Now assume that $\overline{k}$ is perfect. Then we get a better estimate of gen^s_{spl} as we show below. For a finite separable extension F/k, denote by $\widetilde{F}$ the maximal unramified subfield of F/k. We start with a lemma.

Lemma 6.13. *Let $[I] \in \mathrm{IBr}(k)$ and let F/k be a finite separable extension. Assume that $\overline{k}$ is perfect. Then F splits I if and only if $\widetilde{F}$ splits I.*

Proof. Suppose F splits I. Note that $[I \otimes_k \widetilde{F}] \in \mathrm{IBr}(\widetilde{F})$. Now

$$I \otimes_k F \simeq (I \otimes_k \widetilde{F}) \otimes_{\tilde{F}} F$$

Since $\overline{k}$ is perfect, $\overline{F} = \overline{\widetilde{F}}$ and therefore $F/\widetilde{F}$ is totally ramified. By Lemma 6.1, $\widetilde{F}$ splits I. □

Theorem 6.14. *Suppose $\overline{k}$ is perfect. Let I be unramified and N be NSR division algebras over k. Then*

$$\mathrm{gen}^s_{spl}([I \otimes_k N]) \subseteq \{[I' \otimes_k N'] \mid [I'] \in \mathrm{gen}^s_{spl}([I]) \text{ and } [N'] \in \mathrm{gen}^s_{spl}([N])\}$$

Proof. The proof is similar to the proof of Theorem 6.12, where F is replaced with $\widetilde{F}$. We leave the details to the reader. □

Now Theorem 6.12, Theorem 6.14 together with Proposition 6.5 and Proposition 6.3 yields:

Corollary 6.15. *Let $[D] \in \mathrm{SBr}(k)$. Write $D \sim I \otimes_k N$ where I is inertial and N is NSR. Then any element in $\mathrm{gen}^s_{spl}([D])$ is of the form $[I' \otimes_k N^{\otimes j}]$ for some $(j, e_D) = 1$ where $[I'] \in \mathrm{gen}^s([I])$. The algebra I' is the (unique) inertial lift of some division algebra whose class lies in $\mathrm{gen}^s([\bar{I}])$. In particular*

$$|\mathrm{gen}^s_{spl}([D])| \leq |\mathrm{gen}^s([\bar{I}])| \cdot \phi(e_D)$$

where ϕ denotes the Euler's Totient function. If moreover $\overline{k}$ is perfect, then I' above is the (unique) inertial lift of some division algebra whose class lies in $\mathrm{gen}^s_{spl}([\bar{I}])$ and

$$|\mathrm{gen}^s_{spl}([D])| \leq |\mathrm{gen}^s_{spl}([\bar{I}])| \cdot \phi(e_D)$$

As an easy corollary for the case of ${}_2\mathrm{Br}(k)$ we get:

Corollary 6.16. *Let $\overline{k}$ satisfy the property that the genus is trivial for any element in ${}_2\mathrm{Br}(\overline{k})$. Then $\mathrm{gen}^s_{spl}([D])$ is trivial for any $[D] \in {}_2\mathrm{SBr}(k)$.*

Proof. By [17, Corollary 6.10], for any $[D] \in {}_2\mathrm{Br}(k)$, $e_D | \operatorname{ord}([D])$. Hence $e_D = 2$ and the result follows from Corollary 6.15. ☐

Example 6.17. Since the genus of any element in ${}_2\mathrm{Br}(\mathbb{Q})$ (this element is necessarily the class of some quaternion) is trivial ([6, §3.6]), by recursively applying Corollary 6.16, we see that the $\mathrm{gen}^s_{spl}([D])$ is trivial for any $[D] \in {}_2\mathrm{SBr}(\mathbb{Q}((t_1))((t_2))\ldots((t_n)))$

Question 6.18. Does the statement of Corollary 6.15 hold if we replace gen^s_{spl} with *gen*?

Remark 6.19. If the answer to the above question is positive, then the stronger version of Corollary 6.16 where $\mathrm{gen}^s_{spl}([D])$ is replaced with $\mathrm{gen}^s([D])$ holds for any $[D] \in {}_2\mathrm{SBr}(k)$.

7. The Genus of quaternion algebras

In this section K is an arbitrary field of char $\neq 2$ with a set of discrete valuations V. As before, we denote the completion of K with respect to v and its residue field respectively by K_v and $\overline{K_v}$.

Definition 7.1. For a division algebra D over K, we say that D (or its class in $\mathrm{Br}(K)$) is tame with respect to V if $D_v := D \otimes_K K_v$ is tame for every $v \in V$. We denote that set of tame elements of $\mathrm{Br}(K)$ with respect to V by $\mathrm{SBr}(K, V)$.

Remark 7.2. If the exponent of D_v is coprime to char $\overline{K_v}$ or when $\overline{K_v}$ is perfect, D_v is tame ([36, Theorem 1, Exercise 3(b), Chapter XII]. In particular, every $[D] \in {}_2\mathrm{Br}(K)$ is tame over V if char $\overline{K_v} \neq 2$ or if $\overline{K_v}$ is perfect for every $v \in V$.

For a field K, let

$$_2\text{Ш}^{\mathrm{Br}}(K, V) := \ker\left({}_2\mathrm{Br}(K) \longrightarrow \prod_{v \in V} {}_2\mathrm{Br}(K_v)\right)$$

Theorem 7.3. *Let $[Q] \in \mathrm{SBr}(K, V)$ be the class of a quaternion division algebra over K. Suppose for every $v \in V$, the genus of any quaternion division algebra over $\overline{K_v}$ is trivial. Then*

$$\mathrm{gen}^s([Q]) \subseteq [Q] + {}_2\text{Ш}^{\mathrm{Br}}(K, V)$$

In particular,

$$|\mathrm{gen}^s([Q])| \leq |{}_2\text{Ш}^{\mathrm{Br}}(K, V)|$$

Proof. Let $[P] \in \mathrm{gen}^s([Q])$. By [34, Corollary 2.4], $[P_v] \in \mathrm{gen}^s([Q_v])$ for every $v \in V$. Now by Corollary 6.7, $[P_v] \sim [Q_v]$ for every $v \in V$. Therefore $[P \otimes_K Q] \in {}_2\text{Ш}^{\mathrm{Br}}(K, V)$ and the theorem follows. □

As an easy corollary of the theorem, we get:

Corollary 7.4. *Assume that K satisfies local-global principle on the 2-torsion part of the Brauer group with respect to V i.e., ${}_2\text{Ш}^{\mathrm{Br}}(K, V)$ is trivial. Suppose for every $v \in V$, the genus of any class of quaternion division algebra in $\mathrm{Br}(\overline{K_v})$ is trivial, then the genus of any class of quaternion division algebra in $\mathrm{SBr}(K, V)$ is trivial.*

Corollary 7.5. *If K is one of the following fields, the genus of any quaternion division algebra over K is trivial*

1. *Higher local fields where the final residue field has characteristic $\neq 2$*
2. *Iterated Laurent series $k((t_1))((t_2))\ldots((t_n))$ (resp. their finite extensions) where char $k \neq 2$ and every quaternion division algebra over k (resp. every finite extension of k) has trivial genus*
3. *Function fields of one variable over fields in (1)*
4. *Function fields of one variable any real closed field*
5. *Function fields of one variable over fields in (2) where for any curve C over k (where k is as in (2)), every quaternion algebra over every finite extension of k and over $k(C)$ has trivial genus (for example, $k((t_1))((t_2))\ldots((t_n))(C)$ where one can take k to be any real closed field by (4))*

Proof. For each K as above, the proof below considers the discrete valuations V where residue fields are of characteristic $\neq 2$, so all the division algebras over these fields are tame by Remark 7.2.

For (1) set $V = v$ to be the canonical discrete valuation on K. Then by induction on $\mathrm{cdv.dim}(K)$ and observing that the final residue field is finite, the result follows from Corollary 7.4. The proof of (2) is similar by using induction on n.

For (3), let $K = F(C)$ where F is a field in (1). We will use induction on $d := \mathrm{cdv.dim}(F)$ of the higher local field F. When $d = 0$, F is a finite field and K is a global field and hence the genus of any quaternion over K is trivial ([6, §3.6]). Now assume that the property is satisfied by K for every F with $\mathrm{cdv.dim}(F) \leq d$. Let $\mathrm{cdv.dim}(F) = d+1$ and let V be discrete valuations corresponding to codimension 1 points in a regular proper model of K over $\mathcal{O}_F$. Then by [10, Theorem 4.2 and Theorem 4.3 (ii)], ${}_2\text{Ш}^{\mathrm{Br}}(K, V)$ is trivial. Moreover, in this case, the residue fields corresponding to $v \in V$ are either finite extensions of F which is again a higher local field (Section 4.3) or of the form $F'(C')$ with $\mathrm{cdv.dim}(F') \leq d$. So by (1) and induction hypothesis, the result now follows from Corollary 7.4.

For (4), let C be a smooth projective curve over a real closed field R with function field K. Such a curve exists by [38, Theorem 53.2.6 and Lemma 53.2.8]. By in [9, Theorem 2.3.1], the specialization map $\mathrm{Br}(C) \to \prod_{P \in C(R)} \mathrm{Br}(k_P)$ is injective where k_P is the residue field at P. Taking V to be the discrete valuations corresponding to the points in $C(R)$, Lemma A.3 yields triviality of ${}_2\text{Ш}^{\mathrm{Br}}(K, V)$ and so the result follows from Corollary 7.4.

The proof of (5) is again using an induction argument similar to (3). □

7.1. Curves over global fields with rational point. Let C be a smooth projective geometrically integral curve over a global field k with a k-rational point. In this section, we discuss the relation between the genus of quaternion algebras over the function field of C and the 2-torsion subgroup of the Tate–Shafarevich group of the Jacobian of C. We deduce that the size of the genus is bounded above by the size of the 2-torsion component of the Tate–Shafarevich group of the Jacobian. Then we specialize to the case of elliptic curves and demonstrate that this bound is better than the one known before.

Let C be a smooth projective geometrically integral curve over a global field k with $C(k) \neq \emptyset$ and let J_C denote its Jacobian. Recall that the Tate–Shafarevich group of J_C, denoted by $\text{Ш}(J_C)$ is defined as

$$\text{Ш}(J_C) = \ker\left(H^1(k, J_C(k^{sep}) \longrightarrow \prod_{v \in P(k)} H^1(k_v, J_C(k_v^{sep})\right)$$

where $P(k)$ denotes the set of places of k. Let $C_v := C \times_{\mathrm{Spec}\,k} \mathrm{Spec}\,k_v$ and let $\text{Ш}^{\mathrm{Br}}(C)$ be the kernel of the local global map on the Brauer group of

C with respect to $P(k)$, i.e.,

$$Ш^{\mathrm{Br}}(C) := \ker\left(\mathrm{Br}(C) \longrightarrow \prod_{v\in P(k)} \mathrm{Br}(C_v)\right)$$

By [29, §2(B)], we have the following isomorphism[3]:

$$Ш^{\mathrm{Br}}(C) \simeq Ш(J_C) \tag{7.1}$$

Let $S := \operatorname{Spec} \mathcal{O}_k$ where $\mathcal{O}_k$ is the ring of integers of k when $\operatorname{char} k = 0$ and a smooth complete curve over its field of constants with function field k when $\operatorname{char} k \neq 0$. Suppose $\mathcal{C} \to S$ is a regular proper model for C. Let $V_{\mathcal{C}}$ denote the valuations on $k(C)$ corresponding to codimension one points $\mathcal{C}^{(1)}$ of $\mathcal{C}$. We define (see Appendix A for details),

$$Ш^{\mathrm{Br}}(C/\mathcal{C}) := Ш^{\mathrm{Br}}(k(C), V_{\mathcal{C}}) = \ker\left(\mathrm{Br}(k(C)) \longrightarrow \prod_{x\in \mathcal{C}^{(1)}} \mathrm{Br}(k(C)_x)\right)$$

where $k(C)_x$ denotes the completion of $k(C)$ with respect to the valuation corresponding to the codimension 1 point $x \in \mathcal{C}^{(1)}$.

Then by the isomorphism in (7.1) together with Theorem A.1, we conclude

Theorem 7.6. *With notations as above, we have*

$$Ш^{\mathrm{Br}}(C/\mathcal{C}) \simeq Ш(J_C)$$

In particular, when $C = E$ is an elliptic curve with a regular proper model $\mathcal{E}$,

$$Ш^{\mathrm{Br}}(E/\mathcal{E}) \simeq Ш(E)$$

Remark 7.7. Recall that *index = period* for any element in $Ш(E)$ ([3, Theorem 1.3]). Therefore the same holds for $Ш^{\mathrm{Br}}(E/\mathcal{E})$. In particular, the elements in ${}_2Ш^{\mathrm{Br}}(E/\mathcal{E})$ correspond to quaternions.

The residue fields of $K := k(C)$ corresponding to $V_{\mathcal{C}}$ are global fields and therefore the genus of any class of quaternion division algebra in ${}_2\mathrm{Br}(\overline{K_v})$ is trivial for every $v \in V$ ([6, §3.6]). So by Theorem 7.3 and Theorem 7.6, we conclude

Theorem 7.8. *With notations as above, let $[Q] \in {}_2\mathrm{SBr}(K, V)$ be the class of a quaternion algebras where $K = k(C)$ and $V = V_{\mathcal{C}}$. Then*

$$|\mathrm{gen}^s([Q])| \leq |{}_2Ш(J_C)|$$

Therefore when $C = E$ is an elliptic curve, we have

$$|\mathrm{gen}^s([Q])| \leq |{}_2Ш(E)| \tag{7.2}$$

[3] Although the isomorphism (7.1) is shown for number fields in [29], the same arguments show that the isomorphism holds for global fields as well.

In particular, we get $|\mathrm{gen}^s([Q])|$ *is trivial whenever* ${}_2Ш(E)$ *is trivial.*

Remark 7.9. Comparing with [6, Theorem 4.1, Corollary 4.11], we see that the bound in (7.2) gives a better estimate of the genus of any class of quaternion algebra $[Q] \in \mathrm{SBr}(K, V)$ since for a regular proper model $\mathcal{E}$ of E, $Ш(E) \simeq Ш^{\mathrm{Br}}(E/\mathcal{E}) \subseteq \mathrm{Br}(E)_{V_0 \cup V_1}$ where V_0 and V_1 are the sets of valuations on K given in [6, §4]. See also Example 7.10.

One can extensively use the arithmetic properties of a given elliptic curve E over a number field together with Theorem 7.8 to compute bounds on the genus. We will demonstrate this below.

For the rest of the section, E denotes an elliptic curve over a number field k and $[Q]$ is an arbitrary class of quaternion division algebra in $\mathrm{SBr}(K, V)$ where $K = k(E)$, $V = V_{\mathcal{E}}$ for a regular proper model $\mathcal{E}$ of E.

Example 7.10. Let E be given by $y^2 = x^3 - x$ over $\mathbb{Q}$. We have ${}_2Ш(E) = 0$ (one can verify using MAGMA) and therefore from Theorem 7.8, we conclude that $\mathrm{gen}^s([Q])$ is trivial. Compare with [6, Example 4.12].

Example 7.11. Let p be an odd prime and let E be given by $y^2 = x^3 + px$ over $\mathbb{Q}$. By Theorem 7.8 and [37, Proposition X.6.2 (c)], we get

$$|\mathrm{gen}^s([Q])| \leq \begin{cases} 1 & \text{if } p \equiv 7, 11 \pmod{16} \\ 2 & \text{if } p \equiv 3, 5, 13, 15 \pmod{16} \\ 4 & \text{if } p \equiv 1, 9 \pmod{16} \end{cases}$$

We will now explicitly show triviality of the genus of some quaternion division algebras over $\mathbb{Q}(E)$ using arithmetic properties of E even when ${}_2Ш(E)$ is not trivial. Given an isogeny $\phi : E \to E'$ over k with $\widehat{\phi}$ denoting the dual isogeny, let $S^{\phi}(E)$ and $S^{\hat{\phi}}(E')$ denote the respective Selmer groups.

We have an exact sequence of groups [37, Chapter X, Theorem 4.2]

$$0 \longrightarrow E'(k)/\phi(E(k)) \longrightarrow S^{\phi}(E/k) \longrightarrow {}_{\phi}Ш(E/k) \longrightarrow 0$$

where ${}_{\phi}Ш(E/k)$ denotes the ϕ-torsion subgroup of $Ш(E/k)$. Let $S \subset P(k)$ denote the union of the set of infinite places, the set of finite primes at which E has bad reduction, and the set of finite primes dividing 2. Recall [37, Chapter X, Proposition 1.4 and Proposition 4.9] that if ϕ is of degree 2, $S^{\phi}(E/k)$ and $S^{\hat{\phi}}(E'/k)$ are subgroups of $k(S, 2)$ where

$$k(S, 2) = \{b \in k^*/(k^*)^2 : \mathrm{ord}_v(b) \equiv 0 \pmod 2 \ \forall\, v \notin S\}$$

Example 7.12. Let E be given by $y^2 = x^3 - 113x$ over $k := \mathbb{Q}$. There is 2-isogeny ([37, X.6])

$$\phi : E \longrightarrow E'$$

where E' is given by $y^2 = x^3 + 452x$. Let $E'^{(d)}$ denote the quadratic twist of E' by d. We observe that $E'^{(-1)} \simeq E'$. Let $l := \mathbb{Q}(\sqrt{-1})$. One can verify that $E'_{tors}(l) = E'_{tors}(k)$ and $\operatorname{rank}(E'(k)) = 0$ using Magma. Moreover by [37, Chapter X, Exercise 10.16], $\operatorname{rank}(E'(l)) = \operatorname{rank}(E'(k)) + \operatorname{rank}(E'^{(-1)}(k)) = 2 \cdot \operatorname{rank}(E'(k)) = 0$. Therefore, $E'(l) = E'(k)$.

Now let $\widehat{\phi}$ denote the dual isogeny. From the computations given in [29, Lemma 3.4], we get

$$S^{\phi}(E/k) = \{1, 113, 2, 226\} \pmod{(k^*)^2}$$
$$S^{\hat{\phi}}(E'/k) = \{\pm 1 \pm 113\} \pmod{(k^*)^2}$$

We have $|{}_{\phi}Ш(E/k)| = |{}_{\hat{\phi}}Ш(E'/k)| = 2$ and ${}_{2}Ш(E'/k) = {}_{\phi}Ш(E/k) \oplus {}_{\hat{\phi}}Ш(E'/k)$. The non-trivial element in ${}_{\phi}Ш(E/k)$ and ${}_{\hat{\phi}}Ш(E'/k)$ are respectively given by the principal homogeneous spaces corresponding to 2 and -1. Let D_2 and D_{-1} denote the respective (non-split) quaternion algebras under the isomorphism (7.1). Let us compute $\operatorname{gen}^s([D_{-1}])$. It is easy to see that D_{-1} is split by l since the class of -1 is trivial in $S^{\hat{\phi}}(E'/l)$. We also have the following commutative diagram where the vertical maps are induced by restrictions

$$\begin{array}{ccccccccc}
0 & \longrightarrow & E'(k)/\phi(E(k)) & \xrightarrow{i_k} & S^{\phi}(E/k) & \longrightarrow & {}_{\phi}Ш(E/k) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle\theta} & & \downarrow & & \\
0 & \longrightarrow & E'(l)/\phi(E(l)) & \xrightarrow{i_l} & S^{\phi}(E/l) & \longrightarrow & {}_{\phi}Ш(E/l) & \longrightarrow & 0
\end{array}$$

By Theorem 7.3,

$$\operatorname{gen}^s([D_{-1}] \subseteq \{[D_{-1}], [D_2], [D_{-1} \otimes D_2]\}$$

We will show that $\operatorname{gen}^s([D_{-1}])$ is trivial. So it suffices to show that D_2 is not split by l or equivalently that $\theta(2)$ is not in the image of i_l. Now suppose $\theta(2)$ is in the image of i_l, say $i_l(P) = \theta(2)$ for some $P \in E'(l) = E'(k)$. Then

$$\theta(i_k(P)) = \theta(2)$$
$$\implies 2 \cdot i_k(P) \in \ker(\theta) = S^{\phi}(E/k) \cap \{1, -1\} \pmod{(k^*)^2} = 1 \pmod{(k^*)^2}$$

Therefore $i_k(P) = 2 \pmod{(k^*)^2}$. This contradicts the fact that 2 is non-trivial in ${}_{\phi}Ш(E/k)$ and concludes that $\operatorname{gen}^s([D_{-1}]) = \{[D_{-1}]\}$.

For a quadratic extension l/k, the techniques used in the above example can be generalized to compute genus of classes in $\operatorname{Br}(l/k)$ whenever $E(l) = E(k)$. We demonstrate this below.

We will assume that E is split, i.e., E is given by a Weierstrass equation $y^2 = f(x)$ where $f(x)$ has three distinct roots over k. In this case $E[2] \subseteq$

$E(k)$ and the 2-Selmer group $S^{(2)}(E/k)$ is a subgroup of $k(S,2) \times k(S,2)$. Now let $l = k(\sqrt{a})$ be a quadratic extension. Denote by

$$\theta : S^{(2)}(E/k) \longrightarrow S^{(2)}(E/l)$$

the natural map induced by restriction. Identifying $S^{(2)}(E/k)$ as a subgroup of $k(S,2) \times k(S,2)$, we have

$$\text{(7.3)} \quad H := \ker(\theta) = S^{(2)}(E/k) \cap \{(1,1),(a,1),(1,a),(a,a)\} \pmod{(k^*)^2}$$

Given $\gamma \in S^{(2)}(E/k)$, let D_γ denote the image of γ under the composition

$$S^{(2)}(E/k) \longrightarrow {}_2\text{Ш}(E/k) \xrightarrow{\simeq} {}_2\text{Ш}^{\text{Br}}(E/k)$$

where the last arrow comes from the isomorphism (7.1).

Lemma 7.13. *Assume that E is split. Let $l = k(\sqrt{a})$ be a quadratic extension such that $E(l) = E(k)$. If $[Q]$ is split over $l(E)$, then*

$$\text{gen}^s([Q]) \subseteq \{[Q \otimes D_\gamma] | \gamma \in H\}$$

where H is given by (7.3)*. In particular,*

$$|\text{gen}^s([Q])| \leq 4$$

and $\text{gen}^s([Q])$*is trivial if H is trivial.*

Proof. We have the following commutative diagram of exact sequences where the vertical arrows are induced by restrictions.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E(k)/2E(k) & \xrightarrow{i_k} & S^{(2)}(E/k) & \longrightarrow & {}_2\text{Ш}(E/k) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow \theta & & \downarrow & & \\ 0 & \longrightarrow & E(l)/2E(l) & \xrightarrow{i_l} & S^{(2)}(E/l) & \longrightarrow & {}_2\text{Ш}(E/l) & \longrightarrow & 0 \end{array}$$

By Theorem 7.3 and the above diagram, any element in $\text{gen}^s([Q])$ is of the form $[Q \otimes D_\gamma]$ for some $\gamma \in S^{(2)}(E/k)$. So it suffices to show that $\gamma \in i_k(P)H$ for some $P \in E(k)$. Since $[Q]$ is split over $l(E)$, for any $[Q \otimes D_\gamma] \in \text{gen}^s([Q])$, D_γ is split over $l(E)$ which implies $\theta(\gamma) = i_l(P)$ for some $P \in E(l) = E(k)$. Therefore by commutativity of the above diagram, we have

$$\theta(i_k(P)) = \theta(\gamma) \implies \gamma \in i_k(P)H$$

which finishes the proof. $\square$

Remark 7.14. If the answer to Question 6.18 is positive, then by Remark 6.19, all the results and examples of this section hold if quaternions are replaced by division algebras of exponent two

Appendix A. Curves over global fields

Throughout this section k denotes a global field. Let S denote the Dedekind scheme given by $\operatorname{Spec} \mathcal{O}_k$ where $\mathcal{O}_k$ is the ring of integers of k when $\operatorname{char} k = 0$ and a smooth complete curve over its field of constants with function field k when $\operatorname{char} k \neq 0$. Let C denote a smooth geometrically integral projective curve over k. By a theorem of Lipman ([22, §10.1.1 and Corollary 8.3.51]) there exists a regular projective model for C over S i.e., there exists a regular fibered surface $\mathcal{C}$ over S such that the generic fiber is isomorphic to C i.e., $C \simeq \mathcal{C} \times_S \operatorname{Spec} k$. Since the map $\mathcal{C} \to S$ is dominant, $k(\mathcal{C}) \simeq k(C)$. Therefore, the codimension 1 points $\mathcal{C}^{(1)}$ in $\mathcal{C}$ give rise to discrete valuations on $k(C)$. For $x \in \mathcal{C}^{(1)}$, let $k(C)_x$ denote the completion of $k(C)$ with respect to this valuation. Let

$$\text{Ш}^{\mathrm{Br}}(C/\mathcal{C}) := \ker\left(\operatorname{Br}(k(C)) \longrightarrow \prod_{x \in \mathcal{C}^{(1)}} \operatorname{Br}(k(C)_x\right)$$

be the kernel of the local global map on the Brauer group with respect to the valuations from the codimension 1 points on $\mathcal{C}$. One also has the local global map on $\operatorname{Br}(C)$ with respect to the places $P(k)$ of k whose kernel will be denoted by $\text{Ш}^{\mathrm{Br}}(C)$.

$$\text{Ш}^{\mathrm{Br}}(C) := \ker\left(\operatorname{Br}(C) \longrightarrow \prod_{v \in P(k)} \operatorname{Br}(C_v)\right)$$

Since C is a smooth integral curve, the natural map $\operatorname{Br}(C) \to \operatorname{Br}(k(C))$ is injective by Corollary 1.10 in [16]. So we consider $\operatorname{Br}(C)$ as a subgroup of $\operatorname{Br}(k(C))$. One may ask how are the groups $\text{Ш}^{\mathrm{Br}}(C/\mathcal{C})$ and $\text{Ш}^{\mathrm{Br}}(C)$ related as subgroups of $\operatorname{Br}(k(C))$. The main result of this section is the following theorem:

Theorem A.1. *With notations as above,*

$$\text{Ш}^{\mathrm{Br}}(C/\mathcal{C}) = \text{Ш}^{\mathrm{Br}}(C)$$

as subgroups of $\operatorname{Br}(k(C))$.

In order to prove the theorem we need a few lemmas. Although the following lemmas are well known, we could not find any explicit references stating these results. Therefore, we include them here for completeness and for the reader's convenience.

Lemma A.2. *Let $x \in X$ be a closed point on a scheme X locally of finite type over k with residue field $k(x)$. For a place $v \in P(k)$, let $P_v(k(x))$ denote the valuations on $k(x)$ extending v. Then the set of completions*

$\{k(x)_w | w \in P_v(k(x))\}$ *is equal to the residue fields of the points lying above* x *under the base change map*

$$\phi_v : X_v := X \times_k k_v \longrightarrow X$$

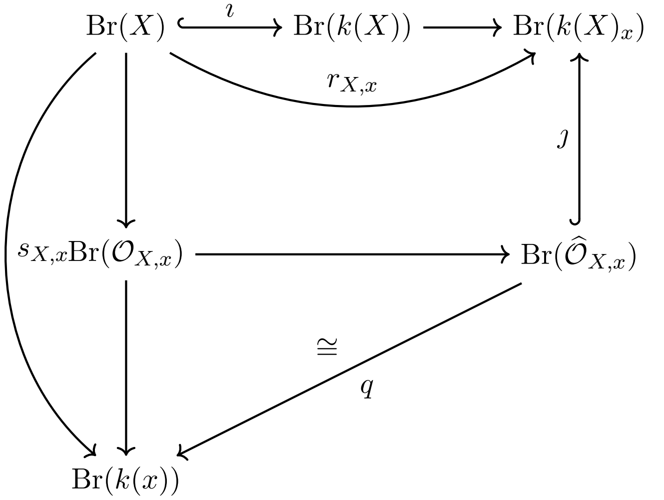
Proof. The closed point x corresponds to $\operatorname{Spec} k(x) \hookrightarrow X$ where $k(x)$ is a finite extension of k. The residue fields of points lying above x in X_v are precisely $\{L_i\}$ where $k(x) \otimes_k k_v \simeq \prod L_i$. When $k(x)/k$ is separable, by [24, Propositon 8.1 and 8.2], $\{L_i\}$ is the set of completions of $k(x)$ with respect to possible extensions of v to $k(x)$ which proves the lemma. When $k(x)/k$ is purely inseparable, v extends uniquely to $k(x)$ ([20, Theorem 4.1]) and the lemma follows. The general case is derived using the above two cases. $\square$

Let X be a regular integral scheme with function field $k(X)$. For a codimension 1 point $x \in X^{(1)}$, let $k(x)$ denote the residue field of x and $k(X)_x$ denote the completion of $k(X)$ with respect to the discrete valuation associated to x. For $\alpha \in \operatorname{Br}(X)$ and $x \in X^{(1)}$, let α_x and $\alpha(x)$ denote respectively the image of α under the restriction maps given by

$$\begin{aligned} r_{X,x} : \operatorname{Br}(X) \longrightarrow \operatorname{Br}(k(X)) &\longrightarrow \operatorname{Br}(k(X)_x) \\ \alpha &\longmapsto \alpha_x \\ s_{X,x} : \operatorname{Br}(X) \longrightarrow \operatorname{Br}(\mathcal{O}_{X,x}) &\longrightarrow \operatorname{Br}(k(x)) \\ \alpha &\longmapsto \alpha(x) \end{aligned}$$

Lemma A.3. *With notations as above, for* $\alpha \in \operatorname{Br}(X)$*,* $\alpha_x = 0 \iff \alpha(x) = 0$

Proof. This follows from the commutativity of the following diagram where $\widehat{\mathcal{O}}_{X,x}$ denotes the completion of $\mathcal{O}_{X,x}$. The maps $\imath$, $\jmath$ are injective (Corollary 1.10 in [16]) and q is an isomorphism ([2, Theorem 31]).



$\square$

We are now ready to prove the main theorem of this section.

Proof of Theorem A.1. First we will show the inclusion

$$Ш^{\mathrm{Br}}(C/\mathcal{C}) \supseteq Ш^{\mathrm{Br}}(C)$$

Let $\alpha \in \mathrm{Br}(C)$ be such that $\alpha_v = 0$ for every $v \in P(k)$, where α_v is the image of α under $\mathrm{Br}(C) \to \mathrm{Br}(C_v)$. We need to show that $\forall\, x \in \mathcal{C}^{(1)}$, α_x is zero.

Suppose x corresponds to a vertical divisor in the fiber corresponding to $v \in P(k)$. Then, $k_v(C) \subseteq k(C)_x$ which yields the required result.

On the other hand, if x corresponds to a horizontal divisor of $\mathcal{C}$, then x is a closed point in C. By Lemma A.3, we need to show that $\alpha(x) \in \mathrm{Br}(k(x))$ is trivial. Let w be a valuation on in $k(x)$ that restricts to the valuation v on k. Denote by $x_w \in \mathcal{C}_v$, the point corresponding to $k(x)_w$ (Lemma A.2). Then $\alpha(x)_w = \alpha_v(x_w) = 0$ since $\alpha_v = 0$ for every $v \in P(k)$. Therefore $\alpha(x)$ is in the kernel of the local-global map

$$\mathrm{Br}(k(x)) \longrightarrow \prod_{v \in P(k(x))} \mathrm{Br}(k(x)_v)$$

which is trivial by the classical Albert–Brauer–Hasse–Noether Theorem for the number field case and its generalization by Hasse for global fields ([14, Corollary 6.5.4]).

We will now show

$$Ш^{\mathrm{Br}}(C/\mathcal{C}) \subseteq Ш^{\mathrm{Br}}(C)$$

Suppose $\alpha \in Ш^{\mathrm{Br}}(C/\mathcal{C})$. Then note that α is unramified with respect to every $x \in \mathcal{C}^{(1)}$ and hence $\alpha \in \mathrm{Br}(\mathcal{C})$. Let $v \in P(k)$ be non-archimedean. Now $\mathcal{C}_v := \mathcal{C} \otimes_{\mathcal{O}_k} \mathcal{O}_v$ is a regular projective model for C_v over $\mathcal{O}_v$ ([38, Tag 0BG4, Theorem 54.11.2]). Then $\alpha \otimes_{\mathcal{C}} C_v \simeq (\alpha \otimes_{\mathcal{C}} \mathcal{C}_v) \otimes_{\mathcal{C}_v} C_v = 0$ since Brauer group of a regular proper curve over a complete discrete valued ring with finite residue field is trivial ([15, Theorem 3.1 and Remark 2.5 (b)]).

Now let v be archimedean. If $k_v \simeq \mathbb{C}$, then since $\mathrm{Br}(C_v) = 0$ by Tsen's theorem, $\alpha \otimes_{\mathcal{C}} C_v = 0$. Suppose $k_v \simeq \mathbb{R}$. Let $\widetilde{k_v}$ denote the real closure of k with respect to v. It suffices to show that $\widetilde{\alpha}_v := \alpha \otimes C_{\tilde{k_v}} = 0$. By [9, Theorem 2.3.1], this is equivalent to showing that $\widetilde{\alpha}_v(y) = 0$ for every real closed point $y \in C_{\tilde{k_v}}$. Now let x be the image of y under the canonical map $C_{\tilde{k_v}} \to C$. Note that x is a closed point of codimension 1. Since $\alpha \in Ш^{\mathrm{Br}}(C/\mathcal{C})$, $\alpha_x = 0$. Hence by Lemma A.3, $\alpha(x) = 0$. Therefore $\widetilde{\alpha}_v(y) \simeq \alpha(x) \otimes_{k(x)} k(y) = 0$. ☐

Appendix B. Totally ramified separable subfields in NSR algebras

Let k be a complete discrete valued field with value group $\mathbb{Z}$ and residue field $\bar{k}$. Let v denote the valuation on k. Let N be a NSR division algebra over k. The main goal of this section is to show that N contains a totally ramified separable maximal subfield. If $\operatorname{char} k = 0$ or is coprime to $\deg(N)$, the every subfield of N is separable and the claim follows since N contains a totally ramified maximal subfield by definition. So we may assume that $(\operatorname{char} k, \deg(N)) = p$.

Recall from Proposition 4.1 that N is a cyclic algebra and hence contains some cyclic maximal subfield. The next theorem sheds some light on the ramification.

Theorem B.1. *Let D be a NSR division algebra over k containing a totally ramified purely inseparable maximal subfield. Then D contains a totally ramified cyclic maximal subfield.*

Proof. Since D contains a purely inseparable maximal subfield, $\deg(D) = p^n$ for some n where $p = \operatorname{char} k$. Let F be a totally ramified purely inseparable maximal subfield of D. Then there exists an element $y \in F$ with $v(y) = \frac{1}{p^n}\mathbb{Z}$. Therefore y generates F over k i.e., $F \simeq k(y)$ where $y^{p^n} = a$ for some $a \in k^*$ with $(v(a), p) = 1$. By ([1, Chapter VII, Theorem 26]), D is isomorphic to the symbol algebra $[\omega, a)$ where $\omega = (\omega_1, \omega_2, \dots, \omega_n) \in W_n(k)$, the group of truncated Witt vectors of length n. By symbol manipulation techniques ([23, Proposition 1], [39], [42, Satz 15, 16]), we may assume that $v(a) < min(0, v(\omega_1))$ and

$$D \cong [\omega', a) := [(\omega + (a, 0, 0 \dots 0), a)]$$

where $\omega' = (\omega_1 + a, \omega'_2, \dots, \omega'_n) \in W(k)$ for some $\omega'_2, \dots, \omega'_n \in k$. Consider the Artin–Schreier–Witt extension L corresponding to ω'. Then $L \simeq k(x_1, x_2, \dots, x_n)$ where $(x_1^p, x_2^p, \dots, x_n^p) - (x_1, x_2, \dots, x_n) = \omega'$ in $W(L)$. We claim that L is totally ramified which will yield the theorem. To see this, let $\bar{L}$ denote the residue field of L. Suppose L/k is not totally ramified, then $\bar{L} \subseteq \bar{D}$ is a non-trivial extension of $\bar{k}$. But D is NSR, so $\bar{D}$ is a field that is a cyclic extension ([17, Lemma 5.1]) of $\bar{k}$ of degree p^n. Let $m \subset \bar{L} \subseteq \bar{D}$ be the unique degree p extension over $\bar{k}$ and let $M \subset L$ be the inertial lift of m [36, Chapter III, §5, Corollary 2]). Therefore M is an unramified degree p extension of k inside L. But L/k is cyclic and contains the unique degree p extension $k(x_1)$ defined by

$$x_1^p - x_1 = \omega_1 + a$$

By the assumption on $v(a)$, we have $v(x_1) = \frac{1}{p}v(a)$. Since $(v(a), p) = 1$, the extension $k(x_1)$ is ramified over k leading to contradiction. Therefore L/k is totally ramified. □

Corollary B.2. *Let N be a NSR division algebra over k. Then N contains a totally ramified separable maximal subfield.*

Proof. Let $L \subset N$ be a totally ramified maximal subfield. Then there is a tower of extensions $k \subseteq F \subseteq L$ where F/k is separable and L/F is purely inseparable ([38, Lemma 030K]). Note that both F/k and L/F are totally ramified. Let $D := C_N(F)$ be the centralizer of F in N. Then D is a division algebra over F and $D \simeq N \otimes_k F$([30, §13.3, Lemma]). Let K/k be an unramified splitting field of N. Then FK/F is unramified ([21, Chapter II, §4, Proposition 8 (ii)]) and splits D. Therefore D is tame. Now D contains the totally ramified purely inseparable subfield L which is a maximal subfield by the Double Centralizer Theorem ([30, §12.7 Theorem]). Therefore by Theorem B.1, D contains a cyclic totally ramified maximal subfield M/F. Since F/k is separable, M/k is a totally ramified separable maximal subfield in N. □

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