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Stability of Rankin–Selberg local γ-factors for split classical groups: The symplectic case

Stability of Rankin–Selberg local γ-factors for split classical groups: The symplectic case

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Résumé. Nous établissons la stabilité des facteurs γ locaux de Rankin–Selberg attachés aux représentations génériques des groupes symplectiques et des groupes généraux linéaires sur les corps p-adiques. Notre approche utilise la méthode de Langlands–Shahidi et fournit une nouvelle preuve qui ne repose pas sur les résultats de stabilité précédemment connus. Les innovations clés incluent la maîtrise de la géométrie des espaces d'orbites avec des stabilisateurs non triviaux et le développement de nouvelles techniques pour analyser les intégrales de Bessel partielles dans ce cadre. Ces résultats ont des implications importantes pour le programme de Langlands, en particulier pour les théorèmes de réciprocité locaux et la conjecture de fonctorialité.

Abstract. We establish the stability of Rankin–Selberg local γ-factors attached to generic representations of symplectic groups and general linear groups over p-adic fields. Our approach uses the Langlands–Shahidi method and provides a new proof that does not rely on previously known stability results. The key innovations involve handling the geometry of orbit spaces with non-trivial stabilizers and developing new techniques for analyzing partial Bessel integrals in this setting. The results have important implications for the Langlands program, particularly for local converse theorems and the functoriality conjecture.

1. Introduction

Let M be a connected reductive group defined over a p-adic field F, π an irreducible admissible representation of $M(F)$. Fix a non-trivial additive character ψ of F. In the study of Langlands functoriality, or more precisely the L-function theory, one needs to consider local γ-factors $\gamma(s, \pi, r, \psi)$ for any finite dimensional complex representation r of the Langlands L-group $^LM = \widehat{M} \rtimes \Gamma_F$, where $\Gamma_F = \mathrm{Gal}(\overline{F}/F)$, or W_F, the local Weil group of F. The general definition of $\gamma(s, \pi, r, \psi)$ is unknown so far, but in special cases there are many methods of constructing them. A very important method is the Langlands–Shahidi method [10], [12]. It defines the local γ-factors

$\gamma^{\mathrm{sh}}(s, \pi, r, \psi)$ when π is ψ-generic, namely, π admits a non-zero Whittaker model, M appears as a Levi subgroup of a larger reductive group G, and r an irreducible constituent of the adjoint action of ^{L}M on the Lie algebra of ^{L}N, the L-group of N where N is the unipotent radical of a parabolic $P = MN$.

The local γ-factors $\gamma(s, \pi, r, \psi)$, once defined, are related to the local ϵ- and L-factors in the following way:

$$\gamma(s, \pi, r, \psi) = \epsilon(s, \pi, r, \psi) \frac{L(1 - s, \widetilde{\pi}, r)}{L(s, \pi, r)},$$

where $\widetilde{\pi}$ is the contragredient representation of π.

Suppose π corresponds to ρ under the local Langlands correspondence, then it is expected that

$$\gamma(s, r \circ \rho, \psi) = \gamma(s, \pi, r, \psi).$$

We call $\gamma(s, r \circ \rho, \psi)$ and $\gamma(s, \pi, r, \psi)$ the arithmetic and analytic γ-factors respectively.

The local arithmetic and analytic γ-factors are expected to satisfy some stable equality under highly ramified twists, called the arithmetic and analytic stability respectively. The arithmetic stability is fully proved by P. Deligne [7], but the analytic stability is only known for certain cases. Let us first explain the precise meaning of analytic stability, which is the main concern of the present paper, then we give a list of references to known cases.

Given irreducible admissible representations π_1 and π_2 of $M(F)$ with the same central character, then for a highly ramified character χ of $F^{\times}$, i.e., a continuous character of $F^{\times}$ with sufficiently large conductor, regarded as a character of $M(F)$ via $m \mapsto \chi(\det \mathrm{Ad}_{\mathfrak{n}}(m))$, where $\mathrm{Ad} : M \to \mathrm{GL}(\mathfrak{n})$ is the adjoint representation and $\mathfrak{n} := \mathrm{Lie}(N)$, we expect to have

$$\gamma(s, \pi_1 \otimes \chi, r, \psi) = \gamma(s, \pi_2 \otimes \chi, r, \psi).$$

The analytic stability serves as an important intermediate step to prove many crucial results in the Langlands program such as the local converse theorems, the functoriality conjecture, and the local Langlands correspondence. Many results are known in these areas.

The analytic stability of local γ-factors is known in the following cases:

- For the Rankin–Selberg γ-factors of GL_n and GL_m, by Jacquet & Shalika [8];
- For the standard γ-factors of $M = \mathrm{SO}_{2n+1}$, i.e. r is the natural embedding $\widehat{M} \simeq \mathrm{Sp}_{2n}(\mathbb{C}) \hookrightarrow \mathrm{GL}_{2n}(\mathbb{C})$, by Cogdell & Piatetski-Shapiro [3];

- For the standard γ-factors of an F-split classical group M, either of symplectic or orthogonal type, by Rallis & Soudry [9] via the doubling method;
- For the standard γ-factors of GSpin groups, by Wagh [17], based on [9];
- For the symmetric and exterior square γ-factors of GL_n, by Cogdell, Shahidi & Tsai [6];
- For the twisted symmetric and exterior square γ-factors of GL_n, by She [15];
- For Asai γ-factors, by Shankman [13];
- For the symmetric cube γ-factors of GL_2, by Shankman & She [14];
- For Rankin–Selberg γ-factors given by the GL_1-twists of Sp_{2n}, $\widetilde{\mathrm{Sp}}_{2n}$, and $U(n, n)$, by Zhang [18];
- For γ-factors attached to a general quasi-split reductive group M, by Cogdell, Piatetski-Shapiro & Shahidi [5], but under the assumptions that $\dim(U_M \backslash N) = 2$ and $\mathrm{rank}\{Z_G \backslash T_w\} = 2$, where U_M is a chosen unipotent radical in a Borel subgroup of M, Z_G is the center of G and T_w is the subtorus defined in (3.6) of [5].

In this paper, we establish the stability for the Rankin–Selberg local γ-factor attached to generic representations of a symplectic group and a general linear group. The construction of such γ-factors by Langlands–Shahidi method will be explained in Section 2 of this paper. Our main result is the following:

Theorem 1.1. *Let $M = \mathrm{GL}_r \times \mathrm{Sp}_{2m}$ be split over a p-adic field F. Suppose $r \geq 1$, σ is an irreducible ψ-generic representation of $\mathrm{GL}_r(F)$ and $\tau_i, (i = 1, 2)$ are irreducible ψ-generic representations of $\mathrm{Sp}_{2m}(F)$. Let $\pi_i = \sigma \boxtimes \tau_i (i = 1, 2)$, then π_i is a representation of $M(F)$. Take a continuous character $\chi : F^\times \to \mathbb{C}^\times$, regarded as a character of $M(F)$ via $(m_1, m_2) \mapsto \chi(\det(m_1))$ for $m_1 \in \mathrm{GL}_r(F)$ and $m_2 \in \mathrm{Sp}_{2m}(F)$. Assume χ is highly ramified, i.e., the conductor of χ is sufficiently large, then we have*

$$\gamma(s, (\sigma \times \tau_1) \otimes \chi, \psi) = \gamma(s, (\sigma \times \tau_2) \otimes \chi, \psi)$$

We first remark that the joint work of Cai, Friedberg, and Kaplan [2] also constructed the Rankin–Selberg local γ-factors attached to classical groups and general linear groups using the generalized doubling method, and showed that they are equal to the ones constructed by Langlands–Shahidi method [2, Corollary 4.5] when the representations are generic while their γ-factors apply to all representations. They also proved a stability result [2, Lemma 7.3] which relies on the previous known results of stability.

We will give another proof of Theorem 1.1 in this paper without assuming any known results on stability. The strategy is to use Langlands–Shahidi

method to define the local γ-factors, and reduce the stability to the stability of the corresponding local coefficient. Similar to [6], the local coefficient admits an integral representation as the Mellin transform of certain partial Bessel functions, whose asymptotic expansion via relevant Bruhat cells breaks into a sum of two parts: the first part depends only on the central character, the second part is a uniform smooth function on certain subtorus. Hence when twisted by a highly ramified character, the second part becomes zero. This gives the stability of the local coefficient. We believe that our method which generalizes the method of Cogdell–Shahidi–Tsai may lead to a uniform proof of the stability for all Langlands–Shahidi γ-factors.

We point out two main differences in our cases compared to the known cases using this method. Firstly, the action of U_M on N has non-trivial stabilizers, and consequently the geometry of the orbit space $U_M\backslash N$ becomes more subtle. Secondly, the orbit space $U_M\backslash N$, over which the integral of the partial Bessel function is taken, is no longer isomorphic to a torus in our cases. Therefore a similar argument to [6] cannot be directly applied. Instead, we separate the toric part out of the orbit space $U_M\backslash N$, and show that only the toric part accounts for the proof of stability. We also observe that the map $n \mapsto m$ via the Bruhat decomposition $\dot{w}_0^{-1}n = mn'\bar{n}$ is finite étale onto its image with covering group isomorphic to finitely many copies of $\mathbb{Z}/2\mathbb{Z}$. This is a new phenomenon with sufficient generality and we believe it may show us one possible direction towards a uniform proof of stability of Langlands–Shahidi γ-factors in our future work. We also generalize the asymptotic analysis for partial Bessel integrals in [6] and [15]. Lastly, we remark that the cases where we replace the symplectic groups by odd orthogonal groups are very similar. A slight modification of our arguments would also give a proof of the same result for odd orthogonal cases.

Finally, our constructions and computations of the orbit space representatives of $U_M\backslash N$ and invariant measures in Section 4, as well as the part of Section 6 concerning the Bruhat decompositions are inspired by the computations for low rank cases with the help of SageMath, Mathematica and GP/PARI.

2. Construction of the Rankin–Selberg local γ-factors

Consider the split reductive group $G = G(n) = \mathrm{Sp}_{2n}$ defined over a p-adic field F which is a finite extension $\mathbb{Q}_p$. We realize G as

$$\mathrm{Sp}_{2n} = \{h \in \mathrm{GL}_{2n} : {}^t h J' h = J'\}$$

where

$$J' = J'_{2n} := \begin{bmatrix} & J_n \\ -{}^tJ_n & \end{bmatrix} \quad \text{and} \quad J_n = \begin{bmatrix} & & & 1 \\ & & -1 & \\ & \cdot^{\displaystyle\cdot^{\displaystyle\cdot}} & & \\ (-1)^{n-1} & & & \end{bmatrix}.$$

It is a semi-simple group of type C_n. Fix a Borel subgroup $B = TU$ consisting of the upper triangular matrices in Sp_{2n}, then

$$T = \{t = \mathrm{diag}(t_1, \ldots, t_n, t_n^{-1}, \ldots, t_1^{-1}) : t_i \in \mathbb{G}_m\}.$$

Following Bourbaki's labeling [1] of the root systems, the set of positive roots is given by

$$\Phi^+ = \begin{cases} \alpha_1, \ldots, \alpha_n, & \\ \displaystyle\sum_{i \le k < j} \alpha_k & (1 \le i < j \le n), \\ \displaystyle\sum_{i \le k < j} \alpha_i + 2\sum_{j \le k < n} \alpha_k + \alpha_n & (1 \le i < j \le n), \\ 2\displaystyle\sum_{i \le k < n} \alpha_k + \alpha_n & (1 \le i \le n) \end{cases}$$

with the set of simple roots

$$\Delta = \{\alpha_1, \ldots, \alpha_n\}.$$

Note that all the standard maximal parabolic subgroups of Sp_{2n} are self-associate. Indeed, let $\Delta_r = \Delta - \{\alpha_r\}$, $P_{\Delta_r} = M_{\Delta_r}N_{\Delta_r}$, and $w_0 = w_G w_{M_{\Delta_r}}^{-1}$ where w_G and $w_{M_{\Delta_r}}$ are the long Weyl group element of G and M_{Δ_r} respectively, then $w_0(\Delta_r) = \Delta_r$, and $w_0(\alpha_r) < 0$. For simplicity, we write $P = MN$ from now on. The Levi components of the maximal parabolic subgroups of $G(n)$ are of the form $\mathrm{GL}_r \times \mathrm{Sp}_{2m}$, with $r + m = n$, obtained by removing the simple root α_r $(1 \le r \le n)$ from the set of simple roots. (When $r = n$, the corresponding maximal Levi is isomorphic to GL_n and we regard Sp_0 as the trivial group 1.) We realize $M \simeq \mathrm{GL}_r \times \mathrm{Sp}_{2m}$ via

$$m = \begin{bmatrix} m_1 & & \\ & m_2 & \\ & & J_r{}^tm_1^{-1}J_r^{-1} \end{bmatrix} \longmapsto (m_1, m_2)$$

where $m_1 \in \mathrm{GL}_r$, $m_2 \in \mathrm{Sp}_{2m}$. Let $\mathrm{Mat}_{r \times m}$ be the space of $r \times m$ matrices with coefficients in F. For $X \in \mathrm{Mat}_{r \times 2m}, Y \in \mathrm{Mat}_{r \times r}$, we have a unique $Z \in \mathrm{Mat}_{2m \times r}$ such that

$$n(X, Y) := \begin{bmatrix} I_r & X & Y \\ & I_{2m} & Z \\ & & I_r \end{bmatrix} \in \mathrm{Sp}_{2n},$$

then we have

$$N = \{n = n(X, Y) : X \in \mathrm{Mat}_{r \times 2m}, Y \in \mathrm{Mat}_{r \times r}\}.$$

We apply the local theory of Langlands–Shahidi method to construct the local Rankin–Selberg γ-factors in our cases. Fix a non-trivial additive character ψ of F. Let $U_M = U \cap M$, then U_M acts on N by conjugation. We obtain from ψ a generic character χ for $U(F)$: we fix an F-splitting for G, i.e., a collection of root vectors $\{x_\alpha : \alpha \in \Delta\}$ which induces a homomorphism

$$\phi : U \longrightarrow \prod \mathbb{G}_a$$

where the product is taken over Δ. For $u \in U(F)$, we set

$$\chi(u) := \psi(\phi(u)).$$

Via restriction we obtain a character for $U_M(F)$ which we still denote by χ. Suppose σ and τ are irreducible representations of $\mathrm{GL}_r(F)$ and $\mathrm{Sp}_{2m}(F)$ respectively. Set $\pi \simeq \sigma \boxtimes \tau$, then its central character $\omega_\pi \simeq \omega_\sigma \boxtimes \omega_\tau$. We assume that π is ψ-generic, in the sense that the Whittaker functional space $\mathrm{Hom}_{U_M(F)}(\pi, \chi)$ is non-zero.

Let $\lambda \in \mathrm{Hom}_{U_M(F)}(\pi, \chi)$ be a non-zero element. Then

$$\lambda(\pi(u)v) = \chi(u)\lambda(v)$$

for any $v \in \pi$ and $u \in U_M(F)$. Suppose that $\dot{w}_0 \in G(F)$ is a representative of $w_0 = w_G \cdot w_M^{-1}$, such that $\dot{w}_0 = \dot{w}_G \dot{w}_M^{-1}$ is compatible with ψ, in the sense that $\psi(\dot{w}_0 u \dot{w}_0^{-1}) = \psi(u)$ for all $u \in U_M(F)$. For $\nu \in \mathfrak{a}_{P,\mathbb{C}}^*$, where $\mathfrak{a}_P^* = X^*(M)_F \otimes \mathbb{R}$, and $\mathfrak{a}_{P,\mathbb{C}}^* = \mathfrak{a}_P^* \otimes_\mathbb{R} \mathbb{C}$, let

$$I(\nu, \pi) = \mathrm{Ind}_{M(F)N(F)}^{G(F)} \pi \otimes q_F^{\langle \nu + \rho_P, H_P(\cdot) \rangle} \otimes \mathbf{1}_{N(F)}$$

be the normalized parabolic induction, where ρ_P is the half sum of positives roots in N, and $H_P : M(F) \to \mathfrak{a}_P = \mathrm{Hom}(X^*(M)_F, \mathbb{R})$ is the Harish–Chandra map defined by $q_F^{\langle \chi, H_M(m) \rangle} = |\chi(m)|$ for all $\chi \in X^*(M)_F$, the lattice of F-rational characters of M. Then the functional

$$\lambda_\psi(\nu, \pi) : I(\nu, \pi) \longrightarrow \mathbb{C}$$

$$\lambda_\psi(\nu, \pi)f \longmapsto \int_{N_1(F)} \lambda(f(\dot{w}_0^{-1}n'))\overline{\chi}(n')\mathrm{d}n'$$

where $N_1 = \dot{w}_0 \overline{N} \dot{w}_0^{-1}$ and $\overline{\chi}$ denotes the complex conjugate of χ. The linear functional $\lambda_\psi(\nu, \pi)$ defines a non-zero Whittaker functional on $I(\nu, \pi)$. Here $\overline{N}$ is the opposite of N, i.e., the unipotent subgroup of G generated by the root subgroups $x_{-\alpha}(F)(\alpha \in \Phi_+)$ such that $x_\alpha(F)$ is contained in N.

Suppose that $w \in W(G,T)$, such that $\Theta' = w(\Theta) \subset \Delta$ with $\dot{w} \in G(F)$ is chosen to be compatible with ψ, then

$$A(\nu, \pi, \dot{w}) : I(\nu, \pi) \longrightarrow I(\dot{w}(\nu), \dot{w}(\pi))$$

$$A(\nu, \pi, \dot{w})f \longmapsto \left(g \mapsto \int_{N_{\dot{w}(F)}} f(\dot{w}^{-1} ng) \mathrm{d}n \right)$$

where $N_{\dot{w}} = U \cap \dot{w} \overline{N} \dot{w}^{-1}$, defines an intertwining operator between the induced representations. Consequently, $\lambda_\psi(\dot{w}(\nu), \dot{w}(\pi)) \circ A(\nu, \pi, \dot{w})$ is another non-zero Whittaker functional on $I(\nu, \pi)$. By uniqueness of local Whittaker functionals, there exists a non-zero constant $C_\psi(\nu, \sigma, \dot{w})$, called the Shahidi local coefficient, such that

$$\lambda_\psi(\nu, \pi) = C_\psi(\nu, \pi, \dot{w}) \lambda_\psi(\dot{w}(\nu), \dot{w}(\pi)) \circ A(\nu, \pi, \dot{w}).$$

In our cases, $P = MN = P_{\Delta_r} = M_{\Delta_r} N_{\Delta_r}$ is self-associate for all $1 \leq r \leq n$. Let $\nu = s\tilde{\alpha}$, where $s \in \mathbb{C}$ and $\tilde{\alpha} = \langle \rho, \alpha \rangle^{-1} \rho$, where ρ is half sum of the positive roots in N. Fix

$$\dot{w}_G = \begin{bmatrix} & & J_r \\ & (-1)^r J'_{2m} & \\ -^t J_r & & \end{bmatrix},$$

$$\dot{w}_M = \begin{bmatrix} J_r & & \\ & (-1)^r J'_{2m} & \\ & & J_r{}^t J_r^{-1} J_r^{-1} \end{bmatrix} = \begin{bmatrix} J_r & & \\ & (-1)^r J'_{2m} & \\ & & J_r \end{bmatrix}$$

since $^t J_r^{-1} = J_r$. Then we pick the representative $\dot{w}_0$ to be

$$\dot{w}_0 = \dot{w}_G \dot{w}_M^{-1} = \begin{bmatrix} & & I_r \\ & I_{2m} & \\ (-1)^r I_r & & \end{bmatrix}$$

A straightforward computation shows that $\psi(\dot{w}_0 u \dot{w}_0^{-1}) = \psi(u)$ for all $u \in U_M$, i.e., our choice of $\dot{w}_0$ is compatible with the generic character ψ. Denote by $A(s, \pi) := A(s\tilde{\alpha}, \pi, \dot{w}_0)$ and $I(s, \pi) := I(s\tilde{\alpha}, \pi)$ then

$$A(s, \pi) : I(s, \pi) \longrightarrow I(-s, \dot{w}_0(\pi))$$

and the local coefficient $C_\psi(s, \pi) := C_\psi(s\tilde{\alpha}, \pi, \dot{w}_0)$ is a product of two γ-factors, namely,

$$C_\psi(s, \pi) = \gamma(s, \tilde{\sigma} \times \tilde{\tau}, \psi^{-1}) \cdot \gamma(s, \tilde{\sigma}, \wedge^2, \psi^{-1})$$

where $\tilde{\sigma}$ is the contragredient of σ and $\gamma(s, \tilde{\sigma}, \wedge^2, \psi^{-1})$ is the exterior square local factor of GL_r attached to σ, which is defined in [6, (4.4)] also by Langlands–Shahidi method. Therefore we have defined the Rankin–Selberg local factor $\gamma(s, \sigma \times \tau, \psi)$ for the reductive groups GL_r and Sp_{2m}. Moreover, since the analytic stability of $\gamma(s, \sigma, \wedge^2, \psi)$ is established in [6], it reduces

the stability of $\gamma(s, \sigma \times \tau, \psi)$ to the stability of the corresponding local coefficient $C_\psi(s, \pi)$.

3. A Bruhat decomposition

From the construction of the local Whittaker functionals on the space of induced representations, we would like to write $\dot{w}_0^{-1} n \in P\overline{N}$ where $\overline{N} = \dot{w}_0 N \dot{w}_0^{-1}$ is the opposite of N. This decomposition does not hold for all $n \in N$, but holds for a Zariski open dense subset N' of N. The existence of such an open dense subset is simply due to the fact that $Pw_0 N \cap \overline{N}$ is a non-empty open subset of $\overline{N}$. However, we will need the explicit decomposition for several purposes: first, it will be used in Section 4 to construct orbit space representatives and invariant measure on the space of U_M-orbits of N', where $U_M = U \cap M$, by which we obtain an explicit formula of the local coefficient at the end of Section 5; second, it will be discussed in Section 6 to understand the geometry of the map $n \mapsto m$ passed to U_M-orbits induced from the decomposition $\dot{w}_0^{-1} n = mn'\overline{n}$ with $m \in M$, $n' \in N$, and $\overline{n} \in \overline{N}$.

Suppose $\dot{w}_0^{-1} n = mn'\overline{n}$, with $m \in M$, $n' \in N$, and $\overline{n} \in \overline{N}$. A typical element in $\overline{N} = \dot{w}_0 N \dot{w}_0^{-1}$ is of the form

$$\overline{n} = \overline{n}(X_1, Y_1) = \begin{bmatrix} I_r & & \\ (-1) J'_{2m}{}^t X_1 J_r & I_{2m} & \\ (-1)^r Y_1 & (-1)^r X_1 & I_r \end{bmatrix}$$

for some $X_1 \in \mathrm{Mat}_{r \times 2m}, Y_1 \in \mathrm{Mat}_{r \times r}$. Let

$$m = (m_1, m_2) = \begin{bmatrix} m_1 & & \\ & m_2 & \\ & & J_r{}^t m_1^{-1} J_r \end{bmatrix} \in M,$$

and

$$n' = n(X', Y') = \begin{bmatrix} I_r & X' & Y' \\ & I_{2m} & (-1)^r J'_{2m}{}^t X' J_r \\ & & I_r \end{bmatrix} \in N.$$

One computes that

$$mn'\overline{n} = \begin{bmatrix} m_1 & & \\ & m_2 & \\ & & J_r{}^t m_1^{-1} J_r^{-1} \end{bmatrix} \begin{bmatrix} I_r & X' & Y' \\ & I_{2m} & (-1)^r J'_{2m}{}^t X' J_r \\ & & I_r \end{bmatrix} \begin{bmatrix} I_r & & \\ (-1)^r J'_{2m}{}^t X_1 J_r & I_{2m} & \\ (-1)^r Y_1 & (-1)^r X_1 & I_r \end{bmatrix}$$

$$= \begin{bmatrix} m_1 + (-1)^r m_1 X' J'_{2m}{}^t X_1 J_r + (-1)^r m_1 Y' Y_1 & m_1 X' + (-1)^r m_1 Y' X_1 & m_1 Y' \\ (-1)^r m_2 J'_{2m}{}^t X_1 J_r + m_2 J'_{2m}{}^t X' J_r Y_1 & m_2 + m_2 J'_{2m}{}^t X' J_r X_1 & (-1)^r m_2 J'_{2m}{}^t X' J_r \\ (-1)^r J_r{}^t m_1^{-1} J_r^{-1} Y_1 & (-1)^r J_r{}^t m_1^{-1} J_r^{-1} X_1 & J_r{}^t m_1^{-1} J_r^{-1} \end{bmatrix}.$$

On the other hand, let $n = n(X, Y) \in N$, we have

$$\dot{w}_0^{-1} n = \begin{bmatrix} & & (-1)^r I_r \\ & I_{2m} & (-1)^r J'_{2m}{}^t X J_r \\ I_r & X & Y \end{bmatrix}.$$

Compare both sides we obtain the following equalities:

(1) $m_1 + (-1)^r m_1 X' J'_{2m}{}^t X_1 J_r + (-1)^r m_1 Y' Y_1 = m_1 X' + (-1)^r m_1 Y' X_1 = (-1)^r m_2 J'_{2m}{}^t X_1 J_r + m_2 J'_{2m}{}^t X' J_r Y_1 = 0,$

(2) $m_1 Y' = (-1)^r I_r = J_r {}^t m_1^{-1} J_r^{-1} Y_1,$

(3) $m_2 + m_2 J'_{2m}{}^t X' J_r X_1 = I_{2m},$

(4) $m_2 J'_{2m}{}^t X' J_r = J'_{2m}{}^t X J_r,$

(5) $(-1)^r J_r {}^t m_1^{-1} J_r^{-1} X_1 = X,$

(6) $J_r {}^t m_1^{-1} J_r^{-1} = Y.$

The outer automorphism $\theta_r(g) := J_r {}^t g^{-1} J_r^{-1}, g \in \mathrm{GL}_r$ defines a involution of GL_r. Assume $\det(Y) \neq 0$, then

$$(6) \iff m_1 = J_r {}^t Y^{-1} J_r^{-1} = \theta_r(Y),$$

$$(5) \iff X_1 = (-1)^r \theta_r(m_1)^{-1} X = (-1)^r Y^{-1} X,$$

$$(2) \iff Y_1 = (-1)^r Y^{-1}, Y' = (-1)^r \theta_r(Y)^{-1}.$$

The second equality of (1) is equivalent to

$$X' = (-1)^{r-1} \theta_r(Y^{-1}) Y^{-1} X.$$

Plug this in (3), we have

$$m_2(I_{2m} + J'_{2m}{}^t X' J_r X_1) = m_2(I_{2m} - J'_{2m}{}^t X^t Y^{-1} \theta_r({}^t Y^{-1}) J_r Y^{-1} X),$$
$$= m_2(I_{2m} - J'_{2m}{}^t X^t Y^{-1} J_r X)$$
$$= m_2(I_{2m} + (-1)^{r-1} \theta_{r,m}(Y^{-1} X) X) = I_{2m},$$

where

$$\theta_{r,m} : \mathrm{Mat}_{r \times 2m} \longrightarrow \mathrm{Mat}_{2m \times r},$$
$$X \longmapsto {}^t(J_r X J'_{2m}) = {}^t J'_{2m}{}^t X^t J_r = (-1)^r J'_{2m}{}^t X J_r.$$

If we further assume that $I_{2m} - J'_{2m}{}^t X^t Y^{-1} J_r X \in \mathrm{GL}_{2m}$, which is a Zariski open dense condition on N, then

$$m_2 = (I_{2m} - J'_{2m}{}^t X^t Y^{-1} J_r X)^{-1} = (I_{2m} + (-1)^{r-1} \theta_{r,m}(Y^{-1} X) X)^{-1}.$$

One easily checks that other equalities in (1) are automatically satisfied for our solutions of $m_1, m_2, X', Y', X_1, Y_1$. To check that (4) holds, since $m_2 \in \mathrm{Sp}_{2m}$, i.e., ${}^t m_2 J'_{2m} m_2 = J'_{2m}$, (4) is equivalent to $X' m_2^{-1} = X$. So we need to check that

$$(-1)^{r-1} \theta_r(Y^{-1}) Y^{-1} X (I_{2m} - J'_{2m}{}^t X^t Y^{-1} J_r X) = X,$$

hence it suffices to check that

$$(-1)^{r-1} J_r {}^t Y J_r^{-1} Y^{-1} + (-1)^r J_r {}^t Y J_r^{-1} Y^{-1} X J'_{2m}{}^t X^t Y^{-1} J_r = I_r.$$

Multiplying $Y J_r {}^t Y^{-1} J_r^{-1}$ on the left and $J_r^{-1}{}^t Y$ on the right, one simplifies to get that $J_r {}^t Y - Y^t J_r + (-1)^r X J'_{2m}{}^t X = 0$, which holds automatically by the structure of Sp_{2n}.

4. The orbit space and its invariant measure

The Bruhat decomposition $\dot{w}_0^{-1} n = mn'\bar{n}$ discussed in Section 3 induces an M-equivariant map $n \mapsto m$, where the action on N is by conjugation and the action on M is the $\mathrm{Int}(w_0^{-1})$-twisted conjugation. This can be observed from that for any $x \in M$, we have

$$\dot{w}_0^{-1} x n x^{-1} = \dot{w}_0^{-1} x \dot{w}_0 m x^{-1} (x n' x^{-1})(x \bar{n} x^{-1}).$$

In particular, it is U_M-equivariant. When the parabolic subgroup P is self-associate, under some mild assumptions, Theorem 6.2 of [11] represents the local coefficient as the Mellin transform of certain partial Bessel functions over the orbit space $U_M \backslash N'$. We will show later in Section 5 that these assumptions are valid in our cases, so that Theorem 6.2 of [11] can be applied to obtain an integral representation of the local coefficient in our cases. The geometry of the orbit space $U_M \backslash N'$ is not fully understood in general, and its importance is emphasized in [5] towards the proof of stability for general quasi-split groups. The main goal of this section is to establish the orbit space representatives of $U_M \backslash N'$ and its invariant measure in our cases, which will be needed in Section 6 to understand the geometry of $U_M \backslash N'$, the map $n \mapsto m$, and the analysis in Sections 6 and 7.

We will use the same notations as in Section 3. A simple calculation shows that U_M acts on N by

$$U_M \times N \longrightarrow N$$

$$(u, n(X, Y)) \longmapsto n(u_1 X u_2^{-1}, u_1 Y J_r {}^t u_1 J_r^{-1}) = n(u_1 X u_2^{-1}, u_1 Y \theta_r(u_1^{-1}))$$

where we identify $u = \mathrm{diag}(u_1, u_2, J_r {}^t u_1^{-1} J_r^{-1}) \in U_M$ with $(u_1, u_2) \in U_{\mathrm{GL}_r} \times U_{\mathrm{Sp}_{2m}}$, where U_{GL_r} and $U_{\mathrm{Sp}_{2m}}$ are the maximal unipotent subgroups of GL_r and Sp_{2m} respectively.

Since $n(X, Y) \in N$, X and Y are related by $J_r {}^t Y - Y^t J_r + (-1)^r X J'_{2m} {}^t X = 0$. Set $Z = J_r {}^t Y + \frac{(-1)^r X J'_{2m} {}^t X}{2}$, thus

$$J_r {}^t Y - Y^t J_r + (-1)^r X J'_{2m} {}^t X = 0 \Longleftrightarrow Z = {}^t Z.$$

Then the action $(X, Y) \mapsto (u_1 X u_2^{-1}, u_1 Y \theta_r(u_1^{-1}))$ is equivalent to $X \mapsto u_1 X u_2^{-1}, Z \mapsto u_1 Z {}^t u_1$. The advantage of this change of variable is that now X and Z are independent. Denote the space of $k \times k$ symmetric matrices with coefficients in F as Sym^k. We have the following description on the orbit space representatives and measures for $U_M \backslash N$:

Proposition 4.1. *There exists a Zariski open dense subset $N' \subset N$, such that the Bruhat decomposition $\dot{w}_0^{-1} n = mn'\bar{n}$ holds, and the orbit space $U_M \backslash N'$ admits a set of representatives of the form*

$$R = R_X \times R_Z$$

where R_X consists of matrices of the form

$$\begin{cases} \begin{cases} x_{r-k,1+k} \ (0 \le k \le r-1), \\ x_{i,j} \ (2m-r+1 \le j-i, j \le 2m) \end{cases}, & \text{if } r \le m, \\[2em] \begin{cases} i+j = r+1 \ (j \le m+1), \\ x_{i,j} : \text{or } r-m-l \le i \le r-m+l, \\ \qquad j = m+l+1 \ (1 \le l \le r-m-1) \end{cases}, & \text{if } m < r < 2m, \\[2em] \begin{cases} i+j = r+1 \ (j \le m+1), \\ x_{i,j} : \text{or } r-m-l \le i \le r-m+l, \\ \qquad j = m+l+1 \ (1 \le l \le m-1) \end{cases}, & \text{if } r \ge 2m, \end{cases}$$

where the entries which are not indicated are understood to be 0, and

$$R_Z = \begin{cases} \mathrm{Sym}^r, & \text{if } r < 2m, \\ \{(z_{ij}) \in \mathrm{Sym}^r \,|\, z_{ij} = 0 \text{ if } i,j \le r-2m \text{ and } i \ne j\} & \text{if } r \ge 2m. \end{cases}$$

Set $d_X = \dim R_X$, and $d_Z = \dim R_Z$. Then

$$d_X = \begin{cases} \frac{r(r+1)}{2}, & \text{if } r \le m, \\ 2rm - \frac{r(r-1)}{2} - m^2, & \text{if } m < r < 2m, \\ m(m+1), & \text{if } r \ge 2m, \end{cases}$$

$$d_Z = \begin{cases} \frac{r(r+1)}{2} & \text{if } r < 2m, \\ (2m+1)(r-m) & \text{if } r \ge 2m. \end{cases}$$

Note that when $r < m$, the action has a non-trivial but base point free stabilizer isomorphic to $U_{\mathrm{Sp}_{2(m-r)}}$ and when $r \ge m$ the stabilizer is always trivial. Moreover, the corresponding invariant measure $\mathrm{d}\mu$ on R is given by $\mathrm{d}\mu = \mathrm{d}\mu_X \wedge \mathrm{d}\mu_Z$, where

$$\mathrm{d}\mu_X = \begin{cases} |x_{r,1}^{r+2m-2} x_{r-1,2}^{r+2m-5} x_{r-2,3}^{r+2m-8} \cdots x_{1,r}^{2m-2r+1}| \prod \mathrm{d}x_{ij}, & \text{if } r \le m, \\[1em] |x_{r,1}^{r+2m-2} x_{r-1,2}^{r+2m-5} x_{r-2,3}^{r+2m-8} \cdots x_{r-m+1,m}^{r-m+1} x_{r-m,m+1}^{r-m-1} \\ \qquad x_{r-m-1,m+2}^{r-m-2} \cdots x_{2,r-1}| \prod \mathrm{d}x_{ij}, & \text{if } m < r < 2m, \\[1em] |x_{r,1}^{r+2m-2} x_{r-1,2}^{r+2m-5} x_{r-2,3}^{r+2m-8} \cdots x_{r-m+1,m}^{r-m+1} x_{r-m,m+1}^{r-m-1} \\ \qquad x_{r-m-1,m+2}^{r-m-2} \cdots x_{r-2m+1,2m}^{r-2m}| \prod \mathrm{d}x_{ij}, & \text{if } r \ge 2m. \end{cases}$$

and

$$\mathrm{d}\mu_Z = \begin{cases} \prod_{i,j} \mathrm{d}z_{i,j} & \text{if } r \le 2m, \\ |z_{r-2m,r-2m}^{r-2m-1} z_{r-2m-1,r-2m-1}^{r-2m-2} \cdots z_{2,2}| \prod \mathrm{d}z_{ij} & \text{if } r > 2m, \end{cases}$$

where the product runs over all (i,j)'s such that $x_{i,j}$ and $z_{i,j}$ are non-zero in each case. In addition, if we use (X,Y) instead of (X,Z) to parameterize the orbit space, the corresponding invariant measures are related by

$$\mathrm{d}\mu_X \wedge \mathrm{d}\mu_Y = \mathrm{d}\mu_X \wedge (\mathrm{d}\mu_Z \cdot J_r).$$

Proof. Based on the arguments before the statement of the proposition, it suffices to study the action $X \mapsto u_1 X u_2^{-1}, Z \mapsto u_1 Z {}^t u_1$, by induction of the size of X and Z. If $r = 1$, then $u_1 = 1$, the action degenerates as $(X, Z) \mapsto (X u_2^{-1}, Z)$. Assume $x_{1,1} \neq 0$, then it is easy to see that the orbit space representative can be given as $((x_{1,1}, 0, \ldots, 0), Z) \in \mathbb{A}^{2m} \times \mathrm{Sym}^r$.

Suppose $r > 1$. We study the action on X first. Write

$$u_1 = \begin{bmatrix} v_1 & {}^t \delta_1 \\ & 1 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 1 & \delta_2 \\ & v_2' \end{bmatrix}, \quad X = \begin{bmatrix} {}^t \alpha & X_1' \\ x_{r,1} & \beta \end{bmatrix}$$

with $\alpha, \delta_1 \in \mathbb{A}^{r-1}$, $\beta, \delta_2 \in \mathbb{A}^{2m-1}$, $v_1 \in U_{\mathrm{GL}_{r-1}}, v_2' \in U_{\mathrm{GL}_{2m-1}}$, and $X_1' \in \mathrm{Mat}_{(r-1) \times (2m-1)}$. Then

$$\begin{aligned}
u_1 X u_2^{-1} &= \begin{bmatrix} v_1 & {}^t \delta_1 \\ & 1 \end{bmatrix} \begin{bmatrix} {}^t \alpha & X_1' \\ x_{r,1} & \beta \end{bmatrix} \begin{bmatrix} 1 & -\delta_2 (v_2')^{-1} \\ & (v_2')^{-1} \end{bmatrix} \\
&= \begin{bmatrix} v_1 {}^t \alpha + {}^t \delta_1 x_{r,r+1} & v_1 X_1' + {}^t \delta_1 \beta \\ x_{r,1} & \beta \end{bmatrix} \begin{bmatrix} 1 & -\delta_2 (v_2')^{-1} \\ & (v_2')^{-1} \end{bmatrix} \\
&= \begin{bmatrix} v_1 {}^t \alpha + {}^t \delta_1 x_{r,1} & -v_1 {}^t \alpha \delta_2 (v_2')^{-1} - x_{r,1} {}^t \delta_1 \delta_2 (v_2')^{-1} + v_1 X_1' (v_2')^{-1} + {}^t \delta_1 \beta (v_2')^{-1} \\ x_{r,1} & -x_{r,1} \delta_2 (v_2')^{-1} + \beta (v_2')^{-1} \end{bmatrix}.
\end{aligned}$$

Assume $x_{r,1} \neq 0$, choose δ_1 and δ_2 such that $v_1 {}^t \alpha + {}^t \delta_1 x_{r,1} = -x_{r,1} \delta_2 (v_2')^{-1} + \beta (v_2')^{-1} = 0$, i.e. $\delta_1 = -\frac{\alpha {}^t v_1}{x_{r,1}}$, $\delta_2 = \frac{\beta}{x_{r,1}}$, then

$$\begin{aligned}
-v_1 {}^t \alpha \delta_2 (v_2')^{-1} &- x_{r,1} {}^t \delta_1 \delta_2 (v_2')^{-1} + v_1 X_1' (v_2')^{-1} + {}^t \delta_1 \beta (v_2')^{-1} \\
&= v_1 X_1' (v_2')^{-1} - \frac{v_1 {}^t \alpha \beta (v_2')^{-1}}{x_{r,1}}.
\end{aligned}$$

Let $X_1'' = X_1' - \frac{{}^t \alpha \beta}{x_{r,1}}$, then

$$v_1 X_1' (v_2')^{-1} - \frac{v_1 {}^t \alpha \beta (v_2')^{-1}}{x_{r,1}} = v_1 X_1'' (v_2')^{-1}.$$

We have $X_1'' \in \mathrm{Mat}_{(r-1) \times (2m-1)}$, $v_1 \in U_{\mathrm{GL}_r}$ and $v_2' \in U_{\mathrm{GL}_{2m-1}}$. Now if $m = 1$ then $v_2' = 1$, and the action degenerates as $X_1'' \mapsto v_1 X_1''$ with $X_1'' \in \mathrm{Mat}_{(r-1) \times 1}$. So it is clear that if we assume that the $(r-1, 2)$-th entry of X_1'' is non-zero, we obtain an orbit space representative of the action on X given by $R_X = \begin{bmatrix} 0 & 0 \\ \vdots & \vdots \\ 0 & x_{r-1,2} \\ x_{r,1} & 0 \end{bmatrix}$. Therefore, we assume that $m > 1$ from now on.

Write

$$u_2 = \begin{bmatrix} 1 & \delta_2 \\ & v_2' \end{bmatrix} = \begin{bmatrix} 1 & \gamma_2 & x \\ & v_2 & {}^t \gamma_2' \\ & & 1 \end{bmatrix},$$

where $\delta_2 = (\gamma_2, x)$ and $v_2' = \begin{bmatrix} v_2 & {}^t\gamma_2' \\ & 1 \end{bmatrix}$, with $\gamma_2, \gamma_2' \in \mathbb{A}^{2m-2}$. Moreover, by the structure of Sp_{2m}, a simple calculation shows that $\gamma_2' = -\gamma_2 v_2^{-1} J_{2m-2}'$, $\gamma_2'{}^t J_{2m-2}'{}^t\gamma_2' = 0$, x is free, and ${}^t v_2 J_{2m-2}' v_2 = J_{2m-2}'$, hence $v_2 \in U_{\mathrm{Sp}_{2m-2}}$. Note that since δ_2 is determined by the above process, so is γ_2. We also write $X_1'' = \begin{bmatrix} X_1 & {}^t\gamma \end{bmatrix}$ with $X_1 \in \mathrm{Mat}_{(r-1)\times(2m-2)}$ and $\gamma \in \mathbb{A}^{r-1}$. Therefore we can write

$$
v_1 X_1'(v_2')^{-1} = v_1 \begin{bmatrix} X_1 & {}^t\gamma \end{bmatrix} \begin{bmatrix} v_2^{-1} & -v_2^{-1}{}^t\gamma_2' \\ & 1 \end{bmatrix}
$$

$$
= \begin{bmatrix} v_1 X_1 v_2^{-1} & v_1 X_1 v_2^{-1}{}^t\gamma_2' + v_1{}^t\gamma \end{bmatrix}
$$

$$
= \begin{bmatrix} v_1 X_1 v_2^{-1} & v_1 X_1 v_2^{-1}(-{}^t J_{2m-2}'{}^t v_2^{-1}{}^t\gamma_2) + v_1{}^t\gamma \end{bmatrix}
$$

$$
= \begin{bmatrix} v_1 X_1 v_2^{-1} & -v_1 X_1({}^t v_2 J_{2m-2}' v_2)^{-1}{}^t\gamma_2 + v_1{}^t\gamma \end{bmatrix}
$$

$$
= \begin{bmatrix} v_1 X_1 v_2^{-1} & v_1 X_1 J_{2m-2}'{}^t\gamma_2 + v_1{}^t\gamma \end{bmatrix}
$$

$$
= \begin{bmatrix} v_1 X_1 v_2^{-1} & (v_1 X_1 v_2^{-1}) v_2 J_{2m-2}'{}^t\gamma_2 + v_1{}^t\gamma \end{bmatrix}.
$$

From this observation we see that it suffices to study the action

$$
(U_{\mathrm{GL}_{r-1}} \times U_{\mathrm{Sp}_{2m-2}}) \times (\mathrm{Mat}_{(r-1)\times(2m-2)} \times \mathbb{A}^{r-1}) \longrightarrow \mathrm{Mat}_{(r-1)\times(2m-2)} \times \mathbb{A}^{r-1}
$$

$$
((v_1, v_2), (X_1, \gamma)) \longmapsto (v_1 X_1 v_2^{-1}, v_1{}^t\gamma).
$$

Set $X_0 := X$, $r' = r - 1$ and $m' = m - 1$, replace X by X_1, u_1 by v_1, u_2 by v_2, r by r', and m by m'. Continue with the above process, we can construct X_i's ($i \geq 0$) inductively. Since the relative size of r and $2m$ will affect the inductive process, we will have to discuss three cases separately:

Case (1): $r \leq m$. Then $r' \leq m'$ for all r' and m' in the inductive process. Note that in this case all the entries of v_1, hence of u_1, are chosen in the inductive process. Consequently we do not have free variables in v_1 when considering the action $\gamma \mapsto v_1{}^t\gamma$. We further assume that all the entries of $v_1{}^t\gamma$ are non-zero. We eventually obtain that $X_{r-1} = (\widetilde{x}_{1,r}, *, \ldots, *)$ is a vector in $\mathbb{A}^{2(m-r)+2}$ and we are left to consider the right-action of $U_{\mathrm{Sp}_{2(m-r)+2}}$. Write the action as $X_{r-1} \mapsto X_{r-1} u^{-1}$ with

$$
u = \begin{bmatrix} 1 & \delta \\ & v' \end{bmatrix} = \begin{bmatrix} 1 & \gamma & a \\ & v & {}^t\gamma' \\ & & 1 \end{bmatrix} \in U_{\mathrm{Sp}_{2(m-r)+2}},
$$

where $\delta = (\gamma, a) \in \mathbb{A}^{2(m-r)+1}$, $\gamma, \gamma' \in \mathbb{A}^{2(m-r)}$, and v is a unipotent matrix of size $2(m - r)$. Then a similar calculation as above shows that $\gamma' = -\gamma v^{-1} J_{2(r-m)}'$, a is free, and $v \in U_{\mathrm{Sp}_{2(m-r)}}$. Write $X_{r-1} = (\widetilde{x}_{r,1}, \alpha')$ with $\alpha' \in \mathbb{A}^{2(m-r)+1}$. By assuming $\widetilde{x}_{1,r} \neq 0$, we choose $\delta = (\gamma, a)$ such that

$\widetilde{x}_{r,1}\delta + \alpha'v = 0$, then $X_{r-1}u^{-1} = (\widetilde{x}_{r,1},0,\dots,0)$. So we obtain our orbit space R_X in this case as the space consisting of matrices

$$
\begin{bmatrix}
0 & 0 & \dots & 0 & x_{1,r} & 0 \dots 0 & x_{1,r+2(m-r)+2} & x_{1,r+2(m-r)+3} & \cdots & x_{1,2m} \\
0 & 0 & \dots & x_{2,r-1} & 0 & 0 \dots 0 & 0 & x_{2,r+2(m-r)+3} & \cdots & x_{2,2m} \\
\vdots & \vdots & \ddots & \vdots & \vdots & 0 \dots 0 & \vdots & \vdots & \ddots & \vdots \\
0 & x_{r-1,2} & \cdots & 0 & 0 & 0 \dots 0 & 0 & 0 & \cdots & x_{r-1,2m} \\
x_{r,1} & 0 & \dots & 0 & 0 & 0 \dots 0 & 0 & 0 & \dots & 0
\end{bmatrix},
$$

where there are $2(m-r)+1$ zero columns in the middle. From the last step above, we also see that in this case the action $X \mapsto u_1 X u_2^{-1}$ has stablizer $U_{\mathrm{Sp}_{2(m-r)}}$ given by v, provided that $m > r$. Moreover, one computes easily that $\dim R_X = r + \frac{r(r-1)}{2} = \frac{r(r+1)}{2}$.

Case (2): $r \geq 2m$. Then $r' \geq 2m'$ for all r' and m' in the inductive process. Perform the inductive process to the m-th step, we obtain that

$$
X_m = \begin{bmatrix}
* & & * \\
\vdots & & \vdots \\
* & & x_{r-m,m+1} \\
x_{r-m+1.m} & & *
\end{bmatrix}.
$$

Now $v_1 = \begin{bmatrix} v_1' & {}^t\delta_1 \\ & 1 \end{bmatrix} \in U_{\mathrm{GL}_{r-m}}$, and $v_2 = \begin{bmatrix} 1 & x \\ & 1 \end{bmatrix}$. Choose δ_1 and x accordingly we can make X_m to be of the form $\begin{bmatrix} 0 & & * \\ \vdots & & \vdots \\ 0 & & x_{r-m,m+1} \\ x_{r-m+1,m} & & 0 \end{bmatrix}$. From the $(m+1)$-th step on the action degenerates as the left action of $U_{\mathrm{GL}_{r-m-1}}$ on $(X, {}^t\gamma)$. By determining the last column of v_1 each time to make the first column but the last entry of X zero, we eventually obtain a vector of the form ${}^t(*,\dots,*,\widetilde{x}_{r-2m+1,2m}) \in \mathbb{A}^{r-2m+1}$. During this process, all the entries of u_2 together with all the last $2m-1$ columns of u_1 are determined, and we are left to consider the left-action of $U_{\mathrm{GL}_{r-2m+1}}$ on $\mathbb{A}^{2m-r+1}$. Assume $\widetilde{x}_{r-2m+1,2m} \neq 0$, we pick the orbit representative in the last step as ${}^t(0,\dots,0,\widetilde{x}_{r-2m+1,2m})$. As a result, we obtain our orbit space R_X as the

space consisting of matrices

$$
\begin{bmatrix}
0 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 & 0 \\
& & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 & x_{r-2m+1,2m} \\
0 & 0 & \ldots & 0 & 0 & 0 & 0 & x_{r-2m+2,2m-1} & * \\
& & \vdots & 0 & 0 & 0 & \iddots & \vdots & \vdots \\
0 & 0 & \ldots & 0 & 0 & x_{r-m-1,m+2} & \ldots & * & * \\
0 & 0 & \ldots & 0 & x_{r-m,m+1} & x_{r-m,m+2} & \ldots & * & * \\
0 & 0 & \ldots & x_{r-m+1,m} & 0 & x_{r-m+1,m+2} & \ldots & * & * \\
& \iddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \ldots & 0 & 0 & 0 & 0 & x_{r-2,2m-1} & * \\
0 & x_{r-1,2} & \ldots & 0 & 0 & 0 & 0 & 0 & x_{r-1,2m} \\
x_{r,1} & 0 & \ldots & 0 & 0 & 0 & 0 & 0 & 0,
\end{bmatrix}
$$

where there are $r - 2m$ zero rows on the top. Observe that in this case the action $X \mapsto u_1 X u_2^{-1}$ has a stablizer $U_{\mathrm{GL}_{r-2m}}$ given by the upper-left unipotent submatrix of u_1 of size $r - 2m$, if $r > 2m$. We also have $\dim R_X = m + (m + \frac{m(m-1)}{2} \cdot 2) = m(m+1)$.

Case (3): $m < r < 2m$. In this case it is possible that there exists some $k \geq 0$ such that $r' = r - k \geq 2(m - k) = 2m'$, i.e., $2m - r \leq k \leq m$. If this happens, continue with our algorithm in case (2). Specifically, note that when $k = 2m - r > 0$, $r' = 2m' = 2(r - m)$, we can conclude as in the last step of case (2) that the action $X \mapsto u_1 X u_2^{-1}$ has trivial stablizer. If for all r' and m' in the inductive process, we have $r' < 2m'$, we perform a similar inductive argument as in case (1). Since $r' < 2m'$ for all r', m' in the inductive process, by a similar argument as in case (1), we have exhausted all the possible choices of entries in u_1. On the other hand, since $r' > m'$, when r' decreases to 1, so does m', hence we also exhaust all the possible choices of entries in u_2. Consequently, the action $X \mapsto u_1 X u_2^{-1}$ has trivial stablizer. Eventually one obtains the orbit space R_X in this case as the space consisting of matrices

$$
\begin{bmatrix}
0 & \ldots & 0 & 0 & 0 & \ldots & 0 & x_{1,r} & x_{1,r+1} & \ldots & x_{1,2m} \\
0 & \ldots & 0 & 0 & 0 & \ldots & x_{2,r-1} & x_{2,r} & x_{2,r+1} & \ldots & x_{2,2m} \\
\vdots & & \vdots & \vdots & \vdots & \iddots & \vdots & \vdots & \vdots & \ldots & \vdots \\
0 & \ldots & 0 & 0 & x_{r-m-1,m+2} & \ldots & * & * & * & \ldots & * \\
0 & \ldots & 0 & x_{r-m,m+1} & x_{r-m,m+2} & \ldots & * & * & * & \ldots & * \\
0 & \ldots & x_{r-m+1,m} & 0 & x_{r-m+1,m+2} & \ldots & * & * & * & \ldots & * \\
\vdots & & \vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \vdots & \ldots & \vdots \\
0 & \ldots & 0 & 0 & 0 & \ldots & 0 & x_{2(r-m)-1,r} & * & \ldots & * \\
0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 0 & x_{2(r-m),r+1} & \ldots & * \\
0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & * \\
\vdots & \iddots & \vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & x_{r-1,2m} \\
x_{r,1} & \ldots & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0
\end{bmatrix}.
$$

We have dim $R_X = m + (r-m)^2 + \frac{(3r-2m-1)(2m-r)}{2} = 2rm - \frac{r(r-1)}{2} - m^2$.

Note that only in case (2) we also have to consider the action of $U_{\mathrm{GL}_{r-2m}}$ on $Z \in \mathrm{Sym}^r$. Write $Z = \begin{bmatrix} Z_1 & h \\ {}^t h & Z_2 \end{bmatrix}$, with $Z_1 \in \mathrm{Sym}^{r-2m}, Z_2 \in \mathrm{Sym}^{2m}$, $h \in \mathrm{Mat}_{(r-2m)\times 2m}$, and $u_1 = \begin{bmatrix} u_1' & w \\ & u_1'' \end{bmatrix}$ with $u_1' \in U_{\mathrm{GL}_{r-2m}}$, $u_1'' \in U_{\mathrm{GL}_{2m}}$ and $w \in \mathrm{Mat}_{(r-2m)\times 2m}$. A simple calculation gives

$$u_1 Z^t u_1 = \begin{bmatrix} u_1' Z_1 {}^t u_1' + w^t h^t u_1' + (u_1' h + w Z_2)^t w & u_1' h^t u_1'' + w Z_2 {}^t u_1'' \\ u_1'' {}^t h^t u_1' + u_1'' Z_2 {}^t w & u_1'' Z_2 {}^t u_1'' \end{bmatrix}$$

Note that both w and u_1'' are fixed during the inductive process of the action $X \mapsto u_1 X u_2^{-1}$, so it suffices to consider the action $(u_1', (Z_1, h)) \mapsto (u_1' Z_1 {}^t u_1', u_1' h)$. We further write $u_1' = \begin{bmatrix} \tilde{u}_1 & {}^t \alpha_1 \\ & 1 \end{bmatrix}$, and $Z_1 = \begin{bmatrix} Z_1' & {}^t \delta_1' \\ \delta_1' & z \end{bmatrix}$. Then

$$u_1' Z_1 {}^t u_1' = \begin{bmatrix} \tilde{u}_1 Z_1' {}^t \tilde{u}_1 + {}^t \alpha_1 \delta_1' {}^t \tilde{u}_1 + (\tilde{u}_1 {}^t \delta_1' + {}^t \alpha_1 z)\alpha_1 & \tilde{u}_1 {}^t \delta_1' + {}^t \alpha_1 z \\ \delta_1' {}^t \tilde{u}_1 + z\alpha_1 & z \end{bmatrix}$$

Assume $z \neq 0$ we can choose α_1 such that $\tilde{u}_1 {}^t \delta_1' + {}^t \alpha_1 z = 0$, then the matrix becomes $\begin{bmatrix} \tilde{u}_1 \left(Z_1' - \frac{{}^t \delta_1' \delta_1'}{2} \right){}^t \tilde{u}_1 & 0 \\ 0 & z \end{bmatrix}$. Replace Z_1 by $Z_1' - \frac{{}^t \delta_1' \delta_1'}{2}$, u_1' by $\tilde{u}_1$ and continue with the above process, we exhaust all possible choices of entries in u_1. Hence there is no need to consider $h \mapsto u_1' h$. Assume that after our choices of all entries in u_1', we have $u_1' h \neq 0$. We obtain a orbit space R_Z in this case as the space consisting of matrices

$$\begin{bmatrix} z_{11} & & & z_{1,r-2m+1} & * & \cdots & * & z_{1,r} \\ & \ddots & & \vdots & \vdots & \cdots & \vdots & \vdots \\ & & z_{r-2m,r-2m} & z_{r-2m,r-2m+1} & * & \cdots & * & * \\ z_{1,r-2m+1} & \cdots & z_{r-2m,r-2m+1} & z_{r-2m+1,r-2m+1} & * & \cdots & * & * \\ * & \cdots & * & * & * & \cdots & * & * \\ \vdots & \cdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ * & \cdots & * & * & * & \cdots & z_{r-1,r-1} & * \\ z_{1,r} & \cdots & * & * & * & \cdots & * & z_{r,r} \end{bmatrix}.$$

It follows that dim $R_Z = r - 2m + (r-2m)2m + \frac{(2m+1)2m}{2} = (2m+1)(r-m)$.

Next, we study the invariant measure on our chosen orbit space representatives. We will first work on R_X. Denote by U_0 the stabilizer in $U_M \simeq U_{\mathrm{GL}_r} \times U_{\mathrm{Sp}_{2m}}$, which is independent of the base point as can be observed from our previous discussion. The action is given by

$$(U_{\mathrm{GL}_r} \times U_{\mathrm{Sp}_{2m}})/U_0 \times R_X \longrightarrow \mathrm{Mat}_{r \times 2m}$$

$$((u_1, u_2), X_0) \longmapsto u_1 X_0 u_2^{-1}$$

Without loss of generality we assume $m > 2$. Write $u_1 = \begin{bmatrix} v_1 & {}^t\delta_1 \\ & 1 \end{bmatrix}$, $u_2 = \begin{bmatrix} 1 & \delta_2 \\ & v_2' \end{bmatrix} = \begin{bmatrix} 1 & \gamma_2 & x \\ & v_2 & {}^t\gamma_2' \\ & & 1 \end{bmatrix}$, $X_0 = \begin{bmatrix} 0 & X_1' \\ x_{r,1} & 0 \end{bmatrix} = \begin{bmatrix} 0 & X_1 & {}^t\gamma \\ x_{r,1} & 0 & 0 \end{bmatrix}$. By the previous argument, $\gamma_2' = -\gamma_2 v_2^{-1} J_{2m-2}'$. Then

$$u_1 X_0 u_2^{-1} = \begin{bmatrix} {}^t\delta_1 x_{r,1} & -x_{r,1}{}^t\delta_1\delta_2(v_2')^{-1} + v_1 X_1'(v_2')^{-1} \\ x_{r,1} & -x_{r,1}\delta_2(v_2')^{-1} \end{bmatrix}$$

$$= \begin{bmatrix} g^t\delta_1 x_{r,1} & -x_{r,1}{}^t\delta_1\gamma_2 v_2^{-1} + v_1 X_1 v_2^{-1} & x_{r,1}{}^t\delta_1\gamma_2 J_{2m-2}'{}^t\gamma_2 - x_{r,1}{}^t\delta_1 x - v_1 X_1 J_{2m-2}'{}^t\gamma_2 + v_1{}^t\gamma \\ x_{r,1} & -x_{r,1}\gamma_2 v_2^{-1} & x_{r,1}\gamma_2 J_{2m-2}'{}^t\gamma_2 - x_{r,1}x \end{bmatrix}.$$

We write this map as

$$(\delta_1, \gamma_2, x, x_{r,1}, \gamma, v_1, v_2, X_1)$$
$$\longmapsto \big(x_{r,1}\delta_1, -x_{r,1}\gamma_2 v_2^{-1}, x_{r,1}\gamma_2 J_{2m-2}'{}^t\gamma_2 - x_{r,1}x, x_{r,1},$$
$$x_{r,1}{}^t\delta_1\gamma_2 J_{2m-2}'{}^t\gamma_2 - x_{r,1}{}^t\delta_1 x - v_1 X_1 J_{2m-2}'{}^t\gamma_2 + v_1{}^t\gamma,$$
$$- x_{r,1}{}^t\delta_1\gamma_2 v_2^{-1} + v_1 X_1 v_2^{-1}\big).$$

The Jacobian matrix of this map is given by

$$\begin{bmatrix} x_{r,1}I_{r-1} & 0 & 0 & \delta_1 & 0 & 0 & 0 & 0 \\ 0 & -x_{r,1}v_2^{-1} & 0 & \gamma_2 v_2^{-1} & 0 & 0 & \frac{\partial(-x_{r,1}\gamma_2 v_2^{-1})}{\partial v_2} & 0 \\ 0 & * & -x_{r,1} & * & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ * & * & * & * & \frac{\partial(v_1{}^t\gamma)}{\partial\gamma} & * & 0 & * \\ * & * & 0 & * & 0 & \frac{\partial(v_1 X_1 v_2^{-1})}{\partial v_1} & \frac{\partial(-x_{r,1}{}^t\delta_1\gamma_2 v_2^{-1})}{\partial v_2} + \frac{\partial(v_1 X_1 v_2^{-1})}{\partial v_2} & \frac{\partial(v_1 X_1 v_2^{-1})}{\partial X_1} \end{bmatrix}.$$

Multiplying the second row by $-\delta_1$ and adding it to the last row, one computes that the determinant of the Jacobian matrix is equal to

$$|x_{r_1}|^{r-2m-2} \cdot \left|\frac{\partial(v_1{}^t\gamma)}{\partial\gamma}\right| \cdot \left|\det\begin{bmatrix} \frac{\partial(v_1 X_1 v_2^{-1})}{\partial v_1} & \frac{\partial(v_1 X_1 v_2^{-1})}{\partial v_2} & \frac{\partial(v_1 X_1 v_2^{-1})}{\partial X_1} \end{bmatrix}\right|$$

From the form of R_X we observe that

$$\left|\frac{\partial(v_1{}^t\gamma)}{\partial\gamma}\right| = \begin{cases} 1, & \text{if } r \leq m, \\ |x_{r-2m+1,2m}|^{r-2m}, & \text{if } r \geq 2m. \end{cases}$$

And the last term $\left|\det\begin{bmatrix} \frac{\partial(v_1 X_1 v_2^{-1})}{\partial v_1} & \frac{\partial(v_1 X_1 v_2^{-1})}{\partial v_2} & \frac{\partial(v_1 X_1 v_2^{-1})}{\partial X_1} \end{bmatrix}\right|$ gives the Jacobian of the same type of action but with rank drop by 1. Proceed by induction we obtain the invariant measure on R_X. When $r < m < 2m$, $\frac{\partial(v_1{}^t\gamma)}{\partial\gamma} = 1$ until our induction procedure goes to the $(2m-r+1)$-th step, where $\gamma = (x_{2,r-1}, *, \ldots, *)$, in which case we have $\left|\frac{\partial(v_1{}^t\gamma)}{\partial\gamma}\right| = |x_{2,r-1}|$. Then the induction continues as case (2). Consequently, we obtain the invariant

measure $\mathrm{d}\mu_X$ on R_X as

$$\mathrm{d}\mu_X = \begin{cases} |x_{r,1}^{r+2m-2}x_{r-1,2}^{r+2m-5}x_{r-2,3}^{r+2m-8}\ldots x_{1,r}^{2m-2r+1}|\prod \mathrm{d}x_{ij}, & \text{if } r \leq m, \\[2mm] |x_{r,1}^{r+2m-2}x_{r-1,2}^{r+2m-5}x_{r-2,3}^{r+2m-8}\ldots x_{r-m+1,m}^{r-m+1}x_{r-m,m+1}^{r-m-1} \\ \qquad x_{r-m-1,m+2}^{r-m-2}\ldots x_{2,r-1}|\prod \mathrm{d}x_{ij}, & \text{if } m < r < 2m, \\[2mm] |x_{r,1}^{r+2m-2}x_{r-1,2}^{r+2m-5}x_{r-2,3}^{r+2m-8}\ldots x_{r-m+1,m}^{r-m+1}x_{r-m,m+1}^{r-m-1} \\ \qquad x_{r-m-1,m+2}^{r-m-2}\ldots x_{r-2m+1,2m}^{r-2m}|\prod \mathrm{d}x_{ij}, & \text{if } r \geq 2m. \end{cases}$$

where the product runs over all (i,j) such that $x_{i,j} \neq 0$.

Recall that when $r \geq 2m$, we will also need to consider the action of $U_{\mathrm{GL}_{r-2m}}$ on Sym^r. The corresponding invariant measure on R_Z can also be proved inductively. Write $u_1 = \begin{bmatrix} u_1' & w \\ & u_1'' \end{bmatrix}$, where $u_1' \in U_{\mathrm{GL}_{r-2m}}$, $u_1'' \in U_{\mathrm{GL}_{2m}}$, $w \in \mathrm{Mat}_{(r-2m)\times 2m}$, and $Z_0 = \begin{bmatrix} Z_1 & h \\ {}^t h & Z_2 \end{bmatrix}$ with

$$Z_1 = \mathrm{diag}\{z_{1,1}, \ldots, z_{r-2m,r-2m}\},$$

$Z_2 \in \mathrm{Sym}^{2m}$, $h \in \mathrm{Mat}_{(r-2m)\times 2m}$. As we see from the proof of R_Z above, it suffices to only consider the action $(U_1', Z_1) \mapsto u_1' Z_1 {}^t u_1'$. Without loss of generality, we assume $r - 2m > 0$. Then we can write $u_1' = \begin{bmatrix} \tilde{u}_1 & \alpha_1 \\ & 1 \end{bmatrix}$, with $\tilde{u}_1 \in U_{\mathrm{GL}_{r-2m-1}}$, $\alpha_1 \in \mathbb{A}^{r-2m-1}$, and $Z_1 = \begin{bmatrix} Z_1' & \\ & z_{r-2m,r-2m} \end{bmatrix}$ where $Z_1' = \mathrm{diag}\{z_{1,1}, \ldots, z_{r-2m-1,r-2m-1}\}$. Then the map $(u_1', Z_1) \mapsto u_1' Z_1 {}^t u_1'$ can be written as

$$(z_{r-2m,r-2m}, \alpha_1, \tilde{u}_1, Z_1')$$
$$\longmapsto (z_{r-2m,r-2m}, z_{r-2m,r-2m}\alpha_1, \tilde{u}_1 Z_1' {}^t \tilde{u}_1 + {}^t\alpha_1 z_{r-2m,r-2m}\alpha_1),$$

whose Jacobian matrix if of the form

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & z_{r-2m,r-2m}I_{r-2m-1} & 0 & 0 \\ * & * & \frac{\partial(\tilde{u}_1 Z_1' {}^t \tilde{u}_1)}{\partial \tilde{u}_1} & \frac{\partial(\tilde{u}_1 Z_1' {}^t \tilde{u}_1)}{\partial Z_1'} \end{bmatrix}.$$

So the absolute value of its determinant is equal to

$$|x_{r-2m,r-2m}^{r-2m-1}| \cdot \left|\det\begin{bmatrix} \frac{\partial(\tilde{u}_1 Z_1' {}^t \tilde{u}_1)}{\partial \tilde{u}_1} & \frac{\partial(\tilde{u}_1 Z_1' {}^t \tilde{u}_1)}{\partial Z_1'} \end{bmatrix}\right|$$

where the second term is the absolute value of the Jacobian matrix of the same type of action with rank drop by 1. Hence we can proceed by induction. As a result, we obtain the invariant measure $\mathrm{d}\mu_Z$ on R_Z in this case given by

$$\mathrm{d}\mu_Z = |z_{r-2m,r-2m}^{r-2m-1}z_{r-2m-1,r-2m-1}^{r-2m-2}\ldots z_{2,2}|\prod \mathrm{d}z_{ij}$$

where the product runs over all (i,j) such that $z_{i,j} \neq 0$. For the other two cases, $\mathrm{d}\mu_Z = \prod_{i,j} \mathrm{d}z_{i,j}$ where the product runs over all (i,j).

Finally, since $Z = J_r{}^t Y + \frac{(-1)^r X J'_{2m}{}^t X}{2}$, and ${}^t Z = Z$, we see that

$$Y = {}^t\left(J_r^{-1}\left(Z - \frac{(-1)^r X J'_{2m}{}^t X}{2} \right)\right)$$

$$= {}^t Z J_r + \frac{(-1)^r X J'_{2m}{}^t X J_r}{2} = Z J_r + \frac{X \theta_{r,m}(X)}{2}.$$

Let $R_Y = \{Z J_r + \frac{X \theta_{r,m}(X)}{2} : (X, Z) \in R_X \times R_Z\}$. Since the map $(X, Y) \mapsto (X, Z)$ is bijective and U_M-equivariant, $R_X \times R_Y$ gives a set of orbit space representatives for the action $(X, Y) \mapsto (u_1 X u_2^{-1}, u_1 Y \theta_r(u_1^{-1}))$. If we denote the corresponding measure on R_Y as $\mathrm{d}\mu_Y$, since $\theta_{r,m}$ is linear, we have

$$\mathrm{d}\mu_Y = \mathrm{d}\mu_Z \cdot J_r + \frac{\mathrm{d}\mu_X \cdot \theta_{r,m}(X) + X \cdot \theta_{r,m}(\mathrm{d}\mu_X)}{2}$$

Therefore

$$\mathrm{d}\mu_X \wedge \mathrm{d}\mu_Y = \mathrm{d}\mu_X \wedge (\mathrm{d}\mu_Z \cdot J_r).$$

This shows that we replace integrals over $R_X \times R_Y$ by the ones over $R_X \times R_Z$, without changing the corresponding invariant Haar measures. $\square$

We will apply Theorem 6.2 of [11] which represents the local coefficient as the Mellin transform of certain partial Bessel functions, the following lemma is to show the validity of assumption 5.1 in [11]:

Lemma 4.2. *There exists an injection $\alpha^\vee : F^\times \to Z_G \backslash Z_M$, s.t., $\alpha(\alpha^\vee(t)) = t$, for $\forall\, t \in F^\times$, where $\alpha = \alpha_r$.*

Proof. Since $Z_M = \bigcap_{i \neq r} \ker(\alpha_i) = \{\mathrm{diag}(\underbrace{t, \ldots, t}_{r}, \pm 1, \ldots, \pm 1, \underbrace{t^{-1}, \ldots, t^{-1}}_{r})\}$,

and $Z_G = \bigcap_{i=1}^n \ker(\alpha_i) = \{\pm 1\}$, the quotient $Z_G \backslash Z_M$ can be identified as $\{\mathrm{diag}(\underbrace{t, \ldots, t}_{r}, 1, \ldots, 1, \underbrace{t^{-1}, \ldots, t^{-1}}_{r}) : t \in \mathbb{G}_m\}$. Define

$$\alpha^\vee(t) = \mathrm{diag}(\underbrace{t, \ldots, t}_{r}, 1, \ldots, 1, \underbrace{t^{-1}, \ldots, t^{-1}}_{r}),$$

then clearly we have $\alpha(\alpha^\vee(t)) = t$ for $\forall\, t \in F^\times$. $\square$

Let Z_M^0 denote the image of $\alpha^\vee$. Notice that Z_M^0 is the identity component of the center Z_M of M. As we will see later in Section 5, the test functions in the local coefficient formula are compactly supported modulo Z_M, so it is necessary to consider the action of Z_M^0 on $U_M \backslash N$. As a result, the local coefficient is an integral of certain partial Bessel function over the space $Z_M^0 U_M \backslash N$. The corresponding orbit space representatives and invariant measures can be given by the following proposition:

Proposition 4.3. *The action of Z_M^0 on $R = R_X \times R_Z = U_M \backslash N$ admits a set of orbit space representatives of the form $R_{X'} \times R_{Z'} := \{(X', Z') \in R_X \times R_Z | x'_{r,1} = 1\}$ where we use the notation $X' = (x'_{i,j})$ in the definition. The corresponding invariant measure is given by $\mathrm{d}\mu' = \mathrm{d}\mu_{X'} \wedge \mathrm{d}\mu_{Z'}$, where*

$$
\mathrm{d}\mu_{X'} =
\begin{cases}
|x'^{r+2m-5}_{r-1,2} x'^{r+2m-8}_{r-2,3} \dots x'^{2m-2r+1}_{1,r}| \prod \mathrm{d}x'_{ij}, & \text{if } r \leq m, \\[2ex]
|x'^{r+2m-5}_{r-1,2} x'^{r+2m-8}_{r-2,3} \dots x'^{r-m+1}_{r-m+1,m} x'^{r-m-1}_{r-m,m+1} \\
\qquad x'^{r-m-2}_{r-m-1,m+2} \dots x'_{2,r-1}| \prod \mathrm{d}x'_{ij}, & \text{if } m < r < 2m, \\[2ex]
|x'^{r+2m-5}_{r-1,2} x'^{r+2m-8}_{r-2,3} \dots x'^{r-m+1}_{r-m+1,m} x'^{r-m-1}_{r-m,m+1} \\
\qquad x'^{r-m-2}_{r-m-1,m+2} \dots x'^{r-2m}_{r-2m+1,2m}| \prod \mathrm{d}x'_{ij}. & \text{if } r \geq 2m.
\end{cases}
$$

and

$$
\mathrm{d}\mu_{Z'} =
\begin{cases}
\prod_{i,j} \mathrm{d}z'_{i,j} & \text{if } r \leq 2m, \\[1.5ex]
|z'^{r-2m-1}_{r-2m,r-2m} z'^{r-2m-2}_{r-2m-1,r-2m-1} \dots z'_{2,2}| \prod \mathrm{d}z'_{ij} & \text{if } r > 2m.
\end{cases}
$$

Proof. For $z = \alpha^\vee(t)$, we have $zn(X,Y)z^{-1} = n(tX, t^2Y)$ for $(X,Y) \in R_X \times R_Y$. This is equivalent to say that the action of Z_M^0 on $U_M \backslash N' \simeq R_X \times R_Z$ is given by

$$
(t, (X, Z)) \longmapsto (tX, t^2 Z), \quad (X, Z) \in R_X \times R_Z.
$$

From this we identify $Z_M^0 U_M \backslash N$ with $R_{X'} \times R_{Z'}$ where X' is given by setting $x_{r,1} = 1$ in R_X, and Z' is of the same form with Z. We will construct the invariant measure $\mathrm{d}\mu'$ on the orbit space $R' := Z_M^0 U_M \backslash N' \simeq F^\times \backslash (R_X \times R_Z) = R_{X'} \times R_{Z'}$ such that it is compatible with our invariant measure $\mathrm{d}\mu = \mathrm{d}\mu_X \wedge \mathrm{d}\mu_Z$, in the sense that

$$
\int_{R'(F)} \int_{F^\times} f(tX', t^2 Z') q^{\langle 2\rho, H_M(\alpha^\vee(t)) \rangle} \frac{\mathrm{d}t}{|t|} \mathrm{d}\mu' = \int_{R(F)} f(X, Z) \mathrm{d}\mu_X \wedge \mathrm{d}\mu_Z
$$

for any integrable function f on $R = R_X \times R_Z$, where ρ is half sum of the positive roots in N. Observe that $\mathrm{d}x_{i,j} = |t| \mathrm{d}x'_{i,j}$, $\mathrm{d}z_{i,j} = |t|^2 \mathrm{d}z_{i,j}$, and

$$
2\rho = \sum_{1 \leq i \leq r, r+1 \leq j \leq n} (e_i \pm e_j) + \sum_{1 \leq i \leq j \leq r} (e_i + e_j)
$$

$$
= (2m+2) \sum_{i=1}^r e_i + \sum_{1 \leq i < j \leq r} (e_i + e_j) = (2m + r + 1) \sum_i^r e_i
$$

Hence $q^{\langle 2\rho, H_M(\alpha^\vee(t)) \rangle} = |t|^{r(2m+r+1)}$. The above observation implies that the measures on $R_{X'}$ and $R_{Z'}$ must be of the form $\mathrm{d}\mu_{X'} = \prod |x'_{ij}|^{k_{ij}} \mathrm{d}x'_{ij}$, and

$\mathrm{d}\mu_{Z'} = \prod |z'_{st}|^{l_{st}}\mathrm{d}z'_{st}$. Therefore

$$\int_{R'(F)}\int_{F^\times} f(tX',t^2Z')q^{\langle 2\rho,H_M(\alpha^\vee(t))\rangle}\frac{\mathrm{d}t}{|t|}\mathrm{d}\mu'$$

$$= \int_{F^\times \times R'(F)} f(tX',t^2Z')|t|^{r(2m+r+1)}\frac{\mathrm{d}t}{|t|}(\mathrm{d}\mu_{X'}\wedge\mathrm{d}\mu_{Z'})$$

$$= \int_{R(F)} f(X,Z)(|x_{r,1}|^{r(2m+r+1)-1}$$

$$\cdot \prod_{(i,j)\neq(r,1)} |x_{r,1}|^{-k_{ij}-1}|x_{ij}|^{k_{ij}}\mathrm{d}x_{ij})\wedge\left(\prod_{(s,t)}|x_{r,1}|^{-l_{st}-2}|z_{st}|^{l_{st}}\mathrm{d}z_{st}\right).$$

Compare this with the formulas for R_X and R_Z we obtained, if forces that all the powers of $x'_{i,j}$ and $z'_{i,j}$ remain the same as those of $x_{i,j}$ and $z_{i,j}$ respectively, except $x_{r,1}$, i.e. the invariant measure on $Z^0_M U_M\backslash N$ can be given as $\mathrm{d}\mu' = \mathrm{d}\mu_{X'}\wedge\mathrm{d}\mu_{Z'}$, where

$$\mathrm{d}\mu_{X'} = \begin{cases} |x'^{r+2m-5}_{r-1,2}x'^{r+2m-8}_{r-2,3}\cdots x'^{2m-2r+1}_{1,r}|\prod \mathrm{d}x'_{ij}, & \text{if } r\leq m, \\[2mm] |x'^{r+2m-5}_{r-1,2}x'^{r+2m-8}_{r-2,3}\cdots x'^{r-m+1}_{r-m+1,m}x'^{r-m-1}_{r-m,m+1} \\ \quad x'^{r-m-2}_{r-m-1,m+2}\cdots x'_{2,r-1}|\prod \mathrm{d}x'_{ij}, & \text{if } m<r<2m, \\[2mm] |x'^{r+2m-5}_{r-1,2}x'^{r+2m-8}_{r-2,3}\cdots x'^{r-m+1}_{r-m+1,m}x'^{r-m-1}_{r-m,m+1} \\ \quad x'^{r-m-2}_{r-m-1,m+2}\cdots x'^{r-2m}_{r-2m+1,2m}|\prod \mathrm{d}x'_{ij}. & \text{if } r\geq 2m. \end{cases}$$

and

$$\mathrm{d}\mu_{Z'} = \begin{cases} \prod_{i,j}\mathrm{d}z'_{i,j} & \text{if } r\leq 2m, \\ |z'^{r-2m-1}_{r-2m,r-2m}z'^{r-2m-2}_{r-2m-1,r-2m-1}\cdots z'_{2,2}|\prod \mathrm{d}z'_{ij} & \text{if } r>2m. \end{cases}$$

Finally, to match the power of $x_{r,1}$ in the expression of $\mathrm{d}\mu_X$, one simply verifies that the following identities hold in each cases:

If $r\leq m$, then

$$r(r+2m+1)-1-\sum_{k=1}^{r-1}((r+2m-2-3k)+1)-\sum_{k=1}^{r-1}k-2\sum_{k=1}^{r}k$$

$$= r+2m-2;$$

If $m < r < 2m$, then

$$r(r+2m+1) - 1 - \sum_{k=1}^{m-1}((r+2m-2-3k)+1) - \sum_{k=1}^{r-m-1}((r-m-k)+1)$$

$$- ((r-m)^2 - (r-m-1)) - \frac{(3r-2m-1)(2m-r)}{2} - 2\sum_{k=1}^{r}k$$

$$= r + 2m - 2;$$

If $r \geq 2m$, then

$$r(r+2m+1) - 1 - \sum_{k=1}^{m-1}((r+2m-2-3k)+1)$$

$$- \sum_{k=1}^{m}((r-m-k)+1) - m(m-1) - 2\sum_{k=1}^{r-2m}((k-1)+2)$$

$$- 2\left(\frac{r(r+1)}{2} - \frac{(r-2m)(r-2m-1)}{2}\right) = r + 2m - 2. \quad \square$$

5. Local coefficient as Mellin transforms of partial Bessel functions

In this section we apply Theorem 6.2 of [11] to represent the local coefficient in our case as the Mellin transform of certain partial Bessel integrals. Recall that given irreducible ψ-generic representations σ and τ of $\mathrm{GL}_r(F)$ and $\mathrm{Sp}_{2m}(F)$ respectively, then $\pi = \sigma \otimes \tau$ is a ψ-generic representation of $M(F) \simeq \mathrm{GL}_r(F) \times \mathrm{Sp}_{2m}(F)$. The corresponding local coefficient is a product of two γ-factors:

$$C_\psi(s,\pi) = \gamma(s, \sigma \times \tau, \psi)\gamma(s, \sigma, \wedge^2, \psi)$$

As the stability of $\gamma(s, \sigma, \wedge^2, \psi)$ is proved in [6], the stability of $\gamma(s, \sigma \times \tau, \psi)$ is equivalent to the stability of the local coefficient $C_\psi(s,\pi)$. Let $n \in N(F)$ such that the Bruhat decomposition $\dot{w}_0^{-1}n = mn'\bar{n}$ holds. Denote by

$$U_{M,n} := \{u \in U_M : unu^{-1} = n\},$$

$$U'_{M,m} := \{u \in U_M : mum^{-1} \in U_M \text{ and } \psi(mum^{-1}) = \psi(u)\}.$$

By [16], except for a set of measure zero on $N(F)$, we have $U_{M,n} = U'_{M,m}$. Together with Lemma 4.2, this implies that the assumptions for Theorem 6.2 in [11] are satisfied. Bessel functions are defined on Bruhat cells. Suppose in the decomposition $\dot{w}_0^{-1}n = mn'\bar{n}$, m lies in some Bruhat cell $C_M(w) = B_M w B_M$, where $B_M = B \cap M$. We define the Bessel function associated to π and w as

$$j_{\pi,w}(m) = \int_{U_M} W(mu)\psi^{-1}(u)\mathrm{d}u = \int_{U_M} W(mu^{-1})\psi(u)\mathrm{d}u$$

where $W \in W(\pi, \psi)$ is a Whittaker model defined by a Whittaker functional λ on π as $W(m) = \lambda(\pi(m)v)$ for certain $v \in \pi$, normalized so that $W(e) = 1$, where e is the identity element of $M(F)$. Then it is immediate that

$$j_{\pi,w}(u_1 m u_2) = \psi(u_1 u_2) j_{\pi,w}(m)$$

for any $u_1, u_2 \in U_M(F)$. It follows by the Bruhat decomposition that the Bessel functions are essentially functions on wT, where T is the maximal (split) torus of G. We refer the readers to [4] for partial Bessel funcitons for quasi-split groups.

However, the local coefficient is represented by an integral of certain partial Bessel function, not the Bessel function itself. To define that, we need an exhaustive sequence of open compact subgroups $\overline{N}_{0,\kappa} \subset \overline{N}(F)$ such that $\alpha^\vee(t) \overline{N}_{0,\kappa} \alpha^\vee(t)^{-1}$ depends only on $|t|$. For a matrix X of size $l \times k$, let

$$\varphi_\kappa(X) = \begin{cases} 1 & |X_{i,j}| \leq q^{((k-i)+(l-j)+1)\kappa}, \\ 0 & \text{otherwise.} \end{cases}$$

Set

$$\overline{N}_{0,\kappa} = \{\overline{n}(X,Y) : \varphi_\kappa(\varpi^{-(d+g)}X) \cdot \varphi_\kappa(\varpi^{-2(d+g)}Y) = 1\}$$

where $d = \mathrm{Cond}(\psi)$ denotes the conductor of ψ and $g = \mathrm{Cond}(\omega_\pi^{-1}(w_0(\omega_\pi)))$. As we have seen before that in our case $\alpha^\vee(t)\overline{n}(X,Y)\alpha^\vee(t)^{-1} = \overline{n}(tX, t^2Y)$, it follows that our definition of $\overline{N}_{0,\kappa}$ makes $\alpha^\vee(t)\overline{N}_{0,\kappa}\alpha^\vee(t)^{-1}$ depends only on $|t|$. Denote by $\varphi_{\overline{N}_{0,\kappa}}$ the characteristic function of $\overline{N}_{0,\kappa}$. Then the partial Bessel function is a function on $M(F) \times Z_M^0(F)$ defined as

$$j_{\overline{N}_{0,\kappa},\pi,w}(m,z) := \int_{U_{M,n} \backslash U_M} W(mu)\varphi_{\overline{N}_{0,\kappa}}(zu^{-1}\overline{n}uz^{-1})\psi^{-1}(u)\mathrm{d}u.$$

Given that $\dot{w}_0^{-1}n = mn'\overline{n}$, let

$$j_{\overline{N}_{0,\kappa}\pi,w}(n) := j_{\overline{N}_{0,\kappa},\pi,w}(m, \alpha^\vee(\varpi^{d+g}u_{\alpha_r}(\dot{w}_0\overline{n}\dot{w}_0^{-1}))).$$

Suppose σ and τ are irreducible ψ-generic representations of $\mathrm{GL}_r(F)$ and $\mathrm{Sp}_{2m}(F)$ respectively, and $L = \prod_{k=1}^d \mathrm{GL}_{n_k} \times \prod_{j=1}^l \mathrm{GL}_{t_j} \times \mathrm{Sp}_{2m'}$ with $\sum_{k=1}^d n_k = r$ and $\sum_{j=1}^l n_j + m' = m$. Then we can find ψ-generic supercuspidal representations σ_k', $(1 \leq k \leq d)$, σ_j'', $(1 \leq j \leq l)$, and τ' of $\mathrm{GL}_{n_k}(F)$, $\mathrm{GL}_{t_j}(F)$, and $\mathrm{Sp}_{2m'}(F)$ respectively, so that $\pi = \sigma \boxtimes \tau \hookrightarrow \mathrm{Ind}_{L(F)N(F)}^{M(F)} \sigma' \otimes \mathbf{1}_{N(F)}$, where $\sigma' = \bigotimes_{k=1}^d \sigma_k' \otimes \bigotimes_{j=1}^l \sigma_j'' \otimes \tau'$. By the multiplicativity of Langlands–Shahidi local γ-factors in our cases, see Theorem 3.1 of [10] or Theorem 8.3.2 of [12], we obtain that

$$\gamma(s, \sigma \times \tau, \psi) = \prod_{k=1}^d \prod_{j=1}^l \gamma(s, \sigma_k' \times \sigma_j'', \psi)\gamma(s, \sigma_k \times \widetilde{\sigma}_j'', \psi) \prod_{k=1}^d \gamma(s, \sigma_k' \times \tau', \psi)$$

where $\widetilde{\sigma}_j''$ is the contragredient of σ_j''. Consequently, it suffices to prove stability for ψ-generic supercuspidal representations. Suppose σ and τ are ψ-generic supercuspidal representations of $\mathrm{GL}_n(F)$ and $\mathrm{Sp}_{2m}(F)$ respectively. Set $\pi = \sigma \boxtimes \tau$, then its central character $\omega_\pi = \omega_\sigma \boxtimes \omega_\tau$. In this case, one can find a function $f \in C_c^\infty(M(F); \omega_\pi)$, the space of smooth functions of compact support modulus the center on $M(F)$, such that $f(zm) = \omega_\pi(m)f(z)$ for all $z \in Z_M(F)$, and $m \in M(F)$. Then $W(m) = W_f(m) := \int_{U_M(F)} f(xm)\psi^{-1}(x)\mathrm{d}x$ defines a non-zero Whittaker model. We normalize it so that $W_f(1) = 1$. For simplicity, we omit w and denote $j_{\pi,\kappa,f}(n) := j_{\overline{N}_{0,\kappa},\pi,w}(n)$ where we replace W by W_f. Let $\alpha = \alpha_r$ and $\widetilde{\alpha} = \langle \rho, \alpha \rangle^{-1}\rho$, where ρ is half sum of the positive roots in N. Then by Theorem 6.2 of [11], we have

Proposition 5.1. *Let σ and τ be ψ-generic supercuspidal representations of $\mathrm{GL}_r(F)$ and $\mathrm{Sp}_{2m}(F)$ respectively, set $\pi = \sigma \boxtimes \tau$, then*

$$C_\psi(s,\pi)^{-1} = \gamma(2\langle \widetilde{\alpha}, \alpha^\vee \rangle s, \omega_\pi(\dot{w}_0 \omega_\pi^{-1}), \psi^{-1})$$
$$\int_{Z_M^0(F)U_M(F)\backslash N(F)} j_{\pi,\kappa,f}(\dot{n})\omega_{\pi_s}^{-1}(\dot{w}_0 \omega_{\pi_s})(x_\alpha)q_F^{\langle s\widetilde{\alpha}+\rho, H_M(m)\rangle}\mathrm{d}\dot{n}$$

for sufficiently large κ, where m is the image of $\dot{n} \mapsto m$ with $\dot{n}$ a representative of the orbit of $n \in Z_M^0(F)U_M(F)\backslash N(F)$ in $N(F)$ via $\dot{w}_0^{-1}\dot{n} = mn'\overline{n}$, which holds off a subset of measure zero on $N(F)$. Here $\mathrm{d}\dot{n}$ is the invariant measure on the orbit space $Z_M^0(F)U_M(F)\backslash N(F)$, $\pi_s = \pi \otimes q^{\langle s\widetilde{\alpha}, H_M(\cdot)\rangle}$, $x_\alpha = u_{\alpha_r}(\dot{w}_0\overline{n}\dot{w}_0^{-1})$, and $\gamma(2\langle \widetilde{\alpha}, \alpha^\vee \rangle s, \omega_\pi(\dot{w}_0\omega_\pi^{-1}), \psi^{-1})$ is the Abelian γ-factor depending only on ω_π.

Let us simplify this formula in our cases. First, recall that we have $2\rho = (r + 2m + 1)\sum_{i=1}^r e_i$, $\alpha = \alpha_r = e_r - e_{r+1}$. So

$$\langle \rho, \alpha \rangle = \frac{2(\rho, \alpha)}{(\alpha, \alpha)} = \frac{((r + 2m + 1)\sum_{i=1}^r e_i, e_r - e_{r+1})}{(e_r - e_{r+1}, e_r - e_{r+1})} = \frac{r + 2m + 1}{2}$$

and therefore $\widetilde{\alpha} = \langle \rho, \alpha \rangle^{-1}\rho = \sum_{i=1}^r e_i$. It follows that

$$\langle \widetilde{\alpha}, \alpha^\vee \rangle = \left\langle \sum_{i=1}^r e_i, \sum_{i=1}^r e_i^* \right\rangle = r,$$

and if we write $m = \mathrm{diag}\{m_1, m_2, \theta_r(m_1)\}$, then

$$q^{\langle s\widetilde{\alpha}, H_M(m)\rangle} = |\det(m_1)|^s, \quad q^{\langle s\widetilde{\alpha}+\rho, H_M(m)\rangle} = |\det(m_1)|^{s + \frac{r+2m+1}{2}}.$$

Moreover,

$$\omega_\pi(\dot{w}_0\omega_\pi^{-1})(\alpha^\vee(t)) = \omega_\pi(\alpha^\vee(t))\omega_\pi^{-1}(\dot{w}_0^{-1}\alpha^\vee(t)\dot{w}_0)$$

$$= \omega_\pi(\mathrm{diag}(\underbrace{t,\ldots,t}_{r},1,\ldots,1,\underbrace{t^{-1},\ldots,t^{-1}}_{r}))$$

$$\cdot \omega_\pi^{-1}(\mathrm{diag}(\underbrace{t^{-1},\ldots,t^{-1}}_{r},1,\ldots,1,\underbrace{t,\ldots,t}_{r}))$$

$$= \omega_\pi^2(\alpha^\vee(t)).$$

Hence $\omega_\pi(\dot{w}_0\omega_\pi^{-1}) \circ \alpha^\vee = \omega_\pi^2 \circ \alpha^\vee$. Similarly,

$$\omega_{\pi_s}^{-1}(\dot{w}_0\omega_{\pi_s})(\alpha^\vee(t)) = \omega_{\pi_s}^{-2}(\alpha^\vee(t)) = \omega_\pi^{-2}(\alpha^\vee(t))|t|^{-rs}.$$

From our previous calculation of the Bruhat decomposition $\dot{w}_0^{-1}n = mn'\overline{n}$ with $n = n(X,Y)$, we see that

$$\dot{w}_0\overline{n}\dot{w}_0^{-1} = \begin{bmatrix} & & I_r \\ & I_{2m} & \\ (-1)^rI_r & & \end{bmatrix} \begin{bmatrix} I_r & & \\ J'_{2m}{}^tX^tY^{-1}J_r & I_{2m} & \\ Y^{-1} & Y^{-1}X & I_r \end{bmatrix} \begin{bmatrix} & & (-1)^rI_r \\ & I_{2m} & \\ I_r & & \end{bmatrix}$$

$$= \begin{bmatrix} I_r & Y^{-1}X & Y^{-1} \\ & I_{2m} & (-1)^rJ'_{2m}{}^tX^tYY^{-1}J_r \\ & & I_r \end{bmatrix} = n(Y^{-1}X,Y^{-1}).$$

Hence $u_{\alpha_r}(\dot{w}_0\overline{n}\dot{w}_0^{-1})$ is the lower-left entry of $Y^{-1}X$. When we take our orbit space representative

$$R = R_X \times R_Y \simeq R_X \times R_Z \simeq U_M\backslash N,$$

and $n = n(X,Y)$ with $(X,Y) \in R_X \times R_Y$, this is just $\frac{y^*_{rr}x_{r,1}}{\det Y}$, where y^*_{rr} is the (r,r)-th entry of the adjoint matrix of $Y = ZJ_r + \frac{X\theta_{r,m}(X)}{2}$, which is a polynomial function of (X,Z), we denote it as $P(X,Z)$. When passing to the orbit of $Z^0_M(F)$, we obtain that $u_{\alpha_r}(\dot{w}_0\overline{n}(X',Y')\dot{w}_0^{-1}) = \frac{P(X',Z')}{\det Y'}$.

From now on, we will also use $n(X,Z)$(resp. $m(X,Z)$) to denote $n(X,Y)$ (resp. $m(X,Y)$) if we emphasis that it is parameterized by (X,Z) instead of (X,Y). With the discussion of the orbit space structure of $Z^0_MU_M\backslash N$ and its invariant measure in Proposition 4.3, we obtain the following result:

Proposition 5.2. *Let σ and τ be ψ-generic supercuspidal representations of $\mathrm{GL}_r(F)$ and $\mathrm{Sp}_{2m}(F)$ respectively, set $\pi = \sigma \boxtimes \tau$, then*

$$C_\psi(s,\pi)^{-1} = \gamma(2rs,\omega_\pi^2,\psi^{-1}) \int_{R_{X'}\times R_{Z'}} j_{\pi,\kappa,f}(n(X',Z'))$$

$$\cdot \omega_\pi^{-2}\left(\frac{P(X',Z')}{\det(Z'J_r + \frac{X'\theta_{r,m}(X')}{2})}\right)\left|\frac{P(X',Z')}{\det(Z'J_r + \frac{X'\theta_{r,m}(X')}{2})}\right|^{-rs}$$

$$\cdot |\det(m_1(X',Z'))|^{s+\frac{r+2m+1}{2}}\,\mathrm{d}\mu_{X'} \wedge \mathrm{d}\mu_{Z'}$$

for sufficiently large κ, where $m(X', Z')$ is the image of $n(X', Z') \mapsto m(X', Z')$ via the Bruhat decomposition $\dot{w}_0^{-1}\dot{n} = mn'\bar{n}$, which holds off a subset of measure zero on $N(F)$, $m = \mathrm{diag}\{m_1, m_2, \theta_r(m_1)\}$ with $m_1 \in \mathrm{GL}_r(F)$, $m_2 \in \mathrm{Sp}_{2m}(F)$, and $\gamma(2rs, \omega_\pi^2, \psi^{-1})$ is the Abelian γ-factor depending only on ω_π.

To study the stability of local coefficient, we need to consider $C_\psi(s, \pi \otimes \chi)$ with χ a highly ramified character of $F^\times$, regarded as a character of $M(F) \simeq \mathrm{GL}_r(F) \times \mathrm{Sp}_{2m}(F)$ via $\chi(m_1, m_2) := \chi(\det(m_1))$ in our case. Therefore it is necessary to choose the open compact subgroups $\{\overline{N}_{0,\kappa}\}_\kappa$ of $\overline{N}(F)$ to be independent of χ. As in the proof of Theorem 6.2 of [11], given an irreducible ψ-generic representation π of $M(F)$ with ramified central character ω_π, we choose a section $h \in I(s, \pi) = \mathrm{Ind}_{P(F)}^{G(F)}(\pi \otimes q^{\langle s\tilde\alpha + \rho, H_M(\cdot)\rangle}) \otimes \mathbf{1}_{N(F)}$, such that h is compactly supported modulus $P(F)$, and use it to obtain the integral representation of the local coefficient formula as above. So we choose a sufficiently large open compact subgroup $\overline{N}_0$ of $\overline{N}(F)$ such that $\mathrm{Supp}(h) \subset P(F)\overline{N}_0$. In our situation, we fix a character χ_0 of $F^\times$ such that $\omega_{\pi \otimes \chi_0} = \omega_\pi \chi_0^r$ is ramified, and choose a κ_0 and $h_0 \in I(s, \pi)$ such that $\mathrm{Supp}(h_0) \subset P(F)\overline{N}_{0,\kappa_0}$. Since $\overline{N}_{0,\kappa} \subset \overline{N}_{0,\kappa_0}$ for all $\kappa \geq \kappa_0$, $\overline{N}_{0,\kappa}$ is independent of χ_0 if $\kappa \geq \kappa_0$. Suppose χ is any character of $F^\times$ such that $\omega_{\pi \otimes \chi}$ is ramified, choose $h_\chi \in I(s, \pi \otimes \chi)$ such that $\mathrm{Supp}(h_\chi) \subset P(F)\overline{N}_{0,\chi}$ for some open compact subgroup $\overline{N}_{0,\chi}$ of $\overline{N}(F)$. If $\overline{N}_{0,\chi} \subset \overline{N}_{0,\kappa_0}$, we are fine as we just discussed. If not, we replace h_χ by a right shift $R(\alpha^\vee(t))h_\chi : g \mapsto h_\chi(g\alpha^\vee(t))$, then $R(\alpha^\vee(t)h_\chi)$ is supported in $\alpha^\vee(t)\overline{N}_{0,\chi}\alpha^\vee(t)^{-1} \bmod P(F)$, as $\alpha^\vee(t) \in M(F)$. Since $\alpha^\vee(t)\bar{n}(X, Y)\alpha^\vee(t)^{-1} = \bar{n}(tX, t^2Y)$, we choose $|t|$ to be small enough so that $\alpha^\vee(t)\overline{N}_{0,\chi}\alpha^\vee(t)^{-1} \subset \overline{N}_{0,\kappa_0}$, then we are done. So we obtain the following stronger version of the above proposition:

Proposition 5.3. *Let σ and τ be ψ-generic supercuspidal representations of $\mathrm{GL}_r(F)$ and $\mathrm{Sp}_{2m}(F)$ respectively, such that ω_π is ramified where $\pi = \sigma \boxtimes \tau$, then there exists a κ_0, such that for all $\kappa \geq \kappa_0$ and all characters χ of $F^\times$ such that $\omega_{\pi \otimes \chi} = \omega_\pi \chi^r$ is ramified, we have*

$$C_\psi(s, \pi \otimes \chi)^{-1} = \gamma(2rs, \omega_\pi^2 \chi^{2r}, \psi^{-1}) \int_{R_{X'} \times R_{Z'}} j_{\pi \otimes \chi, \kappa, f}(n(X', Z'))(\omega_\pi^{-2}\chi^{-2r})$$

$$\cdot \left(\frac{P(X', Z')}{\det(Z'J_r + \frac{X'\theta_{r,m}(X')}{2})}\right) \left|\frac{P(X', Z')}{\det(Z'J_r + \frac{X'\theta_{r,m}(X')}{2})}\right|^{-rs}$$

$$\cdot |\det(m_1(X', Z'))|^{s + \frac{r+2m+1}{2}} \, \mathrm{d}\mu_{X'} \wedge \mathrm{d}\mu_{Z'}$$

where $m(X', Z')$ is the image of $n(X', Z') \mapsto m(X', Z')$ via the Bruhat decomposition $\dot{w}_0^{-1}\dot{n} = mn'\bar{n}$, which holds off a subset of measure zero on

$N(F)$, $m = \mathrm{diag}\{m_1, m_2, \theta_r(m_1)\}$ *with* $m_1 \in \mathrm{GL}_r(F)$, $m_2 \in \mathrm{Sp}_{2m}(F)$, *and* $\gamma(2rs, \omega_\pi^2 \chi^{2r}, \psi^{-1})$ *is the Abelian γ-factor depending only on ω_π and χ.*

6. Asymptotics of partial Bessel integrals

So far we obtained our local coefficient as the Mellin transform of certain partial Bessel functions. The partial Bessel functions appeared in our discussions can be reformulated using partial Bessel integrals, which admit nice asymptotic expansion formulas, as well as some uniform smooth properties. These properties are crucial for the proof of stability using our method. In this section, we discuss some general properties of partial Bessel integrals and obtain the asymptotic formula and uniform smoothness in our case. In particular, different from the known results using this method, we observe that the orbit space $Z_M^0 U_M \backslash N$ is no longer isomorphic to a torus, and we will separate its "toric"-part out, which plays the same role as the torus over which the integral representing the local coefficient is taken in the known cases. This is a new phenomenon and we believe it can be generalized in our future work.

6.1. Some general properties of partial Bessel integrals.
We begin by introducing partial Bessel integrals and some of its important properties. The structure and results in this subsection is the same as Section 6 of [15] but are under a more general setting, so we will reprove some of the important results. Let M be a connected reductive quasi-split group defined over a local field F. Fix a Borel subgroup $B_M = AU_M$, where A is the maximal (split) torus and U_M is the unipotent radical of B_M. Suppose π is a ψ-generic supercuspidal representation of $M(F)$ with central character ω_π. Take a matrix coefficient f of π, then $f \in C_c^\infty(M(F); \omega_\pi)$, the space of smooth functions of compact support on $M(F)$ modulo the center, such that $f(zm) = \omega_\pi(z)f(m)$ for all $z \in Z_M(F)$, $m \in M(F)$. Then the integral $W_f(m) = \int_{U_M(F)} f(xm)\psi^{-1}(x)\mathrm{d}x$ converges since $Z_M(F)U_M(F)m$ is closed in $M(F)$, and hence defines a non-zero Whittaker model attached to π. We normalize it so that $W_f(e) = 1$, where e is the identity element of $M(F)$.

From now on, we will not distinguish algebraic group and the group of its F-points unless it is necessary in order to simplify the notations, we hope this will not cause any troubles for the readers. Given an F-involution $\Theta_M : M \to M$, we define the partial integral as

$$B_\varphi^M(m, f) := \int_{U_{M,m}^{\Theta_M} \backslash U_M} W_f(mu)\varphi(\Theta_M(u^{-1})m'u)\psi^{-1}(xu)\mathrm{d}u$$

$$= \int_{U_{M,m}^{\Theta_M} \backslash U_M} \int_{U_M} f(xmu)\varphi(\Theta_M(u^{-1})m'u)\psi^{-1}(xu)\mathrm{d}x\mathrm{d}u$$

where $U_{M,m}^{\Theta_M} = \{m \in U_M : \Theta_M(u^{-1})mu = m\}$ is the twisted centralizer of m in U_M, and φ is certain characteristic function of a subset of $M(F)$. And m' is obtained from m by stripping off the center, i.e. $m = zm'$ with $z \in Z_M$. Similarly one can define partial Bessel integrals on any Levi subgroup L of M by an involution Θ_L of L.

Let Δ_M denote the set of simple roots in M, $W(M, A)$ the Weyl group, and $w_M \in W(M, A)$ the long Weyl group element. The following objects needed for our study of the asymptotic expansion of partial Bessel integrals.

B(M): Following [6], the subset of Weyl group elements that supports Bessel functions is given by $B(M) = \{w \in W(M, A) : \alpha \in \Delta_M$ s.t. $w\alpha > 0 \Rightarrow w\alpha \in \Delta_M\}$, there is a bijection

$$B(M) \longleftrightarrow \{L : \text{Levi of standard parabolic subgroups of } M\}$$

by $w \mapsto L = Z_M(\bigcap_{\alpha \in \theta_{M,w}^+} \ker \alpha)$, where $\theta_{M,w}^+ = \{\alpha \in \Delta_M : w\alpha > 0\}$, and conversely $L \mapsto w = w_M w_L^{-1}$.

$\boldsymbol{U_{M,w}^+, U_{M,w}^-}$**:** For each $w \in W(M, A)$, define $U_{M,w}^+ = \{u \in U_M : wuw^{-1} \in U_M\}$ and $U_{M,w}^- = \{u \in U_M : wuw^{-1} \in U_M^-\}$, where U_M^- is the opposite of U_M. Then $U_M = U_{M,w}^+ U_{M,w}^-$ and $U_{M,w}^+$ normalizes $U_{M,w}^-$. In particular, if $w \in B(M)$ with $\dot{w} = \dot{w}_M \dot{w}_L^{-1}$, then $U_{M,w}^+ = U_L := U_M \cap L, U_{M,w}^- = N_L$, the unipotent radical of the standard parabolic subgroup of M with Levi component L. In the extreme cases, if $w = w_L$, then $U_{M,w_L}^+ = N_L, U_{M,w_L}^- = U_L$. If $w = w_M$, then $U_{M,w_M}^+ = \{e\}, U_{M,w_M}^- = U_M$.

Bessel distance: For $w, w' \in B(M)$ with $w > w'$ define

$$d_B(w, w') = \max\{m : \exists\, w_i \in B(M) \text{ s.t. } w = w_m > w_{m-1} > \cdots > w_0 = w'\}.$$

Bruhat order: For $w, w' \in W(M, A)$, $w \leq w' \Longleftrightarrow C(w) \subset \overline{C(w')}$.

The relevant torus $\boldsymbol{A_w}$: For $w \in B(M)$, define $A_w = \{a \in A : a \in \bigcap_{\alpha \in \theta_{M,w}^+} \ker \alpha\}^\circ \subset A$. Note that it is also the connected center of $L_w = Z_M(\bigcap_{\alpha \in \theta_{M,w}^+} \ker \alpha)$.

The relevant Bruhat cell $\boldsymbol{C_r(\dot{w})}$: We call $C_r(\dot{w}) = U_M \dot{w} A_w U_{M,w}^-$ the relevant part of the Bruhat cell $C(w)$. Note that $C_r(\dot{w})$ depends on the choice of the representative $\dot{w}$ of w. We choose the representative $\dot{w}$ of w so that it is compatible with ψ in the sense that $\psi(\dot{w}u\dot{w}^{-1}) = \psi(u)$ for all $u \in U_{M,w}^+$, as in Proposition 5.1 of [6].

Transverse tori: For $w, w' \in B(M)$, let $L = L_w$ and $L' = L_{w'}$ be their associated Levi subgroups respectively. Suppose $w' \leq w$. Then $L \subset L'$ and $A_{w'} \supset A_w$. Let $A_w^{w'} = A_w \cap L'_{der} = Z_L \cap L'_{der}$. In particular $A_w^{w'} \cap A_{w'} = A_{w'}^{w'} = Z_{L'} \cap L'_{der}$ is finite and the subgroup

$A_w^{w'} A_{w'} \subset A_w$ is open and of finite index. So this decomposition is essentially a "transfer principal" for relevant tori.

Let $\{x_\alpha\}_{\alpha \in \Delta_M}$ be a set of 1-parameter subgroups, i.e. $x_\alpha : \mathbb{G}_a \simeq U_\alpha$, Then $(B_M, A, \{x_\alpha\}_{\alpha \in \Delta_M})$ defines an F-splitting of M. We make the following assumptions from now on:

(1) Θ_M preserves the F-splitting and is compatible with Levi subgroups and ψ. To be precise, if $L \subset M$ is a Levi subgroup, we require that $\tau_L^M \circ \Theta_M|_L = \Theta_L$, where $\tau_L^M = \mathrm{Int}((\dot{w}_L^M)^{-1})$. And Θ_L also preserves the F-splitting $\{B_L := B_M \cap L, A, \{x_\alpha\}_{\alpha \in \Delta_L}\}$. Here $\mathrm{Int}(w)(m) = wmw^{-1}$, and we denote $\dot{w}_L^M := \dot{w}_M \dot{w}_L^{-1}$. Consequently $\Theta_M(U_M) = U_M$, $\Theta_L(U_L) = U_L$. The representatives $\dot{w}_L^M$ are chosen to be compatible with the generic character ψ as in Proposition 5.1 of [6], i.e., $\psi(\tau_L^M(u)) = \psi(\dot{w}_L^M u(\dot{w}_L^M)^{-1}) = \psi(u)$ for all $u \in U_L(F)$. By Θ_M is compatible with ψ, we mean that $\psi(\Theta_M(u)) = \psi(u)$ for all $u \in U_M(F)$. As a result, if $u \in U_L(F)$, then $\psi(\Theta_L(u)) = \psi(\tau_L^M \circ \Theta_M(u)) = \psi(\Theta_M(u)) = \psi(u)$.

(2) φ is the characteristic function of some subset of $M(F)$ so that it is invariant under the Θ_M-twisted conjugate action by some open compact subgroup U_0 of $U_M(F)$ i.e. $\varphi(\Theta_M(u^{-1})mu) = \varphi(m)$ if $u \in U_0$.

These assumptions guarantees some nice behavior of partial Bessel functions, and are satisfied in all the known cases. We will see later in the beginning of Section 6.3 that they also hold in our cases.

We would like to first understand the structure of twisted centralizer in the definition of partial Bessel integrals, in order to get a descent formula for partial Bessel integrals to those defined over Levi subgropus of M. This serves as an important step to prove the general asymptotic expansion formula for partial Bessel integrals.

Lemma 6.1. *For* $m = u_1 \dot{w} a u_2 \in C(w_L^M)$, *then* $U_{M,m}^{\Theta_M} \subset u_2^{-1} U_L u_2 = u_2^{-1} U_{M,w_L^M}^+ u_2$.

Proof.

$$u \in U_{M,m}^{\Theta_M} \iff \Theta_M(u^{-1}) u_1 w_M w_L^{-1} a u_2 u = u_1 w_M w_L^{-1} a u_2$$
$$\iff w_M^{-1} u_1^{-1} \Theta_M(u^{-1}) u_1 w_M = w_L^{-1} a u_2 u^{-1} u_2^{-1} a^{-1} w_L.$$

Since $\Theta_M(U_M) = U_M$, the left hand side of the last equality lies in U_M^-. Hence $a u_2 u^{-1} u_2^{-1} a^{-1} \in U_{w_L}^- = U_L$, therefore $u_2 u^{-1} u_2^{-1} \in a^{-1} U_L a = U_L$, so $u \in u_2^{-1} U_L u_2$. $\square$

Lemma 6.2. *Let* $H \subset L \subset M$ *be Levi subgroups of* M, *then for* $m = u_1 \dot{w}_H^M a u_2 \in C_r(\dot{w}_H^M)$, *we have*

$$U_{M,m}^{\Theta_M} = (\widetilde{u}_1^-)^{-1} U_{L,l}^{\Theta_L} \widetilde{u}_1^- \cap (u_2^-)^{-1} U_{L,l}^{\Theta_L} u_2^-$$

where $u_1 = u_1^- u_1^+$ *with* $u_1^- \in U_{M,w^{-1}}^-$ *and* $u_1^+ \in U_{M,w^{-1}}^+$, $u_2 = u_2^+ u_2^-$ *with* $u_2^+ \in U_{M,w}^+$ *and* $u_2^- \in U_{M,w}^-$, $\widetilde{u}_1^- = \tau_L^M \circ \Theta_M \circ \tau_L^M((u_1^-)^{-1})$, $l = w^{-1} u_1^+ w \dot{w}_H^L a u_2^+ \in L$, *where* $w = w_L^M \in B(M)$.

Proof. By Lemma 6.1 of [15], $m = u_1 \dot{w}_H^M a u_2 \in C_r(\dot{w}_H^M) \subset \Omega_w = U_{M,w^{-1}}^- \times wL \times U_{M,w}^-$, where $\Omega_w := \bigsqcup_{w \leq w'} C(w')$, and the decomposition is unique. Decompose $u_1 = u_1^- u_1^+$, $u_2 = u_2^+ u_2^-$ as stated in the Lemma. Then we can write $m = u_1^- \dot{w}(\dot{w}^{-1} u_1^+ \dot{w} \dot{w}_H^L a u_2^+) u_2^-$ and it is easy to see that $l = \dot{w}^{-1} u_1^+ \dot{w} \dot{w}_H^L a u_2^+ \in L$. Then

$$u \in U_{M,m}^{\Theta_M}$$
$$\iff \Theta_M(u^{-1}) m u = m$$
$$\iff (\Theta_M(u^{-1}) u_1^- \dot{w} \Theta_L(u^+) \dot{w}^{-1}) \dot{w} \Theta_L((u^+)^{-1}) l u^+ ((u^+)^{-1} u_2^- u^+ u^-)$$
$$= u_1^- \dot{w} l u_2^-$$

where we decompose $u = u^+ u^-$ with $u^+ \in U_{M,w}^+ = U_L$ and $u^- \in U_{M,w}^- = N_L$. We claim that

$$\Theta_M(u^{-1}) u_1^- \dot{w} \Theta_L(u^+) \dot{w}^{-1} \in U_{M,w^{-1}}^-.$$

To see this, note that it is equivalent to $\dot{w}^{-1} \Theta_M(u^{-1}) u_1^- \dot{w} \Theta_L(u^+) \in U_M^-$. But

$$\dot{w}^{-1} \Theta_M(u^{-1}) u_1^- \dot{w} \Theta_L(u^+)$$
$$= \dot{w}^{-1} \Theta_M((u^-)^{-1}) \dot{w} (\dot{w}^{-1} \Theta_M((u^+)^{-1}) \dot{w}) (\dot{w}^{-1} u_1^- \dot{w}) \Theta_L(u^+).$$

Since $(u^-)^{-1} \in N_L$, and by our assumption, $\tau_L^M \circ \Theta_M$ preserves the F-splitting when restricted to L, so $\Theta_M(N_L) = N_L$. In addition, since $\dot{w} = \dot{w}_L^M$, we have $\dot{w}^{-1} N_L \dot{w} \cap U_L = \{1\}$. As a result, $\dot{w}^{-1} \Theta_M((u^-)^{-1}) \dot{w} \in U_M^-$. This also implies that $\tau_L^M \circ \Theta_M|_{N_L} \subset U_M^-$. Moreover, $\dot{w}^{-1} \Theta_M((u^+)^{-1}) \dot{w} = \tau_L^M \circ \Theta_L((u^+)^{-1}) = \Theta_L((u^+)^{-1}) \in U_L$ by our assumption on Θ_L. We also have $\dot{w}^{-1} U_{M,w^{-1}}^- \dot{w} = (U_{M,w}^-)^- = N_L^-$, so $\dot{w}^{-1} u_1^- \dot{w} \in N_L^-$. Since U_L normalizes N_L^-, $(\dot{w}^{-1} \Theta_M((u^+)^{-1}) \dot{w})(\dot{w}^{-1} u_1^- \dot{w}) \Theta_L(u^+) \in N_L^- \subset U_M^-$. This proves the claim. It follows from the claim and the uniqueness of the decomposition $\Omega_w = U_{M,w^{-1}}^- \times \dot{w} L \times U_{M,w}^-$ that

(1) $\Theta_M(u^{-1}) u_1^- \dot{w} \Theta_L(u^+) \dot{w}^{-1} = u_1^-$,
(2) $\Theta_L((u^+)^{-1}) l u^+ = l$,
(3) $(u^+)^{-1} u_2^- u^+ u^- = u_2^-$.

Note that

$$(1) \Longleftrightarrow \dot{w}^{-1}(u_1^-)\Theta_M(u^{-1})u_1^-\dot{w} = \Theta_L(u^+)^{-1}$$
$$\Longleftrightarrow \tau_L^M(u_1^-)\tau_L^M \circ \Theta_M(u^{-1})\tau_L^M(u_1^-) = \Theta_L(u^+)^{-1}.$$

Apply $\tau_L^M \circ \Theta_M$ on both sides, and use $\tau_L^M \circ \Theta_M|_L = \Theta_L$, then take inverse on both sides, we obtain that $\widetilde{u}_1^- u(\widetilde{u}_1^-)^{-1} = u^+$, where $\widetilde{u}_1^- = \tau_L^M \circ \Theta_M \circ \tau_L^M((u_1^-)^{-1})$. (2) is saying that $u^+ \in U_{L,l}^{\Theta_L}$. Therefore (1) and (2) imply that $U_{M,m}^{\Theta_M} \subset (\widetilde{u}_1^-)^{-1}U_{L,l}^{\Theta_L}\widetilde{u}_1^-$. On the other hand, (3) $\Leftrightarrow u_2^- u(u_2^-)^{-1} = u^+$, then (2) and (3) imply that $U_{M,m}^{\Theta_M} \subset (u_2^-)^{-1}U_{L,l}^{\Theta_L}u_2^-$. Hence

$$U_{M,m}^{\Theta_M} \subset (\widetilde{u}_1^-)^{-1}U_{L,l}^{\Theta_L}\widetilde{u}_1^- \cap (u_2^-)^{-1}U_{L,l}^{\Theta_L}u_2^-.$$

Conversely, if $u = (\widetilde{u}_1^-)^{-1}u'\widetilde{u}_1^- = (u_2^-)^{-1}u''u_2^-$ with $u', u'' \in U_{L,l}^{\Theta_L}$, then $u = u^+u^- = u'(u'^{-1}(\widetilde{u}_1^-)^{-1}u'\widetilde{u}_1^-) = u''((u'')^{-1}(u_2^-)^{-1}u''u_2')$. One can see from the above argument that $\widetilde{u}_1^-, u_2^- \in N_L$. Since $U_L \cap N_L = \{1\}$, and U_L normalizes N_L, this implies that $u^+ = u' = u''$, this is equivalent to (2). Replace u' and u'' by u^+ we obtain that $\widetilde{u}_1^- u(\widetilde{u}_1^-)^{-1} = u^+$ and $u_2^- u(u_2^-)^{-1} = u^+$, which are equivalent to (1) and (3) respectively. Reverse the above process we obtain that (1), (2), (3) hold simultaneously, hence the reversed inclusion. $\qquad\square$

By Lemma 5.9 of [6], we have a surjective map $C_c^\infty(\Omega_w; \omega_\pi) \twoheadrightarrow C_c^\infty(L; \omega_\pi)$ given by $f \mapsto h_f$ where

$$h_f(l) = \int_{U_{M,w}^-} \int_{U_{M,w^{-1}}^-} f(x^-\dot{w}lu^-)\psi^{-1}(x^-u^-)\mathrm{d}x^-\mathrm{d}u^-.$$

We have the following descent formula for partial Bessel integrals:

Proposition 6.3. *Suppose $H \subset L \subset M$ are Levi subgroups of M, and φ is invariant under the Θ_M-twisted conjugate action of some open compact subgroups of $U_M(F)$. Let $w = w_L^M \in B(M)$. Then for any fixed $m \in C_r(\dot{w}_H^M)$, decompose m as $m = u_1^-\dot{w}lu_2^-$, with $u_1^- \in U_{M,w^{-1}}^-$, $u_2^- \in U_{M,w}^-$, $l \in L$, we have*

$$B_\varphi^M(m, f) = \psi(u_1^- u_2^-)B_\varphi^L(u_1^-, u_2^-, l.h_f)$$

where

$$B_\varphi^L(u_1^-, u_2^-, l, h_f) = \int_{U_{L,l}\cap n_0 U_{L,l}n_0^{-1}\backslash U_L} \int_{U_L} h_f(x'lu')\varphi(\Theta_M((n_0u')^{-1}\dot{w}l'u')$$
$$\cdot\psi^{-1}(x')\psi^{-1}(u')\mathrm{d}x'\mathrm{d}u'$$

in which $n_0 = \widetilde{u}_1^-(u_2^-)^{-1} \in N_L$, $\widetilde{u}_1^- = \tau_L^M \circ \Theta_M \circ \tau_L^M((u_1^-)^{-1})$, and $l = zl'$ with $z \in Z_M$.

Proof. By Lemma 6.2, $U_{M,m}^{\Theta_M} = (\tilde{u}_1^-)^{-1}U_{L,l}^{\Theta_L}\tilde{u}_1^- \cap (u_2^-)^{-1}U_{L,l}^{\Theta_L}u_2^-$. Denote $n_1 = (\tilde{u}_1^-)^{-1}, n_2 = (u_2^-)^{-1}$, and $n_0 = \tilde{u}_1^-(u_2^-)^{-1}$, then $n_1, n_2 \in U_{M,w}^- = N_L \subset U_M$, so $U_M = n_1^{-1}U_M n_1$. Hence if we decompose $x = x^-x^+$ with $x^- \in U_{M,w^{-1}}^-$, $x^+ \in U_{M,w^{-1}}^+$, make a change of variable $u \mapsto n_1 u n_1^{-1}$, and then decompose $u = u^+u^-$ with $u^+ \in U_{M,w}^+ = U_L$, $u^- \in U_{M,w}^- = N_L$ in the integral

$$B_\varphi^M(m,f) = \int_{U_{M,m}^{\Theta_M}\backslash U_M} \int_{U_M} f(xmu)\varphi(\Theta_M(u^{-1})m'u)\psi^{-1}(xu)\mathrm{d}x\mathrm{d}u,$$

we obtain

$$B_\varphi^M(m,f) = \int_{n_1(U_{L,l}^{\Theta_L}\cap n_0 U_{L,l}^{\Theta_L}n_0^{-1})n_1^{-1}\backslash n_1 U_L n_1^{-1}} \int_{U_{M,w^{-1}}^+} \int_{U_{M,w}^-} \int_{U_{M,w^{-1}}^-}$$
$$f(x^-x^+u_1^-\dot{w}lu_2^-n_1u^+u^-n_1^{-1})$$
$$\cdot \varphi(\Theta_M(n_1(u^-)^{-1}(u^+)^{-1}n_1^{-1})u_1^-\dot{w}l'u_2^-n_1u^+u^-n_1^{-1})$$
$$\cdot \psi^{-1}(x^+x^-)\psi^{-1}(n_1u^+u^-n_1^{-1})\,\mathrm{d}x^-\mathrm{d}u^-\mathrm{d}x^+\mathrm{d}u^+.$$

Since $n_1^{-1} = \tilde{u}_1^- = \tau_L^M \circ \Theta_M \circ \tau_L^M((u_1^-)^{-1})$, we see that $\Theta_M(n_1^{-1}) = (\Theta_M \circ \tau_L^M)^2((u_1^-)^{-1}) = (u_1^-)^{-1}$. Let $y^- = x^-x^+u_1^-(x^+)^{-1}$, $x' = \dot{w}^{-1}x^+\dot{w}$, $v^- = (u^+)^{-1}u_2^-n_1u^+u^-n_1^{-1}$, and $u' = u^+$, then $y^- \in U_{M,w^{-1}}^-$, $x' \in U_{M,w}^+ = U_L$, and $v^- \in U_{M,w}^- = N_L$. By the compatibility of ψ with $\dot{w}$ by [6, Proposition 5.1], we have $\psi(x') = \psi(x^+)$. Therefore we can rewrite the above integral as

$$B_\varphi^M(m,f) = \psi(u_1^-u_2^-) \int_{U_{L,l}^{\Theta_L}\cap n_0 U_{L,l}^{\Theta_L}n_0^{-1}\backslash U_L} \int_{U_L} \int_{U_{M,w^{-1}}^-} \int_{U_{M,w}^-} f(y^-\dot{w}x'lu'v^-)$$
$$\cdot \varphi(\Theta_M((n_0u'v^-)^{-1})\dot{w}l'u'v^-)\psi^{-1}(v^-y^-x'u')\mathrm{d}v^-\mathrm{d}y^-\mathrm{d}x'\mathrm{d}u'.$$

Since $f \in C_c^\infty(\Omega_w; \omega_\pi)$, $\Omega_w = U_{M,w^{-1}}^- \times \dot{w}L \times U_{M,w}^-$, $U_{M,w^{-1}}^-$ and $U_{M,w}^-$ are closed in Ω_w, so there exists compact subgroups U_1 of $U_{M,w}^-$ and U_2 of $U_{M,w^{-1}}^-$ such that $f(y^-\dot{w}x'lu'v^-) \neq 0$ implies that $y^- \in U_1$ and $v^- \in U_2$. On the other hand, by our assumption on φ, φ is invariant under the Θ_M-twisted conjugate action by some open compact subgroup U_0 of $U_M(F)$. We shrink U_2 if necessary so that v^- lies in U_0. It follows that

$$\varphi(\Theta_M(n_0u'v^-)^{-1}\dot{w}l'u'v^-) = \varphi(\Theta_M(n_0u')^{-1}\dot{w}l'u').$$

So we obtain that

$$B_\varphi^M(m,f) = \psi(u_1^- u_2^-) \int_{U_{L,l}^{\Theta_L} \cap n_0 U_{L,l}^{\Theta_L} n_0^{-1} \backslash U_L} \int_{U_L} \int_{U_{M,w^{-1}}^-} \int_{U_{M,w}^-}$$

$$f(y^- \dot{w} x' l u' v^-) \psi^{-1}(v^- y^-) \mathrm{d}v^- \mathrm{d}y^-$$

$$\cdot \varphi(\Theta_M((n_0 u')^{-1}) \dot{w} l' u') \psi^{-1}(x' u') \mathrm{d}x' \mathrm{d}u'$$

$$= \psi(u_1^- u_2^-) \int_{U_{L,l}^{\Theta_L} \cap n_0 U_{L,l}^{\Theta_L} n_0^{-1} \backslash U_L} \int_{U_L}$$

$$h_f(x' l u') \varphi(\Theta_M((n_0 u')^{-1}) \dot{w} l' u') \psi^{-1}(x' u') \mathrm{d}x' \mathrm{d}u'$$

$$= \psi(u_1^- u_2^-) B_\varphi^L(u_1^-, u_2^-, l, h_f).$$

To show the last integral is well-defined, note that if $v'' = n_0 u'' n_0^{-1}$, with $u'', v'' \in U_{L,l}^{\Theta_L}$, then

$$l' = \Theta_L((v'')^{-1}) l' v''$$

$$= \Theta_L((u''^{-1} n_0 u'' n_0^{-1})^{-1}) \Theta_L((u'')^{-1}) l' u''(u''^{-1} n_0 u'' n_0^{-1})$$

$$= \tau_L^M \circ \Theta_M((u''^{-1} n_0 u'' n_0^{-1})^{-1}) l'(u''^{-1} n_0 u'' n_0^{-1}),$$

so

$$l'^{-1} \tau_L^M \circ \Theta_M(u''^{-1} n_0 u'' n_0^{-1}) l' = u''^{-1} n_0 u'' n_0^{-1}.$$

Since $\Theta_M(U_M) = U_M$, $U_M = U_L N_L$, and $\tau_L^M(N_L) = \overline{N_L}$, the opposite of N_L. The left hand side of the above equality lies in $\overline{N_L}$, while the right hand side lies in N_L, as $N_L \cap \overline{N_L} = \{1\}$ it forces that $u''^{-1} n_0 u'' n_0^{-1} = 1$, i.e. $u'' n_0 = n_0 u''$, and $v'' = u''$. It follows that

$$\Theta_M((n_0 v'' u')^{-1}) \dot{w} l' v'' u' = \Theta_M((u'' n_0 u')^{-1}) \dot{w} \Theta_L(u'')(\Theta_L(u'') l' u'') u'$$

$$= \Theta_M((n_0 u')^{-1}) \Theta_M((u'')^{-1}) \dot{w} \Theta_L(u'') l' u'$$

$$= \Theta_M((n_0 u')^{-1}) \dot{w} l' u'$$

where we used $\dot{w}^{-1} \Theta_M(u'') \dot{w} = \tau_L^M \circ \Theta_M(u'') = \Theta_L(u'')$. $\square$

Remark 6.4. If we impose a stronger condition on the cut-off function φ, i.e, we require that

$$\varphi(\Theta_M(x_1^{-1}) m x_2) = \varphi(m)$$

for $x_1, x_2 \in U_0$, then we can show that

$$B_\varphi^M(m,f) = \psi(u_1^- u_2^-) \mathrm{Vol}(U_{L,l}^{\Theta_L} \cap n_0 U_{L,l}^{\Theta_L} n_0^{-1} \backslash U_{L,l}^{\Theta_L}) B_{\varphi_{\dot{w}}}^L(l, h_f)$$

for all $m \in C_r(\dot{w})$ where $\varphi_{\dot{w}}(\cdot) = \varphi(\dot{w} \cdot)$ is the left translation of φ by $\dot{w}$. But notice that in general the subset consists of the image $n \mapsto m$ via $\dot{w}_0^{-1} n = m n' \overline{n}$ is not invariant under the action $m \mapsto \Theta_M(x_1^{-1}) m x_2$ for $x_1, x_2 \in U_M$. In particular, one checks easily from our calculation of the Bruhat decomposition in Section 3 that in our case the set of elements of

the form $m(X,Y)$ is not invariant under this action. So we do not use this formula, although it is of a nicer form.

Remark 6.5. We observe easily that in particular, if $n_0 = 1$, i.e. $\widetilde{u}_1^- = u_2^-$, we have

$$B_\varphi^M(m, f) = B_{\varphi_{\dot{w}}}^L(l, h_f).$$

6.2. Asymptotic expansion and uniform smoothness. The main idea to prove stability is that once we relate our local coefficient formula with the Mellin transform of the partial Bessel integrals, there are two parts in the asymptotic expansion formula of the partial Bessel integrals, one depends only on the central character of π, the other is certain uniform smooth function. Therefore under highly ramified twists, the second part vanishes.

To obtain the asymptotic expansion of partial Bessel integrals, we study its boundary behavior on Ω_w with $w \in B(M)$, i.e. on Bruhat cells of smallest possible dimension, and do induction on Bessel distance. This is a standard procedure as in [6] or [15]. We begin with the small cell $C(e) = B_M$ of M. First note that $L_e = M, A_e = Z_M, U_{M,e}^+ = U_M$, and $\Omega_e = M$. Take $\dot{e} = I$, the identity matrix in M. Recall that π is ψ-generic supercuspidal and we pick an matrix coefficient $f \in C_c^\infty(M; \omega_\pi)$, normalized with $W_f(e) = 1$. Then we have

Proposition 6.6. *Fix an auxiliary function $f_0 \in C_c^\infty(M; \omega_\pi)$ with $W_{f_0}(e) = 1$. Then for each $f \in C_c^\infty(M; \omega_\pi)$ with $W_f(e) = 1$ and $w' \in B(M)$ with $d_B(e, w') = 1$, there exists a function $f_{w'} \in C_c^\infty(\Omega_{w'}; \omega_\pi)$ such that for any $w \in B(M)$ and $m \in M$, we have*

$$B_\varphi^M(m, f) = B_\varphi^M(m, f_1) + \sum_{w' \in B(M), d_B(w', e) = 1} B_\varphi^M(m, f_{w'})$$

where $f_1(m) := \sum_{m = m_1 c} f_0(m_1) B^M(\dot{e}c, f) = \sum_{m = m_1 c} f_0(m_1) \omega_\pi(c)$, the sum runs over all possible decompositions $m = m_1 c$ with $m_1 \in M_{der}$, and $c \in A_e = Z_M$.

Proof. Decompose $M = M_{der} A_e = M_{der} Z_M$, where M_{der} is the derived group of M. Write $m = m_1 c$ with $m_1 \in M_{der}$ and $c \in Z_M$, then there are only finitely many such decompositions which are indexed by elements in the transverse torus $A_e^e = M_{der} \cap Z_M$. Define

$$f_1(m) := \sum_{m = m_1 c} f_0(m_1) B^M(\dot{e}c, f) = \sum_{m = m_1 c} f_0(m_1) \omega_\pi(c)$$

Then $f_1(m) \in C_c^\infty(M; \omega_\pi)$. For $a \in A_e = Z_M$, $a' = e$, therefore

$$B_\varphi^M(\dot{e}a, f_1) = \omega_\pi(a) \int_{U_{M,e}^{\Theta_M} \backslash U_M} \int_{U_M} f_1(xu) \varphi(\Theta_M(u^{-1})u) \psi^{-1}(xu) \mathrm{d}x \mathrm{d}u$$

$$= \omega_\pi(a) W_{f_1}(e) \int_{U_{M,e}^{\Theta_M} \backslash U_M} \varphi(\Theta_M(u^{-1})u) \mathrm{d}u.$$

On the other hand, $W_{f_1}(e) = \int_U f_1(x) \psi^{-1}(x) \mathrm{d}x$, while $x \in U_M \subset M_{der}$ so $f_1(x) = f_0(x)$, thus $W_{f_1}(e) = W_{f_0}(e) = 1$, hence we obtain that $B_\varphi^M(\dot{e}a, f_1) = B_\varphi^M(\dot{e}a, f)$ for all $a \in A_e$. Therefore $B_\varphi^M(\dot{e}a, f - f_1) = 0$ for all $a \in A_e$. Apply [6, Lemma 5.13], there exists $f_2' \in C_c^\infty(\Omega_e^\circ, ; \omega_\pi)$, such that $B_\varphi^M(m, f - f_1) = B_\varphi^M(m, f_2')$ for all $m \in M$, where $\Omega_e^\circ = \Omega_e - C(e) = M - B_M$. Let $\Omega_1 = \bigcup_{w \in B(M), w \neq e} \Omega_w = \bigcup_{w \in B(M), d_B(w, e) = 1} \Omega_w$, and $\Omega_0 = M - C(e) = \Omega_e^\circ$. By [6, Lemma 5.14], there exists $f_2 \in C_c^\infty(\Omega_1; \omega_\pi)$, so that $B_\varphi^M(m, f_2) = B_\varphi^M(m, f_2') = B_\varphi^N(m, f - f_1)$, for a sufficiently large φ depending only on f_1. Then by a partition of unity argument, for each $w' \in B(M)$, $d_B(w', e) = 1$, we can find $f_{w'} \in C_c^\infty(\Omega_{w'}; \omega_\pi)$, such that $f_2 = \sum_{w' \in B(M), d_B(w', e) = 1} f_{w'}$. $\square$

We would like to do the same type of analysis on each $B_\varphi^M(m, f_{w'})$ and obtain an asymptotic expansion formula for $B_\varphi^M(m, f)$ indexed by Weyl group elements that supports Bessel functions, and for all m lying in the relevant part of Bruhat cells. To be precise we will show that

Proposition 6.7. *Fix an auxiliary function $f_0 \in C_c^\infty(M; \omega_\pi)$ with $W_{f_0}(e) = 1$. Then for each $f \in C_c^\infty(M; \omega_\pi)$ with $W_f(e) = 1$ and $w' \in B(M)$ with $d_B(e, w') \geq 1$, there exists a function $f_{w'} \in C_c^\infty(\Omega_{w'}; \omega_\pi)$ such that for any $m = u_1 \dot{w} a u_2 \in C_r(\dot{w})$ with $\dot{w} = \dot{w}_L^M \in B(M)$, we have*

$$B_\varphi^M(m, f) = \sum_{a = bc} \omega_\pi(c) B_\varphi^M(u_1 \dot{w} b u_2, f_0) + \sum_{w' \in B(M), d_B(w', e) \geq 1} B_\varphi^M(m, f_{w'})$$

where $a = bc$ runs over the possible decompositions of $a \in A_w$ with $b \in A_w^e$ and $c \in A_e = Z_M$.

Proof. Note that in particular if $m = u_1 \dot{w} a u_2 \in C_r(\dot{w})$, by Lemma 6.1, $U_{M,m}^{\Theta_M} \subset u_2^{-1} U_{M.w}^+ u_2$, we can write

$$U_M = u_2^{-1} U_{M,w}^+ U_{M,w}^- u_2^{-1} = u_2^{-1} U_{M,w}^+ u_2 (u_2^{-1} U_{M,w}^- u_2).$$

Let $u = u'(u_2^{-1}u^- u_2)$ where $u' = u_2^{-1}u^+ u_2$ with $u^+ \in U_{M,w}^+$, $u^- \in U_{M,w}^-$, we have

$$B_\varphi^M(m, f_1) = \int_{U_{M,m}^{\Theta_M}\backslash u_2^{-1}U_{M,w}^+ u_2} \int_{U_{M,w}^-} \int_{U_M} f_1(xu_1\dot{w}au_2u'u_2^{-1}u^- u_2)$$

$$\cdot \varphi(\Theta_M(u_2^{-1}(u^-)^{-1}u_2 u'^{-1})u_1\dot{w}a'u_2u'u_2^{-1}u^- u_2)$$

$$\cdot \psi^{-1}(xu'u_2^{-1}u^- u_2)\mathrm{d}x\mathrm{d}u^-\mathrm{d}u'.$$

As $u^+ = u_2u'u_2^{-1} \in U_{M,w}^+$ and $a \in A_w$, we have $xu_1\dot{w}au_2^{-1}u'u_2^{-1}u^- u_2 = xu_1(\dot{w}u_2u'u_2^{-1}\dot{w}^{-1})\dot{w}u^- u_2$. Let $x' = xu_1(\dot{w}u_2u'u_2^{-1}\dot{w}^{-1}) \in U_{M,w}^+$, $v^- = u^- u_2 \in U_{M,w}^- = U_{M,w}^-$, and by compatibility of ψ with $\dot{w}$, we obtain that

$$B_\varphi^M(m, f_1) = \psi(u_1u_2) \int_{U_{M,m}^{\Theta_M}\backslash u_2^{-1}U_{M,w}^+ u_2} \int_{U_{M,w}^-} \int_{U_M} f_1(x'\dot{w}av^-)$$

$$\cdot \varphi(\Theta_M((v^-)^{-1}u_2u'^{-1})u_1\dot{w}a'u_2u'u_2^{-1}v^-)\psi^{-1}(x'v^-)\mathrm{d}x'\mathrm{d}v^-\mathrm{d}u'$$

So by the construction of f_1, we need to decompose $x'\dot{w}av^- = m_1c$ with $m_1 \in M_{der}$ and $c \in A_e = Z_M$, which is equivalent to

$$ac^{-1} = \dot{w}^{-1}(x')^{-1}m_1(v^-)^{-1}.$$

Since we pick $\dot{w} \in M_{der}$, and $x', v^- \in U_M \subset M_{der}$, this is saying that $b := ac^{-1} \in M_{der} \cap A_w = A_w^e$. It follows that

$$f_1(x'\dot{w}av^-) = \sum_{a=bc} f_0(x'\dot{w}bv^-)\omega_\pi(c),$$

and consequently

$$B_\varphi^M(m, f_1) = \sum_{a=bc} \omega_\pi(c)B_\varphi^M(u_1\dot{w}bu_2, f_0)$$

where $a = bc$ runs over the possible decompositions of $a \in A_w$ with $b \in A_w^e$ and $c \in A_e = Z_M$.

We continue to obtain expansions for $B_\varphi^M(m, f_{w'})$. Suppose $w' = w_L^M \in B(M)$. Let $h = h_{f_{w'}} \in C_c^\infty(L; \omega_\pi)$ be the image of $f_{w'}$ under $C_c^\infty(\Omega_{w'}; \omega_\pi) \twoheadrightarrow C_c^\infty(L; \omega_\pi)$. Pick $h_0 \in C_c^\infty(L; \omega_\pi)$ normalized so that $B_{\varphi_{\dot{w}'}}^L(\dot{e}, h_0) = \frac{1}{|Z_L \cap A_{w'}^L|}$, and $B^M(b, h_0) = 0$ for $b \in A_{w'}^{w'}$ but $b \notin Z_M \cap A_{w'}^{w'}$. Similar to the construction of f_1 from f, let $h_1(l) = \sum_{l=l_1c} h_0(l_1)B^L(c, h)$, where the decomposition is taken over all possible $l = l_1c$ where $l_1 \in L_{der}$ and $c \in A_{w'} = Z_L$. By [6, Proposition 5.4], $B_{\varphi_{\dot{w}'}}^L(a, h_1) = B_{\varphi_{\dot{w}'}}^L(a, h)$, for all $a \in A_{w'} = Z_L$. Choose f_1 which maps to h_1 under $C_c^\infty(\Omega_{w'}, \omega_\pi) \twoheadrightarrow C_c^\infty(L; \omega_\pi)$, apply Proposition 6.3 to case when $H = L$, and $m = w'a$, $a \in A_{w'} = Z_L$, we get $B_\varphi^M(w'a, f_1) = B_{\varphi_{\dot{w}'}}^L(a, h_1) = B_{\varphi_{\dot{w}'}}^L(a, h) = B_\varphi^M(w'a, f_{w'})$. So $B_\varphi^M(w'a, f_{w'} - f_1) = 0$, for all $a \in A_w$. Apply [6, Lemma 5.13, 5.14] again, together with a partition of

unity argument, there exists $f_{w',w''} \in C_c^\infty(\Omega_{w''}; \omega_\pi)$ for $w' < w'' \in B(M)$ such that $d_B(w', w'') = 1$, and

$$
B_\varphi^M(m, f_{w'}) = B_\varphi^M(m, f_1) + \sum_{w'' \in B(M), w'' > w', d_B(w'', w') = 1} B_\varphi^M(m, f_{w''}).
$$

Proceed by repeating this process and do induction on the Bessel distance, and combine with the above argument on $B_\varphi^M(m, f_1)$, we obtain the asymptotic expansion formula as desired. $\qquad\square$

A smooth function on a p-adic group is *uniform smooth* if it is uniformly locally constant, i.e., there exists an open compact subgroup K_0 such that the function is constant on aK_0 for any a in the group. Note that the first sum in Proposition 6.7 depends only on ω_π and a fixed auxiliary function f_0. The next important step is to show that each $B_\varphi^M(m, f_{w'})$ satisfy some uniform smooth property. Recall that for $h \in C_c^\infty(\Omega_{w'}; \omega_\pi)$ with $w' = w_L^M \in B(M)$, we constructed $h_1(l) = \sum_{l=l_1 c} h_0(l_1) B^L(c, h)$. Suppose $m = u_1^- \dot{w}' l u_2^-$ and write $l = zl'$ with $z \in Z_M$, $n_0 = \tilde{u}_1^-(u_2^-)^{-1}$, where $\tilde{u}_1^- = \tau_L^M \circ \Theta_M \circ \tau_L^M((u_1^-)^{-1})$, we have $B_\varphi^L(u_1^-, u_2^-, l, h_1) = \omega_\pi(z) B_\varphi^L(u_1^-, u_2^-, l', h_1)$. Since

$$
B_\varphi^L(u_1^-, u_2^-, l', h_1) = \int_{U_{L,l}^{\Theta_L} \cap n_0 U_{L,l}^{\Theta_L} n_0^{-1} \backslash U_L} \int_{U_L} h_1(xl'u)
$$
$$
\cdot \varphi(\Theta_M((n_0 u)^{-1}) \dot{w}' l' u) \psi^{-1}(xu) dx du,
$$

we need to write $xl'u = l_1 c$ with $l_1 \in L_{der}$, $c \in Z_L$. Suppose $l = v_1 \dot{w} a v_2 \in C^L(\dot{w})$ a Bruhat cell in L with $\dot{w} \in L_{der}$, then $l' = v_1 \dot{w} a' v_2$ where $a = za'$. As $v_1, v_2, \dot{w} \in L_{der}$, so to write $xl'u = xv_1 \dot{w} a' v_2 = l_1 c$ is equivalent to write $a' = bc$ with $b \in A \cap L_{der}$ and $c \in Z_L$. Let $b = z_b b'$, $c = z_c c'$ with $z_b, z_c \in Z_M$, $b' \in A'$, $c' \in Z_L'$, then $a' = z_b z_c b' c' \Rightarrow z_b z_c = 1$, and $a' = b' c'$. Let $l_b = v_1 \dot{w} b v_2$, and similarly $l_{b'} = v_1 \dot{w} b' v_2$, then $h_1(xl'u) = \sum_{a'=b'c'} h_0(xl_{b'} u) B^L(c', h)$. Since $U_{L,l'}^{\Theta_L} = \{u \in U_L : \Theta_L(u^{-1}) l' u = l'\} = \{u \in U_L : \Theta_L(u^{-1}) l_{b'} c' u = l_{b'} c'\} = \{u \in U_L : \Theta_L(u^{-1}) l_{b'} u c' = l_{b'} c'\} = \{u \in U_L : \Theta_L(u^{-1}) l_{b'} u = l_{b'}\} = U_{L,l_{b'}}^{\Theta_L}$, we obtain that

$$
B_\varphi^L(u_1^-, u_2^-, l', h_1) = \sum_{a'=b'c'} B^L(c', h) \int_{U_{L,l'}^{\Theta_L} \cap n_0 U_{L,l'}^{\Theta_L} n_0^{-1} \backslash U_L} \int_{U_L} h_0(xl_{b'} u)
$$
$$
\cdot \varphi(\Theta_M((n_0 u)^{-1}) l_{b'} u c') \psi^{-1}(xu) dx du
$$
$$
= \sum_{a'=b'c'} B^L(c', h) B_{\varphi^{c'}}^L(u_1^-, u_2^-, l_{b'}, h_0)
$$

where $\varphi^{c'}(\,\cdot\,) := \varphi(\,\cdot\, c')$ is the right shift of φ by c'.

Recall that A is the maximal split torus of M. We say a unipotent element $u \in U_M$ is *rationally parameterized* by $(a, d) \in A \times \mathbb{A}^k$ if all the entries in the 1-parameter subgroups corresponding to u are rational functions of (a, d),

here $\mathbb{A}^k$ is the affine space of dimension k, i.e., if one has $u = \prod_\alpha u_\alpha(x_\alpha)$ with $x_\alpha \in \mathbb{G}_a$, then each x_α is a rational function of (a, d).

Proposition 6.8. *Let $H \subset L \subset M$ be Levi subgroups of M, suppose $m = u_1^- \dot{w}' l u_2^- = m(a, d) = u_1(a, d)\dot{w}au_2(a, d) \in C_r(\dot{w})$ is rationally parameterized by $(a, d) \in A_w^{w'} A_{w'} \times \mathbb{A}^k \subset A \times \mathbb{A}^k$ for some $k \geq 0$ with $w = w_H^M, w' = w_L^M \in B(M)$. Assume that the rational functions that parameterize u_1 and u_2 have no singularities on $A \times \mathbb{A}^k$. Then if one writes $a = bc$ with $b \in A_w^{w'}$, $c \in A_{w'}$ and $c = zc'$ with $z \in Z_M$, $c \in A'_{w'}$, where $A_{w'} = Z_M A'_{w'}$, we have that*

$$B_\varphi^M(m, f_1) = \omega_\pi(z)\psi(u_1^-(bc'z, d), u_2^-(bc'z, d))$$
$$\cdot \sum_{a'=b'c'} B^L(c', h) B_{\varphi^{c'}}^L(u_1^-(bc'z), u_2^-(bc'z), l_{b'}(bc'z, d), h_0)$$

is uniformly smooth as a function of $c' \in A'_{w'}$, where $f_1 \mapsto h_1$ via

$$C_c^\infty(\Omega_{w'}; \omega_\pi) \twoheadrightarrow C_c^\infty(L; \omega_\pi).$$

Proof. We have

$$B_\varphi^M(m, f_1) = \psi(u_1^-, u_2^-) B_\varphi^L(u_1^-, u_2^-, l, h_1)$$
$$= \psi(u_1^-, u_2^-) \sum_{a=bc} B^L(c, h) B_{\varphi^c}^L(u_1^-(bc, d), u_2^-(bc, d), l_b, h_0)$$

by Proposition 6.3 and the above argument. By fixing a decomposition $a = bc$ with $b \in A_{w_M}^{w'}$, $c \in A_{w'}$, all such decompositions are of the form $a = (b\xi^{-1})(\xi c)$ with $\xi \in A_w^{w'} \cap A_{w'} = A_{w'}^{w'} = Z_L \cap L_{der}$, a finite set, so $|\xi| = 1$. Therefore

$$B_\varphi^L(u_1^-, u_2^-, l, h_1) = \sum_{\xi \in A_{w'}^{w'}} B^L(\xi c, h) B_{\varphi^{\xi c}}^L(u_1^-(bc, d), u_2^-(bc, d), l_{b\xi^{-1}}, h_0)$$

where $l_{b\xi^{-1}} = \dot{w}'^{-1} u_1^+ \dot{w}' \dot{w}_H^L b\xi^{-1} u_2^+$. As φ depends only on the absolute value, $\varphi^{\xi c} = \varphi^c$. On the other hand,

$$B^L(\xi c, h) = \int_{U_M(F)} h(x\xi c)\psi^{-1}(x)\mathrm{d}x = \omega_\pi(\xi_1 z) \int_{U_M(F)} h(x\xi'c')\psi^{-1}(x)\mathrm{d}x$$

where we write $\xi = \xi_1 \xi'$ with $\xi_1 \in Z_M$ and $\xi' = \xi\xi_1^{-1}$. As $h \in C_c^\infty(L; \omega_\pi)$ is smooth of compact support modulo Z_M, and the small Bruhat cell $C^L(e_L) = AU_L = Z_M A'U_L$ of L is closed in L, and $Z'_L \subset A'$ is closed, there exists compact subsets $K_1 \subset Z'_L$ and $U_1 \subset U_M$ such that

$h(x\xi'c') \neq 0 \Rightarrow \xi'c' \in K_1$ and $x \in U_1$. Writing $c = zc'$, we see that

$$B_\varphi^L(u_1^-, u_2^-, l, h_1)$$
$$= \omega_\pi(z) \sum_{\xi \in A_{w'}^{w'}} B^L(\xi c', h) B_{\varphi^c}^L(u_1^-(bc'z, d), u_2^-(bc'z, d), l_{b\xi^{-1}}, h_0)$$

does not vanish unless $c' \in K := \bigcup_{\xi'} \xi'^{-1} K_1$, a compact subset. From this we also see that the support of h in c' is compact and independent of the decomposition $a = bc$.

Moreover, since the support of h in c' is compact, and h is smooth, there exists an open compact subgroup $C_0 \subset Z_L'$ such that $h(x\xi'c'c_0) = h(x\xi'c')$ for all $x \in U_1$, $c' \in Z_L'$, and $c_0 \in C_0$. By our assumption the entries in the 1-parameter subgroups corresponding to $u_i(a, d)$, and hence $u_i^\pm(a, d) = u_i^\pm(bc'z, d)$ $(i = 1, 2)$ are rational functions in (b, c', z, d) without singularities, it follows that $u_i^\pm$ and hence also $\psi(u_1^-, u_2^-)$ are smooth functions of (b, c', z, d) with no singularities, and the quotient space $U_{L,l}^{\Theta_L} \cap n_0 U_{L,l}^{\Theta_L} n_0^{-1} \backslash U_L$ is also smoothly parameterized by (b, c', z, d). Therefore $B_\varphi^M(m(bc'z, d), f_1)$ is zero when $c' \notin K$. So for each fixed z, b, d, we simultaneously choose C_0, depending on z, b, d, so that $u_i^\pm(zbc'c_0, d) = u_i^\pm(zbc', d)$ for all $c_0 \in C_0, c' \in K$. Finally, We shrink C_0 so that $C_0 \subset Z_L'(\mathcal{O}_F)$ if necessary, then $\varphi^{cc_0} = \varphi^c$ for all $c_0 \in C_0$ and $c \in A_{w'} = Z_L$. Consequently, there exists an open compact subgroup $C_0 \subset Z_L'$ such that

$$B_\varphi^M(m(zbc'c_0, d), f_1) = B_\varphi^M(m(zbc', d), f_1)$$

for all $a = bc$ and $c_0 \in C_0 \subset Z_L' = A_{w'}'$, i.e. $B_\varphi^M(m, f_1)$ is uniformly smooth in $c' \in A_{w'}'$. $\qquad\square$

6.3. The final local coefficient formula and the separation of the toric part.

We will relate the partial Bessel functions defined in Section 5 with partial Bessel integrals introduced in Section 6, so that the local coefficient formula in Proposition 5.2 can be restated in terms of Mellin transforms of partial Bessel integrals. With nice asymptotic expansion formulas in Proposition 6.7 and uniform smoothness in Proposition 6.8, we will be able to show our desired analytic stability.

For our applications, we set the involution $\Theta_M : m \mapsto \dot{w}_0^{-1} m \dot{w}_0$, and it is easily verified that it satisfies the assumption that it preserves the F-splitting and is compatible with Levi subgroups and the generic character ψ. Recall that for a matrix X of size $k \times l$, we defined

$$\varphi_\kappa(X) = \begin{cases} 1 & |X_{i,j}| \leq q^{((k-i)+(l-j)+1)\kappa} \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\overline{N}_{0,\kappa} = \{\bar{n}(X,Y) : \varphi_\kappa(\varpi^{-(d+g)}X) \cdot \varphi_\kappa(\varpi^{-2(d+g)}Y) = 1\},$$

where $d = \mathrm{Cond}(\psi)$ and $g = \mathrm{Cond}(\omega_\pi^{-1}(w_0(\omega_\pi)))$. We also denoted by $\varphi_{\overline{N}_{0,\kappa}}$ the characteristic function of $\overline{N}_{0,\kappa}$. Suppose $n(X,Y) \mapsto m(X,Y)$ via the Bruhat decomposition $\dot{w}_0^{-1}n = mn'\bar{n}$. Define

$$\varphi(m(X,Y)) := \varphi_\kappa((Y^{-1}X)_{r1}X)\varphi_\kappa(((Y^{-1}X)_{r1})^2Y)$$

where $(Y^{-1}X)_{r1}$ is the $(r,1)$-th entry of $Y^{-1}X$. We will show that φ satisfies the assumption that it is invariant under the Θ_M-twisted conjugate action by some open compact subgroup U_0 of $U_M(F)$. Consequently, the assumptions on Θ_M and φ in the beginning of Section 6 are both satisfied and we have

Proposition 6.9. *Let $m(X',Z')$ be the image of $n(X',Z')$ under the map $n \mapsto m$ via $\dot{w}_0^{-1}n = mn'\bar{n}$. Then*

$$j_{\overline{N}_{0,\kappa},\pi,w}(n(X',Z')) = B_\varphi^M(m(X',Z'),f)$$

where $\varphi_{\overline{N}_{0,\kappa}}$ and φ are defined as above and $f \in C_c^\infty(M(F);\omega_\pi)$ such that $W = W_f$, normalized so that $W_f(e) = 1$.

Proof. Given the Bruhat decomposition $\dot{w}_0^{-1}n = mn'\bar{n}$, for $z = \alpha^\vee(t) \in Z_M^0$, if $u \mapsto zu^{-1}nuz^{-1}$, then

$$m \longmapsto \Theta_M(zu^{-1})muz^{-1} = \Theta_M(u)^{-1}\Theta_M(z)z^{-1}mu = \Theta_M(u^{-1})\alpha^\vee(t^{-2})mu,$$

as $\Theta_M(z), z \in Z_M$.

In our cases, where $n = n(X,Y)$, then

$$m = m(X,Y) = \mathrm{diag}\{m_1(X,Y), m_2(X,Y), \theta_r(m_1(X,Y))\}.$$

So

$$\Theta_M(m(X,Y)) = \mathrm{diag}(\theta_r(m_1(X,Y)), m_2(X,Y), m_1(X,Y))$$

It shows that Θ_M induces θ_r. Moreover, for $z = \alpha^\vee(t) \in Z_M^0$, $u = \mathrm{diag}\{u_1, u_2, \theta_r(u_1)\} \in U_M$, we have

$$zu^{-1}\bar{n}(X,Y)uz^{-1} = \bar{n}(u_1^{-1}(tX)u_2, u_1^{-1}(t^2Y)\theta_r(u_1)) = u^{-1}\bar{n}(tX, t^2Y)u.$$

Hence the action $\bar{n}(X,Y) \mapsto zu^{-1}\bar{n}(X,Y)uz^{-1}$ is given by

$$X \longmapsto u_1^{-1}tXu_2, \quad Y \longmapsto u_1^{-1}t^2Y\theta_r(u_1).$$

On the other hand,

$$\Theta_M(zu^{-1})muz^{-1} = \Theta_M(u)^{-1}\alpha^\vee(t^{-2})m(X,Y)$$

$$= \begin{bmatrix} \theta_r(u_1^{-1}) & & \\ & u_2^{-1} & \\ & & u_1^{-1} \end{bmatrix} \begin{bmatrix} t^{-2}\theta_r(Y) & & \\ & (I_{2m}-J'_{2m}{}^tX^tY^{-1}J_rX)^{-1} & \\ & & t^2Y \end{bmatrix} \begin{bmatrix} u_1 & & \\ & u_2 & \\ & & \theta_r(u_1) \end{bmatrix}$$

$$= \begin{bmatrix} \theta_r(u_1^{-1})\theta_r(t^2Y)u_1 & & \\ & (I_{2m} - J'_{2m}{}^t(u_1^{-1}tXu_2)^t(u_1^{-1}t^2Y\theta_r(u_1))J_r(u_1^{-1}tXu_2))^{-1} & \\ & & u_1^{-1}t^2Y\theta_r(u_1) \end{bmatrix}.$$

This verifies that for $z = \alpha^\vee(t)$, $u = \mathrm{diag}(u_1, u_2, \theta_r(u_1)) \in U_M$, we have

$$\Theta_M(zu^{-1})m(X,Y)uz^{-1} = m(u_1^{-1}tXu_2, u_1^{-1}t^2Y\theta_r(u_1)).$$

Let $U_{0,\kappa} = U_{1,\kappa} \times U_{2,\kappa}$ with $U_{1,\kappa} \subset U_{\mathrm{GL}_r}(F)$ and $U_{2,\kappa} \subset U_{\mathrm{Sp}_{2m}}(F)$ given by

$$U_{1,\kappa} := \{(u_{1,ij}) \in U_{\mathrm{GL}_r(F)} : |u_{1,ij}| \le q^{(j-i)\kappa}\},$$
$$U_{2,\kappa} := \{(u_{2,ij}) \in U_{\mathrm{Sp}_{2m}(F)} : |u_{2,ij}| \le q^{(j-i)\kappa}\}.$$

Then $U_{0,\kappa}$ are open compact subgroups of $U_M(F)$. For $u = (u_1, u_2) \in U_{0,\kappa}$, one checks easily that $\Theta_M(u^{-1})\overline{N}_{0,\kappa}u \subset \overline{N}_{0,\kappa}$, i.e, $\overline{N}_{0,\kappa}$ is invariant under the twisted conjugate action by $U_{0,\kappa}$.

Let $\varphi(m(X,Y)) := \varphi_\kappa((Y^{-1}X)_{r1}X)\varphi_\kappa(((Y^{-1}X)_{r1})^2Y)$. So φ is defined on the subset of M which lie in the image of $n(X,Y)$ under the map $n \mapsto m$ via $\dot{w}_0^{-1}n = mn'\bar{n}$, where $(Y^{-1}X)_{r1}$ is the $(r,1)$-th entry of $Y^{-1}X$. We will show that φ is invariant under the Θ_M-twisted conjugate action $U_{0,\kappa}$. Suppose $u = (u_1, u_2) \in U_{0,\kappa}$, then

$$\varphi(\Theta_M(u^{-1})m(X,Y)u) = \varphi(m(u_1^{-1}Xu_2, u_1^{-1}Y\theta_r(u_1)))$$
$$= \varphi_\kappa((\theta_r(u_1^{-1})Y^{-1}Xu_2)_{r1})u_1^{-1}Xu_2)\varphi_\kappa((\theta_r(u_1^{-1})Y^{-1}Xu_2)_{r1})^2u_1^{-1}Y\theta_r(u_1))$$
$$= \varphi_\kappa((Y^{-1}X)_{r1})u_1^{-1}Xu_2)\varphi_\kappa(((Y^{-1}X)_{r1})^2u_1^{-1}Y\theta_r(u_1)).$$

The last equality holds since $(Y^{-1}X)_{r1}$ is clearly invariant under the left or right translation by an upper triangular unipotent matrix. On the other hand, since $u_1(Y^{-1}X)_{r1}u_1^{-1} = (Y^{-1}X)_{r1}$,

$$\varphi_\kappa((Y^{-1}X)_{r1}u_1^{-1}Xu_2) = 1 \Longleftrightarrow (Y^{-1}X)_{r1}u_1^{-1}Xu_2 \in X(\kappa)$$
$$\Longleftrightarrow X \in u_1(Y^{-1}X)_{r1}u_1^{-1}(u_1X(\kappa)u_2^{-1}) = (Y^{-1}X)_{r1}X(\kappa)$$
$$\Longleftrightarrow \varphi_\kappa((Y^{-1}X)_{r1}X) = 1$$

where $X(\kappa) = \{X = (x_{ij}) \in \mathrm{Mat}_{r\times 2m} : |x_{ij}| \le q^{((r-i)+(2m-j)+1)\kappa}\}$, which can be checked by direct calculation that it is invariant under the left-action by $U_{1,\kappa}$ and right-action by $U_{2,\kappa}$. Similarly we have

$$\varphi_\kappa(((Y^{-1}X)_{r1})^2u_1^{-1}Y\theta_r(u_1)) = \varphi_\kappa(((Y^{-1}X)_{r1})^2Y).$$

It follows that

$$\varphi(\Theta_M(u^{-1})m(X,Y)u) = \varphi(m(X,Y))$$

for all $u \in U_{0,\kappa}$.

Then for $(X', Z') \in R_{X'} \times R_{Z'}$, $z = \alpha^\vee(\varpi^{d+g}u_{\alpha_r}(\dot{w}_0 \bar{n}(X',Z')\dot{w}_0^{-1})) = \alpha^\vee(\varpi^{d+g} \frac{y'^*_{rr}}{\det(Y')}))$, where y'^*_{rr} is the adjoint matrix of the (r,r)-th entry of Y', we have

$$\varphi_{\overline{N}_{0,\kappa}}(zu^{-1}\bar{n}(X',Z')uz^{-1})$$

$$= \varphi_\kappa\left(u_1^{-1} \frac{y'^*_{rr}}{\det(Y')} X' u_2\right)\varphi_\kappa\left(u_1^{-1}\left(\frac{y'^*_{rr}}{\det Y'}\right)^2 Y'\theta_r(u_1)\right)$$

$$= \varphi_\kappa(u_1^{-1}(X'^{-1}Y')_{r1}X'u_2)\varphi_\kappa(u_1^{-1}((Y'^{-1}X')_{r1})^2 Y'\theta_r(u_1))$$

$$= \varphi(\Theta_M(u^{-1})m(X',Z')u).$$

Since the uniqueness of the Bruhat decomposition $\dot{w}_0^{-1}n = mn'\bar{n}$ implies that when $n \mapsto u^{-1}nu$, then $m \mapsto \Theta_M(u)^{-1}mu$. By our calculation, for $u = (u_1, u_2) \in U_M$, $n = n(X,Y) \mapsto m = m(X,Y)$, we have $u^{-1}n(X,Y)u = n(u_1^{-1}Xu_2, u_1^{-1}Y\theta(u_1))$, $\Theta_M(u)^{-1}m(X,Y)u = m(u_1^{-1}Xu_2, u_1^{-1}Y\theta(u_2))$. So both the actions are given by $X \mapsto u_1^{-1}Xu_2$, $Y \mapsto u_1^{-1}Y\theta(u_1)$ Therefore,

$$U^{\Theta_M}_{M,m(X,Y)} = U_{M,n(X,Y)}$$

$$= \{u = (u_1, u_2) \in U_M : u_1^{-1}Xu_2 = X, u_1^{-1}Y\theta(u_1) = Y\}$$

This shows that the centralizer of $n(X,Y)$ in U_M agrees with the Θ_M-twisted centralizer of $m(X,Y)$ in U_M. Compare the definitions of

$$j_{\overline{N}_{0,\kappa},\pi,w}(n(X',Z')) \text{ and } B^M_\varphi(m(X',Z'),f),$$

we obtain the desired equality. $\qquad\square$

Finally, by Proposition 5.2 and Proposition 6.8, we can restate Proposition 5.2 as:

Proposition 6.10. *Let σ and τ be ψ-generic supercuspidal representations of $\mathrm{GL}_r(F)$ and $\mathrm{Sp}_{2m}(F)$ respectively, such that ω_π is ramified where $\pi = \sigma \boxtimes \tau$, then there exists a κ_0, such that for all $\kappa \geq \kappa_0$ and all characters χ of $F^\times$ such that $\omega_{\pi \otimes \chi} = \omega_\pi \chi^r$ is ramified, we have*

$$C_\psi(s, \pi \otimes \chi)^{-1} = \gamma(2rs, \omega_\pi^2 \chi^{2r}, \psi^{-1}) \int_{R_{X'} \times R_{Z'}} B^M_\varphi(m(X', Z'), f)(\omega_\pi^{-2}\chi^{-2r})$$

$$\cdot \left(\frac{P(X', Z')}{\det(Z'J_r + \frac{X'\theta_{r,m}(X')}{2})}\right) \left|\frac{P(X', Z')}{\det(Z'J_r + \frac{X'\theta_{r,m}(X')}{2})}\right|^{-rs}$$

$$\cdot |\det(m_1(X', Z'))|^{s + \frac{r+2m+1}{2}} \, d\mu_{X'} \wedge d\mu_{Z'}$$

where $m(X', Z')$ is the image of $n(X', Z') \mapsto m(X', Z')$ via the Bruhat decomposition $\dot{w}_0^{-1}\dot{n} = mn'\bar{n}$, which holds off a subset of measure zero on $N(F)$, $m = \mathrm{diag}\{m_1, m_2, \theta_r(m_1)\}$ with $m_1 \in \mathrm{GL}_r(F)$, $m_2 \in \mathrm{Sp}_{2m}(F)$, and $\gamma(2rs, \omega_\pi^2 \chi^{2r}, \psi^{-1})$ is the Abelian γ-factor depending only on ω_π and χ.

One important observation in our case is that our orbit space $R' = R_{X'} \times R_{Z'}$ is no longer isomorphic to a torus, and attempts to reparameterize $m(X, Z)$ by the maximal torus together with non-torus parts fail since the map $(X, Z) \mapsto (m_1, m_2)$ is not surjective in general. So in order to apply the asymptotic analysis of partial Bessel integrals, we will need to study the toric action on the orbit space and separate its toric part out from the integral. First we would like to understand the action of the maximal torus $A \simeq A_1 \times A_2$ on $R = R_X \times R_Z = U_M \backslash N$, where A_1 is the maximal torus of GL_r and A_2 is the maximal torus of Sp_{2m}. Recall that through the Bruhat decomposition $\dot{w}_0^{-1} n = mn'\bar{n}$, for $s \in A$, if $n \mapsto sns^{-1}$, then $m \mapsto \Theta_M(s)ms^{-1}$. In our case, for $m = (m_1, m_2)$ with $m_1 \in \mathrm{GL}_r$ and $m_2 \in \mathrm{Sp}_{2m}$, the action is given by $m_1 \mapsto s'm_1\theta_r(s'^{-1})$, and $m_2 \mapsto s''m_2s''^{-1}$, which is equivalent to $X \mapsto s'Xs''^{-1}$, $Y \mapsto s''Y\theta_r(s'')^{-1}$, or to $X \mapsto s'Xs''^{-1}$, $Z \mapsto s'Zs'$, where $s = (s', s'')$ with $s' \in A_1$ and $s'' \in A_2$. Write $s' = \mathrm{diag}(s_1, s_2, \ldots, s_r) \in A_1, s' = \mathrm{diag}(s_{r+1}, s_{r+2}, \ldots, s_{r+m}, s_{r+m}{}^{-1}, \ldots, s_{r+1}{}^{-1}) \in A_2$. Define the subspace $T_{X,Z}$:

$$\begin{cases} \{(x_{r,1}, x_{r-1,2} \ldots, x_{r-m+1,m}, z_{1,1}, \ldots, z_{r,r})\}, & \text{if } r \geq m, \\ \{(x_{r,1}, x_{r-1,2}, \ldots, x_{1,r}, z_{1,1}, \ldots, z_{r,r})\}, & \text{if } r < m. \end{cases}$$

in which the corresponding $(X, Z) \in R_X \times R_Z$. Note that in the definition of $T_{X,Z}$ we always use the notation $X = (x_{i,j})$ and $Z = (z_{i,j})$ and the entries which are not indicated should understood to be zero.

Proposition 6.11. *Let $\Phi : R_X \times R_Z \to M \to A = A_1 \times A_2$ be the map given by $(X, Z) \mapsto m = m(X, Z) = (m_1, m_2) \mapsto (t_1, t_2)$, where $m(X, Z)$ is the image of $n(X, Z)$ under the map $n \mapsto m$ via $\dot{w}_0^{-1}n = mn'\bar{n}$, and $m_1 = u_1\dot{w}_1 t_1 u_2$, $m_2 = v_1\dot{w}_2 t_2 v_2$ are the Bruhat decomposition of m_1 and m_2 respectively. Then Φ is compatible with the toric action, i.e., $\Phi(s'Xs''^{-1}, s'Zs') = (m_1(s'Xs''^{-1}, s'Zs'), m_2(s'Xs''^{-1}, s'Zs'))$ for $s = (s', s'') \in A$. The toric action on $R_X \times R_Z$ is completely determined by its restriction on $T_{X,Z}$, up to the square of each s_i ($1 \leq i \leq r + \min\{r, m\}$). Moreover, the restriction $\Phi|_{T_{X,Z}}$ is a finite étale map onto its image in A.*

Proof. The compatibility of Φ with the toric action follows directly from

$$\Theta_M(s)m(X, Z)s^{-1} = m(s'Xs''^{-1}, s'Zs').$$

Recall that we also write $m = m(X, Z) = m(X, Y)$ where $Y = ZJ_r + \frac{X\theta_{r,m}(X)}{2}$. We claim that if we inductively write $Y = Y_r = \begin{bmatrix} \alpha & y_{1,r} \\ Y_{r-1} & {}^t\beta \end{bmatrix}$ and

$Y = Y^{(r)} = \begin{bmatrix} {}^t\alpha' & Y^{(r-1)} \\ y_{r,1} & \beta' \end{bmatrix}$, and assume that $\det(Y_i)$'s and $\det(Y^{(j)})$'s are all non-zero, then the image $m(X, Y)$ of $n(X, Y)$ lies in the big cell of M. If we decompose $m_1 = u_1 \dot{w}_1 t_1 u_2$, $m_2 = v_1 \dot{w}_2 t_2 v_2$ with $\dot{w}_1 = J_r$, $\dot{w}_2 = J'_{2m}$, the long Weyl group elements of GL_r and Sp_{2m} respectively, we will show that

$$t_1 = \mathrm{diag}\left((-1)^{r-1}\frac{\det(Y_{r-1})}{\det(Y_r)}, (-1)^{r-2}\frac{\det(Y_{r-2})}{\det(Y_{r-1})}, \ldots, \frac{1}{\det(Y_1)} \right)$$

and $t_2 = \mathrm{diag}(\bar{t}_2, \bar{t}_2^{-1})$ with $\bar{t}_2$ the diagonal vector given by:

- if $r < m$,

$$\bar{t}_2 = \left(-\frac{\det(Y^{(r-1)})}{\det(Y^{(r)})} x_{r,1}^2, (-1)^2 \frac{\det(Y^{(r-2)})}{\det(Y^{(r-1)})} x_{r-1,2}^2, \ldots, (-1)^{r-1}\frac{1}{\det(Y^{(1)})} x_{1,r}^2, 1, \ldots, 1 \right),$$

- if $r \geq m$,

$$\bar{t}_2 = \left(-\frac{\det(Y^{(r-1)})}{\det(Y^{(r)})} x_{r,1}^2, (-1)^2 \frac{\det(Y^{(r-2)})}{\det(Y^{(r-1)})} x_{r-1,2}^2, \ldots, (-1)^m \frac{\det(Y^{(r-m)})}{\det(Y^{(r-m+1)})} x_{r-m+1,m}^2 \right).$$

Let us first prove this for t_1 by induction on r. If $r = 1$ then there is nothing to show. Assume $r > 1$ and write $Y_r = \begin{bmatrix} \alpha & y_{1,r} \\ Y_{r-1} & {}^t\beta \end{bmatrix}$. Take the representative of the long Weyl group element in GL_r as J_r. Then $m_1 = \theta_r(Y) = u_1 J_r t_1 u_2 \Leftrightarrow \tilde{u}_1 Y \tilde{u}_2 = t_1^{-1} J_r$, where $\tilde{u}_i = \theta_r(u_i)^{-1} (i = 1, 2)$. Write $\tilde{u}_1 = \begin{bmatrix} 1 & \delta_1 \\ & u'_1 \end{bmatrix}$, and $\tilde{u}_2 = \begin{bmatrix} u'_2 & {}^t\delta_2 \\ & 1 \end{bmatrix}$, with $u'_i \in U_{\mathrm{GL}_{r-1}}$, $\delta_i \in \mathbb{A}^{r-1}$. Then

$$\tilde{u}_1 Y \tilde{u}_2 = \begin{bmatrix} (\alpha + \delta_1 Y_{r-1})u'_2 & (\alpha + \delta_1 Y_{r-1})^t\delta_2 + (y_{1,r} + \delta_1{}^t\beta) \\ u'_1 Y_{r-1} u'_2 & u'_1 Y_{r-1}{}^t\delta_2 + u'_1\beta \end{bmatrix}.$$

Assume $\det(Y_{r-1}) \neq 0$, $y_{1,r} \neq 0$, choose $\delta_1 = -\alpha Y_{r-1}^{-1}$, ${}^t\delta_2 = -Y_{r-1}^{-1}\beta$, write $t_1 = \mathrm{diag}\{a_1, \ldots, a_r\}$ then $\tilde{u}_1 Y \tilde{u}_2 = t_1^{-1} J_r \Leftrightarrow \begin{bmatrix} 0 & y_{1,r} - \alpha Y_{r-1}^{-1}{}^t\beta \\ u'_1 Y_{r-1} u'_2 & 0 \end{bmatrix} = \begin{bmatrix} & & a_1^{-1} \\ & \cdot^{\cdot^{\cdot}} & \\ (-1)^{r-1}a_r & & \end{bmatrix}$. By induction hypothesis, it suffices to show that $y_{1,r} - \alpha Y_{r-1}^{-1}{}^t\beta = a_1^{-1}$. Note that

$$\det(Y_r) = \det\begin{bmatrix} \alpha & y_{1,r} \\ Y_{r-1} & {}^t\beta \end{bmatrix} = (-1)^{r-1}\det\begin{bmatrix} Y_{r-1} & {}^t\beta \\ \alpha & y_{1,r} \end{bmatrix}.$$

On the other hand, $\begin{bmatrix} Y_{r-1}^{-1} & \\ & 1 \end{bmatrix}\begin{bmatrix} Y_{r-1} & {}^t\beta \\ \alpha & y_{1,r} \end{bmatrix} = \begin{bmatrix} I_{r-1} & Y_{r-1}^{-1}{}^t\beta \\ \alpha & y_{1,r} \end{bmatrix}$. Taking determinant on both sides, we obtain that

$$(-1)^{r-1}\det(Y_r)\det(Y_{r-1})^{-1} = \det\begin{bmatrix} I_{r-1} & Y_{r-1}^{-1}{}^t\beta \\ \alpha & y_{1,r} \end{bmatrix}$$

$$= \det\begin{bmatrix} I_{r-1} & Y_{r-1}^{-1}{}^t\beta \\ 0 & -\alpha Y_{r-1}^{-1}{}^t\beta + y_{1,r} \end{bmatrix} = y_{1,r} - \alpha Y_{r-1}^{-1}{}^t\beta,$$

i.e., $a_1 = (-1)^{r-1} \frac{\det(Y_{r-1})}{\det(Y_r)}$.

Next, we work on the formula for t_2. Let us first assume that $r \geq m$ and we prove the formula by induction on m. When $m = 1$, the result follows from direct calculation. Assume $m > 1$ and suppose $m_2 = v_1 J'_{2m} t_2 v_2$, then $m_2^{-1} = I_{2m} - J'_{2m}{}^t X^t Y^{-1} J_r X = -v_2^{-1} t_2^{-1} J'_{2m} v_1^{-1}$, which is equivalent to $v_2(I_{2m} - J'_{2m}{}^t X^t Y^{-1} J_r X)v_1 = -t_2^{-1} J'_{2m}$. Write $v_i = \begin{bmatrix} 1 & \gamma_i & b_i \\ & v'_i & \gamma_i^* \\ & & 1 \end{bmatrix}, (i = 1, 2)$, $X = \begin{bmatrix} 0 & X_1 & {}^t\gamma \\ x_{r,1} & 0 & 0 \end{bmatrix}$, $Y = Y^{(r)} = \begin{bmatrix} {}^t\alpha' & Y^{(r-1)} \\ y_{r,1} & \beta' \end{bmatrix}$. Assume that each $Y^{(j)}, (1 \leq j \leq r)$ is invertible, $y_{r,1} \neq 0$, and set ${}^t(Y^{(r)})^{-1} = \begin{bmatrix} {}^t\delta'_1 & H \\ x & \delta'_2 \end{bmatrix}$. Then one computes that

$$I_{2m} - J'_{2m}{}^t X^t Y^{-1} J_r X$$

$$= I_{2m} - \begin{bmatrix} & & 1 \\ & -J'_{2m-2} & \\ -1 & & \end{bmatrix} \begin{bmatrix} 0 & x_{r,1} \\ {}^t X_1 & 0 \\ \gamma & 0 \end{bmatrix} \begin{bmatrix} {}^t\delta'_1 & H \\ x & \delta'_2 \end{bmatrix} \begin{bmatrix} & & 1 \\ & -J_{r-1} & \\ \end{bmatrix} \begin{bmatrix} 0 & X_1 & {}^t\gamma \\ x_{r,1} & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 - \gamma^t\delta'_1 x_{r,1} & \gamma H J_{r-1} X_1 & \gamma H J_{r-1}{}^t\gamma \\ J'_{2m-2}{}^t X_1 {}^t\delta'_1 x_{r,1} & I_{2m-2} - J'_{2m-2}{}^t X_1 H J_{r-1} X_1 & -J'_{2m-2}{}^t X_1 H J_{r-1}{}^t\gamma \\ x_{r,1}^2 x & -x_{r,1}\delta'_2 J_{r-1} X_1 & 1 - x_{r,1}\delta'_2 J_{r-1}{}^t\gamma \end{bmatrix}$$

By definition of ${}^t(Y^{(r)})^{-1}$, we have

$$\alpha'^t\delta'_1 + y_{r,1}x = 1, \qquad \alpha' H + y_{r,1}\delta'_2 = 0,$$
$${}^t Y^{(r-1)t}\delta'_1 + {}^t\beta' x = 0, \qquad {}^t Y^{(r-1)} H + {}^t\beta'\delta'_2 = I_{r-1}.$$

Multiply by v_2 on the left and v_1 on the right, and compare with

$$-t_2^{-1} J'_{2m} = \begin{bmatrix} & & a_{r+1}^{-1} \\ & -t_2'^{-1} J'_{2m-2} & \\ -a_{r+1} & & \end{bmatrix},$$

where we write

$$t_2 = \operatorname{diag}\{a_{r+1}, \ldots, a_n, a_n^{-1}, \ldots, a_{r+1}^{-1}\} = \operatorname{diag}\{a_{r+1}, t_2', a_{r+1}^{-1}\}.$$

The middle entry of $v_2(I_{2m} - J'_{2m}{}^t X^t Y^{-1} J_r X)v_1$ gives

$$v_2'(I_{2m-2} - J'_{2m-2}{}^t X_1 {}^t(Y^{(r-1)})^{-1} J_{r-1} X_1)v_1'$$
$$+ v_2' J'_{2m-2}{}^t X_1 {}^t(Y^{(r-1)})^{-1t}\beta'\delta'_2 J_{r-1} X_1 v_1' + \gamma_2^*(-x_{r,1}\delta'_2 J_{r-1} X_1)v_1'$$
$$= -t_2'^{-1} J'_{2m-2}.$$

Therefore we have to show that

$$(*) \quad v_2' J'_{2m-2}{}^t X_1 {}^t(Y^{(r-1)})^{-1t}\beta'\delta'_2 J_{r-1} X_1 v_1' + \gamma_2^*(-x_{r,1}\delta'_2 J_{r-1} X_1)v_1' = 0$$

in order to continue with our induction procedure. Since ${}^t Y^{(r-1)t}\delta'_1 + {}^t\beta' x = 0$, we have ${}^t\delta'_1 + {}^t(Y^{(r-1)})^{-1t}\beta' x = 0$, thus ${}^t\delta'_1\delta'_2 + {}^t(Y^{(r-1)})^{-1t}\beta'\delta'_2 x = 0$.

We also note that $\gamma_2^* = v_2' J_{2m-2}'{}^t\gamma_2$ by the structure of Sp_{2m-2}. So the left hand side of $(*)$ is equal to

$$v_2' J_{2m-2}'(-{}^t X_1 {}^t\delta_1' \delta_2' - {}^t\gamma_2 x_{r,1}\delta_2')J_{r-1}X_1 v_1'.$$

On the other hand, one computes that the $(2,1)$-th entry of $v_2(I_{2m} - J_{2m}'{}^t X^t Y^{-1} J_r X)v_1$ gives

$$v_2' J_{2m-2}'({}^t X_1 {}^t\delta_1' x_{r,1} + {}^t\gamma_2 x_{r,1}^2 x) = 0,$$

hence ${}^t X_1 {}^t\delta_1' x_{r,1} + {}^t\gamma_2 x_{r,1}^2 x = 0$. As a result,

$$ {}^t X_1 {}^t\delta_1' \delta_2' x_{r,1} + {}^t\gamma_2 x_{r,1}^2 \delta_2' x = 0.$$

Since $x_{r,1} \neq 0$, this shows that the left hand side of $(*)$ is equal to 0.

By induction hypothesis, it suffices to show that $a_{r+1} = -x_{r,1}^2 \frac{\det(Y^{(r-1)})}{\det(Y^{(r)})}$. Note that the $(3,1)$-th entry of $v_2(I_{2m} - J_{2m}'{}^t X^t Y^{-1} J_r X)v_1$ implies that $x_{r,1}^2 x = -a_{r+1}$. One also checks by direct calculation that other identities in the comparison of entries in $v_2(I_{2m} - J_{2m}'{}^t X^t Y^{-1} J_r X)v_1$ and $-t_2^{-1}J_{2m}'$ are automatically satisfied. On the other hand, $\alpha'^t\delta_1' + y_{r,1}x = 1$ together with ${}^t\delta_1 = -{}^t(Y^{(r-1)})^{-1t}\beta' x$ imply that $x(y_{r,1} - \alpha'^t(Y^{(r-1)})^{-1t}\beta') = 1$. We have

$$\frac{\det(Y^{(r)})}{\det(Y^{(r-1)})} = \det\left(Y^{(r)}\begin{bmatrix} (Y^{(r-1)})^{-1} & \\ & 1 \end{bmatrix}\right)$$

$$= \det\left(\begin{bmatrix} Y^{(r-1)} & {}^t\alpha' \\ \beta' & y_{r,1} \end{bmatrix}\begin{bmatrix} (Y^{(r-1)})^{-1} & \\ & 1 \end{bmatrix}\right) = \det\begin{bmatrix} I_{r-1} & {}^t\alpha_1 \\ \beta'(Y^{(r-1)})^{-1} & y_{r,1} \end{bmatrix}$$

$$= \det\begin{bmatrix} I_{r-1} & {}^t\alpha' \\ 0 & -\beta'(Y^{(r-1)})^{-1t}\alpha' + y_{r,1} \end{bmatrix} = y_{r,1} - \alpha'^t(Y^{(r-1)})^{-1t}\beta'.$$

Hence by assuming that $y_{r,1} - \alpha'^t(Y^{(r-1)})^{-1t}\beta' \neq 0$, we obtain that $a_{r+1} = -x_{r,1}^2 \frac{\det(Y^{(r-1)})}{\det(Y^{(r)})}$ as desired.

When $r < m$, recall that the toric action has a stabilizer isomorphic to the maximal torus of $\mathrm{Sp}_{2(m-r)}$, given by the entries $(s_{2r+1}, \ldots, s_{r+m})$. So the above induction process terminates at the r-th step. As a result, the formula directly follows from the same procedure.

Given the explicit formula for t_1 and t_2 in the Bruhat decomposition, when restricted to $T_{X,Z}$, the map Φ is completely determined by $Y \mapsto \det(Y_i)$ $(1 \leq i \leq r)$, $Y \mapsto \det(Y^{(i)})$ $(1 \leq i \leq r)$, $X \mapsto x_{r-j+1,j}^2$ $(1 \leq j \leq \min(r,m))$, which are clearly finite *étale* maps since we assumed that $\det(Y^i)$'s, $\det(Y^{(i)})$'s, and $x_{r-j+1,j}$'s are all non-zero. The restriction of the toric action $X \mapsto s'Xs''^{-1}$, $Z \mapsto s'Zs'$, or equivalently, $X \mapsto s'Xs''^{-1}$, $Y \mapsto s'Y\theta_r(s')^{-1}$, on $T_{X,Z}$ is given by $x_{r-i+1,i} \mapsto s_{r-i+1}s_{r+i}^{-1}x_{r-i+1,i}$, $z_{j,j} \mapsto s_j^2 z_{j,j}$ for $1 \leq i \leq m$ if $r \geq m$, $1 \leq i \leq r$ if $r < m$, and $1 \leq j \leq r$.

Moreover, when $r < m$, the action has a non-trivial stabilizer isomorphic to the maximal torus of $\mathrm{Sp}_{2(m-r)}$, given by the variables $(s_{2r+1}, \ldots, s_{r+m})$. A simple calculation shows that it takes $\det(Y_i)$ to $\prod_{k=1}^r s_{r-k+1}^2 \det(Y_i)$, $\det(Y^i)$ to $\prod_{k=1}^i s_i^2 \det(Y^{(i)})$, and $x_{r-k+1,k}$ to $\frac{s_{r-k+1}}{s_{r+k}} x_{r-k+1}$. Together with the compatibility of the toric action, it implies that

$$t_1 \longmapsto s'^{-2} t_1, t_2' \longmapsto \begin{cases} \mathrm{diag}\{s_{r+1}^{-2}, \ldots, s_{2r}^{-2}, 1, \ldots, 1, \} t_2', & \text{if } r < m, \\ s''^{-2} t_2', & \text{if } r \geq m. \end{cases}$$

Consequently, the toric action on $R_X \times R_Z$ is completely determined by its action on $T_{X,Z}$ up to the square of each s_i ($1 \leq i \leq r + \min\{r, m\}$). It follows that the covering group is isomorphic to finitely many copies of $\mathbb{Z}/2\mathbb{Z}$. $\qquad\square$

Finally, in order to apply our uniform smooth results for partial Bessel integrals, we need to verify the non-singular condition in Proposition 6.8:

Proposition 6.12. *In the Bruhat decompositions $m_1 = u_1 \dot{w}_1 t_1 u_2$, and $m_2 = v_1 \dot{w}_2 t_2 v_2$, the entries in the 1-parameter subgroups for u_1, u_2, v_1, v_2 are rational functions of the image of $T_{X,Z}$ under Φ without singularities.*

Proof. Based on the calculation in Proposition 6.11, it suffices to show that the denominator of entries in u_1, u_2, v_1, v_2 are polynomial functions of $\det(Y^{(i)})$, $\det(Y_i)$ and $x_{r-i+1,i}$ without singularities. In fact, we show that they are monomials of these factors. We start with $m_1 = u_1 w_1 t_1 u_2$. Denote $u_1 = (u_{ij}^1), u_2 = (u_{ij}^2), Y = (y_{ij})$. Let

$$Y_{i,j} = Y[r - j + 1, r - j + 2, \ldots, \widehat{r - i + 1} \ldots, r; 1, \ldots, j - 1],$$

the matrix of size $(j-1) \times (j-1)$ with elements from the indicated columns and rows of Y. We will show by induction on r that

$$u_{ij}^1 = \frac{\det(Y_{i,j})}{\det(Y_{j-1})}.$$

When $r = 1$ there is nothing to show. Suppose $r > 1$, write $u_1 = \begin{bmatrix} u_1 & {}^t\delta_1 \\ & 1 \end{bmatrix}$, $u_2 = \begin{bmatrix} 1 & \delta_2 \\ & u_2' \end{bmatrix}$, $t = \mathrm{diag}\{a_1, \ldots, a_r\} = \mathrm{diag}\{a_1, t_1'\}$, and $Y = \begin{bmatrix} \alpha & y_{1,r} \\ Y_{r-1} & {}^t\beta \end{bmatrix}$. Note that $m_1 = \theta_r(Y) = u_1 J_r t_1 u_2 \Leftrightarrow Y = Y_r = J_r {}^t u_1^{-1} J_r^{-1} t_1^{-1} {}^t u_2^{-1} J_r$. We

compute that

$$\begin{bmatrix} \alpha & y_{1,r} \\ Y_{r-1} & {}^t\beta \end{bmatrix} = \begin{bmatrix} & 1 \\ -J_{r-1} & \end{bmatrix} \begin{bmatrix} {}^t u'^{-1}_1 & \\ -\delta_1 {}^t u'^{-1}_1 & 1 \end{bmatrix} \begin{bmatrix} & -J^{-1}_{r-1} \\ 1 & \end{bmatrix}$$

$$\cdot \begin{bmatrix} a^{-1}_1 & \\ & t'^{-1}_1 \end{bmatrix} \begin{bmatrix} 1 & \\ -{}^t u'^{-1}_2 {}^t \delta_2 & {}^t u'^{-1}_2 \end{bmatrix} \begin{bmatrix} & 1 \\ -J_{r-1} & \end{bmatrix}$$

$$= \begin{bmatrix} -\delta_1 {}^t u'^{-1}_1 J^{-1}_{r-1} t'^{-1}_1 {}^t u'^{-1}_2 J_{r-1} & a^{-1}_1 - \delta_1 {}^t u'^{-1}_1 J^{-1}_{r-1} t'^{-1}_1 {}^t u'^{-1}_2 {}^t \delta_2 \\ -J_{r-1} {}^t u'^{-1}_1 J^{-1}_{r-1} t'^{-1}_1 {}^t u'^{-1}_2 J_{r-1} & -J_{r-1} {}^t u'^{-1}_1 J^{-1}_{r-1} t'^{-1}_1 {}^t u'^{-1}_2 {}^t \delta_2 \end{bmatrix}.$$

Hence $Y_{r-1} = -J_{r-1} {}^t u'^{-1}_1 J^{-1}_{r-1} t'^{-1}_1 {}^t u'^{-1}_2 J_{r-1}$. Compare this with the expression for Y_r, we replace $Y = Y_r$ by $-Y_{r-1}$ and continue by induction. Suppose we show that $u^1_{i,r} = \frac{\det(Y_{i,r})}{\det(Y_{r-1})}$, $(1 \le i \le r-1)$, then the induction hypothesis would imply that

$$u^1_{i,r-1} = u'^1_{i,r-1} = \frac{\det((-Y_{r-1})_{i,r-1})}{\det((-Y_{r-1})_{r-2})}$$

$$= \frac{(-1)^{r-2} \det(Y_{i,r-1})}{(-1)^{r-2} \det(Y_{r-2})} = \frac{\det(Y_{i,r-1})}{\det(Y_{r-2})}, \ (1 \le i \le r-2),$$

since

$$(Y_{r-1})_{i,r-1} = Y_{r-1}[1, \ldots, \widehat{(r-1)-i+1}, \ldots, r-1; 1, \ldots, r-2]$$

$$= Y_r[2, \ldots, \widehat{r-i+1}, \ldots, r; 1, \ldots, r-2] = Y_{i,r-1}.$$

Therefore it suffices to show that $u^1_{i,r} = \frac{\det(Y_{i,r})}{\det(Y_{r-1})}$ for $1 \le i \le r-1$. From the above calculation we also have $\alpha = -\delta_1 J^{-1}_{r-1} Y_{r-1}$, ${}^t\beta = -Y_{r-1} J^{-1}_{r-1} \delta_2$. So

$$\delta_1 = -\alpha Y^{-1}_{r-1} J_{r-1} = (u^1_{1,r}, u^1_{2,r}, \ldots, u^1_{r-1,r}),$$

$$\delta_2 = -\beta^t Y^{-1}_{r-1} J^{-1}_{r-1} = (u^2_{1,2}, u^2_{1,3}, \ldots, u^2_{1,r}).$$

We will show the formula for u_1 first. By the construction of $Y_{i,j}$, we expand the determinant of Y along the last column to get

$$\det(Y_r) = (-1)^{r+1} y_{1,r} \det(Y_{r-1}) + (-1)^{r+2} y_{2,r} \det(Y_{r-1,r})$$

$$+ \cdots + (-1)^{2r} y_{r,r} \det(Y_{1,r}).$$

Since $\beta = (y_{2,r}, \ldots, y_{r-1,r})$, if we set

$$\delta = \left((-1)^{r+2} \frac{\det(Y_{r-1,r})}{\det(Y_{r-1})}, (-1)^{r+3} \frac{\det(Y_{r-2,r})}{\det(Y_{r-2})}, \right.$$

$$\left. \ldots, (-1)^{r+l+1} \frac{\det(Y_{r-l,r})}{\det(Y_{r-1})}, \ldots, (-1)^{2r} \frac{\det(Y_{1,r})}{\det(Y_{r-1})} \right),$$

then the above formula is equivalent to $\det(Y_r) = (-1)^{r+1} y_{1,r} \det(Y_{r-1}) + \det(Y_{r-1}) \delta^t \beta$. On the other hand, we have

$$\begin{bmatrix} Y_{r-1}^{-1} & \\ & 1 \end{bmatrix} \begin{bmatrix} Y_{r-1} & {}^t\beta \\ \alpha & y_{1,r} \end{bmatrix} = \begin{bmatrix} I_{r-1} & Y_{r-1}^{-1}{}^t\beta \\ \alpha & y_{1,r} \end{bmatrix}.$$

Taking determinant on both sides, we obtain that

$$(-1)^{r-1} \det(Y_r) \det(Y_{r-1})^{-1} = \det \begin{bmatrix} I_{r-1} & Y_{r-1}^{-1}{}^t\beta \\ \alpha & y_{1,r} \end{bmatrix}$$

$$= \det \begin{bmatrix} I_{r-1} & Y_{r-1}^{-1}{}^t\beta \\ 0 & -\alpha Y_{r-1}^{-1}{}^t\beta + y_{1,r} \end{bmatrix} = y_{1,r} - \alpha Y_{r-1}^{-1}{}^t\beta.$$

Hence $\det(Y_r) = (-1)^{r-1} y_{1,r} \det(Y_{r-1}) + (-1)^r \det(Y_{r-1}) \alpha Y_{r-1}^{-1}{}^t\beta$. Consequently, $((-1)^r \alpha Y_{r-1}^{-1} - \delta)^t \beta = 0$. Note that our formula works for any β, therefore $\delta = (-1)^r \alpha Y_{r-1}^{-1}$. As $\delta_1 = -\alpha Y_{r-1}^{-1} J_{r-1}$, we have $\delta = (-1)^{r-1} \delta_1 J_{r-1}^{-1} = \delta_1 J_{r-1}$. It follows that

$$(-1)^{r+l+1} \frac{\det(Y_{r-l,r})}{\det(Y_{r-1})} = (-1)^{r-l-1} u_{r-l,r}^1$$

for each $1 \le l \le r-1$. i,e., $u_{i,r}^1 = \frac{\det(Y_{i,r})}{\det(Y_{r-1})}$ for $1 \le i \le r-1$. This completes the proof for the formula for u_1.

Now we turn to the formula for u_2. Let

$$Y'_{ij} = Y[(i+1), \ldots, r; 1, \ldots, \widehat{r-j+1}, \ldots, r-i+1]$$

which is of size $(r-i) \times (r-i)$. We will show that

$$u_{ij}^2 = \frac{\det(Y'_{ij})}{\det(Y_{r-i})}.$$

Similar to the proof for u_1, since $Y'_{1,1} = Y_{r-1}$, if we expand $\det(Y)$ along the first row, we get

$$\det(Y) = \det(Y_r) = \sum_{j=1}^{r} (-1)^{j+1} y_{1,j} \det(Y'_{1,r-j+1})$$

$$= (-1)^{r+1} y_{1,r} \det(Y_{r-1}) + \sum_{j=1}^{r-1} y_{1,j} \det(Y'_{1,r-j+1}).$$

Set $\delta' = ((-1)^{j+1} \frac{\det(Y'_{1,r-j+1})}{\det(Y_{r-1})})_{j=1}^{r-1}$, then $\frac{\det(Y_r)}{\det(Y_{r-1})} = (-1)^{r+1} y_{1,r} + \alpha^t \delta'$.

On the other hand, by the above argument we still have

$$(-1)^{r-1} \det(Y_r) \det(Y_{r-1})^{-1} = y_{1,r} - \alpha Y_{r-1}^{-1}{}^t\beta.$$

So $\alpha((-1)^{rt}\delta' - Y_{r-1}^{-1}{}^t\beta) = 0$. Since this works for any α, we obtain that $(-1)^{rt}\delta' = Y_{r-1}^{-1}\beta$. Note that the formula $\delta_2 = -\beta^t Y_{r-1}^{-1} J_{r-1}^{-1}$ is equivalent to $\delta' = (-1)^{r-1}\delta_2 J_{r-1}$. As $\delta_2 = (u_{1,2}, \ldots, u_{1,r})$, this means that $(-1)^{j+1}\frac{\det(Y'_{1,r-j+1})}{\det(Y_{r-1})} = (-1)^j u_{1,r-j+1}^2$, $(1 \le j \le r-1)$ and therefore $u_{1,j}^2 = \frac{\det(Y'_{1,j})}{\det(Y_{r-1})}$, $(1 \le j \le r-1)$. Then a similar induction argument as the proof of formulas for entries of u_1 implies that $u_{ij}^2 = \frac{\det(Y'_{i,j})}{\det(Y_{r-1})}$ for all (i, j).

For entries of v_1 and v_2, recall that in the proof of Proposition 6.11, we wrote $v_i = \begin{bmatrix} 1 & \gamma_i & b_i \\ v'_i & \gamma_i^* \\ & & 1 \end{bmatrix}$, $(i = 1, 2)$, $X = \begin{bmatrix} 0 & X_1 & {}^t\gamma \\ x_{r,1} & 0 & 0 \end{bmatrix}$, $Y = Y^{(r)} = \begin{bmatrix} {}^t\alpha' & Y^{(r-1)} \\ y_{r,1} & \beta' \end{bmatrix}$. We assumed that $\det(Y^{(j)}) \ne 0$, $(1 \le j \le r)$, $y_{r,1} \ne 0$, and wrote ${}^t(Y^{(r)})^{-1} = \begin{bmatrix} {}^t\delta'_1 & H \\ x & \delta'_2 \end{bmatrix}$. It follows immediately from the computation of $v_2(I_{2m} - J'_{2m}{}^tX^tY^{-1}J_rX)v_1 = -t_2^{-1}J'_{2m}$ that

(6.1) $\qquad 1 - \gamma^t\delta'_1 x_{r,1} + \gamma_2 J'_{2m-2}{}^tX_1{}^t\delta'_1 x_{r,1} + b_2 x_{r,1}^2 x = 0,$

(6.2) $\gamma H J_{r-1} X_1 + \gamma_2(I_{2m-2} - J'_{2m-2}{}^tX_1 H J_{r-1} X_1) - b_2 x_{r,1}\delta'_2 J_{r-1} X_1 = 0.$

Multiplying (6.1) by $\delta'_2 J_{r-1} X_1$, (6.2) by $x_{r,1}x$, and taking their sum, we get

$$\gamma_2(J'_{2m-2}{}^tX_1({}^t\delta'_1\delta'_2 - Hx)J_{r-1}X_1 x_{r,1} + x_{r,1}x I_{2m-2})$$
$$+ \delta'_2 J_{r-1} X_1 - \gamma({}^t\delta_1{}'\delta'_2 - Hx)J_{r-1}X_1 x_{r,1} = 0.$$

Since ${}^tY^{(r-1)}H + {}^t\beta\delta'_2 = I_{r-1}$, it follows that ${}^t\delta'_1\delta'_2 - Hx = -{}^t(Y^{(r-1)})^{-1}x$. By induction hypothesis, $I_{2m-2} - J'_{2m-2}{}^tX_1{}^tY^{(r-1)-1}J_{r-1}X_1 = -t'_2 J'_{2m-2}$, where we write $-t_2^{-1}J'_{2m} = \begin{bmatrix} & & a_{r+1}^{-1} \\ & -t_2'^{-1}J'_{2m-2} & \\ -a_{r+1} & & \end{bmatrix}$. Therefore

$$\gamma_2 = -x_{r,1}^{-1}x^{-1}(\delta'_2 J_{r-1}X_1 + \gamma^t(Y^{(r-1)})^{-1}J_{r-1}X_1 x x_{r,1})v'_1 t'_2 J'_{2m-2}v'_2.$$

On the other hand, since x is the $(r,1)$-th entry of ${}^tY^{-1}$, and recall that ${}^tY = \begin{bmatrix} {}^t\alpha & {}^tY_{r-1} \\ y_{1,r} & \beta \end{bmatrix}$. By computing the inverse of a matrix using its adjoint matrix, we have $x = (-1)^{r+1}\frac{\det(Y_{r-1})}{\det(Y^{(r)})}$. Since ${}^tY^{-1} = \begin{bmatrix} {}^t\delta'_1 & H \\ x & \delta'_2 \end{bmatrix}$, the common denominator of entries in δ'_2 is still $\det(Y^{(r)})$ for the same reason. Similarly ${}^t(Y^{(r-1)})^{-1} = \frac{({}^tY^{(r-1)})^*}{\det(Y^{(r-1)})}$, where $({}^tY^{(r-1)})^*$ is the adjoint matrix of ${}^tY^{(r-1)}$. In addition, by induction hypothesis and the computation in Proposition 6.11, denominators of entries in v'_1, v'_2, and t'_2 are all monomials of $\det(Y^{(j)}), \det(Y_j), (j \le r-1)$, and $x_{r-i+1,i}, (2 \le i \le \min\{r, m\})$. Moreover, since $\gamma_2^* = v'_2 J'_{2m-2}{}^t\gamma_2$, and by (6.1), with the fact that the common denominators of entries in δ'_1 is still $\det(Y^{(r)})$, we conclude that

the denominators of entries in γ_2, γ_2^*, and b_2 are monomials of $\det(Y_i)$, $\det(Y^{(i)})$, $(1 \le i \le r)$, and $x_{r-i+1,i}$, $(1 \le i \le \min\{r, m\})$.

The same computation of $v_2(I_{2m} - J'_{2m}{}^t X^t Y^{-1} J_r X) v_1 = -t_2^{-1} J'_{2m}$ also implies that

$$(6.3) \qquad x_{r,1}^2 x \gamma_1 - x_{r,1} \delta_2' J_{r-1} X_1 v_1' = 0,$$

$$(6.4) \qquad x_{r,1}^2 x b_1 - x_{r,1} \delta_2' J_{r-1} X_1 \gamma_1^* + 1 - x_{r,1} \delta_2' J_{r-1}{}^t \gamma = 0.$$

From the previous argument on x, δ_1', δ_2', and the fact that $\gamma_1^* = v_1' J'_{2m-2}{}^t \gamma_1$, the formula (6.3) and (6.4) imply the same conclusion for denominators of entries in γ_1, γ_1^*, and b_1. $\qquad \square$

7. Proof of Stability

In the final section, we prove Theorem 1.1 by proving the stability for the corresponding local coefficient. By the structure theorems it suffices to prove stability of local coefficient attached to ψ-generic supercuspidal representations. Given a ψ-generic supercuspidal representation σ of $\mathrm{GL}_r(F)$ and ψ-generic supercuspidal representation τ_1, τ_2 of $\mathrm{Sp}_{2m}(F)$. Suppose χ is a continuous character of $F^\times$. Theorem 1.1 states that

$$\gamma(s, (\sigma_1 \times \tau_1) \otimes \chi, \psi) = \gamma(s, (\sigma \times \tau_2) \otimes \chi, \psi)$$

when the conductor of χ is sufficiently large.

Denote $\pi_i = \sigma \boxtimes \tau_i$, then $\omega := \omega_{\pi_1} = \omega_{\pi_2}$. As stated in the end of Section 2,

$$C_\psi(s, \pi_i) = \gamma(s, \tilde\sigma \times \tilde\tau_i, \psi^{-1}) \cdot \gamma(s, \tilde\sigma, \wedge^2, \psi^{-1})$$

Therefore it suffices to show that

$$C_\psi(s, \pi_1 \otimes \chi) = C_\psi(s, \pi_2 \otimes \chi)$$

when the conductor of χ is sufficiently large.

Choose f_1 and f_2 to be matrix coefficients of π_1 and π_2 respectively, normalized so that $W_{f_1}(e) = W_{f_2}(e) = 1$. We also choose κ sufficiently large so that Proposition 5.3 and Proposition 6.7 hold for both f_1, f_2 and our fixed auxiliary function f_0 in Proposition 6.7, where $\varphi_{\overline{N}_0, \kappa}$ and φ are related via Proposition 6.9. By Proposition 6.10,

$$C_\psi(s, \pi_1 \otimes \chi)^{-1} - C_\psi(s, \pi_2 \otimes \chi)^{-1} = \gamma(2rs, \omega^2 \chi^{2r}, \psi^{-1}) D_\chi(s)$$

where

$$D_\chi(s) = \int_{R_{X'} \times R_{Z'}} (B_\varphi^M(m(X',Z'), f_1) - B_\varphi^M(m(X',Z'), f_2))$$

$$\cdot (\omega^{-2}\chi^{-2r}) \left(\frac{P(X',Z')}{\det(Z'J_r + \frac{X'\theta_{r,m}(X')}{2})} \right) \left| \frac{P(X',Z')}{\det(Z'J_r + \frac{X'\theta_{r,m}(X')}{2})} \right|^{-rs}$$

$$\cdot |\det(m_1(X',Z'))|^{s + \frac{r+2m+1}{2}} \, d\mu_{X'} \wedge d\mu_{Z'}.$$

By Proposition 6.7, we can find $f_{1,w'}, f_{2,w'} \in C_c^\infty(\Omega_{w'}; \omega)$, for each $w' \in B(M)$ with $d_B(e, w') \geq 1$, such that

$$B_\varphi^M(m(X',Z'), f_1) - B_\varphi^M(m(X',Z'), f_2)$$
$$= \sum_{w' \in B(M), d_B(w', e) \geq 1} (B_\varphi^M(m(X',Z'), f_{1,w'}) - B_\varphi^M(m(X',Z'), f_{2,w'})).$$

Since the toric action is determined by its restriction to $T_{X,Z}$ up to squares of torus entries according to Proposition 6.11, we call $T_{X,Z}$ the toric part of the orbit space $R_{X,Z} = R_X \times R_Z$. Denote by $N_{X,Z}$ the subset of $R_{X,Z}$ given by the variables not in $T_{X,Z}$, we call $N_{X,Z}$ the non-toric part of $R_{X,Z}$. Then by setting $x_{r,1} = 1$ and pass to $R_{X'} \times R_{Z'}$, we can separate our integral over $R_{X',Z'} = R_{X'} \times R_{Z'}$ as a double integral over $N_{X',Z'}$ and $T_{X',Z'}$. We will write the inner integral over $T_{X',Z'}$ and show that it is the part that accounts for stability.

Proposition 6.12 implies that the assumption in Proposition 6.8 is satisfied, since the denominators of rational functions in the 1-parameter subgroups for u_1, u_2, v_1, v_2 in the Bruhat decomposition $m_1 = u_1 \dot{w}_1 t_1 u_2$, $m_2 = v_1 \dot{w}_2 t_2 u_2$ with $m = m(X',Z') = (m_1(X',Z'), m_2(X',Z'))$ are monomials of the image of ϕ on $T_{X',Z'}$. The integral over $R_{X'} \times R_{Z'}$ breaks up into a double integral over $N_{X',Z'}$ and $T_{X',Z'}$. By Proposition 6.11, the restriction $\Phi|_{T_{X',Z'}}$ is finite étale onto its image in A', where $A = Z_M A'$, so the integral over $T_{X',Z'}$ can be written as a finite sum of integrals over $\Phi(T_{X',Z'}) \subset A'$. In addition, since $A_{w_M}^{w'} A_{w'}'$ is open of finite index in A', we can further write the integral over $\Phi(T_{X',Z'})$ as a double integral over $\Phi(T_{X',Z'}) \cap A_{w_m}^{w'}$ and $\Phi(T_{X',Z'}) \cap A_{w'}'$, so we have

$$D_\chi(s) = \sum_{\text{finite}} \sum_{1 \leq d_B(e, w')} \int_{N_{X',Z'}} \int_{\Phi(T_{X',Z'}) \cap A_{w_M}^{w'}} \int_{\Phi(T_{X',Z'}) \cap A_{w'}'}$$
$$(B_\varphi^M(m(X',Z'), f_{1,w'}) - B_\varphi^M(m(X',Z'), f_{2,w'}))$$

$$\cdot \, (\omega^{-2}\chi^{-2r}) \left(\frac{P(X',Z')}{\det(Z'J_r + \frac{X'\theta_{r,m}(X')}{2})} \right) \left| \frac{P(X',Z')}{\det(Z'J_r + \frac{X'\theta_{r,m}(X')}{2})} \right|^{-rs}$$

$$\cdot \, |\det(m_1(X',Z'))|^{s + \frac{r+2m+1}{2}} \, d\mu_{X'} \wedge d\mu_{Z'},$$

where the first finite sum comes from the finite *étale* map $\Phi|_{T_{X',Z'}}$ onto its image in A'.

By the proof of Proposition 6.8, the functions $B_\varphi^M(m(X',Z'), f_{i,w'})$, $(i = 1, 2)$, are compactly supported in the subtorus $A'_{w'} = Z'_L$ where $w' = w_L^M \in B(M)$. By the compatibility of the toric action as in Proposition 6.11, there exists an open compact subgroup $Z'_{L,0}$ of $Z'_L(F)$, such that given $m = m(X',Z')$, for $s = (s', s'') \in Z'_{L,0}$, we have

$$B_\varphi^M(\Theta_M(s)m(X',Z')s^{-1}, f_{i,w'})$$
$$= B_\varphi^M(m(s'X's''^{-1}, s'Z's'), f_{i,w'}) = B_\varphi^M(m(X',Z'), f_{i,w'})$$

for all κ where $\varphi_{\overline{N}_{\kappa,0}}$ is related to φ via Proposition 6.9. In our cases, $L \simeq \prod_{i=1}^k \mathrm{GL}_{r_i} \times \prod_{j=1}^t \mathrm{GL}_{m_j} \times \mathrm{Sp}_{2m'}$ with $\sum_{i=1}^k r_i = r$ and $\sum_{j=1}^t m_j + m' = m$. So we can pick $s = (s', s'') \in Z'_{L,0}$ with

$$s' = \mathrm{diag}\{\underbrace{s_1, \ldots, s_1}_{r_1}, \underbrace{s_2, \ldots, s_2}_{r_2}, \ldots, \underbrace{s_k, \ldots, s_k}_{r_k}\},$$

$$s'' = \mathrm{diag}\{\underbrace{s_{k+1}, \ldots, s_{k+1}}_{m_1}, \ldots, \underbrace{s_{k+l}, \ldots, s_{k+1}}_{m_l}, \underbrace{1, \ldots, 1}_{2m'},$$
$$\underbrace{s_{k+l}^{-1}, \ldots, s_{k+1}^{-1}}_{m_l}, \ldots, \underbrace{s_{k+1}^{-1}, \ldots, s_{k+1}^{-1}}_{m_1}\}.$$

Recall that the term $\frac{P(X',Z')}{\det(Z'J_r + \frac{X'\theta_{r,m}(X')}{2})}$ is just $\frac{y'^*_{rr}}{\det(Y')}$, where y'^*_{rr} is the (r,r)-th entry of the adjoint matrix of Y'. Here $Y' = Z'J_r + \frac{X'\theta_{r,m}(X')}{2}$. One computes that the toric action takes $\frac{y'^*_{rr}}{\det(Y')}$ to $\frac{1}{s_k^2}\frac{y'^*_{rr}}{\det(Y')}$, and takes $\det(m_1(X',Z'))$ to $\prod_{i=1}^k s_i^2 \det(m_1(X',Z'))$, since $m_1(X',Z') = \theta_r(Y')$. The action of $Z_{L,0}$ preserves $\Phi(T_{X',Z'}) \cap A'_{w'}$, therefore if we change variables

under the toric action, the inner integral is equal to

$$(\omega\chi^r)(s_k^4)|s_k|^{rs}\prod_{i=1}^{k}|s_i|^{2s+r+2m+1}\int_{\Phi(T_{X',Z'})\cap A'_{w'}}(B_\varphi^M(m(X',Z'),f_{1,w'})$$

$$-B_\varphi^M(m(X',Z'),f_{2,w'}))(\omega^{-2}\chi^{-2r})\left(\frac{P(X',Z')}{\det(Z'J_r+\frac{X'\theta_{r,m}(X')}{2})}\right)$$

$$\cdot\left|\frac{P(X',Z')}{\det(Z'J_r+\frac{X'\theta_{r,m}(X')}{2})}\right|^{-rs}|\det(m_1(X',Z'))|^{s+\frac{r+2m+1}{2}}.$$

So

$$D_\chi(s)=(\omega\chi^r)(s_k^4)|s_k|^{rs}\prod_{i=1}^{k}|s_i|^{2s+r+2m+1}D_\chi(s).$$

Choose χ highly ramified so that $(\omega\chi^r)(s_k^4)|s_k|^{rs}\prod_{i=1}^{k}|s_i|^{2s+r+2m+1}\neq 1$, therefore $D_\chi(s)=0$. We finally conclude that

$$C_\psi(s,\pi_1\otimes\chi)=C_\psi(s,\pi_2\otimes\chi).$$

Consequently,

$$\gamma(s,(\sigma_1\times\tau_1)\otimes\chi,\psi)=\gamma(s,(\sigma_2\times\tau_2)\otimes\chi,\psi)$$

for highly ramified χ.

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References

[1] N. Bourbaki, *Elements of mathematics. Lie groups and Lie algebras. Chapters 4–6.*, Springer, 2008, xi+300 pages.

[2] Y. Cai, S. Friedberg & E. Kaplan, "The generalized doubling method: local theory", *Geom. Funct. Anal.* **32** (2022), p. 1233-1333.

[3] J. W. Cogdell & I. I. Piatetski-Shapiro, "Stability of gamma factors for SO($2n + 1$).", *Manuscr. Math.* **95** (1998), no. 4, p. 437-462.

[4] J. W. Cogdell, I. I. Piatetski-Shapiro & F. Shahidi, "Partial Bessel functions for quasi-split groups", in *Automorphic representations, L-functions and applications: progress and prospects*, Ohio State University Mathematical Research Institute Publications, vol. 11, Walter de Gruyter, 2005, p. 95-128.

[5] ———, "Stability of γ-factors for quasi-split groups", *J. Inst. Math. Jussieu* **7** (2008), no. 1, p. 27-66.

[6] J. W. Cogdell, F. Shahidi & T.-L. Tsai, "Local Langlands correspondence for GL_n and the exterior and symmetric square ε-factors", *Duke Math. J.* **166** (2017), no. 11, p. 2053-2132.

[7] P. Deligne, "Les constantes des équations fonctionnelles des fonctions L", in *Modular functions of one variable, II (Proc. Internat. Summer School, Univ. Antwerp, Antwerp, 1972)*, Lecture Notes in Mathematics, vol. 349, Springer, 1973, p. 501-597.

[8] H. Jacquet & J. Shalika, "A lemma on highly ramified ϵ-factors", *Math. Ann.* **271** (1985), no. 3, p. 319-332.

[9] S. Rallis & D. Soudry, "Stability of the local gamma factor arising from the doubling method", *Math. Ann.* **333** (2005), no. 2, p. 291-313.

[10] F. Shahidi, "A proof of Langlands' conjecture on Plancherel measures; complementary series for p-adic groups", *Ann. Math. (2)* **132** (1990), no. 2, p. 273-330.

[11] ———, "Local coefficients as Mellin transforms of Bessel functions: towards a general stability", *Int. Math. Res. Not.* **2002** (2002), no. 39, p. 2075-2119.

[12] ———, *Eisenstein series and automorphic L-functions*, Colloquium Publications, vol. 58, American Mathematical Society, 2010, vi+210 pages.

[13] D. Shankman, "Local Langlands correspondence for Asai L functions and ϵ-factors", 2018, https://arxiv.org/abs/1810.11852.

[14] D. Shankman & D. She, "Stability of symmetric cube gamma factors for GL(2)", 2019, https://arxiv.org/abs/1911.03428.

[15] D. She, "Local Langlands correspondence for the twisted exterior and symmetric square ϵ-factors of GL_n", *Manuscr. Math.* **173** (2024), no. 1-2, p. 155-201.

[16] R. Sundaravaradhan, "Some structural results for the stability of root numbers", *Int. Math. Res. Not.* **2008** (2008), no. 2, article no. rnm141 (22 pages).

[17] S. Wagh, "Stability of local gamma factors arising from the doubling method for general spin groups", 2021, https://arxiv.org/abs/2104.12263.

[18] Q. Zhang, "Stability of Rankin–Selberg gamma factors for Sp($2n$), $\widetilde{\mathrm{Sp}}(2n)$ and U(n, n)", *Int. J. Number Theory* **13** (2017), no. 9, p. 2393-2432.

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