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On a conjecture of Ramírez Alfonsín and Skałba II

par Yuchen DING, Wenguang ZHAI et Lilu ZHAO

Résumé. Soit $\pi(t)$ le nombre de nombres premiers inférieurs ou égaux à t. Soient $1 < c < d$ deux entiers premiers entre eux et $g_{c,d} = cd - c - d$. En combinant la méthode de Hardy–Littlewood avec le théorème de Siegel–Walfisz, nous démontrons la conjecture, énoncée en 2020 par Ramírez Alfonsín et Skałba, qui affirme que

$$\#\left\{p \le g_{c,d} : p \in \mathcal{P},\ p = cx + dy,\ x, y \in \mathbb{Z}_{\geqslant 0}\right\} \sim \frac{1}{2}\pi\left(g_{c,d}\right)$$

quand $c \to \infty$, où $\mathcal{P}$ et $\mathbb{Z}_{\geqslant 0}$ désignent l'ensemble des nombres premiers et l'ensemble des entiers non négatifs respectivement.

Abstract. Let $\pi(t)$ be the number of primes not exceeding t. Let $1 < c < d$ be two relatively prime integers and $g_{c,d} = cd - c - d$. We confirm, by combining the Hardy–Littlewood method with the Siegel–Walfisz theorem, a 2020 conjecture of Ramírez Alfonsín and Skałba which states that

$$\#\left\{p \le g_{c,d} : p \in \mathcal{P},\ p = cx + dy,\ x, y \in \mathbb{Z}_{\geqslant 0}\right\} \sim \frac{1}{2}\pi\left(g_{c,d}\right)$$

as $c \to \infty$, where $\mathcal{P}$ and $\mathbb{Z}_{\geqslant 0}$ denote the sets of primes and nonnegative integers, respectively.

1. Introduction

Let $1 < c < d$ be two relatively prime integers and $g_{c,d} = cd - c - d$. As early as 1882, Sylvester [6] showed that $g_{c,d}$ is the largest integer which cannot be represented as the form $cx + dy$ $(x, y \in \mathbb{Z}_{\geqslant 0})$. Furthermore, he proved that for any $0 \le s \le g_{c,d}$, exactly one of s and $g_{c,d} - s$ can be written as the form $cx + dy$ $(x, y \in \mathbb{Z}_{\geqslant 0})$. As an immediate consequence,

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we know that exactly half of the integers between the interval $[0, g_{c,d}]$ can be written as the desired form. Actually, Sylvester's results are the first nontrivial case of the diophantine Frobenius problem [4], which asks the largest integer $g_{c_1,\dots,c_n}$ not of the form

$$c_1x_1 + \cdots + c_nx_n \quad (x_1, \dots, x_n \in \mathbb{Z}_{\geqslant 0}),$$

provided that $c_1, \dots, c_n$ are positive integers with $\gcd(c_1, \dots, c_n) = 1$. There are a huge number of literatures related to the diophantine Frobenius problem. For some of these results, see e.g. the excellent monograph [4] of Ramírez Alfonsín.

Motivated by Sylvester's theorems, Ramírez Alfonsín and Skałba [5] considered the diophantine Frobenius problem in primes. Precisely, let $\pi_{c,d}$ be the number of primes not exceeding $g_{c,d}$ with the form $cx+dy$ $(x, y \in \mathbb{Z}_{\geqslant 0})$. By a very enlightening argument, Ramírez Alfonsín and Skałba proved that for any $\varepsilon > 0$, there is a constant $k(\varepsilon) > 0$ such that

$$\pi_{c,d} \geqslant k(\varepsilon)\frac{g_{c,d}}{(\log g_{c,d})^{2+\varepsilon}}.$$

On observing the antisymmetry property of the integers with the form $cx + dy$ $(x, y \in \mathbb{Z}_{\geqslant 0})$ found by Sylvester, they naturally posed the following conjecture.

Conjecture 1.1 (Ramírez Alfonsín and Skałba). *Let $1 < c < d$ be two relatively prime integers, then*

$$\pi_{c,d} \sim \frac{\pi(g_{c,d})}{2} \quad (\textit{as } c \to \infty),$$

where $\pi(t)$ is the number of primes up to t.

Recently, the first named author made some progress on Conjecture 1.1. For real number $N \geqslant 2$, let $1 < c < d$ be two relatively prime integers satisfying $cd \leqslant N$. The first named author [2] proved that for all but at most

$$O\left(N(\log N)^{1/2}(\log\log N)^{1/2+\varepsilon}\right) \tag{1.1}$$

pairs c and d, we have

$$\pi_{c,d} = \frac{\pi(g_{c,d})}{2} + O\left(\frac{\pi(g_{c,d})}{(\log\log(cd))^{\varepsilon}}\right) \sim \frac{\pi(g_{c,d})}{2} \quad (\text{as } c \to \infty).$$

Since the total number of the relatively prime pairs c, d with $1 < c < d$ and $cd \leqslant N$ is $\gg N \log N$, the first named author actually showed that Conjecture 1.1 is true for almost all c and d, on comparing with (1.1).

Though it seems out of reach [5], the complete proof of Conjecture 1.1 follows from an application of the Hardy–Littlewood method as well as a supplement of the Siegel–Walfisz theorem. Perhaps, one of the novel points in our argument is that only the first coefficient of the "singular series" contributes the main term of the asymptotic formula comparing with the usual applications of the Hardy–Littlewood method. The idea of the applications on the Hardy–Littlewood method presented here is in the same spirit as the one developed by a recent article of Chen, Yang and Zhao [1]. We also mention that when c is very small comparing with d, the Hardy–Littlewood method seems not applicable here any more. The Siegel–Walfisz theorem will then be used to complete this remaining case, which is another key ingredient of the argument.

Now, let's record our result as the following theorem.

Theorem 1.2. *Suppose that $d > c$ are two relatively prime integers with c sufficiently large, then we have*

$$\pi_{c,d} \sim \frac{1}{2}\pi(g_{c,d}), \quad \text{as } c \to \infty.$$

As usual, we shall firstly investigate the following weighted form related to Conjecture 1.1, i.e.,

$$\psi_{c,d} = \sum_{\substack{n \le g \\ n=cx+dy \\ x,y \in \mathbb{Z}_{\geqslant 0}}} \Lambda(n),$$

where the von Mangoldt function $\Lambda(n)$ is defined to be

$$\Lambda(n) = \begin{cases} \log p, & \text{if } n = p^{\alpha} \ (\alpha > 0); \\ 0, & \text{otherwise.} \end{cases}$$

We first establish the following weighted formula and then Theorem 1.1 will be proved from it by a fairly standard transition.

Theorem 1.3. *Suppose that $d > c$ are two relatively prime integers with c sufficiently large, then we have*

$$\psi_{c,d} \sim \frac{g_{c,d}}{2}, \quad \text{as } c \to \infty.$$

As an incident product of Theorem 1.3, we have the following corollary which seems to be of some interests.

Corollary 1.4. *Suppose that $d > c$ are two relatively prime integers with c sufficiently large, then we have*

$$\sum_{\substack{y \leqslant c \\ (y,c)=1}} \psi(dy; c, dy) \sim \frac{g_{c,d}}{2}, \quad \text{as } c \to \infty,$$

where

$$\psi(N; q, m) = \sum_{\substack{n \leqslant N \\ n \equiv m \,(\mathrm{mod}\, q)}} \Lambda(n).$$

2. Outline the proof: an application of the Hardy–Littlewood method

We first fix some basic notations to be used frequently. From now on, we write g instead of $g_{c,d}$ for brevity and c is supposed to be sufficiently large. Let Q denote a positive number depending only on g which shall be decided later. The function $e(t)$ is used to denote $e^{2\pi it}$ as usual. Define the major arcs to be

$$\mathfrak{M}(Q)=\bigcup_{1\le q\le Q}\ \bigcup_{\substack{1\le a\le q\\(a,q)=1}}\left\{\alpha:\left|\alpha-\frac{a}{q}\right|\le\frac{Q}{qg}\right\}\tag{2.1}$$

We make a further provision that $Q<\frac{1}{2}g^{1/2}$ so that the above subsets are pairwise disjoint via similar arguments from [3, p. 214]). In addition, we note that

$$\mathfrak{M}(Q)\subseteq\left[\frac{1}{Q}-\frac{Q}{qg},1+\frac{Q}{qg}\right]\subseteq\left[\frac{Q+1}{g},1+\frac{Q}{g}\right].$$

We can now define the minor arcs to be

$$\mathfrak{m}(Q)=\left[\frac{Q+1}{g},1+\frac{Q+1}{g}\right]\setminus\mathfrak{M}(Q).\tag{2.2}$$

For any real α, let

$$f(\alpha)=\sum_{0\le n\le g}\Lambda(n)e(\alpha n)\quad\text{and}\quad h(\alpha)=\sum_{\substack{0\le x\le d\\0\le y\le c}}e(\alpha(cx+dy))$$

Note that

$$\begin{aligned}\int_0^1 f(\alpha)h(-\alpha)\,\mathrm{d}\alpha&=\int_0^1\sum_{0\le n\le g}\Lambda(n)e(\alpha n)\sum_{\substack{0\le x\le d\\0\le y\le c}}e(-\alpha(cx+dy))\,\mathrm{d}\alpha\\&=\sum_{0\le n\le g}\sum_{\substack{0\le x\le d\\0\le y\le c}}\Lambda(n)\int_0^1 e(\alpha(n-cx-dy))\,\mathrm{d}\alpha.\end{aligned}$$

Thus, by orthogonal relation we have

$$\begin{aligned}\psi_{c,d}&=\int_0^1 f(\alpha)h(-\alpha)\,\mathrm{d}\alpha\\&=\int_{\mathfrak{M}(Q)}f(\alpha)h(-\alpha)\,\mathrm{d}\alpha+\int_{\mathfrak{m}(Q)}f(\alpha)h(-\alpha)\,\mathrm{d}\alpha.\end{aligned}\tag{2.3}$$

The remaining parts of our paper will be organized as follows: In the next section, we shall give a suitable bound of the integral on the minor arcs which shows that it contributes the error term of Theorem 1.3. For the integral on the major arcs, we have

$$\int_{\mathfrak{M}(Q)} f(\alpha)h(-\alpha)\,\mathrm{d}\alpha = \sum_{1\le q\le Q}\sum_{\substack{1\le a\le q\\(a,q)=1}}\int_{\frac{a}{q}-\frac{Q}{qg}}^{\frac{a}{q}+\frac{Q}{qg}} f(\alpha)h(-\alpha)\,\mathrm{d}\alpha$$

$$= \sum_{1\le q\le Q}\sum_{\substack{1\le a\le q\\(a,q)=1}}\int_{|\theta|\le\frac{Q}{qg}} f\left(\frac{a}{q}+\theta\right)h\left(-\frac{a}{q}-\theta\right)\mathrm{d}\theta. \tag{2.4}$$

We would prove, in Section 4, that the integral for $q=1$ in (2.2) contributes the main term of Theorem 1.3 and the integrals for $2\le q\le Q$ only contribute the error terms. In Section 5, we will prove our theorems and the corollary.

3. Estimates of the minor arcs

The aim of this section is to prove the following proposition.

Proposition 3.1. *For estimates of the minor arcs, we have*

$$\int_{\mathfrak{m}(Q)} f(\alpha)h(-\alpha)\,\mathrm{d}\alpha \ll \frac{g(\log g)^6}{Q^{1/2}} + g^{4/5}(\log g)^6.$$

We need two lemmas listed below.

Lemma 3.2. *We have*

$$\sup_{\alpha\in\mathfrak{m}(Q)} |f(\alpha)| \ll \frac{g(\log g)^4}{Q^{1/2}} + g^{4/5}(\log g)^4.$$

Proof. By the Dirichlet approximation theorem (see e.g. [7, Lemma 2.1]), there exist $a\in\mathbb{Z}$ and $q\in\mathbb{Z}^+$ such that

$$(a,q)=1,\quad 1\le q\le\frac{g}{Q}\quad\text{and}\quad \left|\alpha-\frac{a}{q}\right|\le\frac{Q}{qg}$$

for any $\alpha\in\mathfrak{m}(Q)$. First of all, we show that $q>Q$ for these $\alpha\in\mathfrak{m}(Q)$. Suppose the contrary, i.e., $q\le Q$, then from $\left|\alpha-\frac{a}{q}\right|\le\frac{Q}{qg}$ we know that

$$\alpha\le\frac{a}{q}+\frac{Q}{qg}\le\frac{Q}{qg}\le\frac{Q}{g}<\frac{Q+1}{g}\quad(\text{if } a\le 0)$$

and

$$\alpha\ge\frac{q+1}{q}-\frac{Q}{qg}=1+\frac{1}{q}\left(1-\frac{Q}{g}\right)\ge 1+\frac{1}{2Q}>1+\frac{Q+1}{g}\quad(\text{if } a\ge q+1),$$

which contradict with (2.2). We are left over to consider the case that $1\le a\le q\le Q$. In this case, we have $\alpha\in\mathfrak{M}(Q)$ by the definition of the major arcs which is still a contradiction. Thus, we have proved that $q>Q$ for these $\alpha\in\mathfrak{m}(Q)$.

We are in a position to introduce a fairly remarkable theorem of Vinogradov (see e.g. [7, Theorem 3.1]) which states that for $(a,q)=1$ with $1\le q\le g$ and $\left|\alpha-\frac{a}{q}\right|\le\frac{1}{q^2}$, we have

$$f(\alpha)\ll\left(\frac{g}{q^{1/2}}+q^{4/5}+g^{1/2}q^{1/2}\right)(\log g)^4.$$

Employing this estimate, we deduce from $Q<q\le g/Q$ that

$$\begin{aligned}\sup_{\alpha\in\mathfrak{m}(Q)}|f(\alpha)|&\ll\left(\frac{g}{Q^{1/2}}+g^{4/5}+g^{1/2}(g/Q)^{1/2}\right)(\log g)^4\\&\ll\frac{g(\log g)^4}{Q^{1/2}}+g^{4/5}(\log g)^4.\end{aligned}$$

This completes the proof of Lemma 3.2 □

Before the statement of Lemma 3.3, one notes that

$$\begin{aligned}(3.1)\quad h(-\alpha)&=\sum_{x\le d}e(-\alpha cx)\sum_{y\le c}e(-\alpha dy)\\&\ll\min\{d,\|c\alpha\|^{-1}\}\min\{c,\|d\alpha\|^{-1}\}\end{aligned}$$

since for any real number α and integers $N_1<N_2$, we have (see e.g. [3, Lemma 4.7])

$$\sum_{n=N_1+1}^{N_2}e(\alpha n)\ll\min\{N_2-N_1,\|\alpha\|^{-1}\}.$$

Lemma 3.3. *We have*

$$\int_0^1|h(-\alpha)|\,\mathrm{d}\alpha\ll(\log g)^2.$$

Proof. Note that $|h(\alpha)|=|h(-\alpha)|=|h(1-\alpha)|$. Hence, in view of (3.1), the lemma will follows once we have established the inequality

$$(3.2)\qquad\int_0^{1/2}\min\{d,\|c\alpha\|^{-1}\}\min\{c,\|d\alpha\|^{-1}\}\,\mathrm{d}\alpha\ll(\log g)^2.$$

For $0\le\alpha\le\frac{1}{cd}$, we have the following trivial estimates that

$$\int_0^{1/(cd)}\min\{d,\|c\alpha\|^{-1}\}\min\{c,\|d\alpha\|^{-1}\}\,\mathrm{d}\alpha\le dc\int_0^{1/(cd)}\,\mathrm{d}\alpha\le 1.$$

It remains to prove that

$$\int_{\frac{1}{cd}}^{1/2}\min\{d,\|c\alpha\|^{-1}\}\min\{c,\|d\alpha\|^{-1}\}\,\mathrm{d}\alpha\ll(\log g)^2.$$

The above interval is contained in the following union of a few disjoint short intervals

$$\bigcup_{1\le \ell\le \left\lfloor \frac{cd}{2}\right\rfloor}\left[\frac{\ell}{cd}-\frac{1}{2cd},\frac{\ell}{cd}+\frac{1}{2cd}\right].$$

By the inclusion relation, it suffices to show that

$$\sum_{1\le \ell\le \left\lfloor \frac{cd}{2}\right\rfloor}\int_{\frac{\ell}{cd}-\frac{1}{2cd}}^{\frac{\ell}{cd}+\frac{1}{2cd}}\min\{d,\|c\alpha\|^{-1}\}\min\{c,\|d\alpha\|^{-1}\}\,\mathrm{d}\alpha\ll(\log g)^2.$$

Making changes of the variables $\alpha=\frac{\ell}{cd}+\theta$, it is equivalent to prove that

$$\sum_{1\le \ell\le \left\lfloor \frac{cd}{2}\right\rfloor}\int_{-\frac{1}{2cd}}^{\frac{1}{2cd}}\min\left\{d,\left\|c\theta+\frac{\ell}{d}\right\|^{-1}\right\}\min\left\{c,\left\|d\theta+\frac{\ell}{c}\right\|^{-1}\right\}\mathrm{d}\theta \ll(\log g)^2. \tag{3.3}$$

We separate the proofs into four cases.

Case I. For $1\le \ell\le \left\lfloor \frac{cd}{2}\right\rfloor$ with $c\nmid \ell$ and $d\nmid \ell$, we have

$$\left\|c\theta+\frac{\ell}{d}\right\|\ge\frac12\left\|\frac{\ell}{d}\right\|\quad\text{and}\quad\left\|d\theta+\frac{\ell}{c}\right\|\ge\frac12\left\|\frac{\ell}{c}\right\|$$

since $|\theta|\le\frac{1}{2cd}$. Therefore, we obtain that

$$\begin{aligned}\sum_{\substack{1\le \ell\le \left\lfloor \frac{cd}{2}\right\rfloor\\ c\nmid\ell\text{ and } d\nmid\ell}}\int_{-\frac{1}{2cd}}^{\frac{1}{2cd}}&\min\left\{d,\left\|c\theta+\frac{\ell}{d}\right\|^{-1}\right\}\min\left\{c,\left\|d\theta+\frac{\ell}{c}\right\|^{-1}\right\}\mathrm{d}\theta\\ &\le 4\sum_{\substack{1\le \ell\le \left\lfloor \frac{cd}{2}\right\rfloor\\ c\nmid\ell\text{ and } d\nmid\ell}}\int_{-\frac{1}{2cd}}^{\frac{1}{2cd}}\left\|\frac{\ell}{d}\right\|^{-1}\left\|\frac{\ell}{c}\right\|^{-1}\mathrm{d}\theta\\ &=\frac{4}{cd}\sum_{\substack{1\le \ell\le \left\lfloor \frac{cd}{2}\right\rfloor\\ c\nmid\ell\text{ and } d\nmid\ell}}\left\|\frac{\ell}{d}\right\|^{-1}\left\|\frac{\ell}{c}\right\|^{-1}.\end{aligned}\tag{3.4}$$

Now, by the Euclidean division we can assume that $\ell = ch + r$ with $1 \leq r \leq c-1$. It then follows that

$$\begin{aligned}
\sum_{\substack{1\leq \ell\leq \left\lfloor \frac{cd}{2}\right\rfloor \\ c\nmid \ell \text{ and } d\nmid \ell}} \left\|\frac{\ell}{d}\right\|^{-1}\left\|\frac{\ell}{c}\right\|^{-1} &\leq \sum_{\substack{0\leq h\leq \frac{d}{2},\ 1\leq r\leq c-1 \\ h\not\equiv -c^{-1}r \,(\mathrm{mod}\, d)}} \left\|\frac{ch+r}{d}\right\|^{-1}\left\|\frac{r}{c}\right\|^{-1} \\
&= \sum_{1\leq r\leq c-1}\left\|\frac{r}{c}\right\|^{-1} \sum_{0\leq h\leq \frac{d}{2},\ h\not\equiv -c^{-1}r \,(\mathrm{mod}\, d)} \left\|\frac{ch+r}{d}\right\|^{-1} \\
&\leq \sum_{1\leq r\leq c-1}\left\|\frac{r}{c}\right\|^{-1} \sum_{1\leq r'\leq d-1}\left\|\frac{r'}{d}\right\|^{-1},
\end{aligned}$$

where $c^{-1}c \equiv 1 \,(\mathrm{mod}\, d)$ and the last inequality follows from the fact that

$$ch + r \not\equiv ch' + r \,(\mathrm{mod}\, d) \quad (\text{for } 0 \leq h \neq h' \leq d/2).$$

It can be seen that

$$\sum_{1\leq r\leq c-1}\left\|\frac{r}{c}\right\|^{-1} \leq 2\sum_{1\leq r\leq \lfloor c/2\rfloor}\left\|\frac{r}{c}\right\|^{-1} = 2\sum_{1\leq r\leq \lfloor c/2\rfloor}\frac{c}{r} \ll c\log c \ll c\log g$$

and similarly

$$\sum_{1\leq r'\leq d-1}\left\|\frac{r'}{d}\right\|^{-1} \ll d\log g,$$

from which we deduce from (3.4) that

$$\sum_{\substack{1\leq \ell\leq \left\lfloor \frac{cd}{2}\right\rfloor \\ c\nmid \ell \text{ and } d\nmid \ell}} \int_{-\frac{1}{2cd}}^{\frac{1}{2cd}} \min\left\{d, \left\|c\theta+\frac{\ell}{d}\right\|^{-1}\right\}\min\left\{c, \left\|d\theta+\frac{\ell}{c}\right\|^{-1}\right\} \mathrm{d}\theta \ll (\log g)^2.$$

Case II. For $1 \leq \ell \leq \left\lfloor \frac{cd}{2}\right\rfloor$ with $c \nmid \ell$ but $d|\ell$, we have

$$\left\|d\theta+\frac{\ell}{c}\right\| \geq \frac{1}{2}\left\|\frac{\ell}{c}\right\|$$

and then it follows that

$$\sum_{\substack{1\le\ell\le\lfloor\frac{cd}{2}\rfloor\\ c\nmid\ell\text{ and }d|\ell}}\int_{-\frac{1}{2cd}}^{\frac{1}{2cd}}\min\left\{d,\left\|c\theta+\frac{\ell}{d}\right\|^{-1}\right\}\min\left\{c,\left\|d\theta+\frac{\ell}{c}\right\|^{-1}\right\}\mathrm{d}\theta$$

$$\le 2\sum_{\substack{1\le\ell\le\lfloor\frac{cd}{2}\rfloor\\ c\nmid\ell\text{ and }d|\ell}}\int_{-\frac{1}{2cd}}^{\frac{1}{2cd}}d\left\|\frac{\ell}{c}\right\|^{-1}\mathrm{d}\theta$$

$$=\frac{2}{c}\sum_{\substack{1\le\ell\le\lfloor\frac{cd}{2}\rfloor\\ c\nmid\ell\text{ and }d|\ell}}\left\|\frac{\ell}{c}\right\|^{-1}.$$

On writing $\ell=d\ell^*$, we find that

$$\sum_{\substack{1\le\ell\le\lfloor\frac{cd}{2}\rfloor\\ c\nmid\ell\text{ and }d|\ell}}\left\|\frac{\ell}{c}\right\|^{-1}\le\sum_{\substack{1\le\ell^*\le\frac{c}{2}\\ c\nmid\ell^*}}\left\|\frac{d\ell^*}{c}\right\|^{-1}\le\sum_{1\le r\le c-1}\left\|\frac{r}{c}\right\|^{-1}\ll c\log g,$$

from which it follows clearly that

$$\sum_{\substack{1\le\ell\le\lfloor\frac{cd}{2}\rfloor\\ c\nmid\ell\text{ and }d|\ell}}\int_{-\frac{1}{2cd}}^{\frac{1}{2cd}}\min\left\{d,\left\|c\theta+\frac{\ell}{d}\right\|^{-1}\right\}\min\left\{c,\left\|d\theta+\frac{\ell}{c}\right\|^{-1}\right\}\mathrm{d}\theta\ll\log g.$$

Case III. For $1\le\ell\le\left\lfloor\frac{cd}{2}\right\rfloor$ with $c|\ell$ but $d\nmid\ell$, we have

$$\sum_{\substack{1\le\ell\le\lfloor\frac{cd}{2}\rfloor\\ c|\ell\text{ and }d\nmid\ell}}\int_{-\frac{1}{2cd}}^{\frac{1}{2cd}}\min\left\{d,\left\|c\theta+\frac{\ell}{d}\right\|^{-1}\right\}\min\left\{c,\left\|d\theta+\frac{\ell}{c}\right\|^{-1}\right\}\mathrm{d}\theta\ll\log g.$$

via the same argument as Case II.

Case IV. For $1\le\ell\le\left\lfloor\frac{cd}{2}\right\rfloor$ with $c|\ell$ and $d|\ell$, we have $cd|\ell$ since $(c,d)=1$, which is certainly a contradiction with $\ell\le\left\lfloor\frac{cd}{2}\right\rfloor$.

Gathering together Cases I to IV, we established (3.3), and hence (3.2). This completes the proof of Lemma 3.3. □

Proof of Proposition 3.1. The treatment of the minor arcs benefits from the following trivial estimates

$$\left|\int_{\mathfrak{m}(Q)}f(\alpha)h(-\alpha)\,\mathrm{d}\alpha\right|\le\sup_{\alpha\in\mathfrak{m}(Q)}|f(\alpha)|\int_0^1|h(-\alpha)|\,\mathrm{d}\alpha$$

together with Lemma 3.2 and Lemma 3.3. □

4. Calculations of the major arcs

We would provide, in this section, the asymptotic formula of the integral on major arcs as the following proposition.

Proposition 4.1. *For $Q < c^{1/3}$, we have*

$$\int_{\mathfrak{M}(Q)} f(\alpha)h(-\alpha)\,\mathrm{d}\alpha = \frac{g}{2} + O\left(\frac{g}{Q}(\log g)^2 + gQ^2\exp\Big(-\kappa_1\sqrt{\log g}\Big) + dQ^3\right).$$

To this aim, we firstly prove several lemmas.

Lemma 4.2 ([7, Lemma 3.1])**.** *For any real number θ, we have*

$$f(\theta) = \sum_{0\le n\le g} e(n\theta) + O\left(g(1+|\theta|g)\exp\Big(-\kappa_1\sqrt{\log g}\Big)\right).$$

Lemma 4.3. *We have*

$$\int_{|\theta|\le\frac{Q}{g}} f(\theta)\,h(-\theta)\,\mathrm{d}\theta = \frac{g}{2} + O\left(\frac{g}{Q}(\log g)^2 + gQ^2\exp\Big(-\kappa_1\sqrt{\log g}\Big)\right).$$

Proof. From Lemma 4.2, we have

$$\int_{|\theta|\le\frac{Q}{g}} f(\theta)\,h(-\theta)\,\mathrm{d}\theta = \int_{|\theta|\le\frac{Q}{g}} \sum_{0\le n\le g} e(n\theta)h(-\theta)\,\mathrm{d}\theta + \mathcal{R}(\theta),$$

where the error term $\mathcal{R}(\theta)$ can be bounded easily by the trivial estimates as

$$\begin{aligned}\mathcal{R}(\theta) &\ll \int_{|\theta|\le\frac{Q}{g}} g(1+|\theta|g)\exp\Big(-\kappa_1\sqrt{\log g}\Big)|h(-\theta)|\,\mathrm{d}\theta\\ &\ll \int_{|\theta|\le\frac{Q}{g}} g(1+|\theta|g)\exp\Big(-\kappa_1\sqrt{\log g}\Big)g\,\mathrm{d}\theta\\ &\ll gQ^2\exp\Big(-\kappa_1\sqrt{\log g}\Big).\end{aligned}$$

On noting that

$$\int_{|\theta|\le\frac{Q}{g}} = \int_{-1/2}^{1/2} - \int_{\frac{Q}{g}}^{1/2} - \int_{-1/2}^{-\frac{Q}{g}},$$

we can obtain

$$\int_{|\theta|\le\frac{Q}{g}} \sum_{0\le n\le g} e(n\theta)h(-\theta)\,\mathrm{d}\theta = \int_{-1/2}^{1/2} \sum_{0\le n\le g} e(n\theta)h(-\theta)\,\mathrm{d}\theta + \mathcal{R}_1(\theta) + \mathcal{R}_2(\theta),$$

where

$$\mathcal{R}_1(\theta) = \int_{-1/2}^{-\frac{Q}{g}} \sum_{0\le n\le g} e(n\theta)h(-\theta)\,\mathrm{d}\theta$$

$$\text{and}\quad \mathcal{R}_2(\theta) = \int_{\frac{Q}{g}}^{1/2} \sum_{0\le n\le g} e(n\theta)h(-\theta)\,\mathrm{d}\theta.$$

It follows from the periodic property of θ and the definition of h that

$$\int_{-1/2}^{1/2} \sum_{0 \le n \le g} e(n\theta) h(-\theta)\,\mathrm{d}\theta = \sum_{\substack{0 \le n \le g \\ 0 \le x \le d,\ 0 \le y \le c}} \int_0^1 e((n - cx - dy)\theta)\,\mathrm{d}\theta = \frac{g+1}{2},$$

where the last equality above follows from the result of Sylvester [6] mentioned in the introduction. For $\frac{Q}{g} \le \theta \le \frac{1}{2}$, using again the estimate

$$\sum_{0 \le n \le g} e(n\theta) \ll \|\theta\|^{-1} = \frac{1}{\theta} \le \frac{g}{Q},$$

we deduce from (3.1) and (3.2) that

$$\mathcal{R}_2(\theta) \ll \frac{g}{Q} \int_{\frac{Q}{g}}^{1/2} \min\{d, \|c\theta\|^{-1}\} \min\{c, \|d\theta\|^{-1}\}\,\mathrm{d}\theta \ll \frac{g}{Q}(\log g)^2.$$

The same argument would also lead to the estimate $\mathcal{R}_1(\theta) \ll \frac{g}{Q}(\log g)^2$. Collecting the estimates above, we obtain that

$$\int_{|\theta| \le \frac{Q}{g}} f(\theta)\, h(-\theta)\,\mathrm{d}\theta = \frac{g}{2} + O\left(\frac{g}{Q}(\log g)^2 + gQ^2 \exp\Big(-\kappa_1 \sqrt{\log g}\Big)\right).$$

This completes the proof of Lemma 4.3. □

Lemma 4.4. *For $Q < c^{1/3}$, we have*

$$\sum_{2 \le q \le Q} \sum_{\substack{1 \le a \le q \\ (a,q)=1}} \int_{|\theta| \le \frac{Q}{qg}} f\left(\frac{a}{q} + \theta\right) h\left(-\frac{a}{q} - \theta\right)\mathrm{d}\theta \ll dQ^3.$$

Proof. Recall that

$$h(-\alpha) = \sum_{x \le d} e(-\alpha c x) \sum_{y \le c} e(-\alpha d y) \ll \min\{d, \|c\alpha\|^{-1}\} \min\{c, \|d\alpha\|^{-1}\},$$

hence we have

$$h\left(-\frac{a}{q} - \theta\right) \ll \min\left\{d, \left\|c\left(-\frac{a}{q} - \theta\right)\right\|^{-1}\right\} \min\left\{c, \left\|d\left(-\frac{a}{q} - \theta\right)\right\|^{-1}\right\}.$$

Recall that c is sufficiently large and $g \sim cd$, we clearly have

$$\left\|c\left(-\frac{a}{q} - \theta\right)\right\| \ge \frac{1}{2q} \quad \text{if} \quad q \nmid c$$

and

$$\left\|d\left(-\frac{a}{q} - \theta\right)\right\| \ge \frac{1}{2q} \quad \text{if} \quad q \nmid d$$

for $(a,q)=1$, $q\geq 2$ and $|\theta|\leq \frac{Q}{qg}$, provided that $Q\leq c^{1/3}$. Since $(c,d)=1$, at least one of the above inequalities is admissible. Therefore, for all $2\leq q\leq Q$, $(a,q)=1$ and $|\theta|\leq \frac{Q}{qg}$ we have

$$h\left(-\frac{a}{q}-\theta\right)\ll qd,$$

from which we can conclude that

$$\begin{aligned}\sum_{2\leq q\leq Q}\sum_{\substack{1\leq a\leq q\\(a,q)=1}}\int_{|\theta|\leq\frac{Q}{qg}} f\left(\frac{a}{q}+\theta\right)h\left(-\frac{a}{q}-\theta\right)\mathrm{d}\theta &\ll \sum_{2\leq q\leq Q}\sum_{\substack{1\leq a\leq q\\(a,q)=1}}\int_{|\theta|\leq\frac{Q}{qg}} gdq\,\mathrm{d}\theta\\ &\ll dQ^3.\end{aligned}$$

This completes the proof of Lemma 4.4. □

Proof of Proposition 4.1. Let's turn back to (2.4):

$$\int_{\mathfrak{M}(Q)} f(\alpha)h(-\alpha)\,\mathrm{d}\alpha=\sum_{1\leq q\leq Q}\sum_{\substack{1\leq a\leq q\\(a,q)=1}}\int_{|\theta|\leq\frac{Q}{qg}} f\left(\frac{a}{q}+\theta\right)h\left(-\frac{a}{q}-\theta\right)\mathrm{d}\theta.$$

The term of the above sum for $a=q=1$ equals

$$\int_{|\theta|\leq\frac{Q}{g}} f\left(1+\theta\right)h\left(-1-\theta\right)\mathrm{d}\theta=\int_{|\theta|\leq\frac{Q}{g}} f\left(\theta\right)h(-\theta)\,\mathrm{d}\theta.$$

Now, our proposition follows immediately from Lemma 4.3 and Lemma 4.4. □

5. Proofs of Theorem 1.2, Theorem 1.3 and Corollary 1.4

Proof of Theorem 1.3. By (2.3), Proposition 3.1 and Proposition 4.1, we have

$$\begin{aligned}\psi_{c,d}=\frac{g}{2}+O\bigg(\frac{g}{Q}(\log g)^2+gQ^2\exp\Big(-\kappa_1\sqrt{\log g}\Big)\\ +dQ^3+\frac{g(\log g)^6}{Q^{1/2}}+g^{4/5}(\log g)^6\bigg),\end{aligned}$$

where $Q\leq c^{1/3}$. We choose $Q=(\log g)^{14}$. Then

$$\psi_{c,d}=\frac{g}{2}+O\left(\frac{g}{\log g}\right), \tag{5.1}$$

provided that $c\geq(\log g)^{43}$.

For the complement of our proof, it remains to consider the case that $c \le (\log g)^{43}$. Following the proof of [2, p. 299], we have

$$\begin{aligned}
\psi_{c,d} &= \sum_{\substack{n=cx+dy\\ n\leqslant g\\ x,y\in\mathbb{Z}_{\geqslant 0}}} \Lambda(n)\\
&= \sum_{\substack{1\le y\leqslant c\\ (y,c)=1}} \sum_{\substack{n\equiv dy\,(\mathrm{mod}\, c)\\ dy\leqslant n\leqslant g}} \Lambda(n) + O(\log g)\\
&= \sum_{\substack{1\le y\leqslant c\\ (y,c)=1}} (\psi(g;c,dy) - \psi(dy;c,dy)) + O(\log g)\\
&= \psi(g) - \sum_{\substack{1\le y\leqslant c\\ (y,c)=1}} \psi(dy;c,dy) + O(\log g)\,. \qquad (5.2)
\end{aligned}$$

Now, for $c \le (\log g)^{43} \ll (\log d)^{43}$, by the Siegel–Walfisz theorem we have

$$\begin{aligned}
\sum_{\substack{1\le y\leqslant c\\ (y,c)=1}} \psi(dy;c,dy) &= \sum_{\substack{1\le y\leqslant c\\ (y,c)=1}} \left(\frac{dy}{\varphi(c)} + O\left(dy\exp\Big(-\kappa_2\sqrt{\log g}\Big)\right)\right)\\
&= \frac{1}{2}cd + O\left(g\exp\Big(-\kappa_3\sqrt{\log g}\Big)\right)\\
&= \frac{1}{2}g + O\left(\frac{g}{c} + g\exp\Big(-\kappa_3\sqrt{\log g}\Big)\right),
\end{aligned}$$

where κ_2 and κ_3 are two positive integers with $\kappa_3 < \kappa_2$. Since

$$\psi(g) = g + O\left(g\exp\Big(-\kappa_4\sqrt{\log g}\Big)\right)$$

by the prime number theorem, we finally conclude from (5.2) that

$$\psi_{c,d} = \frac{1}{2}g + O\left(\frac{g}{c} + g\exp\Big(-\kappa_5\sqrt{\log g}\Big)\right) \qquad (5.3)$$

for $c \le (\log g)^{43}$, where $\kappa_5 = \min\{\kappa_3, \kappa_4\}$. From (5.1) and (5.3), we proved that

$$\psi_{c,d} \sim \frac{1}{2}g, \quad \text{as } c \to \infty.$$

This completes the proof of Theorem 1.3. □

Proof of Theorem 1.2. For $t \leqslant g$, let

$$\vartheta_{a,b}(t) = \sum_{\substack{p=ax+by\\ p\leqslant t\\ x,y\in\mathbb{Z}_{\geqslant 0}}} \log p \quad \text{and} \quad \vartheta_{a,b} = \vartheta_{a,b}(g).$$

Integrating by parts, we obtain that

$$\pi_{a,b} = \sum_{\substack{p=ax+by \\ p\leqslant g \\ x,y\in\mathbb{Z}_{\geqslant 0}}} 1 = \frac{\vartheta_{a,b}}{\log g} + \int_2^g \frac{\vartheta_{a,b}(t)}{t\log^2 t}\,\mathrm{d}t. \tag{5.4}$$

By the Chebyshev estimate, we have

$$\vartheta_{a,b}(t) \leqslant \sum_{p\leqslant t} \log p \ll t,$$

from which it follows that

$$\int_2^g \frac{\vartheta_{a,b}(t)}{t\log^2 t}\,\mathrm{d}t \ll \int_2^g \frac{1}{\log^2 t}\,\mathrm{d}t \ll \frac{g}{(\log g)^2}. \tag{5.5}$$

Again, using the Chebyshev estimate, we have

$$\vartheta_{a,b} = \psi_{a,b} + O(\sqrt{g}). \tag{5.6}$$

Thus, by Theorem 1.3 and (5.4), (5.5), (5.6), we conclude that

$$\pi_{a,b} = \frac{\psi_{a,b}}{\log g} + O\left(\frac{\sqrt{g}}{\log g} + \frac{g}{(\log g)^2}\right) \sim \frac{1}{2}\pi(g),$$

as $c \to \infty$. This completes the proof of Theorem 1.2. □

Proof of Corollary 1.4. It follows clearly from Theorem 1.3, (5.2) and the prime number theorem. □

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References

[1] Y.-G. Chen, Q.-H. Yang & L. Zhao, "On infinite arithmetic progressions in sumsets", *Sci. China, Math.* **66** (2023), no. 12, p. 2669-2682.
[2] Y. Ding, "On a conjecture of Ramírez Alfonsín and Skałba", *J. Number Theory* **245** (2023), p. 292-302.
[3] M. B. Nathanson, *Additive number theory: The classical bases*, Graduate Texts in Mathematics, vol. 164, Springer, 1996, xiv+342 pages.
[4] J. L. Ramírez Alfonsín, *The Diophantine Frobenius Problem*, Oxford Lecture Series in Mathematics and its Applications, vol. 30, Oxford University Press, 2005, xvi+243 pages.
[5] J. L. Ramírez Alfonsín & M. Skałba, "Primes in numerical semigroups", *C. R. Math.* **358** (2020), no. 9-10, p. 1001-1004.
[6] J. J. Sylvester, "On subinvariants i.e. semi-invariants to binary quantics of an unlimited order", *Am. J. Math.* **5** (1882), no. 1, p. 79-136.
[7] R. C. Vaughan, *The Hardy–Littlewood Method*, 2nd ed., Cambridge Tracts in Mathematics, vol. 125, Cambridge University Press, 1997.

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