Journal de Théorie des Nombres de Bordeaux
anciennement Séminaire de Théorie des Nombres de Bordeaux

Guillem TARRACH

The p-arithmetic homology of mod p representations of $\mathrm{GL}_2(\mathbb{Q}_p)$

Tome 37, nº 1 (2025), p. 299-323.
https://doi.org/10.5802/jtnb.1322

CENTRE MERSENNE

Le Journal de Théorie des Nombres de Bordeaux *est membre du Centre Mersenne pour l'édition scientifique ouverte*
http://www.centre-mersenne.org/
e-ISSN : 2118-8572

Journal de Théorie des Nombres de Bordeaux **37** (2025), 299–323

The p-arithmetic homology of mod p representations of $\mathrm{GL}_2(\mathbb{Q}_p)$

par Guillem TARRACH

Résumé. Nous calculons les systèmes non-Eisenstein de valeurs propres de Hecke contribuant à l'homologie p-arithmétique des représentations lisses irréductibles π de $\mathrm{GL}_2(\mathbb{Q}_p)$ modulo p et à la cohomologie de leurs duaux. Nous montrons que, dans la plupart des cas, ces systèmes sont associés à des représentations galoisiennes irréductibles de dimension 2 impaires, dont la composante locale en p correspond, via la correspondance locale de Langlands modulo p, à une représentation lisse contenant π comme sous-représentation.

Abstract. We compute the non-Eisenstein systems of Hecke eigenvalues contributing to the p-arithmetic homology of irreducible smooth mod p representations π of $\mathrm{GL}_2(\mathbb{Q}_p)$ and to the cohomology of their duals. We show that in most cases they are associated to odd irreducible 2-dimensional Galois representations whose local component at p corresponds under the mod p local Langlands correspondence to a smooth representation that contains π as a subrepresentation.

1. Introduction

Let $p \geq 5$ be a prime number and $N \geq 5$ an integer coprime to p. Let $\Gamma_1^p(N)$ be the subgroup of matrices in $\mathrm{GL}_2(\mathbb{Z}[1/p])$ that have positive determinant and are congruent modulo $N\mathbb{Z}[1/p]$ to a matrix of the form $\left(\begin{smallmatrix} * & * \\ 0 & 1 \end{smallmatrix}\right)$. The goal of this article is to compute the systems of Hecke eigenvalues contributing to the homology of $\Gamma_1^p(N)$ with coefficients in the irreducible mod p representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ and the cohomology of their duals. The goal of this article is to compute the systems of Hecke eigenvalues contributing to the homology of $\Gamma_1^p(N)$ with coefficients in the irreducible mod p representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ and the cohomology of their duals. More specifically, we prove the following result.

Theorem 1.1. *Let π be an irreducible smooth mod p representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ over $\overline{\mathbb{F}}_p$, $\pi^\vee$ its abstract $\overline{\mathbb{F}}_p$-dual. Then:*

(1) *The homology $H_*(\Gamma_1^p(N), \pi)$ and cohomology $H^*(\Gamma_1^p(N), \pi^\vee)$ are finite-dimensional and vanish in degrees outside the range $[0, 3]$.*

Manuscrit reçu le 11 octobre 2023, révisé le 19 septembre 2024, accepté le 8 novembre 2024.
2020 *Mathematics Subject Classification.* 11F75, 11F70, 11F80.
Mots-clefs. Local-global compatibility, p-arithmetic cohomology.

(2) *To any system of Hecke eigenvalues in these (co)homology spaces, one can associate a 2-dimensional odd semisimple mod p representation of* $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ *satisfying the usual relations at primes not dividing pN.*

(3) *Let* $\rho\colon \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ *be a 2-dimensional odd irreducible Galois representation. Then, ρ contributes to $H_*(\Gamma_1^p(N), \pi)$ and $H^*(\Gamma_1^p(N), \pi^\vee)$ if and only if N is a multiple of the minimal level $N(\rho)$ attached to ρ by Serre* [14]*, and one of the following is satisfied:*
   (a) *π is a subrepresentation of the representation associated to the restriction of ρ at a decomposition group $\mathcal{G}_p$ at p by the mod p local Langlands correspondence for* $\mathrm{GL}_2(\mathbb{Q}_p)$*. In this case, ρ contributes to (co)homology in degrees 1, 2 and 3, unless π is a twist of the Steinberg representation, in which case ρ contributes to cohomology in degrees 1 and 2.*
   (b) *π is a character, say $\pi = \chi \circ \det$, and $\rho|_{\mathcal{G}_p}$ is an extension of $\chi\omega^{-1}$ by χ, where we have identified χ with a character of $\mathcal{G}_p$ via local class field theory and ω denotes the mod p cyclotomic character. In this case, ρ contributes to (co)homology in degrees 2 and 3.*

We prove this theorem by combining the explicit construction of the irreducible mod p representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ due to Barthel–Livné [1] and Breuil [3], a result relating p-arithmetic homology to arithmetic homology in the spirit of [11] and [15], and classical results on the weight part of Serre's conjecture. These are already enough to prove the generic case where π is supersingular or a principal series representation. The cases of (twists of) the trivial and Steinberg representations require more work, and involve the group cohomological analogue of multiplication of mod p modular forms by the Hasse invariant studied by Edixhoven–Khare [8] and an interpretation of this map in terms of the representation theory of the local group $\mathrm{GL}_2(\mathbb{Q}_p)$.

1.1. Notation and conventions. Write $G = \mathrm{GL}_2(\mathbb{Q}_p)$, $K = \mathrm{GL}_2(\mathbb{Z}_p)$ and Z for the center of G, so that $Z \simeq \mathbb{Q}_p^\times$. Let $\alpha = \left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right) \in G$ and $\beta = \left(\begin{smallmatrix} p & 0 \\ 0 & p \end{smallmatrix}\right) \in Z$. Write also B for the subgroup of upper-triangular matrices in G and $I = K \cap \alpha K \alpha^{-1}$ for the Iwahori subgroup of matrices in K that are upper-triangular modulo p. Let $G^+ \subseteq G$ be the submonoid of matrices whose entries lie in $\mathbb{Z}_p$, and $G^- = (G^+)^{-1}$. We will write ω for the character $\mathbb{Q}_p^\times \to \mathbb{F}_p^\times$ defined by $x \mapsto x|x| \mod p$. Write $k = \overline{\mathbb{F}}_p$.

We normalise local class field theory so that uniformisers correspond to geometric Frobenii, and for each prime ℓ we let Frob_ℓ be the geometric Frobenius corresponding to ℓ. Choose embeddings $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_\ell$ for all ℓ, and let $\mathcal{G}_\ell$ denote the corresponding decomposition group at ℓ in $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. We use our normalisation of local class field theory to identify $k^\times$-valued characters of $\mathcal{G}_\ell$ and $\mathbb{Q}_\ell^\times$ without comment. Write $\varepsilon\colon \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to k^\times$ for the mod p cyclotomic character, it satisfies $\varepsilon(\mathrm{Frob}_\ell) = \ell^{-1} \mod p$ for $\ell \neq p$ and $\varepsilon(\mathrm{Frob}_p) = 1$. Its restriction to $\mathcal{G}_p$ at p corresponds to ω under local class field theory. Write $\mathcal{I}_p \subseteq \mathcal{G}_p$ for the inertia subgroup. Let $\omega_2\colon \mathcal{I}_p \to \mu_{p^2-1}(\overline{\mathbb{Q}}_p^\times) \subseteq \overline{\mathbb{F}}_p^\times$ be Serre's fundamental character of level 2, defined by $\omega_2(g) = (gp^{1/(p^2-1)})/p^{1/(p^2-1)}$, and for $0 \leq s \leq p$ let $\mathrm{Ind}(\omega_2^s)$ be the irreducible representation of $\mathcal{G}_p$ over χ with determinant ω^s and $\mathrm{Ind}(\omega_2^s)|_{\mathcal{I}_p} = \omega_2^s \oplus \omega_2^{ps}$. All irreducible 2-dimensional representations of $\mathcal{G}_p$ over k are of the form $\mathrm{Ind}(\omega_2^s) \otimes \chi$ for some s as above and character χ. Given a two-dimensional odd and irreducible representation ρ of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ over k, we will write $N(\rho)$ for the minimal level attached to ρ by Serre in [14].

Given $b \in k^\times$, we will write unr_b for the k-valued unramified characters of $\mathbb{Q}_p^\times$ and of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ sending p and Frob_p respectively to b. Thus, all continuous characters $\mathbb{Q}_p^\times \to k^\times$ are of the form $\omega^a \mathrm{unr}_b$ for some $b \in k^\times$ and $0 \leq a \leq p-2$. If V is any representation of G (resp. K) and χ is a k-valued continuous character of $\mathbb{Q}_p^\times$ (resp. $\mathbb{Z}_p^\times$), we will write $V \otimes \chi$ instead of $V \otimes (\chi \circ \det)$.

Acknowledgements. The author is grateful to Jack Thorne for suggesting the question of studying the p-arithmetic cohomology of smooth mod p representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ and for his comments on earlier drafts of this article.

2. Preliminaries

2.1. Arithmetic and *p*-arithmetic (co)homology. Let U^p be a compact open subgroup of $\mathrm{GL}_2(\mathbb{A}^{p\infty})$ of the form $\prod_{\ell \nmid pN} \mathrm{GL}_2(\mathbb{Z}_\ell) \times U_N$ for some $N \geq 1$ coprime to p and open compact subgroup $U_N \subseteq \prod_{\ell | N} \mathrm{GL}_2(\mathbb{Q}_\ell)$. Let $\mathrm{GL}_2(\mathbb{R})^\circ$ be the subgroup of $\mathrm{GL}_2(\mathbb{R})$ consisting of matrices with positive determinant. Assume that for all $g \in \mathrm{GL}_2(\mathbb{A}^p) \times \mathrm{GL}_2(\mathbb{R})^\circ$, the group $\mathrm{GL}_2(\mathbb{Q}) \cap g(U^p \times \mathrm{GL}_2(\mathbb{R})^\circ)g^{-1}$ is torsion-free. Consider the left action of $\mathrm{GL}_2(\mathbb{Q}) \times \mathrm{GL}_2(\mathbb{A})$ on $k[\mathrm{GL}_2(\mathbb{A})]$, where $\mathrm{GL}_2(\mathbb{Q})$ acts by multiplication on the left and $\mathrm{GL}_2(\mathbb{A})$ acts by multiplication on the right. Consider also the right action of $\mathrm{GL}_2(\mathbb{Q}) \times \mathrm{GL}_2(\mathbb{A})$ on this space induced by $g \mapsto g^{-1}$ for $g \in \mathrm{GL}_2(\mathbb{Q}) \times \mathrm{GL}_2(\mathbb{A})$. The (arithmetic) homology of level U^pK of a left $k[K]$-module M, is defined as

$$H_i(U^pK, M) := \mathrm{Tor}_i^{k[\mathrm{GL}_2(\mathbb{Q}) \times U^p \times K \times \mathrm{GL}_2(\mathbb{R})^\circ]}(k[\mathrm{GL}_2(\mathbb{A})], M).$$

Here, we view M as a $\mathrm{GL}_2(\mathbb{Q}) \times U^p \times K \times \mathrm{GL}_2(\mathbb{R})^\circ$ module by letting $\mathrm{GL}_2(\mathbb{Q}) \times U^p \times \mathrm{GL}_2(\mathbb{R})^\circ$ act trivially. The (arithmetic) cohomology of M

in level U^pK is defined analogously as

$$H^i(U^pK, M) := \mathrm{Ext}^i_{k[\mathrm{GL}_2(\mathbb{Q})\times U^p\times K\times \mathrm{GL}_2(\mathbb{R})^\circ]}(k[\mathrm{GL}_2(\mathbb{A})], M).$$

Similarly, if M is a $k[G]$-module, the p-arithmetic homology and cohomology of M in level U^p are defined as

$$H_i(U^p, M) := \mathrm{Tor}_i^{k[\mathrm{GL}_2(\mathbb{Q})\times U^p\times \mathrm{GL}_2(\mathbb{Q}_p)\times \mathrm{GL}_2(\mathbb{R})^\circ]}(k[\mathrm{GL}_2(\mathbb{A})], M),$$
$$H^i(U^p, M) := \mathrm{Ext}^i_{k[\mathrm{GL}_2(\mathbb{Q})\times U^p\times \mathrm{GL}_2(\mathbb{Q}_p)\times \mathrm{GL}_2(\mathbb{R})^\circ]}(k[\mathrm{GL}_2(\mathbb{A})], M).$$

For both arithmetic and p-arithmetic homology (and similarly for cohomology), one can canonically define complexes computing them as in [15, Section 5.1], where they were denoted $C^{\mathrm{ad}}_\bullet(U^pK, M)$ and $C^{\mathrm{ad}}_\bullet(U^p, M)$; here we will denote them by $C_\bullet(U^pK, M)$ and $C_\bullet(U^p, M)$ respectively. One can also speak of arithmetic and p-arithmetic hyperhomology and hypercohomology of complexes of $k[K]$ or $k[G]$-modules; these are just the derived tensor products and derived Hom corresponding to the Tor and Ext modules above in their corresponding derived category. In this article we will only be interested in the case where

$$\begin{aligned} U^p &= U_1^p(N) \\ &:= \prod_{\ell\nmid pN} \mathrm{GL}_2(\mathbb{Z}_\ell) \times \prod_{\ell\mid N} \left\{ g \in \mathrm{GL}_2(\mathbb{Z}_\ell) : g \equiv \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \mod \ell^{v_\ell(N)} \right\} \end{aligned}$$

for $N \geq 5$.

Proposition 2.1. *There are canonical isomorphisms*

$$H_*(U_1^p(N)K, -) \simeq H_*(\Gamma_1(N), -), \quad H^*(U_1^p(N)K, -) \simeq H^*(\Gamma_1(N), -),$$
$$H_*(U_1^p(N), -) \simeq H_*(\Gamma_1^p(N), -), \quad H^*(U_1^p(N), -) \simeq H^*(\Gamma_1^p(N), -),$$

where the right-hand sides denote group homology or cohomology, $\Gamma_1(N)$ *is the usual congruence subgroup of* $\mathrm{SL}_2(\mathbb{Z})$ *and* $\Gamma_1^p(N)$ *is the subgroup of* $\mathrm{GL}_2(\mathbb{Z}[1/p])$ *defined in the introduction, which is torsion-free. The arithmetic (resp. p-arithmetic) (co)homology groups are zero in degrees outside the range* $[0, 1]$ *(resp.* $[0, 3]$*).*

Proof. We have that $\mathrm{GL}_2(\mathbb{Q}) \cap U_1^p(N) \times K \times \mathrm{GL}_2(\mathbb{R})^\circ = \Gamma_1(N)$, where the intersection takes place in $\mathrm{GL}_2(\mathbb{A})$. Indeed, the intersection consists of elements in $\mathrm{GL}_2(\mathbb{Z})$ with positive determinant and which are congruent to a matrix of the form $\left(\begin{smallmatrix} * & * \\ 0 & 1 \end{smallmatrix}\right)$ modulo N. The determinant must be 1, so such matrices are congruent to $\left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right)$ modulo N. Similarly, the intersection $\mathrm{GL}_2(\mathbb{Q}) \cap U_1^p(N) \times \mathrm{GL}_2(\mathbb{Q}_p) \times \mathrm{GL}_2(\mathbb{R})^\circ$ is precisely $\Gamma_1^p(N)$.

Let us show that $\Gamma_1^p(N)$ is torsion-free following a well-known argument for $\Gamma_1(N)$. Let $\gamma \in \Gamma_1^p(N)$ be a torsion element. Since $\gamma \in \Gamma_1^p(N)$, its determinant must be a power of p, and since γ is torsion it must be 1. Together with the definition of $\Gamma_1^p(N)$, this implies that the diagonal entries of γ

are both congruent to 1 mod $N\mathbb{Z}[1/p]$. Hence, the trace of γ is congruent to 2 mod $N\mathbb{Z}[1/p]$. On the other hand, if $\gamma \in \Gamma_1^p(N)$ is torsion, then its eigenvalues in $\overline{\mathbb{Q}}$ are roots of unity that are also roots of the characteristic polynomial of γ. The characteristic polynomial must therefore be one of $(x \pm 1)^2, x^2 \pm 1$ or $x^2 \pm x + 1$. Since $N \geq 5$, only $(x-1)^2$ satisfies the congruence condition for the trace above. Thus, the eigenvalues of γ are 1, which implies that $\gamma = 1$ since γ is torsion.

Next, we note that $\mathrm{GL}_2(\mathbb{A}) = \mathrm{GL}_2(\mathbb{Q})(U_1^p(N) \times K \times \mathrm{GL}_2(\mathbb{R})^\circ)$. Indeed, let $\gamma \in \mathrm{GL}_2(\mathbb{A})$ be arbitrary. Since taking determinants gives a surjection $U_1^p(N) \times K \times \mathrm{GL}_2(\mathbb{R})^\circ \to \widehat{\mathbb{Z}}^\times \times \mathbb{R}_{>0}$, we may multiply γ on the right by an element of the source and on the left by an element of $GL_2(\mathbb{Q})$ to assume that γ has determinant 1. Strong approximation for SL_2 implies that such an element belongs to $\mathrm{SL}_2(\mathbb{Q})(\mathrm{SL}_2(\mathbb{A}) \cap U_1^p(N) \times K \times \mathrm{GL}_2(\mathbb{R})^\circ)$, which proves our claim.

Putting everything together, the arguments in [15, Section 5.1] show the existence of the claimed canonical isomorphisms. The claims about dimensions then follow from the corresponding facts for the arithmetic group $\Gamma_1(N)$ and p-arithmetic group $\Gamma_1^p(N)$ [2, Théorème 6.2]. ☐

2.2. Hecke operators. Let H is any locally profinite group, H_0 a compact open subgroup and $H_+ \subseteq H$ a submonoid containing H_0, and write $H_- = H_+^{-1}$. If M is a (left) $k[H_-]$-module (resp. $k[H_+]$-module), then the H_0-coinvariants M_{H_0} (resp. H_0-invariants M^{H_0}) are a right (resp. left) module for the Hecke algebra $\mathcal{H}(H_+, H_0)_k$, the algebra of smooth compactly supported H_0-biinvariant functions $H_+ \to k$ under convolution. Explicitly, given $h \in H_+$, the function $[H_0hH_0] \in \mathcal{H}(H_+, H_0)_k$ supported on H_0hH_0 that maps h to 1 acts on $m \in M_{H_0}$ (resp. $m \in M^{H_0}$) by

$$m[H_0hH_0] = \sum_{xhH_0 \in H_0hH_0/H_0} (xh)^{-1}m$$

$$\left(\text{resp. } [H_0hH_0]m = \sum_{xhH_0 \in H_0hH_0/H_0} xhm\right).$$

This association from modules over the monoids to Hecke algebras is functorial, and hence it defines actions of the Hecke algebras on higher group (co)homology spaces in addition to the (co)invariants.

This discussion applies in particular to arithmetic and p-arithmetic (co)homology. Let U^p and N be as in the previous section. Let $\mathbb{T}(pN)$ denote the abstract unramified Hecke algebra for GL_2 away from pN with coefficients in $\mathbb{Z}$, that is, the restricted tensor product of the local Hecke algebras $\mathcal{H}(\mathrm{GL}_2(\mathbb{Q}_\ell), \mathrm{GL}_2(\mathbb{Z}_\ell))$ with $\ell \nmid pN$. It is a commutative algebra freely generated by Hecke operators T_ℓ, corresponding to the double coset of $\left(\begin{smallmatrix} 1 & 0 \\ 0 & \ell \end{smallmatrix}\right)$, and invertible operators S_ℓ, corresponding to the double coset of $\left(\begin{smallmatrix} \ell & 0 \\ 0 & \ell \end{smallmatrix}\right)$, where $\ell \nmid pN$. Fix also a submonoid $G_+ \subseteq G$ containing K. Let V be a representation of G_- (resp. G_+) on a k-vector space. Then, the arithmetic homology $H_*(U^pK, V)$ (resp. the arithmetic cohomology $H^*(U^pK, V)$) is computed by taking the U^pK-coinvariants (resp. invariants) of a complex of representations of $\mathrm{GL}_2(\mathbb{A}^{p\infty}) \times G_+$, and is thus endowed with commuting actions of $\mathbb{T}(pN)$ and $\mathcal{H}(G_+, K)_k$, the latter being a right (resp. left) action. For us, $\mathcal{H}(G_+, K)_k$ will always be a commutative algebra so we will not distinguish between left and right actions. Similarly, if V is a representation of G on a k-vector space, then the p-arithmetic homology $H_*(U^p, V)$ (resp. the p-arithmetic cohomology $H^*(U^p, V)$) is endowed with an action of $\mathbb{T}(pN)$. If V is a representation of G_- (resp. G) and $V^\vee$ denotes its (abstract) contragredient, then there are $\mathbb{T}(pN) \otimes \mathcal{H}(G_+, K)$-equivariant (resp. $\mathbb{T}(pN)$-equivariant) isomorphisms $H^*(U^pK, V^\vee) \simeq H_*(U^pK, V)^\vee$ (resp. $H^*(U^p, V^\vee) \simeq H_*(U^p, V)^\vee$). In particular, the systems of eigenvalues in both spaces are the same, so all the statements for cohomology in Theorem 1.1 follow from the corresponding statements for homology. For this reason, we will now work almost exclusively with homology.

Let ρ be a k-valued continuous representation of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ unramified outside N, and consider the maximal ideal $\mathfrak{m}_\rho$ of $\mathbb{T}(pN)$ defined by the kernel of the homomorphism $\mathbb{T}(pN) \to k$ defined by $T_\ell \mapsto \operatorname{tr} \rho(\mathrm{Frob}_\ell)$ and $\ell S_\ell \mapsto \det \rho(\mathrm{Frob}_\ell)$ for $\ell \nmid pN$, where Frob_ℓ is a geometric Frobenius at ℓ. For example, $\mathfrak{m}_{1 \oplus \varepsilon^{-1}}$ is generated by $T_\ell - (1+\ell), \ell S_\ell - \ell$. Given a $\mathbb{T}(pN)$-module M, we will say that ρ contributes to, or appears in, M if the localisation of $M_{\mathfrak{m}_\rho}$ is non-zero. We will sometimes write M_ρ instead of $M_{\mathfrak{m}_\rho}$. If V is an irreducible representation of K, then any system of Hecke eigenvalues in $H^*(\Gamma_1(N), V)$ corresponds to a semisimple 2-dimensional Galois representation as above. For such a V, the cohomology $H^0(\Gamma_1(N), V)$ vanishes unless V is the trivial representation, in which case it is one-dimensional and the Hecke operators act via $T_\ell = 1 + \ell, S_\ell = 1$. In particular, the only semisimple Galois representation ρ that can contribute to $H^0(\Gamma_1(N), V)$ is $1 \oplus \varepsilon^{-1}$. In particular, irreducible Galois representations can only contribute to arithmetic cohomology in level $\Gamma_1(N)$ only in degree 1.

Given a character $\chi \colon \mathbb{Q}_p^\times \to k^\times$, write $k(\chi)$ for the $\mathbb{T}(pN)$-module whose underlying module is k and where T_ℓ (resp. S_ℓ) act via $\chi(\ell)$ (resp. $\chi(\ell)^2$). For any $\mathbb{T}(pN)$-module M, write $M(\chi) := M \otimes_k k(\chi)$ for the twist of M by $k(\chi)$.

2.3. Irreducible mod p representations of $\mathrm{GL}_2(\mathbb{Q}_p)$. In this section we will recall the construction of the smooth irreducible mod p representations of G and some facts about them. Given $0 \leq r \leq p-1$, consider the representation $\mathrm{Sym}^r(k^2)^\vee$ of K over k. Note that $\mathrm{Sym}^r(k^2)$ naturally extends to a representation of the monoid G^+ of matrices with entries in

$\mathbb{Z}_p$, and also (perhaps less naturally) to a representation of KZ where β acts trivially. In particular, $\mathrm{Sym}^r(k^2)^\vee$ extends to representations of G^- and of KZ. The compact induction $\text{c-Ind}_K^G(\mathrm{Sym}^r(k^2)^\vee)$ can be identified with the tensor product $k[G] \otimes_{k[K]} \mathrm{Sym}^r(k^2)^\vee$, or the K-coinvariants[1] of $k[G] \otimes_k \mathrm{Sym}^r(k^2)^\vee$. Via the formalism in the previous section, the action of G^- on $\mathrm{Sym}^r(k^2)^\vee$ induces an action of $\mathcal{H}(G^+, K)$ on $\text{c-Ind}_K^G(\mathrm{Sym}^r(k^2)^\vee)$ and the action of KZ induces an action of $\mathcal{H}(KZ, K)$. We let T denote the operator on $\text{c-Ind}_K^G(\mathrm{Sym}^r(k^2)^\vee)$ corresponding to the double coset of α under the first action and S the operator corresponding to the double coset of β under the second, which is an invertible operator. In fact, there are isomorphisms $\mathcal{H}(G^+, K) \simeq k[T]$ and $\mathcal{H}(KZ, K) \simeq k[S^{\pm 1}]$. Unwinding the definitions, these operators are described as follows. Given $g \in G$ and $v \in \mathrm{Sym}^r(k^2)^\vee$, let $[g, v]$ denote the element of $\text{c-Ind}_K^G(\mathrm{Sym}^r(k^2)^\vee)$ that is supported on gK and maps g to v. We will use the same notation for compact inductions from other groups and of other representations. The Hecke operators above are then defined by the formulas

$$T[g, v] = \sum_{x \in K/I} [gx\alpha, \alpha^{-1}x^{-1}v], \qquad S[g, v] = [\beta g, v],$$

Given a continuous character $\chi\colon \mathbb{Q}_p^\times \to k^\times$ and $\lambda \in k$, define

$$\pi(r, \lambda, \chi) := \frac{\text{c-Ind}_K^G(\mathrm{Sym}^r(k^2)^\vee)}{(T - \lambda, S - 1)} \otimes_k \chi\omega^r$$

(we have included the twist by ω^r in order for this notation to match that of the existing literature).

Remark 2.2. We will later need to consider a variation of this definition that is defined in families. Let R be a k-algebra. One can define Hecke operators T and S on $\text{c-Ind}_K^G(\mathrm{Sym}^r(k^2)^\vee \otimes_k \chi)$ for any character $\chi\colon G \to R^\times$ in the same way as for $R = k$ and $\chi = 1$, and the natural isomorphism $\text{c-Ind}_K^G(\mathrm{Sym}^r(k^2)^\vee \otimes_k \chi) \xrightarrow{\sim} \text{c-Ind}_K^G(\mathrm{Sym}^r(k^2)^\vee) \otimes_k \chi$ intertwines T and S on the source with $T \otimes 1$ and $S \otimes 1$ respectively on the target. Moreover, if $b \in R^\times$ and $\lambda \in R$, then there are isomorphisms of representations of G

$$\begin{aligned} \frac{\text{c-Ind}_K^G(\mathrm{Sym}^r(k^2)^\vee \otimes_k R)}{(T - \lambda, S - 1)} \otimes_R (\chi \mathrm{unr}_b) &\xrightarrow{\sim} \frac{\text{c-Ind}_K^G(\mathrm{Sym}^r(k^2)^\vee \otimes_k (\chi \mathrm{unr}_b))}{(T - \lambda, S - 1)} \\ &\xrightarrow{\sim} \frac{\text{c-Ind}_K^G(\mathrm{Sym}^r(k^2)^\vee \otimes_k \chi)}{(T - \lambda b, S - b^2)}. \end{aligned}$$

Let us define for $\tau \in R$, $\sigma \in R^\times$ and $s \in \mathbb{Z}$,

$$\widetilde{\pi}(r, \tau, \sigma, s)_R := \frac{\text{c-Ind}_K^G(\mathrm{Sym}^r(k^2)^\vee \otimes_k \omega^s \otimes_k R)}{(T - \tau, S - \sigma)}.$$

[1] Here, $k \in K$ acts as $k \otimes (g \otimes 1) = gk^{-1} \otimes kv$.

In particular, when $R = k$, then $\pi(r,\lambda,\omega^a\mathrm{unr}_b) \simeq \widetilde{\pi}(r,\tau,\sigma,s)_k$ for $\tau = \lambda b, \sigma = b^2$ and $s = a + r$.

The following results are due to Barthel–Livné [1] and Breuil [3].

Theorem 2.3.

(1) *If* $(r,\lambda) \neq (0,\pm 1),(p-1,\pm 1)$, *then* $\pi(r,\lambda,\chi)$ *is irreducible.*

(2) *If* $\lambda \neq 0$ *and* $(r,\lambda) \neq (0,\pm 1)$ *then there is a canonical isomorphism*

$$\pi(r,\lambda,\chi) \xrightarrow{\sim} \mathrm{Ind}_B^G(\chi\mathrm{unr}_{\lambda^{-1}} \otimes \chi\mathrm{unr}_\lambda\omega^r).$$

Here, $\mathrm{Ind}_B^G(-)$ *denotes smooth parabolic induction. The representation* $\pi(r,\lambda,\chi)$ *is said to be a principal series representation.*

(3) *If* $r = 0$ *and* $\lambda = \pm 1$ *there is a canonical homomorphism*

$$\pi(r,\lambda,\chi) \longrightarrow \mathrm{Ind}_B^G(\chi\mathrm{unr}_{\lambda^{-1}} \otimes \chi\mathrm{unr}_\lambda)$$

with kernel and cokernel isomorphic to $\mathrm{St}\otimes\chi\mathrm{unr}_\lambda$ *and image isomorphic to* $\chi\mathrm{unr}_\lambda\circ\det$. *Here,* St *is the Steinberg representation of* G *over* k, *i.e. the quotient of* $\mathrm{Ind}_B^G(k)$ *by the trivial representation* k.

(4) *If* $\lambda = 0$, *then* $\pi(r,0,\chi)$ *is not isomorphic to a principal series representation; it is said to be supersingular.*

(5) *Any irreducible representation of* G *over* k *is isomorphic to a principal series, a supersingular, a twist of the Steinberg, or a twist of the trivial representation.*

(6) *If* $\lambda \neq 0$, *the only isomorphisms between various of these representations are*

$$\pi(r,\lambda,\chi) \simeq \pi(r,-\lambda,\chi\mathrm{unr}_{-1})$$

and for $\lambda \neq \pm 1$

$$\pi(0,\lambda,\chi) \simeq \pi(p-1,\lambda,\chi).$$

(7) *If* $\lambda = 0$, *the isomorphisms between various of these representations are*

$$\begin{aligned}\pi(r,0,\chi) &\simeq \pi(r,0,\chi\mathrm{unr}_{-1})\\ &\simeq \pi(p-1-r,0,\chi\omega^r)\\ &\simeq \pi(p-1-r,0,\chi\omega^r\mathrm{unr}_{-1}).\end{aligned}$$

In particular, there exists a non-zero map, unique up to scalar, $\pi(p-1,1,\chi) \to \pi(0,1,\chi)$ (resp. $\pi(0,1,\chi) \to \pi(p-1,1,\chi)$), which factors through a twist of the Steinberg (resp. trivial) representation. In Section 4, we will describe these maps explicitly.

2.4. The mod p Langlands correspondence for $\mathbf{GL_2}(\mathbb{Q}_p)$. Throughout this section, we assume $p \geq 5$. We will use the same conventions for the mod p local Langlands correspondence as in [9], however we also include twists of extensions of the cyclotomic character by the trivial character in our discussion. Let MF be Colmez's magical functor, defined by $\mathrm{MF}(\pi) := \mathbf{V}(\pi) \otimes \omega^{-1}$ for $\mathbf{V}$ as in [6].

Theorem 2.4. *Let $\rho\colon \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \to \mathrm{GL}_2(k)$ be a continuous representation. Then, there exists a finite length smooth representation of G over k, unique up to isomorphism, satisfying the following properties:*

1. $\mathrm{MF}(\pi) \simeq \rho$,
2. *π has central character corresponding to $(\det \rho)\omega$ under local class field theory,*
3. *π has no finite-dimensional G-invariant subrepresentations or quotients.*

More specifically, π can be described as follows:

1. *If ρ is irreducible, say $\rho = \mathrm{Ind}(\omega_2^{r+1}) \otimes \chi$ with $0 \leq r \leq p-1$, then $\pi = \pi(r, 0, \chi\omega)$ is supersingular.*
2. *If $\rho \simeq \left(\begin{smallmatrix} \chi_1 & * \\ 0 & \chi_2 \end{smallmatrix}\right)$ with $\chi_1 \neq \chi_2\omega^{\pm 1}$, then π is an extension*
$$0 \longrightarrow \pi(r, \lambda, \chi) \longrightarrow \pi \longrightarrow \pi([p-3-r], \lambda^{-1}, \chi\omega^{r+1}) \longrightarrow 0,$$
where
$$\chi_1 = \omega^r \chi \mathrm{unr}_\lambda, \qquad \chi_2 = \omega^{-1}\chi\mathrm{unr}_{\lambda^{-1}}$$
with $0 \leq r \leq p-1$, and $[p-3-r]$ is the unique integer between 0 and $p-2$ that is congruent to $p-3-r$ modulo $p-1$. This extension is split if and only if ρ is semisimple.
3. *If ρ is a non-split extension $\left(\begin{smallmatrix} \chi & * \\ 0 & \chi\omega^{-1} \end{smallmatrix}\right)$, then π has a unique Jordan–Hölder series, which is of the form*
$$0 \subsetneq \pi_1 \subsetneq \pi_2 \subsetneq \pi$$
where $\pi_1 \simeq \mathrm{St} \otimes \chi$, $\pi_2/\pi_1 \simeq \chi \circ \det$ and $\pi/\pi_2 \simeq \pi(p-3, 1, \chi\omega)$.
4. *If ρ is an extension $\left(\begin{smallmatrix} \chi\omega^{-1} & * \\ 0 & \chi \end{smallmatrix}\right)$, then π is an extension*
$$0 \longrightarrow \pi(p-3, 1, \chi\omega) \longrightarrow \pi \longrightarrow \mathrm{St} \otimes \chi \longrightarrow 0.$$
This extension is split if and only if ρ is semisimple. On both the Galois side and the $\mathrm{GL}_2(\mathbb{Q}_p)$ side, there is a unique class of nontrivial extensions, so this property determines π.

Proof. All the statements follow from the work of Colmez [6, Section VII.4] except for case (4), which follows from the end of the proof of [12, Lemma 10.35] by taking into account that there is only one isomorphism class of Galois representations that are non-split extensions of 1 by ω^{-1}. $\square$

For ρ as in the theorem, we will say that π is the representation corresponding to ρ under the mod p local Langlands correspondence. One could argue (for example, following [5, Remark 7.7]) that the "true" mod p local Langlands correspondence in case (4) should be (up to isomorphism) a non-trivial extension of π as above by two copies of the trivial character. However, we are only interested in the socle of π, which remains the same and can be easily described in general by the following result.

Proposition 2.5. *Let ρ be a representation $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \to \mathrm{GL}_2(k)$ and let π be the corresponding representation of $\mathrm{GL}_2(\mathbb{Q}_p)$. Then, the following statements hold.*

(1) *The representation $\pi(r, 0, \chi)$ is a subrepresentation of π if and only if $\rho \simeq \mathrm{Ind}(\omega_2^{r+1}) \otimes \chi\omega^{-1}$.*

(2) *For $\lambda \neq 0$ and $(r, \lambda) \neq (0, \pm 1), (p-1, \pm 1)$, $\pi(r, \lambda, \chi)$ is a subrepresentation of π if and only if*
$$\rho \simeq \begin{pmatrix} \omega^r \mathrm{unr}_\lambda & * \\ 0 & \omega^{-1}\mathrm{unr}_{\lambda^{-1}} \end{pmatrix} \otimes \chi.$$

(3) $\mathrm{St} \otimes \chi$ *is a subrepresentation of π if and only if*
$$\rho \simeq \begin{pmatrix} 1 & * \\ 0 & \omega^{-1} \end{pmatrix} \otimes \chi.$$

(4) $\chi \circ \det$ *is never a subrepresentation of π.*

Proof. This follows from Theorem 2.4 and Theorem 2.3 (6). ☐

2.5. The weight part of Serre's conjecture. In this section we will recall the answer to the following question: given a continuous representation $\rho\colon \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(k)$, for what $0 \le r \le p-1$ and s does ρ contribute to $H^1(\Gamma_1(N), \mathrm{Sym}^r(k^2) \otimes \omega^{-s})$ and, when it does, what are its eigenvalues for the Hecke operators at p? What we mean by Hecke operators at p is the following. One can define the action of Hecke operators T and S on the arithmetic homology complex $C_\bullet(U^pK, \mathrm{Sym}^r(k^2)^\vee \otimes_k \omega^s)$ for any tame level $U^p \subseteq \mathrm{GL}_2(\mathbb{A}^{p\infty})$ as we explained in Section 2.2 by using the actions of G^- and KZ on $\mathrm{Sym}^r(k^2)^\vee \otimes_k \omega^s$, and similarly for the cohomology of the dual. When $s = 0$, these operators correspond under the Eichler–Shimura isomorphism followed by reduction mod p to the usual Hecke operators T_p and $\langle p \rangle$ respectively for modular forms of level $\Gamma_1(N)$ and weight $r+2$.

The answer to the question is contained in the proof of [4, Theorem 3.17]. We warn the reader that our conventions are different to those in [4]: ρ contributes to the cohomology of a Serre weight V in the sense of this article if and only if $\rho^\vee$ is modular of weight V in the sense of [4] (equivalently, if and only if ρ is modular of weight $V^\vee \otimes \omega^{-1}$ in the sense of [4]). In particular, it follows from [4, Corollary 2.11] that ρ contributes to the cohomology of $V \otimes \omega^a$ if and only if $\rho\varepsilon^a$ contributes to the cohomology of V for any a.

The theorem below also involves results from [7], which follows arithmetic conventions, so we remark that ρ contributes to $H^1(\Gamma_1(N), \mathrm{Sym}^r(k^2))$ if and only if $\rho^\vee$ is modular of weight $r+2$ in the sense of [7]. Moreover, the character $\lambda(b)$ from [7] is the same as our character $\mathrm{unr}_{b^{-1}}$.

Theorem 2.6. *Let* $\rho\colon \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(k)$ *be an odd irreducible Galois representation. Let* $0 \le r \le p-1$ *and* $a \in \mathbb{Z}$. *Let* $\lambda \in k, b \in k^\times$ *and set* $\tau = \lambda b$, $\sigma = b^2$ *and* $s = a + r$. *Then,* ρ *contributes to the* $(T = \tau, S = \sigma)$*-eigenspace in* $H^1(\Gamma_1(N), \mathrm{Sym}^r(k^2) \otimes \omega^{-s})$ *if and only if* $N(\rho)$ *divides* N *and one of the following holds:*

(1) $\lambda = 0$ *and* $\rho|_{\mathcal{G}_p} \simeq \mathrm{Ind}(\omega_2^{r+1}) \otimes \omega^{a-1}\mathrm{unr}_b$,

(2) $\lambda \neq 0$, $(r, \lambda) \neq (0, \pm 1)$ *and*

$$\rho|_{\mathcal{G}_p} \simeq \begin{pmatrix} \omega^r \mathrm{unr}_\lambda & * \\ 0 & \omega^{-1}\mathrm{unr}_{\lambda^{-1}} \end{pmatrix} \otimes \omega^a \mathrm{unr}_b.$$

(3) $r = 0$, $\lambda = \pm 1$ *and*

$$\rho|_{\mathcal{G}_p} \simeq \begin{pmatrix} 1 & * \\ 0 & \omega^{-1} \end{pmatrix} \otimes \omega^a \mathrm{unr}_{\lambda b}.$$

where $*$ *denotes a peu ramifiée extension.*

Proof. According to [7, Theorem 2.5 and Theorem 2.6], only ρ whose restriction to $\mathcal{G}_p$ is irreducible (resp. reducible) can contribute to the eigenspaces where $T = 0$ (resp. $T \neq 0$). By [7, Theorem 2.6], the Galois representations ρ appearing in the $(T = 0, S = b^2)$-eigenspace of $H^1(\Gamma_1(N), \mathrm{Sym}^r(k^2) \otimes \omega^{-s})$ are those satisfying $(\rho\omega^{-s})^\vee|_{\mathcal{I}_p} \simeq \omega_2^{r+1} \oplus \omega_2^{p(r+1)}$ and $(\det \rho)(\mathrm{Frob}_p) = b^2$ (recall that Frob_p is a Frobenius mapping to p under class field theory, so it is a well-defined element of $\mathcal{G}_p^{\mathrm{ab}}$). Note that the first condition is equivalent to $\rho|_{\mathcal{I}_p} \simeq (\omega_2^{r+1} \oplus \omega_2^{p(r+1)}) \otimes \omega^{s-r-1}$. In particular, the restriction to $\mathcal{G}_p$ of such a representation has determinant $\omega^{2s-r-1}\mathrm{unr}_{b^2}$, so it must be isomorphic to

$$\mathrm{Ind}(\omega_2^{r+1}) \otimes \omega^{s-r-1}\mathrm{unr}_b \simeq \mathrm{Ind}(\omega_2^{r+1}) \otimes \omega^{a-1}\mathrm{unr}_b.$$

For the case where $\lambda \neq 0$, [7, Theorem 2.5] states that if ρ contributes to the $(T = \tau, S = \sigma)$-eigenspace of $H^1(\Gamma_1(N), \mathrm{Sym}^r(k^2) \otimes \omega^{-s})$, then

$$(\rho\omega^{-s})^\vee|_{\mathcal{G}_p} \simeq \begin{pmatrix} \omega^{r+1}\mathrm{unr}_{\tau\sigma^{-1}} & * \\ 0 & \mathrm{unr}_{\tau^{-1}} \end{pmatrix}.$$

If $\tau = \lambda b$ and $\sigma = b^2$, this is equivalent to

$$\rho|_{\mathcal{G}_p} \simeq \begin{pmatrix} \omega^r \mathrm{unr}_\lambda & * \\ 0 & \omega^{-1}\mathrm{unr}_{\lambda^{-1}} \end{pmatrix} \otimes \omega^a \mathrm{unr}_b.$$

Conversely, [4, Theorem 3.17] and its proof show that a representation of this form does contribute to $H^1(\Gamma_1(N), \mathrm{Sym}^r(k^2) \otimes \omega^{-s})$ provided that the extension is peu ramifiée whenever $r=0$ and $\lambda=\pm 1$ (and, in this case, it never contributes if the extension is très ramifiée). It remains to show that it contributes to the $(T=\lambda b, S=b^2)$-eigenspace. When $r \neq p-2$ or $\rho|_{\mathcal{G}_p}$ is non-split, [7, Theorems 2.5 and 2.6] again show that this the only eigenspace for Hecke operators at p to which ρ can contribute, so it must do so. In the case when $r=p-2$ and $\rho|_{\mathcal{G}_p} \simeq \omega^{a-1}\mathrm{unr}_{\lambda b} \oplus \omega^{a-1}\mathrm{unr}_{\lambda^{-1}b}$, the same results show that ρ can only appear in the eigenspaces for $(T=\lambda b, S=b^2)$ and $(T=\lambda^{-1}b, S=b^2)$. We know that ρ contributes to at least one of these, and we must show that ρ contributes to both. When $\lambda=\pm 1$ this is clear, since both systems of eigenvalues are actually the same. When $\lambda \neq \pm 1$, this follows from [10, Theorem 13.10]. ☐

3. p-arithmetic homology of $\pi(r,\lambda,\chi)$

3.1. p-arithmetic and arithmetic homology. In this section we will relate the p-arithmetic homology of the representations $\pi(r,\lambda,\chi)$ to the arithmetic homology of Serre weights $\mathrm{Sym}^r(k^2)^\vee \otimes \omega^s$. The argument is the same as that of [15]. As in Remark 2.2, R is a k-algebra.

Lemma 3.1. *For any $0 \leq r \leq p-1$ and $s \in \mathbb{Z}$, $\tau \in R$ and $\sigma \in R^\times$, we have*

$$\mathrm{Tor}_i^{R[T,S]}(R[T,S]/(T-\tau, S-\sigma), \text{c-Ind}_K^G(\mathrm{Sym}^r(k^2)^\vee \otimes_k \omega^s \otimes_k R)) = 0$$

for $i>0$.

Proof. This follows from the fact that $(S-\sigma, T-\tau)$ is a regular sequence for the $R[T,S]$-module $\text{c-Ind}_K^G(\mathrm{Sym}^r(k^2)^\vee \otimes_k \omega^s \otimes_k R)$, which can be seen by studying how T and S modify the support of a function using the Cartan decomposition. See [5, Lemma 4.10] for the details. ☐

This shows that the G-module $\widetilde{\pi}(r,\tau,\sigma,s)_R$ from Remark 2.2 can be written not just as the eigenquotient

$$\begin{aligned}\text{c-Ind}_K^G(\mathrm{Sym}^r(k^2)^\vee \otimes_k \omega^s \otimes_k R)/(T-\tau, S-\sigma) \\ \simeq \frac{R[T,S]}{(T-\tau,S-\sigma)} \otimes_{R[T,S]} \text{c-Ind}_K^G(\mathrm{Sym}^r(k^2)^\vee \otimes_k \omega^s \otimes_k R)\end{aligned}$$

but also as a *derived* eigenquotient: there is an isomorphism in the derived category of (abstract[2]) $R[G][T,S]$-modules

$$\widetilde{\pi}(r,\tau,\sigma,s)_R \simeq \frac{R[G][T,S]}{(T-\tau,S-\sigma)} \otimes^{\mathbb{L}}_{R[G][T,S]} \text{c-Ind}_K^G(\mathrm{Sym}^r(k^2)^\vee \otimes_k \omega^s \otimes_k R).$$

Fix U^p and N as in Section 2.1. We can define an action of Hecke operators T and S in arithmetic homology over R in the same way as described in the

[2]In our arguments involving homological algebra, we will always work with categories of abstract representations (of G or other groups) and never with categories of smooth representations

beginning of Section 2.5, and the arguments in [15, Proposition 5.8] show that we have an isomorphism in the derived category of $\mathbb{T}(pN) \otimes_{\mathbb{Z}} R[T,S]$-modules for the p-arithmetic homology complex

$$C_\bullet(U^p, \widetilde{\pi}(r,\tau,\sigma,s)_R) \simeq C_\bullet(U^pK, \mathrm{Sym}^r(k^2)^\vee \otimes_k \omega^s \otimes_k R) \otimes^{\mathbb{L}}_{R[T,S]} \frac{R[T,S]}{(T-\tau, S-\sigma)}.$$

Moreover,

$$C_\bullet(U^pK, \mathrm{Sym}^r(k^2)^\vee \otimes_k \omega^s \otimes_k R) \simeq C_\bullet(U^pK, \mathrm{Sym}^r(k^2)^\vee \otimes_k \omega^s) \otimes^{\mathbb{L}}_k R.$$

Thus, in fact

$$C_\bullet(U^p, \widetilde{\pi}(r,\tau,\sigma,s)_R) \simeq C_\bullet(U^pK, \mathrm{Sym}^r(k^2)^\vee \otimes_k \omega^s) \otimes^{\mathbb{L}}_{k[T,S]} \frac{R[T,S]}{(T-\tau, S-\sigma)}.$$

Remark 3.2. The reason why we have considered representations in families is the following. Assume that $R = k[\tau,\sigma,\sigma^{-1}]$ for two indeterminate variables τ and σ. Then,

$$\begin{aligned} C_\bullet(U^p, \widetilde{\pi}(r,\tau,\sigma,s)_R) &\simeq C_\bullet(U^pK, \mathrm{Sym}^r(k^2)^\vee \otimes_k \omega^s) \otimes^{\mathbb{L}}_{k[T,S]} \frac{R[T,S]}{(T-\tau, S-\sigma)} \\ &\simeq C_\bullet(U^pK, \mathrm{Sym}^r(k^2)^\vee \otimes_k \omega^s) \otimes^{\mathbb{L}}_{k[T,S,S^{-1}]} \frac{R[T,S,S^{-1}]}{(T-\tau, S-\sigma)} \\ &\simeq C_\bullet(U^pK, \mathrm{Sym}^r(k^2)^\vee \otimes_k \omega^s) \otimes^{\mathbb{L}}_{k[T,S,S^{-1}]} \frac{k[T,S,S^{-1},\tau,\sigma,\sigma^{-1}]}{(T-\tau, S-\sigma)} \\ &\simeq C_\bullet(U^pK, \mathrm{Sym}^r(k^2)^\vee \otimes_k \omega^s), \end{aligned}$$

where the last term is viewed as an R-module by letting τ act as T and σ as S. In other words, the p-arithmetic homology over R coincides with the corresponding arithmetic homology, and not just a (derived) eigenquotient of it.

Proposition 3.3. *There is a (homological) spectral sequence converging to* $H_*(U^p, \widetilde{\pi}(r,\tau,\sigma,s)_R)$ *whose* E^2 *page is*

$$E^2_{i,j} = \mathrm{Tor}_i^{k[T,S]}\left(\frac{R[T,S]}{(T-\tau, S-\sigma)}, H_j(U^pK, \mathrm{Sym}^r(k^2)^\vee \otimes_k \omega^s)\right).$$

In particular, there exists a spectral sequence converging to $H_*(U^p, \pi(r,\lambda, \omega^a\mathrm{unr}_b))$ *whose* E^2 *page is*

$$E^2_{i,j} = \mathrm{Tor}_i^{k[T,S]}\left(\frac{k[T,S]}{(T-\lambda b, S-b^2)}, H_j(U^pK, \mathrm{Sym}^r(k^2)^\vee \otimes_k \omega^{a+r})\right).$$

Remark 3.4. When $R = k$, the $k[T,S]$-module resolution

$$k[T,S] \xrightarrow{(S-\sigma)\oplus(\tau-T)} k[T,S]^2 \xrightarrow{(T-\tau,S-\sigma)} k[T,S]$$

of $k[T,S]/(T-\tau, S-\sigma)$ shows that for any $k[T,S]$-module V that is finite-dimensional as a k-vector space, $\mathrm{Tor}_i^{k[T,S]}(k[T,S]/(T-\tau,S-\sigma),V)$ vanishes in degrees outside the range $[0,2]$, and is isomorphic in degree 2 (resp. degree 0) to the $(T=\tau, S=\sigma)$-eigenspace (resp. eigenquotient) of V. Moreover, the Tor modules in degree 1 lie in a short exact sequence

$$\begin{aligned} 0 \longrightarrow \frac{k[T]}{(T-\tau)} \otimes_{k[T]} \mathrm{Hom}_{k[S]}\left(\frac{k[S]}{(S-\sigma)}, V\right) \\ \longrightarrow \mathrm{Tor}_1^{k[T,S]}\left(\frac{k[T,S]}{(T-\tau,S-\sigma)}, V\right) \\ \longrightarrow \mathrm{Hom}_{k[T]}\left(\frac{k[T]}{(T-\tau)}, \frac{k[S]}{(S-\sigma)} \otimes_{k[S]} V\right) \longrightarrow 0. \end{aligned}$$

In particular, the Tor groups vanish if and only if they vanish in at least one of the degrees 0, 1 or 2.

3.2. Proof of Theorem 1.1 in the generic case. Parts (1) and (2) of Theorem 1.1 follow immediately from Proposition 3.3 for supersingular and principal series representations, as well as their analogue for the reducible representations $\pi(0,\pm1,\chi)$ and $\pi(p-1,\pm1,\chi)$ (by the corresponding results for arithmetic homology).

Moreover, if ρ is an odd irreducible 2-dimensional Galois representations, then the localisation at ρ of the spectral sequence from Proposition 3.3 satisfies $(E^2_{i,j})_\rho = 0$ for $j \neq 1$, so we may conclude that

$$\begin{aligned} &H_{i+1}(\Gamma_1^p(N), \pi(r,\lambda,\omega^a\mathrm{unr}_b))_\rho \\ &\quad \simeq \mathrm{Tor}_i^{k[T,S]}\left(\frac{k[T,S]}{(T-\lambda b, S-b^2)}, H_1(U^pK, \mathrm{Sym}^r(k^2)^\vee \otimes_k \omega^{a+r})_\rho\right). \end{aligned}$$

In particular, taking into account Remark 3.4, the following are equivalent:

(1) ρ contributes to $H_*(\Gamma_1^p(N), \pi(r,\lambda,\omega^a\mathrm{unr}_b))$, and it does exactly in degrees 1, 2 and 3,
(2) ρ contributes to $H_1(\Gamma_1^p(N), \pi(r,\lambda,\omega^a\mathrm{unr}_b))$,
(3) ρ contributes to $H_*(\Gamma_1^p(N), \pi(r,\lambda,\omega^a\mathrm{unr}_b))$,
(4) ρ contributes to the $(T=\lambda b, S=b^2)$-eigenspace of

$$H_1(\Gamma_1(N), \mathrm{Sym}^r(k^2)^\vee \otimes \omega^{a+r}),$$

(5) ρ contributes to the $(T=\lambda b, S=b^2)$-eigenspace of

$$H^1(\Gamma_1(N), \mathrm{Sym}^r(k^2) \otimes \omega^{-a-r}).$$

By Theorem 2.6 and Proposition 2.5, when $\pi(r, \lambda, \omega^a \mathrm{unr}_b)$ is irreducible, these are equivalent to $N(\rho)$ dividing N and this representation appearing in the socle of the smooth representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ associated to $\rho|_{\mathcal{G}_p}$ by the mod p local Langlands correspondence of Theorem 2.4, which proves part (3) of Theorem 1.1 in this case. For later reference, we also record the following proposition, which follows in the same way from Proposition 3.3 and Theorem 2.6.

Proposition 3.5. *Let* $\rho\colon \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(k)$ *be an odd irreducible representation. Then, the space* $H_1(\Gamma_1^p(N), \pi(p-1, 1, \chi))$ *(resp. the space* $H_1(\Gamma_1^p(N), \pi(0, 1, \chi))$*) is finite-dimensional, and any system of Hecke eigenvalues in it has an attached Galois representation. Moreover, ρ contributes to this space if and only if* $N(\rho)$ *divides* N*, and* $\rho|_{\mathcal{G}_p}$ *is isomorphic to an extension (resp. a peu ramifiée extension)*

$$\begin{pmatrix} 1 & * \\ 0 & \omega^{-1} \end{pmatrix} \otimes \chi.$$

4. Preparation for the non-generic cases

In order to deal with the Steinberg and trivial cases, we will need a few preliminaries on the non-zero maps $\pi(p-1, 1, \chi) \to \pi(0, 1, \chi)$ and $\pi(0, 1, \chi) \to \pi(p-1, 1, \chi)$ and the corresponding maps on p-arithmetic homology. They turn out to be related to the degeneracy maps from modular forms of level $\Gamma_1(N)$ to level $\Gamma_1(N) \cap \Gamma_0(p)$ induced by $\tau \mapsto \tau$ and $\tau \mapsto p\tau$ and to the group cohomological avatar of multiplication by the Hasse invariant studied by Edixhoven–Khare in [8]. Our next goal is to study these maps.

4.1. The map $\boldsymbol{\pi(p-1,1,1) \to \pi(0,1,1)}$. Recall from Section 2.3 that there is a unique-up-to-scalars non-zero map $\pi(p-1, 1, \chi) \to \pi(0, 1, \chi)$. The goal of this section is to give an explicit description of this map.

We may assume that $\chi = 1$. First, let us observe that there is a K-module isomorphism $k \oplus \mathrm{Sym}^{p-1}(k^2)^\vee \xrightarrow{\sim} \mathrm{Map}(\mathbb{P}^1(\mathbb{F}_p), k)$, which identifies the trivial representation with the subrepresentation of constant functions $\mathbb{P}^1(\mathbb{F}_p) \to k$ and $\mathrm{Sym}^{p-1}(k^2)^\vee$ with the subrepresentation of functions whose total sum equals 0. As usual, one can also identify $\mathrm{Sym}^{p-1}(k^2)^\vee$ with the space of homogeneous polynomial functions of degree $p-1$ in two variables. The former identification is then given by sending a homogeneous polynomial function Q of degree $p-1$ in two variables to the function $(x:y) \mapsto Q(x, y)$.

In fact, this can be upgraded to a G^--equivariant isomorphism in a natural way. The action of the monoid G^+ on $\mathbb{F}_p^2$ descends to an action on $\mathbb{F}_p^2/\mathbb{F}_p^\times = \mathbb{P}^1(\mathbb{F}_p) \cup \{0\}$. It is easy to check (for example, using the Cartan decomposition of G) that an element $g \in G^+$ can act in three ways:

invertibly (if $g \in K$), by sending everything to 0 (if all the entries of g are multiples of p), or by mapping 0 and one point of $\mathbb{P}^1(\mathbb{F}_p)$ to 0 and all other points of $\mathbb{P}^1(\mathbb{F}_p)$ to another (fixed) point of $\mathbb{P}^1(\mathbb{F}_p)$. Thus, we get an action of G^- on $\mathrm{Map}(\mathbb{P}^1(\mathbb{F}_p) \cup \{0\}, k)$, and it follows from the previous sentence that the subspace of functions such that $f(0) = \sum_{P\in\mathbb{P}^1(\mathbb{F}_p)} f(P)$ is stable under this action. This space can be naturally identified with $\mathrm{Map}(\mathbb{P}^1(\mathbb{F}_p), k)$ by restriction, and the resulting action of G^- on this space makes the isomorphisms in the previous paragraph G^--equivariant. Naturally, these isomorphisms are also KZ-equivariant when we instead extend the action of K to one of KZ by letting β act trivially. There is a K-equivariant isomorphism $\mathrm{Map}(\mathbb{P}^1(\mathbb{F}_p), k) \to \text{c-Ind}_I^K(k)$ given by sending a function f to $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mapsto f(a:c)$, and we will view the target as a G^--module and a KZ-module (whose underlying K-module structures agree) by transport of structure. In particular, compactly inducing to G we obtain actions of Hecke operators T and S as usual, and the maps above induce $k[T,S]$-module isomorphisms

$$\text{c-Ind}_K^G(k) \oplus \text{c-Ind}_K^G(\mathrm{Sym}^{p-1}(k^2)) \xrightarrow{\sim} \text{c-Ind}_K^G(\text{c-Ind}_I^K(k)).$$

Consider the following two maps $\phi_1, \phi_2\colon \text{c-Ind}_I^G(k) \to \text{c-Ind}_K^G(k)$. The first is given simply by $\phi_1([g,a]) = [g,a]$. The second map ϕ_2 is defined by $\phi_2([g,a]) = [g\alpha, a]$. It will be useful to view this map as the composition of the map $[g,a] \mapsto [g,a]\colon \text{c-Ind}_I^G(k) \to \text{c-Ind}_{\alpha K\alpha^{-1}}^G(k)$ and the intertwining isomorphism

$$\begin{aligned} \text{c-Ind}_{\alpha K\alpha^{-1}}^G(k) &\xrightarrow{\sim} \text{c-Ind}_K^G(k) \\ [g,a] &\longmapsto [g\alpha, a]. \end{aligned} \tag{4.1}$$

It is tedious, but straightforward, to check that the resulting maps

$$\text{c-Ind}_K^G(\text{c-Ind}_I^K(k)) \xrightarrow{\sim} \text{c-Ind}_I^G(k) \rightrightarrows \text{c-Ind}_K^G(k)$$

are $k[T,S]$-equivariant. One can also check that the composition

$$\begin{aligned} \text{c-Ind}_K^G(k) \oplus \text{c-Ind}_K^G(\mathrm{Sym}^{p-1}(k^2)^\vee) &\longrightarrow \text{c-Ind}_K^G(\text{c-Ind}_I^K(k)) \\ &\xrightarrow{\phi_1\oplus\phi_2} \text{c-Ind}_K^G(k) \oplus \text{c-Ind}_K^G(k) \end{aligned}$$

is of the form $\left(\begin{smallmatrix} 1 & 0 \\ T & \phi \end{smallmatrix}\right)$, where ϕ is also $k[T,S]$-equivariant.

Lemma 4.1. *The reduction mod* $(T-1, S-1)$ *of the homomorphism* ϕ *is a non-zero map* $\pi(p-1,1,1) \to \pi(0,1,1)$.

Proof. The following argument is a representation theoretic analogue of the proof of [8, Lemma 2] (in fact, this lemma can be deduced literally from *loc. cit.*, for example as a consequence of Proposition 4.3 below). Write ${}^\circ G = \{g \in G : v_p(\det(g)) = 0\}$. Then, by [13, II.1.4 Theorem 3], ${}^\circ G$ is the amalgamated product of K and $\alpha K \alpha^{-1}$ along I. Thus, there is a Mayer-Vietoris exact sequence in the group homology of $k[G]$,

$$0 \longrightarrow \text{c-Ind}_I^G(k) \longrightarrow \text{c-Ind}_K^G(k) \oplus \text{c-Ind}_{\alpha K \alpha^{-1}}^G(k) \longrightarrow \text{c-Ind}_{{}^\circ G}^G(k) \longrightarrow 0.$$

Composing with the intertwining isomorphism (4.1), we obtain an exact sequence

$$(4.2)\quad 0 \longrightarrow \text{c-Ind}_K^G(\text{c-Ind}_I^K(k)) \xrightarrow{\phi_1 \oplus \phi_2} \text{c-Ind}_K^G(k) \oplus \text{c-Ind}_K^G(k) \longrightarrow \text{c-Ind}_{{}^\circ G}^G(k) \longrightarrow 0$$

where the last map is given by $([g_1, a_1], [g_2, a_2]) \mapsto [g_1, a_1] - [g_2 \alpha^{-1}, a_2]$. This exact sequence is $k[T, S]$-equivariant if we endow $\text{c-Ind}_{{}^\circ G}^G(k)$ with the action of T (resp. S) given by acting by α (resp. β). Taking the quotient of the exact sequence above by the ideal $(T-1, S-1)$, we obtain an exact sequence

$$\pi(0,1,1) \oplus \pi(p-1,1,1) \longrightarrow \pi(0,1,1) \oplus \pi(0,1,1) \longrightarrow k \longrightarrow 0,$$

where the first map is given by $\left(\begin{smallmatrix} 1 & 0 \\ 1 & \bar\phi \end{smallmatrix}\right)$, where $\bar\phi$ is the map induced by ϕ. Looking at the Jordan–Hölder constituents of the terms in the exact sequence, it is clear that $\bar\phi$ cannot be zero. $\square$

Let us also remark that if R is a k-algebra, $\tau \in R, \sigma \in R^\times$ and $s \in \mathbb{Z}$, then tensoring ϕ with $\omega^s \otimes_k R$ and quotienting by $(T-\tau, S-\sigma)$ we get a map $\widetilde{\pi}(p-1, \tau, \sigma, s)_R \to \widetilde{\pi}(0, \tau, \sigma, s)_R$.

4.2. The map $\boldsymbol{\pi(0,1,1) \to \pi(p-1,1,1)}$. There is also a unique-up-to-scalar non-zero map $\pi(0,1,\chi) \to \pi(p-1,1,\chi)$, which factors through $\chi \circ \det$. The goal of this section is to show that this map comes from specialising a map $\widetilde{\pi}(0,\tau,\sigma,s)_R \to \widetilde{\pi}(p-1,\tau,\sigma,s)_R$ as in the setting of the end of the previous section. As in the previous section, this is essentially equivalent to the existence of a lift of the map $\pi(0,1,1) \to \pi(p-1,1,1)$ to a $k[T,S]$-equivariant map

$$\text{c-Ind}_K^G(k) \longrightarrow \text{c-Ind}_K^G(\mathrm{Sym}^{p-1}(k^2)^\vee).$$

We will construct such a map by dualising the procedure of the previous section. Consider the composition

$$(4.3)\quad \text{c-Ind}_K^G(k) \oplus \text{c-Ind}_K^G(k) \xrightarrow{\mathrm{id} \oplus (4.1)^{-1}} \text{c-Ind}_K^G(k) \oplus \text{c-Ind}_{\alpha K \alpha^{-1}}^G(k) \longrightarrow \text{c-Ind}_I^G(k),$$

where the last map is the sum of inclusions. It is $k[T,S]$-equivariant and the resulting map

$$\text{c-Ind}_K^G(k) \oplus \text{c-Ind}_K^G(k) \longrightarrow \text{c-Ind}_K^G(k) \oplus \text{c-Ind}_K^G(\mathrm{Sym}^{p-1}(k^2)^\vee)$$

is of the form $\left(\begin{smallmatrix} 1 & S^{-1}T \\ 0 & -\psi \end{smallmatrix}\right)$. Explicitly, $\psi([g,1]) = \sum_{x\in K/I}[\beta^{-1}gx\alpha, e^*]$, where e^* is the element of $\mathrm{Sym}^{p-1}(k^2)^\vee$ corresponding to the polynomial function $Q(x,y) = x^{p-1}$.

Lemma 4.2. *The reduction mod* $(T-1, S-1)$ *of the homomorphism* ψ *is a non-zero map* $\pi(0,1,1) \to \pi(p-1,1,1)$.

Proof. As for Lemma 4.1, this follows from another result for group cohomology (namely, Proposition 4.6 below), but we give another proof in a more representation-theoretic spirit. Note that $\alpha^{-1}x^{-1}e^*$ is equal to e^* for any $x \in K$ which is not in the same left I-coset as $w := \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, and vanishes if $x \in wI$. In particular, $\psi([g,1]) = T[\beta^{-1}, e^*] + [\beta^{-1}w\alpha, e^*]$. Hence, it's enough to show that $[1,e^*] + [w\alpha, e^*]$ defines a non-zero element of $\pi(p-1,1,1)$. To do this, we will check that its image under the map of Theorem 2.3 (2) is non-zero. This map is defined in [1, Section 6.2] and sends $[g,Q]$ to $h \mapsto Q(x(1:0))$ where we have written $g^{-1}h = xb$ with $x \in K$ and $b \in B$. The image of $[1,e^*] + [w\alpha, e^*]$ maps w to 1, so in particular it is non-zero (in fact, as we would expect, it is the constant function with value 1). ☐

4.3. The resulting maps on arithmetic cohomology.

Proposition 4.3. *Let* $R = k[\tau, \sigma, \sigma^{-1}]$ *be as in Remark 3.2 and* $s \in \mathbb{Z}$. *Then, the map* $\widetilde{\phi}\colon H_*(\Gamma_1(N), \mathrm{Sym}^{p-1}(k^2)^\vee \otimes \omega^s) \to H_*(\Gamma_1(N), \omega^s \circ \det)$ *induced from the map*

$$\widetilde{\pi}(p-1,\tau,\sigma,s)_R \longrightarrow \widetilde{\pi}(0,\tau,\sigma,s)_R$$

defined at the end of Section 4.1 is a twist of the dual of the map in [8, Lemma 2]. *In particular, if* $p \geq 5$, *this map is surjective in degree 1.*

Proof. By [4, Corollary 2.11], we may assume $s = 0$. The first sentence follows from the constructions of both maps, and the second follows from [8, Lemma 2]. ☐

As mentioned above, the proof of Lemma 4.1 is a representation-theoretic analogue the proof of [8, Lemma 2]. The latter can then be recovered by taking p-arithmetic homology of the exact sequence (4.2). To see this, we need the following result.

Lemma 4.4. *If* $p \geq 5$, *the* p*-arithmetic homology* $H_1(\Gamma_1^p(N), \text{c-Ind}_{{}^\circ G}^G(k))$ *vanishes.*

Proof. As $G = \Gamma_1^p(N)^\circ G$ and $\Gamma_1^p(N) \cap {}^\circ G = \Gamma_1^p(N) \cap \mathrm{SL}_2(\mathbb{Q})$, the natural restriction map $\text{c-Ind}_{{}^\circ G}^G(k) \to \text{c-Ind}_{\Gamma_1^p(N)\cap \mathrm{SL}_2(\mathbb{Q})}^{\Gamma_1^p(N)}(k)$ is an isomorphism of representations of $\Gamma_1^p(N)$. The lemma follows by Shapiro's lemma and [8, Proof of Lemma 1]. ☐

Thus, when $p \geq 5$, we have a commutative diagram

$$\begin{array}{ccc} H_1(\Gamma_1(N) \cap \Gamma_0(p), k) & \twoheadrightarrow & H_1(\Gamma_1(N), k) \oplus H_1(\Gamma_1(N), k) \\ \downarrow \wr & \nearrow \begin{pmatrix} \mathrm{id} & 0 \\ T & \tilde{\phi} \end{pmatrix} & \\ H_1(\Gamma_1(N), k) \oplus H_1(\Gamma_1(N), \mathrm{Sym}^{p-1}(k^2)^\vee) & & \end{array}$$

In particular, $\widetilde{\phi}$ is surjective, so we have indeed recovered [8, Lemma 2]. Moreover, we see that the kernel of $\widetilde{\phi}$ is isomorphic to the kernel of the map

$$H_1(\Gamma_1(N) \cap \Gamma_0(p), k) \twoheadrightarrow H_1(\Gamma_1(N), k) \oplus H_1(\Gamma_1(N), k). \tag{4.4}$$

This is the homomorphism induced by the maps between open modular curves determined by $\tau \mapsto \tau$ and $\tau \mapsto p\tau$ on the upper-half plane. A generalisation by Wiles of a lemma of Ribet determines the Galois representations that contribute to this kernel.

Proposition 4.5. *Let ρ be a 2-dimensional odd irreducible representation of* $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ *over k such that $N(\rho)$ divides N and* $\rho|_{\mathcal{G}_p} \simeq \left(\begin{smallmatrix} 1 & * \\ 0 & \omega^{-1} \end{smallmatrix}\right) \otimes \mathrm{unr}_b$. *Then, the localisation at the maximal ideal $\mathfrak{m}_\rho$ of $\mathbb{T}(pN)$ corresponding to ρ of the kernel of* (4.4) *is non-zero.*

Proof. If the extension $*$ in the statement is très ramifiée, then it is clear that ρ contributes to the kernel as it contributes to the source but not the target. Therefore, we may assume that the extension is peu ramifiée. Let f be a normalised newform of weight 2, level $\Gamma_1(N)$ and character χ whose associated Galois representation over k is isomorphic to ρ. If a_p is its p-th Fourier coefficient, let α be the root of $x^2 - a_p x - \chi(p)p$ that is a p-adic unit (there should be no confusion with our previous use of the letter α). Then $\alpha^2 \equiv a_p^2 \equiv \chi(p) \mod p$, the second congruence following from [7, Theorem 2.5].

There is an eigenclass in $H_1(\Gamma_1(N), \overline{\mathbb{Z}}_p)$ for the Hecke operators away from N with the same system of eigenvalues as f and whose reduction to $H_1(\Gamma_1(N), k)$ is non-zero. As ρ is irreducible, the localisation of $H_1(\Gamma_1(N), k)$ at $\mathfrak{m}_\rho$ is isomorphic to $(k \otimes_{\mathbb{F}_p} J_1(N)[p])_{\mathfrak{m}_\rho}$, where $J_1(N)$ is the Jacobian of the compactified modular curve of level $\Gamma_1(N)$. Write P for the element of $(k \otimes_{\mathbb{F}_p} J_1(N)[p])_{\mathfrak{m}_\rho}$ corresponding to the reduction of the eigenclass above. Similarly, $H_1(\Gamma_1(N) \cap \Gamma_0(p), k)_{\mathfrak{m}_\rho} \simeq (k \otimes_{\mathbb{F}_p} J_1(N, p)[p])_{\mathfrak{m}_\rho}$, where $J_1(N, p)$ is the Jacobian of the compactified modular curve of level $\Gamma_1(N) \cap \Gamma_0(p)$. Consider now the image of $(P, 0) = (P, -(a_p - \alpha)P)$ under the map

$$(k \otimes_{\mathbb{F}_p} J_1(N)[p])_{\mathfrak{m}_\rho} \oplus (k \otimes_{\mathbb{F}_p} J_1(N)[p])_{\mathfrak{m}_\rho} \longrightarrow (k \otimes_{\mathbb{F}_p} J_1(N, p)[p])_{\mathfrak{m}_\rho}$$

induced by the morphisms between modular curves determined by $\tau \mapsto \tau$ and $\tau \mapsto p\tau$ on the upper-half plane. The image of $(P, 0)$ is then one of the p-stabilisations of P: it is an eigenvector for all Hecke operators T_ℓ for

$\ell \nmid pN$ (with eigenvalues determined by ρ), as well as the operator U_p (resp. $\langle n\rangle$ for any $n \in \mathbb{Z}$ with $n \equiv p \mod N$ and $n \equiv 1 \mod p$) with eigenvalue α (resp. $\chi(p)$). It is also non-zero (for example, by Proposition 4.6 and its proof below, or by the injectivity of [16, (2.10)]). Hence, by [16, Lemma 2.3], it defines (under the isomorphisms above between group homology and p-torsion points in Jacobians) a non-zero element of the localisation at $\mathfrak{m}_\rho$ of the kernel of (4.4). □

Finally, we turn to the map from Section 4.2.

Proposition 4.6. *Let $R = k[\tau, \sigma, \sigma^{-1}]$ be as in Remark 3.2 and $s \in \mathbb{Z}$. Let ρ be a 2-dimensional odd irreducible representation. Then, the map*

$$H_*(\Gamma_1(N), \omega^s \circ \det)_\rho \longrightarrow H_*(\Gamma_1(N), \mathrm{Sym}^{p-1}(k^2)^\vee \otimes_k \omega^s)_\rho$$

induced from the map $\widetilde{\pi}(0,\tau,\sigma,s)_R \to \widetilde{\pi}(p-1,\tau,\sigma,s)_R$ is injective in degree 1. If $N(\rho)$ divides N and $\rho|_{\mathcal{G}_p} \simeq \left(\begin{smallmatrix}1 & * \\ 0 & \omega^{-1}\end{smallmatrix}\right) \otimes \omega^s \mathrm{unr}_b$, then the cokernel is non-zero.

Proof. The proposition follows from Proposition 4.3 and Proposition 4.5 by Poincaré duality. Let us spell out the details. Again, we may assume that $s = 0$. Given a $\mathbb{T}(pN)$-module M, let us write M^* for the base change of M along the ring isomorphism $\mathbb{T}(pN) \to \mathbb{T}(pN)$ mapping $T_\ell \mapsto S_\ell^{-1}T_\ell$ and $S_\ell \mapsto S_\ell^{-1}$. Thus, ρ contributes to M if and only if $\rho^\vee \otimes \varepsilon^{-1}$ contributes to M^*. Let us also write $M^*_\rho := (M^*)_\rho$. As ρ is irreducible, there are Poincaré duality isomorphisms

$$H_1(\Gamma_1(N), k)_\rho \xrightarrow{\sim} H^1(\Gamma_1(N), k)^*_\rho$$
$$H_1(\Gamma_1(N), \mathrm{Sym}^{p-1}(k^2)^\vee)_\rho \xrightarrow{\sim} H^1(\Gamma_1(N), \mathrm{Sym}^{p-1}(k^2)^\vee)^*_\rho$$
$$H_1(\Gamma_1(N) \cap \Gamma_0(p), k)_\rho \xrightarrow{\sim} H^1(\Gamma_1(N) \cap \Gamma_0(p), k)^*_\rho.$$

On the one hand, the k-linear dual of (4.4) is the direct sum of the pullbacks in cohomology of the maps from the modular curve of level $\Gamma_1(N) \cap \Gamma_0(p)$ to the modular curve of level $\Gamma_1(N)$ determined by $\tau \mapsto \tau$ and $\tau \mapsto p\tau$ on the upper-half plane. On the other hand, the map (4.3) induces on p-arithmetic homology a map

$$H_1(\Gamma_1(N), k) \oplus H_1(\Gamma_1(N), k) \longrightarrow H_1(\Gamma_1(N) \cap \Gamma_0(p), k),$$

that is (by construction) the direct sum of the transfer homomorphisms in the homology of the modular curves above induced by the same pair of maps. Now, under Poincaré duality, pullbacks in cohomology correspond to transfer homomorphisms in homology, and thus the two maps correspond (after localising at ρ) under the above Poincaré duality isomorphism.

Fix an isomorphism $\mathrm{Sym}^{p-1}(k^2)^\vee \xrightarrow{\sim} \mathrm{Sym}^{p-1}(k^2)$. It determines also an isomorphism between $\mathrm{Map}(\mathbb{P}^1(\mathbb{F}_p), k)$ and its dual, and by Shapiro's lemma an isomorphism

$$H^1(\Gamma_1(N)\cap\Gamma_0(p),k) \xrightarrow{\sim} H^1(\Gamma_1(N),k)\oplus H^1(\Gamma_1(N),\mathrm{Sym}^{p-1}(k^2)^\vee)$$

that is compatible under Poincaré duality with the similar isomorphism for homology. In conclusion, we have a commutative square

$$\begin{array}{ccc} H_1(\Gamma_1(N),k)_\rho \oplus H_1(\Gamma_1(N),k)_\rho & \longrightarrow & H_1(\Gamma_1(N),k)_\rho \oplus H_1(\Gamma_1(N),\mathrm{Sym}^{p-1}(k^2)^\vee)_\rho \\ \downarrow\wr & & \downarrow\wr \\ H^1(\Gamma_1(N),k)^*_\rho \oplus H^1(\Gamma_1(N),k)^*_\rho & \longrightarrow & H^1(\Gamma_1(N),k)^*_\rho \oplus H^1(\Gamma_1(N),\mathrm{Sym}^{p-1}(k^2)^\vee)^*_\rho \end{array}$$

where the top horizontal map is obtained from taking the p-arithmetic homology of (4.3) and the bottom horizontal map is (the Poincaré dual of) the k-linear dual of (4.4). Thus, the map

$$H_1(\Gamma_1(N),k)_\rho \longrightarrow H_1(\Gamma_1(N),\mathrm{Sym}^{p-1}(k^2)^\vee)_\rho$$

from the statement of the lemma corresponds under these isomorphisms to the dual of the map from Proposition 4.3. Thus, dualising Proposition 4.3 and Proposition 4.5 gives the result. □

5. Proof of Theorem 1.1

In Section 3.2 we have proven the generic case of Theorem 1.1, in this section we are going to deal with the non-generic cases of twists of the trivial and Steinberg representations. We assume throughout that $p \geq 5$.

5.1. The Steinberg case. The case of twists of the Steinberg representation will follow from Proposition 3.5 and Proposition 4.6 and some formal algebraic manipulations. Consider a two-term complex $C_1 \to C_0$ of representations of G such that $H_0(C_\bullet) \simeq H_1(C_\bullet)$. Taking the p-arithmetic hyperhomology of this complex induces two spectral sequences converging to the same abutment, one E with E^2 page given by

$$E^2_{i,j} = H_i(\Gamma^p_1(N), H_j(C_\bullet))$$

and the other $'E$ with $'E^1$ page given by

$$'E^1_{i,j} = H_j(\Gamma^p_1(N), C_i).$$

This spectral sequence degenerates at the $'E^2$ page, so the systems of Hecke eigenvalues appearing in this page are the same as those in the abutment. A spectral sequence argument shows that these systems of Hecke eigenvalues are the same as those appearing in E^2. Indeed, if i_0 is the smallest degree for which a fixed system of eigenvalues appears in $H_{i_0}(\Gamma^p_1(N), H_0(C_\bullet))$, then the localisation at this system of eigenvalues of the $E_{i_0,0}$ term is stable in the localised spectral sequence, since $E^2_{i_0-2,1} \simeq E^2_{i_0-2,0}$ as $H_0(C_\bullet) \simeq H_1(C_\bullet)$. Therefore, the localisation of the abutment in degree i_0 will be non-zero. Moreover, if the homologies $H_j(\Gamma^p_1(N), C_i)$ are finite-dimensional, then so is the abutment, and the same type of spectral sequence argument shows that so are the terms in $E^2_{i,j}$.

Let us now specialise to our case of interest. The complex we will be considering is given by the map

$$C_1 := \pi(0,1,\chi) \longrightarrow \pi(p-1,1,\chi) := C_0$$

from Lemma 4.2, so that $H_0(C_\bullet) \simeq H_1(C_\bullet) \simeq \mathrm{St} \otimes \chi$. Thus, the previous paragraph and Proposition 3.5 show that Theorem 1.1 (1) and (2) are satisfied for twists of the Steinberg representation.

Let us analyse the systems of eigenvalues appearing in homology. Proposition 3.5 shows that the only odd irreducible Galois representations ρ which can contribute to the ${}'E^1$ page above, and hence to $H_*(\Gamma_1^p(N), \mathrm{St} \otimes \chi)$, are those such that $N(\rho)$ divides N and $\rho|_{\mathcal{G}_p}$ is an extension of $\chi\omega^{-1}$ by χ. Fix such a ρ and write $\chi = \omega^a \mathrm{unr}_b$. By the argument in [15, Proposition 5.8], and using that derived tensor products commute with mapping cones, the p-arithmetic hyperhomology of $C_\bullet$ is isomorphic (in the derived category of $\mathbb{T}(pN) \otimes_{\mathbb{Z}} k[T,S]$-modules) to the derived tensor product over $k[T,S]$ of $k[T,S]/(T-b, S-b^2)$ and

$$\begin{aligned}\Big[C_\bullet(\Gamma_1^p(N), \text{c-Ind}_K^G(\omega^a)) \longrightarrow C_\bullet(\Gamma_1^p(N), \text{c-Ind}_K^G(\mathrm{Sym}^{p-1}(k^2)^\vee \otimes \omega^a))\Big] \\ \simeq \Big[C_\bullet(\Gamma_1(N), \omega^a \circ \det) \longrightarrow C_\bullet(\Gamma_1(N), \mathrm{Sym}^{p-1}(k^2)^\vee \otimes \omega^a)\Big],\end{aligned}$$

where $[-]$ denotes mapping cones and the isomorphism follows from Shapiro's lemma [15, Proposition 5.3]. Using that ρ contributes to arithmetic homology only in degree 1, we see that the localisation of the above hyperhomology at ρ is isomorphic to the derived tensor product over $k[T,S]$ of $k[T,S]/(T-b, S-b^2)$ and the cone

$$\Big[H_1(\Gamma_1(N), \omega^a \circ \det)_\rho[1] \longrightarrow H_1(\Gamma_1(N), \mathrm{Sym}^{p-1}(k^2)^\vee \otimes \omega^a)_\rho[1]\Big]$$

of the map of Proposition 4.6 (up to a shift). This localisation is therefore isomorphic to

$$\frac{k[T,S]}{(T-b, S-b^2)} \otimes^{\mathbb{L}}_{k[T,S]} U_{a,\rho}[1]$$

where

$$U_a = \mathrm{coker}\Big(H_1(\Gamma_1(N), \omega^a \circ \det) \longrightarrow H_1(\Gamma_1(N), \mathrm{Sym}^{p-1}(k^2)^\vee \otimes \omega^a)\Big).$$

In conclusion, the abutment of the localisation spectral sequences E and ${}'E$ is given in degree i by

$$\mathrm{Tor}_{i-1}^{k[T,S]}\left(\frac{k[T,S]}{(T-b, S-b^2)}, U_{a,\rho}\right)$$

In particular, analysing the E^2 page shows that

$$H_0(\Gamma_1^p(N), \mathrm{St} \otimes \chi)_\rho = H_3(\Gamma_1^p(N), \mathrm{St} \otimes \chi)_\rho = 0,$$
$$H_1(\Gamma_1^p(N), \mathrm{St} \otimes \chi)_\rho \simeq \frac{k[T,S]}{(T-b, S-b^2)} \otimes_{k[T,S]} U_{a,\rho},$$
$$H_2(\Gamma_1^p(N), \mathrm{St} \otimes \chi)_\rho \simeq \mathrm{Hom}_{k[T,S]}\left(\frac{k[T,S]}{(T-b, S-b^2)}, U_{a,\rho}\right).$$

Moreover, Proposition 4.6 shows that the homology in degrees 1 and 2 is always non-zero, since ρ contributes to U_a and can only contribute to its $(T = b, S = b^2)$-eigenspace by [7, Theorem 2.5]. Together with Proposition 2.5, this completes the proof of Theorem 1.1 (3) in this case.

5.2. The trivial case. The case of twists of the trivial representation is completely analogous to the case of twists of the Steinberg representation, this time applying the above analysis to

$$C_1 := \pi(p-1, 1, \chi) \longrightarrow \pi(0, 1, \chi) := C_0,$$

where the map is that of Lemma 4.1. We have $H_0(C_\bullet) \simeq H_1(C_\bullet) \simeq \chi \circ \det$, so Theorem 1.1 (1) and (2) hold for these representations. The corresponding spectral sequences and Proposition 4.3 show that if $\chi = \omega^a \mathrm{unr}_b$ and ρ is an odd irreducible representation of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$,

$$H_0(\Gamma_1^p(N), \chi \circ \det)_\rho = H_1(\Gamma_1^p(N), \chi \circ \det)_\rho = 0,$$
$$H_2(\Gamma_1^p(N), \chi \circ \det)_\rho \simeq \frac{k[T,S]}{(T-b, S-b^2)} \otimes_{k[T,S]} V_{a,\rho},$$
$$H_3(\Gamma_1^p(N), \chi \circ \det)_\rho \simeq \mathrm{Hom}_{k[T,S]}\left(\frac{k[T,S]}{(T-b, S-b^2)}, V_{a,\rho}\right).$$

where

$$V_a := \ker\left(H_1(\Gamma_1(N), \mathrm{Sym}^{p-1}(k^2)^\vee \otimes \omega^a) \longrightarrow H_1(\Gamma_1(N), \omega^a \circ \det)\right).$$

By Proposition 4.5 (and [4, Corollary 2.11]), the odd irreducible Galois representations ρ contributing to the $(T = b, S = b^2)$-eigenspace of V_a are those such that $N(\rho)$ divides N and $\rho|_{\mathcal{G}_p}$ is an extension of $\chi\omega^{-1}$ by χ. These representations therefore appear in the p-arithmetic homology of $\chi \circ \det$ exactly in degrees 2 and 3. There are no Galois representations satisfying condition (a) of Theorem 1.1 (3), as finite-dimensional representations never appear in the socle of representations in the image of the mod p local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$. Therefore, Theorem 1.1 (3) holds, which completes the proof of Theorem 1.1.

Remark 5.1. In the proof of Proposition 4.6 we showed using Poincaré duality for arithmetic cohomology that $U_a = (V_{-a}^\vee)^*$ (with the notation introduced in that proof). The Poincaré duality isomorphisms in that proof intertwine T with $S^{-1}T$ and S with S^{-1}. This, together with the above computations, implies that for ρ as above there are "Poincaré duality" isomorphisms

$$H^i(\Gamma_1^p(N), \chi \circ \det)_\rho \xrightarrow{\sim} H_{4-i}(\Gamma_1^p(N), \mathrm{St} \otimes \chi)_\rho^*.$$

Remark 5.2. Assume that χ is the trivial character for simplicity. Then, the fact that Galois representations as above contribute to $H_*(\Gamma_1^p(N), k)$ and $H^*(\Gamma_1^p(N), k)$ but not in degree 1 (unlike in the other cases) is related to the fact that, if π is the smooth representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ corresponding to $\rho|_{\mathcal{G}_p}$ under the mod p Langlands correspondence, then $\mathrm{Hom}_{k[G]}(k, \pi) = 0$, but $\mathrm{Ext}^1_{k[G]}(k, \pi) \neq 0$ (at least when $\rho|_{\mathcal{G}_p}$ is non-split). This can be made precise by relating p-arithmetic cohomology to completed cohomology and taking into account Emerton's local-global compatibility results [9], but we do not pursue this here.

References

[1] L. Barthel & R. Livné, "Irreducible modular representations of GL_2 of a local field", *Duke Math. J.* **75** (1994), no. 2, p. 261-292.

[2] A. Borel & J.-P. Serre, "Cohomologie d'immeubles et de groupes S-arithmétiques", *Topology* **15** (1976), no. 3, p. 211-232.

[3] C. Breuil, "Sur quelques représentations modulaires et p-adiques de $\mathrm{GL}_2(\mathbf{Q}_p)$. I", *Compos. Math.* **138** (2003), no. 2, p. 165-188.

[4] K. Buzzard, F. Diamond & F. Jarvis, "On Serre's conjecture for mod ℓ Galois representations over totally real fields", *Duke Math. J.* **155** (2010), no. 1, p. 105-161.

[5] A. Caraiani, M. Emerton, T. Gee, D. Geraghty, V. Paškūnas & S. W. Shin, "Patching and the p-adic Langlands program for $\mathrm{GL}_2(\mathbb{Q}_p)$", *Compos. Math.* **154** (2018), no. 3, p. 503-548.

[6] P. Colmez, "Représentations de $\mathrm{GL}_2(\mathbf{Q}_p)$ et (ϕ, Γ)-modules", in *Représentations p-adiques de groupes p-adiques II: Représentations de* $\mathrm{GL}_2(\mathbb{Q}_p)$ *et* (φ, Γ)*-modules*, Astérisque, vol. 330, Société Mathématique de France, 2010, p. 281-509.

[7] B. Edixhoven, "The weight in Serre's conjectures on modular forms", *Invent. Math.* **109** (1992), no. 3, p. 563-594.

[8] B. Edixhoven & C. Khare, "Hasse invariant and group cohomology", *Doc. Math.* **8** (2003), p. 43-50.

[9] M. Emerton, "Local-global compatibility in the p-adic Langlands programme for $\mathrm{GL}_{2/\mathbb{Q}}$", unpublished.

[10] B. H. Gross, "A tameness criterion for Galois representations associated to modular forms (mod p)", *Duke Math. J.* **61** (1990), no. 2, p. 445-517.

[11] J. Kohlhaase & B. Schraen, "Homological vanishing theorems for locally analytic representations", *Math. Ann.* **353** (2012), no. 1, p. 219-258.

[12] V. Paškūnas, "The image of Colmez's Montreal functor", *Publ. Math., Inst. Hautes Étud. Sci.* **118** (2013), p. 1-191.

[13] J.-P. Serre, *Trees*, Springer, 1980, translated from the French by John Stillwell, ix+142 pages.

[14] ———, "Sur les représentations modulaires de degré 2 de $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$", *Duke Math. J.* **54** (1987), no. 1, p. 179-230.

[15] G. Tarrach, "S-arithmetic (co)homology and p-adic automorphic forms", 2023, `https://arxiv.org/abs/2207.04554v2`.

[16] A. Wiles, "Modular elliptic curves and Fermat's last theorem", *Ann. Math. (2)* **141** (1995), no. 3, p. 443-551.

Guillem Tarrach
Department of Pure Mathematics and Mathematical Statistics
University of Cambridge, Centre for Mathematical Sciences
Wilberforce Road, Cambridge CB3 0WB, UK
E-mail: guillemtarrach@gmail.com