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The 1-Level Density for Zeros of Hecke *L*-Functions of Imaginary Quadratic Number Fields of Class Number 1

par Kristian HOLM

Résumé. Soient $\mathbb{K} = \mathbb{Q}(\sqrt{-d})$ un corps quadratique imaginaire de nombre de classes égal à 1 et $\mathcal{O}_{\mathbb{K}}$ son anneau des entiers. On étudie une famille de fonctions L de Hecke associées à des caractères angulaires sur les idéaux non nuls de $\mathcal{O}_{\mathbb{K}}$. En employant la puissante *Ratios Conjecture* (RC) de Conrey, Farmer et Zirnbauer, on calcule une expression asymptotique conditionnelle pour la densité moyenne de niveau 1 des zéros de cette famille. Cette estimation comprend des termes d'ordre inférieur au terme principal dans la *Density Conjecture* de Katz et Sarnak. En outre, on démontre un résultat inconditionnel sur la densité de niveau 1, qui concorde avec la prédiction de la RC à condition que les fonctions test soient telles que l'intervalle $(-1,1)$ contienne les supports de leurs transformées de Fourier.

Abstract. Let $\mathbb{K} = \mathbb{Q}(\sqrt{-d})$ be an imaginary quadratic number field of class number 1 and $\mathcal{O}_{\mathbb{K}}$ its ring of integers. We study a family of Hecke L-functions associated to angular characters on the non-zero ideals of $\mathcal{O}_{\mathbb{K}}$. Using the powerful Ratios Conjecture (RC) due to Conrey, Farmer, and Zirnbauer, we compute a conditional asymptotic for the average 1-level density of the zeros of this family, including terms of lower order than the main term in the Katz–Sarnak Density Conjecture coming from random matrix theory. We also prove an unconditional result about the 1-level density, which agrees with the RC prediction when our test functions have Fourier transforms with support in $(-1,1)$.

1. Introduction

Many problems in modern number theory await progress due to the difficulty of obtaining exact information about zeros of L-functions. Perhaps not unrelated to this difficulty, the study of the large scale statistics of such zeros has also become a topic of much interest, the underlying philosophy being that a collection of objects is often more regular and well-behaved than the objects themselves. This line of research began with the work of

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Montgomery in the 1970's who famously conjectured [16] that the *pair correlation* of normalized zeros of the Riemann zeta function, quantifying the "probability" of ζ having two zeros within some prescribed distance of each other, is the same as the pair correlation for eigenvalues of random Hermitian matrices. Today, there is a large body of conjectures describing the links between L-functions and random matrices, or formulating properties that should hold for L-functions by analogy with random matrices. In this paper, we will focus on two of these: The Katz–Sarnak Density Conjecture and the L-functions Ratios Conjecture.

The Katz–Sarnak conjecture is a statement about another statistic of the zeros of (a family of) L-functions, namely the 1-level density, which was first studied by Özlük and Snyder in [17]. For our purposes, it can be defined as follows: If $\mathcal{F} = \{L_k : k \geqslant 1\}$ is a family of L-functions indexed by some parameter k, and $\mathcal{F}(K) = \{L_k : 1 \leqslant k \leqslant K\}$, let

$$Z_k := \{z : 0 \leqslant \mathrm{Re}(z) \leqslant 1,\, L_k(z) = 0\}$$

be the set of zeros of L_k in the critical strip. If $\rho \in Z_k$, let $\gamma(\rho) = -i(\rho - 1/2)$. Thus, under the Riemann Hypothesis for the family $\mathcal{F}(K)$, $\gamma(\rho)$ is the imaginary part of the zero ρ of L_k. Furthermore, let $f : \mathbb{R} \to \mathbb{R}$ be an even Schwartz function with the property that its Fourier transform $\widehat{f}$ has compact support. Then the 1-*level density* of the zeros of the family $\mathcal{F}(K)$ is the number

$$D(\mathcal{F}(K); f) := \frac{1}{K} \sum_{k=1}^{K} \sum_{\rho \in Z_k} f(c_K \gamma(\rho)),$$

where the scaling parameter c_K of $\gamma(\rho)$ ensures that the average spacing between the zeros is approximately 1. In practice, c_K is chosen as an average over $\mathcal{F}(K)$ of certain invariants of L_k that govern the asymptotic number of zeros of L_k in a set of interest (see, for example, [13, Theorem 5.8] and Proposition 2.2).

Under the Grand Riemann Hypothesis, the 1-level density measures the average density of the normalized zeros of the family $\mathcal{F}(K)$ *in a weak sense*. That is, for zeros on the critical line $\mathrm{Re}(z) = 1/2$, and especially such zeros close to the real line, $D(\mathcal{F}(K), \cdot)$ is a functional that sees their (scaled) distribution through the lens of a suitable test function. When taking more and more L-functions of the family and their zeros into account, one thus obtains a sequence of distributions. For the purposes of studying this sequence, and in particular its limit, such weak characterizations in fact give a complete picture of the distribution of the zeros of the family. (At least, this holds if one knows the weak characterizations of the distribution for a sufficiently large class of test functions, thanks to classic results in probability theory such as the Portmanteau theorem, cf. [2, Theorem 25.8].)

Regarding the limiting distribution of the zeros of a family of L-functions, Katz and Sarnak conjectured [14, 15] that when K tends to infinity, the functional $D(\mathcal{F}(K), \cdot)$ converges weakly to some integral kernel that arises in the dimensional limit of the 1-level density of eigenangles of random unitary matrices, chosen uniformly at random with respect to Haar measure either from the full unitary group $\mathrm{U}(N)$ or from one of the subgroups $\mathrm{USp}(N)$ (when N is even), $\mathrm{O}(N)$, or $\mathrm{SO}(N)$. Specifically, if the family $\mathcal{F}$ has so-called *unitary symplectic symmetry type*, the *Katz–Sarnak Density Conjecture* states that

$$\lim_{K\to\infty} D(\mathcal{F}(K); f) = \int_{\mathbb{R}} f(x)\left(1 - \frac{\sin(2\pi x)}{2\pi x}\right) \mathrm{d}x \tag{1.1}$$

for any even Schwartz function f whose Fourier transform has compact support. We emphasize that if the family $\mathcal{F}(K)$ has a different symmetry type, the integral kernel which is conjectured to appear in the limit has a different form.

Next, the very powerful L-functions Ratios Conjecture due to Conrey, Farmer, and Zirnbauer [6] asserts that averages of quotients of L-functions evaluated at certain parameters satisfy asymptotics that parallel those of quotients of characteristic polynomials of matrices. (See Section 3 below for a more detailed statement.) Many authors have used the Ratios Conjecture to study statistical aspects of the zeros of L-functions, or various other aspects of such functions. For example, Conrey and Snaith [7] studied the pair correlation of the zeros of the Riemann zeta function. In the same paper, they also studied the 1-level density for zeros of quadratic Dirichlet L-functions. Later, the Ratios Conjecture was used to study zeros of L-functions of a more general class of characters, namely *Hecke characters* of a number field. The Hecke L-functions considered in [19] are those associated to angular characters of the Gaussian integers, and Waxman here used the Ratios Conjecture to compute the 1-level density and identify lower-order terms (compared with the Katz–Sarnak heuristic) in this asymptotic. The goal of this paper is to follow [19] and do such a study for a general imaginary quadratic number field of class number 1. Thus, we consider a family $\mathcal{F}(K)$ of L-functions associated to angular Hecke characters of such fields, which we will describe now.

By the Baker–Heegner–Stark Theorem, a complete list of imaginary quadratic number fields with class number 1 is given by $\mathbb{K} := \mathbb{K}_d := \mathbb{Q}(\sqrt{-d})$, where d is one of the *Heegner numbers*,

$$d = 1, 2, 3, 7, 11, 19, 43, 67, 163.$$

Since the case $d = 1$ has already been treated in [19], we will let d denote any of the eight remaining numbers on the list above. (This restriction will also make certain computations simpler, as the arguments involve several functions defined conditionally on the value of d.) Moreover, we will also let $N \geqslant 1$ denote any fixed positive multiple of $|\mathcal{O}_{\mathbb{K}}^{\times}|$, where $|\mathcal{O}_{\mathbb{K}}^{\times}| < \infty$ is the order of the group of units in the ring $\mathcal{O}_{\mathbb{K}}$. Our family of L-functions is then given by $\mathcal{F} = \{L_k(s) : k \geqslant 1\}$, where

$$L_k(s) := \sum_{\substack{I \subseteq \mathcal{O}_{\mathbb{K}} \\ I \neq 0}} \frac{\psi_k(I)}{\mathbb{N}(I)^s}, \qquad \psi_k(\langle \alpha \rangle) = \left(\frac{\alpha}{\overline{\alpha}}\right)^{Nk},$$

when $\mathrm{Re}(s) > 1$.

We note that such L-functions have been studied for arithmetic purposes on several occasions in the past. To give a few examples, Harman and Lewis considered the functions $L(s, \Xi_k)$, $k \geqslant 1$, with the Hecke character Ξ_k given by $\Xi_k(\alpha) = (\alpha / \overline{\alpha})^{2k}$ for $\alpha \in \mathbb{Z}[i]$, and proved [10, Theorem 1] the existence of infinitely many rational primes p that have a Gaussian prime factor with a small argument (depending on the size of p). Later, in [18] Rudnick and Waxman considered the same family of L-functions and counted Gaussian primes in more general sectors of the complex plane, in a sense quantifying Hecke's classical theorem about the equidistribution of the angles of Gaussian primes on the circle ([11, 12]). In particular, the authors in [18] studied the variance of such smooth counts of Gaussian primes and conjectured an asymptotic ([18, Conjecture 1.2]) for this statistic based on a random matrix model and an analogue with similar counts over function fields. The asymptotic behaviour of this variance was, in fact, investigated quite recently from a different point of view in [4], with a particular point of interest being the nature of the lower order terms in the asymptotic. An important aspect of the work in [4] relied on a study of the Hecke L-functions $L(s, \Xi_k)$ discussed above in combination with the Ratios Conjecture.

We now state our main results. In the formulations of these (and throughout the paper), D denotes the discriminant of our number field $\mathbb{K}$, and $\chi(n) = (-d/n)$ denotes the Dirichlet character coming from the Kronecker symbol (see Section 2.3). Moreover, γ denotes the Euler–Mascheroni constant. Then, with the 1-level density normalized with the scaling parameter $c_K = \log K/\pi$ (cf. Proposition 2.2), we prove the following.

Theorem 1.1. *Suppose that $f : \mathbb{R} \to \mathbb{R}$ is an even Schwartz function with $\operatorname{supp} \widehat{f} \subseteq (-1, 1)$. Then we have*

$$D(\mathcal{F}(K); f) = \int_{\mathbb{R}} f(x)\Big(1 - \frac{\sin(2\pi x)}{2\pi x}\Big)\,\mathrm{d}x + \frac{\ell_0 \widehat{f}(0)}{\log K} + O\Big(\frac{1}{(\log K)^2}\Big),$$

where

$$\ell_0 = -1 + \gamma - \frac{L'(1,\chi)}{L(1,\chi)} - 2 \sum_{\substack{p \geqslant 3 \\ (-d/p) = -1}} \frac{\log p}{p^2 - 1} + \log \sqrt{|D|} - \log 2\pi + \log N \\ - \frac{\sqrt{d} \log d}{d-1} - \frac{2 \log 2}{3} \cdot \mathbb{1}(d \neq 2, 7).$$

By assuming the Grand Riemann Hypothesis (GRH) and the Ratios Conjecture ([6]), we also prove the following result.

Theorem 1.2. *Suppose that $f : \mathbb{R} \to \mathbb{R}$ is an even Schwartz function whose Fourier transform has compact support. Assume the GRH and the Ratios Conjecture. Then*

$$D(\mathcal{F}(K); f) = \int_{\mathbb{R}} f(x) \left(1 - \frac{\sin(2\pi x)}{2\pi x}\right) \mathrm{d}x + \frac{\ell_0}{\log K} \left(\widehat{f}(0) - \widehat{f}(1)\right) \\ + O\left(\frac{1}{(\log K)^2}\right),$$

where ℓ_0 is as in the statement of Theorem 1.1.

Remark 1.3.

(1) Although we give an explicit value for $L'(1,\chi)/L(1,\chi)$ in Lemma 4.5, this expression is rather intricate, and we therefore decided to keep the notation $L'(1,\chi)/L(1,\chi)$ in the statements of the theorems.

(2) We note that both Theorem 1.1 and Theorem 1.2 verify the Katz–Sarnak Density Conjecture, but to different extents: While Theorem 1.1 requires f to have a Fourier transform with very small support, Theorem 1.2 holds without any such assumption.

(3) The appearance of $|\mathcal{O}_{\mathbb{K}}^{\times}|$ in the exponents of the characters ψ_k is very natural, since ψ_k must satisfy a condition related to the units in order to define a Hecke character on the ideals of $\mathcal{O}_{\mathbb{K}}$ (cf. Section 2.1). Compared to the setup in [19], we are considering a more general family of characters since we allow N to be any multiple of the order of the unit group. The reason why we are able to handle this more general case is that we formulate and prove a generalization (Theorem 6.1) of the result [18, Lemma 2.1] relating the arguments and norms of certain elements of $\mathcal{O}_{\mathbb{K}}$.

(4) It follows from Lemma 4.1 that the symmetry in Theorem 1.2 between the terms involving $\widehat{f}(0)$ and $\widehat{f}(1)$ also holds in the analogous conditional result of [19], which deals with the case $\mathbb{K} = \mathbb{Q}(\sqrt{-1})$. More precisely, in the notation of [19, Conjecture 1.1], one has $c = d$.

Our approach is based on [19]. We first prove Theorem 1.2, and this is accomplished in Section 3, where we also describe the Ratios Conjecture in detail for our family $\mathcal{F}$, and in Section 4. In Section 5, we use the explicit formula for our family $\mathcal{F}$ to give an unconditional asymptotic for the 1-level density. Section 6 is then a comparison between this and the conditional asymptotic, which leads to a proof of Theorem 1.1. As mentioned, this comparison is facilitated by Theorem 6.1, which we also state and prove in Section 6.

2. Preliminaries

We will now introduce Hecke characters on imaginary quadratic number fields, describe our concrete family of L-functions in more detail, and mention various standard results that we will need later.

2.1. Hecke Characters in Imaginary Quadratic Number Fields. An equivalent formulation of $\mathbb{K}$ having class number 1 is that its ring of integers $\mathcal{O}_{\mathbb{K}}$ is a principal ideal domain. Explicitly, we have

$$\mathcal{O}_{\mathbb{K}} = \begin{cases} \mathbb{Z}[\sqrt{-d}] & \text{if } d \equiv 1, 2 \pmod 4, \\ \mathbb{Z}[(1+\sqrt{-d})/2] & \text{if } d \equiv 3 \pmod 4. \end{cases} \tag{2.1}$$

This fact can be more succinctly expressed as $\mathbb{K} = \mathbb{Q}(\sqrt{D})$ and $\mathcal{O}_{\mathbb{K}} = \mathbb{Z}[(D+\sqrt{D})/2]$, where $D < 0$ denotes the *discriminant* of $\mathbb{K}$ (see [13, Section 3.8]). By using the fact that any unit in $\mathcal{O}_{\mathbb{K}}$ must have norm 1, one may easily prove that

$$\mathcal{O}_{\mathbb{K}}^{\times} \simeq \begin{cases} \mathbb{Z}/2\mathbb{Z} & \text{if } d = 2 \text{ or } d \geqslant 5, \\ \mathbb{Z}/6\mathbb{Z} & \text{if } d = 3, \end{cases}$$

cf. [13, eq. (3.72)].

Since we will later make use of the lattice structure of $\mathcal{O}_{\mathbb{K}}$, we also describe these rings in the following way.

Lemma 2.1. *Under the identification* $\mathbb{C} \simeq \mathbb{R}^2$*, we have*

$$\mathcal{O}_{\mathbb{K}} = 2^{1/4}\begin{pmatrix} 2^{-1/4} & 0 \\ 0 & 2^{1/4} \end{pmatrix}\mathbb{Z}^2,$$

when $d = 2$*; or, when* $d \geqslant 3$*,*

$$\mathcal{O}_{\mathbb{K}} = d^{1/4}2^{-1/2}\begin{pmatrix} 2^{-1/2}d^{-1/4} & 0 \\ 0 & 2^{1/2}d^{1/4} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1/2 & 1 \end{pmatrix}\mathbb{Z}^2.$$

Proof. When $d = 2$, this is simply a matter of expanding (2.1). When $d \geqslant 3$, (2.1) shows that any $\alpha \in \mathcal{O}_{\mathbb{K}}$ can be written as

$$-a + b/2 + b\sqrt{-d}/2 = s/2 + (s+2a)\sqrt{-d}/2$$

with $s = -2a + b$. The claimed decomposition in this case now follows once we express the elements of $\mathcal{O}_{\mathbb{K}}$ using the variables s and a. $\square$

When $\mathbb{K}$ has class number 1, a *Hecke character* ψ on $\mathbb{K}$ corresponds to a unique pair (χ, χ_∞) consisting of a generalized Dirichlet character χ (modulo some ideal $\mathfrak{m} \subseteq \mathcal{O}_{\mathbb{K}}$) and a unitary character χ_∞ on $\mathbb{C}^\times$. Conversely, given a pair of such characters, their (pointwise) product is a Hecke character provided that $\chi \cdot \chi_\infty$ is constant on $\mathcal{O}_{\mathbb{K}}^\times$, cf. [13, eq. (3.80)]. In the special case where this condition is satisfied with a Dirichlet character χ of modulus $\mathcal{O}_{\mathbb{K}}$, this product coincides with the unitary character χ_∞, and the corresponding Hecke character is then referred to as *angular*. Thus, in order for us to specify a Hecke character, it is enough to specify two characters

$$\chi : (\mathcal{O}_{\mathbb{K}}/\mathfrak{m})^\times \to \mathbb{C}^\times, \qquad \chi_\infty : \mathbb{C}^\times \to S^1$$

satisfying $\chi(u)\chi_\infty(u) = 1$ for all $u \in \mathcal{O}_{\mathbb{K}}^\times$.

We now describe our concrete family of Hecke characters. Let N be any positive integer multiple of $|\mathcal{O}_{\mathbb{K}}^\times|$. Since $\mathcal{O}_{\mathbb{K}}$ is a principal ideal domain, for any $k \geqslant 1$ we can define the unitary character $\chi_{\infty,k}$ by

$$\chi_{\infty,k}(I) = \chi_{\infty,k}(\alpha) = (\alpha/\overline{\alpha})^{Nk}$$

whenever $I = \langle\alpha\rangle$. This is well-defined since any two generators of I will differ by a factor in $\mathcal{O}_{\mathbb{K}}^\times$ where $\chi_{\infty,k}$ is identically equal to 1. To make this into a Hecke character, we also need to specify a Dirichlet character that is compatible with $\chi_{\infty,k}$ in the above sense. However, we can simply take χ to be the trivial generalized Dirichlet character of modulus $\mathfrak{m} = \mathcal{O}_{\mathbb{K}}$. In this way we obtain the family of Hecke characters given by

$$\psi_k(\langle\alpha\rangle) = \psi_k(\alpha) := \chi(\alpha) \cdot \chi_{\infty,k}(\alpha) = \chi_{\infty,k}(\alpha) = \left(\frac{\alpha}{\overline{\alpha}}\right)^{Nk}$$

for $\alpha \in \mathcal{O}_{\mathbb{K}} \setminus \{0\}$. Since conjugation is an automorphism of $\mathbb{C}$, we note the relation $\overline{\psi_k} = \psi_{-k}$, which will be useful later on.

In the literature it is common to write such unitary characters as $\chi_\infty(\alpha) = (\alpha/|\alpha|)^\ell$ for a suitable integer ℓ called the *frequency*. In the case of our character ψ_k, we see that $\psi_k(\alpha) = (\alpha/|\alpha|)^{2Nk}$, so that ψ_k has frequency $2Nk$.

We note that ψ_k can also be described explicitly as a function of the argument of α, which will be convenient at certain points in the paper. Namely, if we write $\alpha = re^{i\theta_\alpha}$, we have

$$\psi_k(\alpha) = \left(\frac{re^{i\theta_\alpha}}{re^{-i\theta_\alpha}}\right)^{Nk} = e^{2iNk\theta_\alpha}.$$

We also wish to speak of the "argument of the ideal $\langle\alpha\rangle$". A priori, this is not well-defined since $\langle u\alpha\rangle = \langle\alpha\rangle$ for any unit $u \in \mathcal{O}_{\mathbb{K}}^\times$. However, since any unit has argument equal to a multiple of $2\pi/|\mathcal{O}_{\mathbb{K}}^{\times}|$, the effect of multiplying α with a unit u is to change θ_α by such a multiple. For this reason, by choosing u appropriately, we can always ensure that the argument of $u\alpha$ lies in $\big[0, 2\pi/|\mathcal{O}_{\mathbb{K}}^{\times}|\big)$. Accordingly, the angle $\theta_{\langle\alpha\rangle}$ of the ideal $\langle\alpha\rangle$ is well-defined when taken in the interval $\big[0, 2\pi/|\mathcal{O}_{\mathbb{K}}^{\times}|\big)$.

2.2. Hecke *L*-Functions and Their Zeros. To each of the characters ψ_k ($k \geqslant 1$) we can associate a *Hecke L-function* given initially by the series and corresponding Euler product

$$L_k(s) = L(s, \psi_k) := \sum_{\substack{I \subseteq \mathcal{O}_{\mathbb{K}} \\ I \neq 0}} \frac{\psi_k(I)}{\mathbb{N}(I)^s} = \prod_{\mathfrak{p}} \frac{1}{1 - \psi_k(\mathfrak{p})/\mathbb{N}(\mathfrak{p})^s}, \qquad \operatorname{Re}(s) > 1.$$

The symbol $\mathbb{N}(I)$ here denotes the *(absolute) norm* of the ideal $I \subseteq \mathcal{O}_{\mathbb{K}}$, which is defined as the number $[\mathcal{O}_{\mathbb{K}} : I]$ of cosets of I in $\mathcal{O}_{\mathbb{K}}$. For a principal ideal $I = \langle\alpha\rangle$, the absolute norm of I agrees with the usual *(field) norm* $\alpha\bar{\alpha}$ of α, which we will therefore also denote by $\mathbb{N}(\alpha)$.

Let us immediately note that $L_k = L_{-k}$. Indeed, if

$$N_j(a, b) := \mathbb{N}(j(a, b)) = \begin{cases} a^2 + db^2 & \text{if } d = 2, \\ a^2 + ab + \frac{d+1}{4} b^2 & \text{if } d \geqslant 3 \end{cases}$$

denotes the norm of an element

$$j(a, b) := \begin{cases} a + i\sqrt{2}b & \text{if } d = 2, \\ a + b(1 + i\sqrt{d})/2 & \text{if } d \geqslant 3, \end{cases}$$

then we note that the map A_d, which is defined implicitly by the relation $\overline{j(a, b)} = j(A_d(a, b))$, defines a bijection on the set

$$\Big\{(a, b) \in \mathbb{Z}^2 : N_j(a, b) \neq 0\Big\} = \mathbb{Z}^2 \setminus \{(0, 0)\},$$

as complex conjugation is a norm-preserving automorphism of $\mathbb{C}$. Therefore, for $\operatorname{Re}(s) > 1$, the trivial identity [19, eq. (2.1)] gives

$$\begin{aligned} L_{-k}(s) &= \frac{1}{|\mathcal{O}_{\mathbb{K}}^{\times}|} \sum_{N_j(a,b) \neq 0} \left(\frac{j(a, b)}{|j(a, b)|}\right)^{-2Nk} N_j(a, b)^{-s} \\ &= \frac{1}{|\mathcal{O}_{\mathbb{K}}^{\times}|} \sum_{(a,b) \neq (0,0)} \left(\frac{|j(a, b)|}{j(a, b)}\right)^{2Nk} N_j(a, b)^{-s} \\ &= \frac{1}{|\mathcal{O}_{\mathbb{K}}^{\times}|} \sum_{(a,b) \neq (0,0)} \left(\frac{\overline{j(a, b)}}{|j(a, b)|}\right)^{2Nk} N_j(a, b)^{-s} \end{aligned}$$

$$= \frac{1}{|\mathcal{O}_{\mathbb{K}}^{\times}|} \sum_{(a,b)\neq(0,0)} \left(\frac{j(A_d(a,b))}{|j(A_d(a,b))|}\right)^{2Nk} N_j(A_d(a,b))^{-s}$$

$$= \frac{1}{|\mathcal{O}_{\mathbb{K}}^{\times}|} \sum_{(a,b)\neq(0,0)} \left(\frac{j(a,b)}{|j(a,b)|}\right)^{2Nk} N_j(a,b)^{-s} = L_k(s).$$

By a theorem of Hecke [13, Theorem 3.8], if $k \neq 0$ (so that ψ_k is not the trivial character), L_k admits an analytic continuation (which we will also denote by L_k) to the entire complex plane, and it satisfies the functional equation

$$\Lambda(s, \psi_k) = \frac{\tau(\psi_k)}{i^{\ell}\sqrt{\mathbb{N}(\mathfrak{m})}} \Lambda\Big(1 - s, \overline{\psi_k}\Big),$$

where $\mathfrak{m} = \mathcal{O}_{\mathbb{K}}$ is the modulus of ψ_k, $\ell = 2Nk$ is the frequency, and $\Lambda(s, \psi_k)$ denotes the *completed L-function*

$$\Lambda(s, \psi_k) = \Lambda_k(s) := L_k(s) \frac{\big(|D|\mathbb{N}(\mathfrak{m})\big)^{s/2}}{(2\pi)^s} \Gamma\big(s + |\ell|/2\big),$$

and where $\tau(\psi_k)$ denotes the Gauss sum

$$\tau(\psi_k) = \psi_k(\gamma)\psi_k(\mathfrak{c})^{-1} \sum_{\alpha \in \mathfrak{c}/\mathfrak{c}\mathfrak{m}} \exp(2\pi i \cdot \mathrm{Tr}(\alpha/\gamma)),$$

cf. [13, eq. (3.86)]. Here $\gamma \in \mathcal{O}_{\mathbb{K}}$ and $\mathfrak{c} \subseteq \mathcal{O}_{\mathbb{K}}$ are arbitrary except for the requirements that $\mathfrak{c}$ should be an ideal, and that γ and $\mathfrak{c}$ should satisfy $(\mathfrak{c}, \mathfrak{m}) = 1$ and $\mathfrak{c}\mathfrak{d}\mathfrak{m} = \langle \gamma \rangle$, where $\mathfrak{d}$ is the *different* of $\mathbb{K}$. Since we have $\mathfrak{m} = \mathcal{O}_{\mathbb{K}}$ and $\mathfrak{d} = \langle \sqrt{D} \rangle$, these conditions are satisfied with $\mathfrak{c} = \mathcal{O}_{\mathbb{K}}$ and $\gamma = \sqrt{D}$. Then it is clear that $\psi_k(\mathfrak{c}) = 1$. Moreover, the sum over $\alpha \in \mathfrak{c}/\mathfrak{c}\mathfrak{m}$ reduces to a single term, namely $\exp(2\pi i \cdot \mathrm{Tr}(1/\sqrt{D}))$, which equals 1 because $1/\sqrt{D}$ is purely imaginary and therefore has vanishing trace. Finally, the fact that the frequency ℓ of ψ_k is $2Nk \equiv 0 \pmod 4$ implies that also $\psi_k(\gamma) = 1$. In conclusion, we therefore have $\tau(\psi_k) = 1$.

In combination with the relation $\overline{\psi_k} = \psi_{-k}$, the above fact means that the *root number* of L_k is 1, and the functional equation assumes the simpler form

$$\Lambda_k(s) = \Lambda_{-k}(1 - s) = \Lambda_k(1 - s), \tag{2.2}$$

where also the completed L-function can be described in the simpler form

$$\Lambda_k(s) = L_k(s) \frac{|D|^{s/2}}{(2\pi)^s} \Gamma\big(s + Nk\big). \tag{2.3}$$

Of course, it is also possible to recast the identity (2.2) as a statement about L_k that does not explicitly involve Λ_k. Doing so, we find that

$$L_k(s) = L_k(1 - s) X_k(s), \tag{2.4}$$

where

$$X_k(s) := \frac{\Gamma(1-s+Nk)}{\Gamma(s+Nk)}|D|^{1/2-s}(2\pi)^{2s-1}. \tag{2.5}$$

If $K \geqslant 1$ is an integer, then as we mentioned in the introduction, we will use the notation

$$\mathcal{F}(K) := \{L_k : 1 \leqslant k \leqslant K\}$$

to denote our family of L-functions. We wish to normalize the zeros of this family so that they have mean spacing 1. This of course warrants an understanding of the asymptotic number $N_k(T)$ of zeros of L_k in the critical strip $0 \leqslant \mathrm{Re}(z) \leqslant 1$ up to a given height T as $k \to \infty$. Estimates of the count $N_k(T)$ for general L-functions are abundant in the literature, but these usually provide an asymptotic as $T \to \infty$. We therefore prove the following result, which gives an asymptotic expression for $N_k(T)$ when it is not the height, but rather the size of our family, that tends to infinity.

Proposition 2.2. *Let $k \geqslant 1$, and assume the Riemann Hypothesis for L_k. For $T > 0$, let*

$$N_k(T) := \#\Big\{z \in \mathbb{C} : L_k(z) = 0,\, 0 \leqslant \mathrm{Re}(z) \leqslant 1,\, -T \leqslant \mathrm{Im}(z) \leqslant T\Big\}$$

be the number of zeros of L_k in the critical strip up to absolute height T. Then as $k \to \infty$,

$$N_k(T) \sim \frac{2T\log k}{\pi}.$$

Proof. Let us write $X := 1/2 + Nk$. By [3, Theorem 5] and [3, eq. (4.1)], the main term of $N_k(T)$ comes from the integral

$$\frac{1}{\pi}\int_{-T}^{T} \mathrm{Re}\bigg(\frac{\Gamma'(X+it)}{\Gamma(X+it)}\bigg)\,\mathrm{d}t = \frac{1}{\pi}\mathrm{Re}\Bigg(\int_{-T}^{T} \frac{\Gamma'(X+it)}{\Gamma(X+it)}\,\mathrm{d}t\Bigg).$$

Indeed, in the notation of that paper, we have

$$L(z,\pi_\infty) = \frac{|D|^{z/2}}{(2\pi)^z}\Gamma(z+Nk);$$

and since the logarithmic derivative of the factor $|D|^{z/2}(2\pi)^{-z}$ is constant, the integral of the logarithmic derivative of $|D|^{(X+it)/2}(2\pi)^{-X-it}$ over the line $-T \leqslant t \leqslant T$ is at most a constant times T.

Since $-i\log\Gamma(X+it)$ is a primitive function for the logarithmic derivative in the integrand above, and since our domain of integration lies in the right half-plane, we can use Stirling's formula $\log\Gamma(z) = z\log z - z - (\log z)/2 +$

$O(\max\{1, 1/z\})$ to obtain

$$\int_{-T}^{T} \frac{\Gamma'(X+it)}{\Gamma(X-it)}\,\mathrm{d}t = -i\cdot X\log\frac{X+iT}{X-iT} + T\log\Big(X^2+T^2\Big) \\ - 2T + i\cdot\frac{1}{2}\log\frac{X+iT}{X-iT} + O(1).$$

The real part of the right-hand side equals

$$T\log\Big(X^2+T^2\Big) - 2T - (1/2 - X)\cdot\mathrm{Im}\Big(\log\frac{X+iT}{X-iT}\Big) + O(1).$$

The imaginary part of the logarithm is proportional to $\mathrm{Im}(\log(1+iT/X))$, which equals the principal value of the argument of $1+iT/X$. Since we have

$$\Big(X-\frac{1}{2}\Big)\cdot\mathrm{Arg}\Big(1+\frac{iT}{X}\Big) = Nk\cdot\arctan\Big(\frac{T}{Nk}\Big) \ll_T 1,$$

whereas $T\log(X^2+T^2) = 2T\log k + O_T(1)$, the term involving the imaginary part contributes negligibly to the real part of the integral. We therefore conclude that

$$\frac{1}{\pi}\int_{-T}^{T}\mathrm{Re}\Big(\frac{\Gamma'(X+it)}{\Gamma(X+it)}\Big)\,\mathrm{d}t = \frac{2T\log k}{\pi} + O_T(1)$$

as $k\to\infty$. This concludes the proof. □

2.3. Splitting Behaviour of Rational Primes. We now record some facts about how a prime $p\in\mathbb{Z}$ behaves in the extension $\mathbb{K}=\mathbb{Q}(\sqrt{-d})$, where we recall that d is one of the numbers $2, 3, 7, 11, 19, 43, 67, 163$. For later purposes it will be useful to note that, except for 2, all of these are prime numbers congruent to 3 (mod 4). Moreover, except for 2 and 7, they are all congruent to 3 (mod 8).

Translating from the language of ideals in [13, Section 3.8] to the language of algebraic integers (as all ideals in $\mathcal{O}_{\mathbb{K}}$ are principal), we say that a rational prime p is *ramified* if $p=q^2u$ for a prime $q\in\mathcal{O}_{\mathbb{K}}$ and $u\in\mathcal{O}_{\mathbb{K}}^{\times}$, *split* if $p=q\bar{q}u$ for a prime $q\in\mathcal{O}_{\mathbb{K}}$ and $u\in\mathcal{O}_{\mathbb{K}}^{\times}$, or *inert* if p is a prime element in $\mathcal{O}_{\mathbb{K}}$. Any rational prime belongs to exactly one of these categories. Moreover, it is well-known (cf. [13, Section 3.8]) that a rational prime p is ramified in $\mathbb{K}$ if and only if $p\mid D$, the discriminant of $\mathbb{K}$. Since

$$D = \begin{cases} -d & \text{if } d\equiv 3 \pmod 4,\\ -4d & \text{if } d\equiv 1,2 \pmod 4,\end{cases} \tag{2.6}$$

we see that p is ramified in $\mathbb{K}$ if and only if $p=d$.

In the remainder of the paper, we will need to work with the "Legendre symbol modulo composite numbers". We therefore introduce the Kronecker symbol, as it will also be a useful tool in the current discussion. If $n=$

$p_1^{e_1} \cdots p_k^{e_k}$ is (the prime factorization of) a positive integer and $a \in \mathbb{Z}$, the *Kronecker symbol* (a/n) is defined as

$$\left(\frac{a}{n}\right) = \prod_{i=1}^{k} \left(\frac{a}{p_i}\right)^{e_i},$$

where the symbol (a/p_i) appearing on the right-hand side is the Legendre symbol if p_i is an odd prime, or otherwise given by

$$\left(\frac{a}{2}\right) = \begin{cases} 0 & \text{if } a \text{ is even,} \\ 1 & \text{if } a \equiv \pm 1 \pmod{8}, \\ -1 & \text{if } a \equiv \pm 3 \pmod{8}. \end{cases}$$

It follows from this definition that the Kronecker symbol is completely multiplicative in its top argument, provided that this argument is non-zero.

One particular instance of the Kronecker symbol is the quadratic Dirichlet character $\chi_D(n) := (D/n)$, called the *field character*, which keeps track of the splitting behavior of a prime in $\mathbb{K}$. More precisely, $\chi_D(p)$ equals 0 if p is ramified, 1 if p splits, or -1 if p is inert, cf. the beginning of [13, Section 3.8]. We prefer to work instead with the character $\chi(n) := (-d/n)$, where d is one of the eight non-trivial Heegner numbers that we are considering. In light of (2.6), χ and χ_D coincide for $d \neq 2$ since the only other possibility, then, is $d \equiv 3 \pmod 4$. On the other hand, for $d = 2$ and any integer n, we compute that

$$\chi(n) = \left(\frac{-2}{n}\right) = \left(\frac{-2}{n}\right)^3 = \left(\frac{-8}{n}\right) = \chi_D(n),$$

so that $\chi = \chi_D$ even in this case.

We sum up the above discussion in the following lemma.

Lemma 2.3. *Let p be a rational prime and $\chi(\cdot)$ be the Kronecker symbol $(-d/\cdot)$. In $\mathbb{K}$,*

$$p \textit{ is } \begin{cases} \textit{ramified} & \textit{if } \chi(p) = 0 \ \textit{(that is, } p = d\textit{)}, \\ \textit{split} & \textit{if } \chi(p) = 1, \\ \textit{inert} & \textit{if } \chi(p) = -1. \end{cases}$$

Remark 2.4. We mention that the case of the prime $p = 2$ is special, in the sense that this is the only instance where the value $\chi(p)$ of the Dirichlet character appearing in Lemma 2.3 is not given by the Legendre symbol. In this case, we note for later use that 2 is ramified in $\mathbb{K}$ if $d = 2$, split if $d = 7$, or otherwise inert (cf. the discussion at the beginning of this subsection).

We end this section with the following elementary lemma whose proof we include for the sake of completeness.

Lemma 2.5. *Let $\mathfrak{p} \subseteq \mathcal{O}_{\mathbb{K}}$ be a prime ideal. Then $\mathfrak{p}$ lies over a rational prime p (that is, $\mathfrak{p} \cap \mathbb{Z} = p\mathbb{Z}$) if and only if $\mathfrak{p} \mid p\mathcal{O}_{\mathbb{K}}$. Hence, the prime ideals with norm equal to a power of p are precisely the prime ideals dividing $p\mathcal{O}_{\mathbb{K}}$.*

Proof. Suppose that $\mathfrak{p}$ lies over p. Then clearly $p \in \mathfrak{p}$, and by the ideal property of $\mathfrak{p}$ we therefore have $p\mathcal{O}_{\mathbb{K}} \subseteq \mathfrak{p}$, which means $\mathfrak{p} \mid p\mathcal{O}_{\mathbb{K}}$.

Conversely, if $\mathfrak{p} \mid p\mathcal{O}_{\mathbb{K}}$, then $\mathfrak{p} \supseteq p\mathcal{O}_{\mathbb{K}}$, and hence $\mathfrak{p} \cap \mathbb{Z} \supseteq p\mathcal{O}_{\mathbb{K}} \cap \mathbb{Z} \supseteq p\mathbb{Z}$. Since $p\mathbb{Z}$ is a maximal ideal in $\mathbb{Z}$ and $\mathfrak{p} \cap \mathbb{Z} \subsetneq \mathbb{Z}$ must be a proper ideal, it follows that $p\mathbb{Z} = \mathfrak{p} \cap \mathbb{Z}$, so $\mathfrak{p}$ lies over p.

Towards the final claim of the lemma, let $\iota : \mathbb{Z} \to \mathcal{O}_{\mathbb{K}}$ denote the ring homomorphism given by the canonical embedding of $\mathbb{Z}$ into $\mathcal{O}_{\mathbb{K}}$. If now $\mathfrak{q} \subseteq \mathcal{O}_{\mathbb{K}}$ is an arbitrary prime ideal, we note that it lies over a unique prime number $q \in \mathbb{Z}$, namely the positive generator of the prime ideal $\iota^{-1}(\mathfrak{q}) \subseteq \mathbb{Z}$. Hence, by the first part of the lemma, $\mathfrak{q}$ divides the ideal $q\mathcal{O}_{\mathbb{K}}$, and accordingly, by the multiplicative property of the norm, $\mathbb{N}(\mathfrak{q})$ divides $\mathbb{N}(q\mathcal{O}_{\mathbb{K}}) = q^2$. Thus, if $\mathfrak{q}$ has norm equal to a power of p, then $\mathfrak{q}$ lies over p and therefore divides the ideal $p\mathcal{O}_{\mathbb{K}}$. Conversely, our considerations also imply that if $\mathfrak{q}$ lies over p, then $\mathbb{N}(\mathfrak{q}) \mid p^2$. □

3. Implications of the Ratios Conjecture

In this section, we describe the L-functions Ratios Conjecture which is due to Conrey, Farmer, and Zirnbauer ([6]), generalizing a conjecture of Farmer about the Riemann zeta function (see [8]). We show how the conjecture implies strong estimates for the 1-level density of the zeros of the family $\mathcal{F}(K)$ as $K \to \infty$.

The *Ratios Conjecture* states that a sum of ratios of (products of) L-functions evaluated at certain parameters should obey a specific asymptotic estimate. We now describe the recipe from [6] for conjecturing such an asymptotic. However, for the sake of simplicity, we do not describe the most general case possible. In the case of two L-functions in the numerator and denominator, one considers

$$Q(s, \boldsymbol{\alpha}, \boldsymbol{\gamma}; \chi) = \frac{L(s+\alpha_1, \chi)L(s+\alpha_2, \chi)}{L(s+\gamma_1, \chi)L(s+\gamma_2, \chi)},$$

where $L(s, \chi)$ is the L-function associated to a character χ and satisfying the functional equation

$$L(s, \chi) = X(s, \chi)L(1-s, \bar{\chi}).$$

The recipe is as follows:

- *Approximate functional equation for L:*
  Replace each L-function in the numerator of $Q(s, \boldsymbol{\alpha}, \boldsymbol{\gamma}; \chi)$ with the two main terms from its approximate functional equation, completely disregarding the remainder term.

- *Infinite series for* $1/L$*:*
  Replace each reciprocal L-function with its expression as an infinite series involving a suitable Möbius function.
- *Extend ranges and regroup factors:*
  Extending the ranges of all series to infinity and multiplying out the resulting expression, write each of the four resulting terms as

  $$\text{(product of root numbers } \varepsilon_\chi) \times (\text{product of } X(\cdot,\chi)\text{-factors}) \\ \times (\text{sum over } n_1, n_2, n_3, n_4),$$

  where n_1, n_2, n_3, n_4 are the indexing variables from the two approximate functional equations and the two infinite series giving the reciprocal L-functions.
- *Average factors over the family:*
  Replace each product of root numbers, each product of $X(\cdot,\chi)$-factors, and each summand in the last factor with their respective averages over the family $\mathcal{F} = \{\chi\}$. Denote the resulting expression by $M(s, \boldsymbol{\alpha}, \boldsymbol{\gamma})$.
- *Statement of the conjecture:*
  The conjecture now states that for any $\varepsilon > 0$,

  $$\sum_{\chi\in\mathcal{F}} Q(s, \boldsymbol{\alpha}, \boldsymbol{\gamma}; \chi) w(q(\chi)) \\ = \sum_{\chi\in\mathcal{F}} M(s, \boldsymbol{\alpha}, \boldsymbol{\gamma}) w(q(\chi)) \Big(1 + O\Big(e^{(-1/2+\varepsilon)q(\chi)}\Big)\Big),$$

  where $q(\chi) := |X'(1/2, \chi)|$ denotes the *log conductor* of χ, and w is a suitable weight function.

A later addition to the Ratios Conjecture [7, Section 2] is that such an asymptotic is expected to hold provided that α and γ satisfy the constraints

$$-\frac{1}{4} < \operatorname{Re}(\alpha) < \frac{1}{4}, \quad \frac{1}{\log K} \ll \operatorname{Re}(\gamma) < \frac{1}{4}, \quad \operatorname{Im}(\alpha), \operatorname{Im}(\gamma) \ll_\varepsilon K^{1-\varepsilon}. \tag{3.1}$$

3.1. Ingredients for the Recipe. To follow the recipe outlined above in the case of the expression

$$R_K(\alpha, \gamma) = \frac{1}{K} \sum_{k=1}^{K} \frac{L_k(1/2+\alpha)}{L_k(1/2+\gamma)}, \tag{3.2}$$

we first need to describe the approximate functional equation for L_k and obtain an expression for the reciprocal L_k^{-1} as an infinite series.

We begin by describing the approximate functional equation, which involves writing L_k in a different way that is more reminiscent of classical Dirichlet L-functions. To this end, we observe that

$$L_k(s) = \sum_{\substack{I \subseteq \mathcal{O}_{\mathbb{K}} \\ I \neq 0}} \frac{\psi_k(I)}{\mathbb{N}(I)^s} = \sum_{n \geqslant 1} \left(\sum_{\mathbb{N}(I)=n} \psi_k(I) \right) n^{-s} = \sum_{n \geqslant 1} \frac{A_k(n)}{n^s}$$

for $\mathrm{Re}(s) > 1$, with

$$A_k(n) := \sum_{\mathbb{N}(I)=n} \psi_k(I), \qquad n \geqslant 1.$$

We note that A_k is real-valued. Indeed, this follows from the definition of ψ_k and the fact that for any $\alpha \in \mathcal{O}_{\mathbb{K}}$,

$$\mathbb{N}(\langle \alpha \rangle) = \alpha \overline{\alpha} = \mathbb{N}(\langle \overline{\alpha} \rangle).$$

Hence, any ideal that contributes to the sum defining $A_k(n)$ is accompanied by its conjugate ideal, provided that these are different. (If they are not, and $\langle \alpha \rangle = \langle \overline{\alpha} \rangle$, then α's contribution to A_k is clearly real.) Complex conjugation of A_k therefore amounts to permuting the terms in the sum, which of course leaves A_k unchanged.

Using the series defining L_k and the functional equation (2.4), we can now describe the approximate functional equation of L_k as

$$\begin{aligned} L_k(s) &\approx \sum_{n<x} \frac{A_k(n)}{n^s} + X_k(s) \sum_{n<y} \frac{\overline{A_k(n)}}{n^{1-s}} \\ &= \sum_{n<x} \frac{A_k(n)}{n^s} + X_k(s) \sum_{n<y} \frac{A_k(n)}{n^{1-s}}, \end{aligned} \tag{3.3}$$

where x and y are positive real parameters.

As mentioned above, the recipe of the Ratios Conjecture also requires us to obtain a formula for the reciprocal function L_k^{-1}. Taking the reciprocal of the Euler product and using the fact that ψ_k and the ideal norm are completely multiplicative, we see that

$$\begin{aligned} \frac{1}{L_k(s)} &= \prod_{\mathfrak{p}} \left(1 - \frac{\psi_k(I)}{\mathbb{N}(I)^s} \right) \\ &= 1 - \sum_{n=\mathfrak{p}_1} \frac{\psi_k(\mathfrak{p}_1)}{\mathbb{N}(\mathfrak{p}_1)^s} + \sum_{n=\mathfrak{p}_1\mathfrak{p}_2} \frac{\psi_k(\mathfrak{p}_1\mathfrak{p}_2)}{\mathbb{N}(\mathfrak{p}_1\mathfrak{p}_2)^s} - \sum_{n=\mathfrak{p}_1\mathfrak{p}_2\mathfrak{p}_3} \frac{\psi_k(\mathfrak{p}_1\mathfrak{p}_2\mathfrak{p}_3)}{\mathbb{N}(\mathfrak{p}_1\mathfrak{p}_2\mathfrak{p}_3)^s} + \cdots \\ &= \sum_{\substack{I \subseteq \mathcal{O}_{\mathbb{K}} \\ I \neq 0}} \frac{\mu(I)\psi_k(I)}{\mathbb{N}(I)^s}, \end{aligned}$$

where μ is the natural ideal analogue of the Möbius function,

$$\mu(I) = \begin{cases} (-1)^n & \text{if } I \text{ is the product of } n \text{ distinct primes,} \\ 0 & \text{otherwise.} \end{cases}$$

If we define the function μ_k by

$$\mu_k(n) := \sum_{\mathbb{N}(I)=n} \mu(I)\psi_k(I),$$

we therefore obtain the formula

$$\frac{1}{L_k(s)} = \sum_{n\geqslant 1} \frac{\mu_k(n)}{n^s}, \qquad \mathrm{Re}(s) > 1. \tag{3.4}$$

Lemma 3.1. *For any $k \geqslant 1$, the functions μ_k and A_k are multiplicative.*

Proof. It suffices to show that if $p \neq q$ are rational primes and $a, b \geqslant 1$ are rational integers, then

$$\mu_k\left(p^a q^b\right) = \mu_k(p^a)\mu_k\left(q^b\right), \qquad A_k\left(p^a q^b\right) = A_k(p^a)A_k\left(q^b\right).$$

We prove the claim for μ_k as the proof of the claim for A_k is analogous.

Since ψ_k and μ are multiplicative, we have

$$\mu_k(p^a)\mu_k\left(q^b\right) = \sum_{\substack{\mathbb{N}(I_1)=p^a \\ \mathbb{N}(I_2)=q^b}} \mu(I_1 I_2)\psi_k(I_1 I_2).$$

It remains to show that all ideals I of norm $p^a q^b$ have the form $I_1 I_2$ for ideals I_1 and I_2 of norm p^a and q^b, respectively. However, this follows immediately from the unique factorization of ideals in $\mathcal{O}_{\mathbb{K}}$ in combination with the fact that $\mathbb{N}$ is multiplicative, and that it only assumes the value of a prime power at a product of ideals lying over the prime in question. □

3.2. Following the Recipe. In accordance with the recipe outlined above, we now use (3.3) and (3.4) and compute

$$\frac{L_k(1/2+\alpha)}{L_k(1/2+\gamma)} \approx \sum_{m,n\geqslant 1} \frac{\mu_k(m)A_k(n)}{m^{1/2+\gamma}n^{1/2+\alpha}} + X_k(1/2+\alpha) \sum_{m,n\geqslant 1} \frac{\mu_k(m)A_k(n)}{m^{1/2+\gamma}n^{1/2-\alpha}}, \tag{3.5}$$

where we extended the ranges of summation to infinity in accordance with the recipe provided above. We now average the individual factors. We begin with the following lemma.

Lemma 3.2. *As $K \to \infty$, we have*

$$\frac{1}{K}\sum_{k=1}^{K} X_k(1/2+\alpha) = \frac{1}{1-2\alpha}\left(\frac{2\pi}{KN\sqrt{|D|}}\right)^{2\alpha} + O_\alpha\left(\frac{1}{K} + \frac{1}{K^{1+2\alpha}}\right). \tag{3.6}$$

Proof. We can argue exactly as in the proof of [19, Lemma 3.1]. The only thing we need to take into account is that we have $\mathbf{k} = 1/2 + Nk$ in the notation of that paper. □

It remains for us to compute the average of the summands in (3.5). Here it is fruitful to rewrite the sums appearing there as products in order to take advantage of the multiplicative nature of the function μ_k. To that end, with the help of Lemma 3.1 we note that

$$\sum_{m,n\geqslant 1} \frac{\mu_k(m)A_k(n)}{m^{1/2+\gamma}n^{1/2+\alpha}} = \prod_{p \text{ prime}} \sum_{m,n\geqslant 0} \frac{\mu_k(p^m)A_k(p^n)}{p^{m(1/2+\gamma)}p^{n(1/2+\alpha)}}. \tag{3.7}$$

At this point, we will describe the values of μ_k and A_k on prime powers more precisely.

Lemma 3.3. *Let p be a rational prime. Then*

$$\mu_k(p^m) = \begin{cases} 1 & \textit{if } m = 0, \\ -A_k(p) & \textit{if } m = 1, \\ -1 & \textit{if } m = 2 \textit{ and } p \textit{ is inert}, \\ 1 & \textit{if } m = 2 \textit{ and } p \textit{ splits}, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. The first two claims follow immediately from the definition of μ_k. As for the third claim, we have $\mu_k(p^2) = -1$ since only the prime ideal $\langle p\rangle$ has norm p^2 by Lemma 2.5.

Turning to the fourth claim, let us suppose that $\langle p\rangle$ has prime factors $\mathfrak{p}_1$ and $\mathfrak{p}_2$. In this case, there is only one squarefree ideal of norm p^2, namely $\langle p\rangle$, since only the prime ideals dividing $\langle p\rangle$ have norms equal to a power of p by Lemma 2.5.

We now prove the fifth and final claim. We first examine the case where $m = 2$ and p is ramified in $\mathcal{O}_{\mathbb{K}}$ with $\langle p\rangle = \mathfrak{q}^2$ for some prime ideal $\mathfrak{q}$. This immediately implies that $\mu_k(p^2) = 0$ since the only ideal of norm p^2 is $\mathfrak{q}^2$ by Lemma 2.5. For the final case, we assume $m \geqslant 3$. To prove the claim, it is enough to show that the norm of a product of distinct prime ideals lying over p is at most p^2. However, this is clear from Lemma 2.5 once we consider the three possible splitting behaviours of p. □

We also need the following description of the function A_k on prime powers.

Lemma 3.4. *A_k assumes the following values on prime powers (where we understand that the case $n=0$ takes precedence over the remaining cases):*

$$A_k(p^n)=\begin{cases}1 & \text{if } n=0,\\ 1 & \text{if } p \text{ is ramified},\\ 1 & \text{if } p \text{ is inert and } n \text{ is even},\\ 0 & \text{if } p \text{ is inert and } n \text{ is odd},\\ \sum_{j=-n/2}^{n/2}\psi_k(q)^{2j} & \text{if } \langle p\rangle=\langle q\rangle\langle\bar{q}\rangle \text{ and } n \text{ is even},\\ \sum_{j=-(n+1)/2}^{(n-1)/2}\psi_k(q)^{2j+1} & \text{if } \langle p\rangle=\langle q\rangle\langle\bar{q}\rangle \text{ and } n \text{ is odd}.\end{cases}$$

Proof. Since only $\langle 1\rangle=\mathcal{O}_{\mathbb{K}}$ has norm 1, the first claim follows immediately from the definition. In the following we suppose that $n\geqslant 1$.

Suppose that p is ramified in $\mathcal{O}_{\mathbb{K}}$ with $\langle p\rangle=\mathfrak{q}^2$, and that $\mathfrak{q}=\langle q\rangle$. Then Lemma 2.5 immediately implies that $q/\bar{q}\in\mathcal{O}_{\mathbb{K}}^{\times}$, and that the only ideal of norm p^n is $\mathfrak{q}^n$. By definition of N we therefore have $A_k(p^n)=(q^n/\bar{q}^n)^{Nk}=1$.

Suppose that p is inert. Then there is no prime ideal of norm p. Moreover, the ideal $\langle p\rangle$ is prime and has norm p^2, and it is the only such prime ideal. It follows that if I is any ideal of norm p^n, then n must be even, and in this case $I=\langle p\rangle^{n/2}=\langle p^{n/2}\rangle$. This yields the third and fourth claims.

Turning to the final claim, we assume that p splits as $q\bar{q}$ in $\mathcal{O}_{\mathbb{K}}$. Then clearly $\langle q\rangle$ and $\langle\bar{q}\rangle$ are the only prime ideals of norm p. Moreover, as we assumed that p is not ramified, it is clear that $\langle q\rangle\neq\langle\bar{q}\rangle$. Similarly as before, we also see that no prime ideals of norm p^2 can exist. It follows that if I has norm p^n, then $I=\langle q\rangle^j\langle\bar{q}\rangle^{n-j}=\langle q^j\bar{q}^{n-j}\rangle$ for some $j=0,\dots,n$. Thus,

$$A_k(p^n)=\sum_{j=0}^{n}\left(\frac{q^j\bar{q}^{n-j}}{\bar{q}^jq^{n-j}}\right)^{Nk}=\sum_{j=0}^{n}\left(\frac{q^{2j-n}}{\bar{q}^{2j-n}}\right)^{Nk},$$

and therefore

$$A_k(p^n)=\begin{cases}\sum_{j=-n/2}^{n/2}\psi_k(q)^{2j} & \text{if } n \text{ is even},\\ \sum_{j=-(n+1)/2}^{(n-1)/2}\psi_k(q)^{2j+1} & \text{if } n \text{ is odd}.\end{cases}$$

This concludes the proof. □

In accordance with the Ratios Conjecture, we now compute the (asymptotic) averages of the function $\mu_k(p^m)A_k(p^n)$ as $k=1,\dots,K$. We will denote this by $\delta_p(m,n)$ so that

$$\delta_p(m,n):=\lim_{K\to\infty}\frac{1}{K}\sum_{k=1}^{K}\mu_k(p^m)A_k(p^n).$$

The existence of this limit will be clear from the consideration of the special cases of p (split, inert, or ramified). In anticipation of this, we furthermore write

$$\delta_p(m,n) = \begin{cases} \delta_{\mathrm{in}}(m,n) & \text{if } p \text{ is inert} \\ \delta_{\mathrm{sp}}(m,n) & \text{if } p \text{ splits} \\ \delta_{\mathrm{ram}}(m,n) & \text{if } p \text{ is ramified.} \end{cases}$$

Lemma 3.5. *We have*

$$\delta_{\mathrm{in}}(m,n) = \begin{cases} 0 & \textit{if } m=1 \textit{ or } m \geqslant 3 \textit{ or } n \textit{ is odd,} \\ 1 & \textit{if } m=0 \textit{ (and } n \textit{ is even),} \\ -1 & \textit{if } m=2 \textit{ (and } n \textit{ is even),} \end{cases}$$

$$\delta_{\mathrm{sp}}(m,n) = \begin{cases} 1 & \textit{if } m=0 \textit{ and } n \textit{ is even,} \\ -2 & \textit{if } m=1 \textit{ and } n \textit{ is odd,} \\ 1 & \textit{if } m=2 \textit{ and } n \textit{ is even,} \\ 0 & \textit{otherwise,} \end{cases}$$

$$\delta_{\mathrm{ram}}(m,n) = \begin{cases} 1 & \textit{if } m=0, \\ -1 & \textit{if } m=1, \\ 0 & \textit{if } m \geqslant 2. \end{cases}$$

Proof. This follows immediately from Lemma 3.3 and Lemma 3.4. □

Using Lemma 3.5, we can describe the limiting average of the right-hand side of (3.7) as follows. If the prime p is inert, we write

$$G_{\mathrm{in}}(p;\alpha,\gamma) := \sum_{m,n\geqslant 0} \frac{\delta_{\mathrm{in}}(m,n)}{p^{m(1/2+\gamma)+n(1/2+\alpha)}},$$

and we define $G_{\mathrm{sp}}(p;\alpha,\gamma)$ and $G_{\mathrm{ram}}(p;\alpha,\gamma)$ analogously. By using Lemma 3.5, we then deduce that

$$\begin{aligned} G_{\mathrm{in}}(p;\alpha,\gamma) &= \sum_{n\geqslant 0} p^{-n(1+2\alpha)} - p^{-(1+2\gamma)} \sum_{n\geqslant 0} p^{-n(1+2\alpha)} \\ &= \left(1 - p^{-(1+2\gamma)}\right)\left(1 - p^{-(1+2\alpha)}\right)^{-1}, \\ G_{\mathrm{sp}}(p;\alpha,\gamma) &= \left(1 - 2p^{-(1+\alpha+\gamma)} + p^{-(1+2\gamma)}\right)\left(1 - p^{-(1+2\alpha)}\right)^{-1}, \\ G_{\mathrm{ram}}(p;\alpha,\gamma) &= \sum_{n\geqslant 0}\left(p^{-n(1/2+\alpha)} - p^{-(1/2+\gamma)} p^{-n(1/2+\alpha)}\right) \\ &= \left(1 - p^{-(1/2+\gamma)}\right)\left(1 - p^{-(1/2+\alpha)}\right)^{-1}. \end{aligned}$$

Moreover, we let

$$F_2(\alpha,\gamma) := \begin{cases} 1, & \text{if } d=2,\\ \Big(1-2^{-(\alpha+\gamma)}+2^{-(1+2\gamma)}\Big)\Big(1-2^{-(1+2\alpha)}\Big)^{-1} & \text{if } d=7,\\ \Big(1-2^{-(1+2\gamma)}\Big)\Big(1-2^{-(1+2\alpha)}\Big)^{-1} & \text{otherwise.} \end{cases}$$

We now see that the product of $G_*(p;\alpha,\gamma)$ over all rational primes equals

$$\begin{aligned} (3.8)\quad G(\alpha,\gamma) &= F_2(\alpha,\gamma)\, G_{\mathrm{ram}}(d;\alpha,\gamma) \prod_{\substack{p\geqslant 3\\ (-d/p)=1}} G_{\mathrm{sp}}(p;\alpha,\gamma) \prod_{\substack{p\geqslant 3\\ (-d/p)=-1}} G_{\mathrm{in}}(p;\alpha,\gamma)\\ &= \Big(1-d^{-(1/2+\gamma)}\Big)\Big(1-d^{-(1/2+\alpha)}\Big)^{-1} F_2(\alpha,\gamma)\\ &\quad\times \prod_{\substack{p\geqslant 3\\ (-d/p)=1}} \Big(1-2p^{-(1+\alpha+\gamma)}+p^{-(1+2\gamma)}\Big)\\ &\quad\times \prod_{\substack{p\geqslant 3\\ (-d/p)=-1}} \Big(1-p^{-(1+2\gamma)}\Big) \prod_{\substack{p\geqslant 3\\ p\neq d}} \Big(1-p^{-(1+2\alpha)}\Big)^{-1}\\ &= \Big(1-\mathbb{1}(d\geqslant 3)\cdot d^{-(1+2\alpha)}\Big)\Big(1-d^{-(1/2+\gamma)}\Big)\\ &\quad\times \Big(1-d^{-(1/2+\alpha)}\Big)^{-1} F_2(\alpha,\gamma)\\ &\quad\times \prod_{\substack{p\geqslant 3\\ (-d/p)=1}} \Big(1-2p^{-(1+\alpha+\gamma)}+p^{-(1+2\gamma)}\Big)\\ &\quad\times \prod_{\substack{p\geqslant 3\\ (-d/p)=-1}} \Big(1-p^{-(1+2\gamma)}\Big) \prod_{p\geqslant 3} \Big(1-p^{-(1+2\alpha)}\Big)^{-1}\\ &= \Big(1+d^{-(1/2+\alpha)}\Big)\Big(1-d^{-(1/2+\gamma)}\Big)\widetilde{F}_2(\alpha,\gamma)\\ &\quad\times \prod_{\substack{p\geqslant 3\\ (-d/p)=1}} \Big(1-2p^{-(1+\alpha+\gamma)}+p^{-(1+2\gamma)}\Big)\\ &\quad\times \prod_{\substack{p\geqslant 3\\ (-d/p)=-1}} \Big(1-p^{-(1+2\gamma)}\Big)\zeta(1+2\alpha), \end{aligned}$$

when $\mathrm{Re}(\alpha) > 0$. Here $\widetilde{F}_2$ is the function obtained by possibly removing the factor $\left(1 - 2^{-(1+2\alpha)}\right)^{-1}$ from F_2, i.e.

$$\widetilde{F}_2(\alpha, \gamma) := \begin{cases} 1 & \text{if } d = 2, \\ 1 - 2^{-(\alpha+\gamma)} + 2^{-(1+2\gamma)} & \text{if } d = 7, \\ 1 - 2^{-(1+2\gamma)} & \text{otherwise.} \end{cases}$$

It will be convenient to introduce convergence factors in the two Euler products and thus bundle together all singularities in a number of zeta functions and Dirichlet L-functions. Doing so, and writing $\chi(p) = (-d/p)$, we find that our expression (3.8) equals

$$\begin{aligned} &\frac{\zeta(1+2\alpha)L(1+2\gamma,\chi)}{\zeta(1+\alpha+\gamma)L(1+\alpha+\gamma,\chi)}\left(1 + d^{-(1/2+\alpha)}\right)\left(1 - d^{-(1/2+\gamma)}\right) \\ &\qquad \times \left(1 - d^{-(1+\alpha+\gamma)}\right)^{-1} H_2(\alpha,\gamma) \\ &\qquad \times \prod_{\substack{p \geqslant 3 \\ (-d/p)=1}} \frac{\left(1 - 2p^{-(1+\alpha+\gamma)} + p^{-(1+2\gamma)}\right)\left(1 - p^{-(1+2\gamma)}\right)}{\left(1 - p^{-(1+\alpha+\gamma)}\right)^2} \\ &\qquad \times \prod_{\substack{p \geqslant 3 \\ (-d/p)=-1}} \frac{\left(1 - p^{-(1+2\gamma)}\right)\left(1 + p^{-(1+2\gamma)}\right)}{\left(1 - p^{-(1+\alpha+\gamma)}\right)\left(1 + p^{-(1+\alpha+\gamma)}\right)}, \end{aligned}$$

where the function H_2, defined by

$$H_2(\alpha,\gamma) = \begin{cases} 1 & \text{if } d = 2, \\ \left(1 - 2^{-(1+2\gamma)}\right)\left(1 - 2^{-(1+\alpha+\gamma)}\right)^{-2} & \\ \quad \times \left(1 - 2^{-(\alpha+\gamma)} + 2^{-(1+2\gamma)}\right) & \text{if } d = 7, \\ \left(1 - 2^{-2(1+2\gamma)}\right)\left(1 - 2^{-2(1+\alpha+\gamma)}\right)^{-1} & \text{otherwise,} \end{cases}$$

keeps track of the contributions from the prime $p = 2$ when $d \neq 2$. Indeed; for example, in case of the L-function $L(1+2\gamma,\chi)$, we see that all of its factors are cancelled out by reciprocal factors in the two Euler products, except for factors corresponding to the primes $p = 2$ and $p = d$. If $d = 2$, then no such factor is missing as $\chi(2) = 0$, and if $d \geqslant 3$, only the factor corresponding to $p = 2$ is missing, namely the factor $1 - \chi(2)2^{-(1+2\gamma)}$.

If we let

$$A_1(\alpha,\gamma) := \prod_{\substack{p\geqslant 3\\ (-d/p)=1}} \frac{\Big(1-2p^{-(1+\alpha+\gamma)}+p^{-(1+2\gamma)}\Big)\Big(1-p^{-(1+2\gamma)}\Big)}{\big(1-p^{-(1+\alpha+\gamma)}\big)^2},$$

$$A_{-1}(\alpha,\gamma) := \prod_{\substack{p\geqslant 3\\ (-d/p)=-1}} \frac{\Big(1-p^{-(1+2\gamma)}\Big)\Big(1+p^{-(1+2\gamma)}\Big)}{\big(1-p^{-(1+\alpha+\gamma)}\big)\big(1+p^{-(1+\alpha+\gamma)}\big)},$$

$$F_d(\alpha,\gamma) := \frac{\Big(1+d^{-(1/2+\alpha)}\Big)\Big(1-d^{-(1/2+\gamma)}\Big)}{1-d^{-(1+\alpha+\gamma)}},$$

we therefore have

$$G(\alpha,\gamma) = \frac{\zeta(1+2\alpha)L(1+2\gamma,\chi)}{\zeta(1+\alpha+\gamma)L(1+\alpha+\gamma,\chi)} \times F_d(\alpha,\gamma)H_2(\alpha,\gamma)A_1(\alpha,\gamma)A_{-1}(\alpha,\gamma). \tag{3.9}$$

Combining this with (3.6), we see that the prediction of the Ratios Conjecture is that with α and γ subject to the conditions (3.1), the average $R_K(\alpha,\gamma)$ is equal to

$$R_K(\alpha,\gamma) = G(\alpha,\gamma) + \frac{1}{1-2\alpha}\left(\frac{2\pi}{KN\sqrt{|D|}}\right)^{2\alpha} G(-\alpha,\gamma) + O_\varepsilon\Big(K^{-1/2+\varepsilon}\Big). \tag{3.10}$$

3.3. The Logarithmic Derivative. If we know that the effective estimate (3.10) is invariant under differentiation (in the sense that the derivatives of the main terms are equal, at least up to an error of roughly the same order of magnitude), we can use (3.10) to describe an asymptotic for the average of the logarithmic derivative of L_k over the family $\mathcal{F}(K)$. Since the logarithmic derivative L_k'/L_k is intimately connected with the zeros and poles of L_k by Cauchy's residue theorem, such an asymptotic estimate will allow us to study the asymptotics of the one-level density of the family $\mathcal{F}(K)$.

To see that (3.10) is, in fact, invariant under differentiation in the sense above, let $\Omega \subseteq \mathbb{C}$ be any open set where the function

$$\alpha \mapsto f(\alpha,\gamma) := R_K(\alpha,\gamma) - G(\alpha,\gamma) - \frac{1}{1-2\alpha}\left(\frac{2\pi}{KN\sqrt{|D|}}\right)^{2\alpha} G(-\alpha,\gamma)$$

is holomorphic. For $\alpha_0 \in \Omega$, choose $\delta > 0$ so that Ω contains the circle C centered at α_0 with radius δ. Then, if we write $f_\alpha = \partial f/\partial\alpha$ and apply

Cauchy's integral formula,

$$\begin{aligned}\left|f_\alpha(\alpha_0,\gamma)\right| &\leqslant \frac{1}{2\pi}\int_C\left|\frac{f(z,\gamma)}{(z-\alpha_0)^2}\right||\mathrm{d}z| \\ &\leqslant \frac{1}{2\pi}\frac{1}{\delta^2}\cdot 2\pi\delta\cdot\sup_{z\in C}|f(z,\gamma)| \\ &= \frac{1}{\delta}\cdot O_\varepsilon\Big(K^{-1/2+\varepsilon}\Big),\end{aligned}$$

and the claim follows by linearity of differentiation.

Since only the numerators in the sum defining $R_K(\alpha,\gamma)$ depend on α, (3.10) and the estimate obtained above by differentiating (3.10) therefore imply that

$$\begin{aligned}&\frac{1}{K}\sum_{k=1}^{K}\frac{L_k'(1/2+r)}{L_k(1/2+r)} \\ &= \frac{\partial}{\partial\alpha}R_K(\alpha,\gamma)\Big|_{\alpha=\gamma=r} \\ &= \frac{\partial}{\partial\alpha}G(\alpha,\gamma)\Big|_{\alpha=\gamma=r} + \frac{\partial}{\partial\alpha}\frac{1}{1-2\alpha}\left(\frac{2\pi}{KN\sqrt{|D|}}\right)^{2\alpha}G(-\alpha,\gamma)\Big|_{\alpha=\gamma=r} \\ &\qquad + \frac{\partial}{\partial\alpha}f(\alpha,\gamma)\Big|_{\alpha=\gamma=r} \\ &= \frac{\partial}{\partial\alpha}G(\alpha,\gamma)\Big|_{\alpha=\gamma=r} + \frac{\partial}{\partial\alpha}\frac{1}{1-2\alpha}\left(\frac{2\pi}{KN\sqrt{|D|}}\right)^{2\alpha}G(-\alpha,\gamma)\Big|_{\alpha=\gamma=r} \\ &\qquad + O_\varepsilon\Big(K^{-1/2+\varepsilon}\Big)\end{aligned} \tag{3.11}$$

for any $r\in\mathbb{C}$ satisfying the conditions

$$\frac{1}{\log K} \ll \mathrm{Re}(r) < 1/4, \qquad \mathrm{Im}(r) \ll_\varepsilon K^{1-\varepsilon}. \tag{3.12}$$

By going through a computation identical to that in the proof of [19, Lemma 3.4] and observing that $F_d(r,r) = H_2(r,r) = A_1(r,r) = A_{-1}(r,r) = 1$, we now see that

$$\begin{aligned}\frac{\partial}{\partial\alpha}G(\alpha,\gamma)\Big|_{\alpha=\gamma=r} &= \frac{\zeta'(1+2r)}{\zeta(1+2r)} - \frac{L'(1+2r,\chi)}{L(1+2r,\chi)} - \frac{d^{r+1/2}\log d}{d^{2r+1}-1} \\ &\qquad + H_2'(r) - 2\sum_{\substack{p\geqslant 3\\ (-d/p)=-1}}\frac{\log p}{p^{4r+2}-1},\end{aligned}$$

where

$$H_2'(r) := \frac{\partial}{\partial\alpha} H_2(\alpha,\gamma)\Big|_{\alpha=\gamma=r} = \begin{cases} 0 & \text{if } d = 2,\, 7 \\ -2\log 2\Big(2^{2(2r+1)} - 1\Big)^{-1} & \text{otherwise.} \end{cases}$$

It remains to compute the other partial derivative in (3.11). However, in the resulting sum only the term coming from differentiating the quotient of zeta- and L-functions survives on account of the pole of $\zeta(s)$ at $s = 1$. Thus, with

$$\begin{aligned} F_d(-r,r) &= 1 + \frac{d^{1/2+r} - d^{1/2-r}}{d-1}, \\ H_2(-r,r) &= \begin{cases} 1 & \text{if } d = 2, \\ 2^{1-2r}\big(1 - 2^{-1-2r}\big) & \text{if } d = 7, \\ \frac{4}{3}\Big(1 - 2^{-2(2r+1)}\Big) & \text{otherwise,} \end{cases} \\ A_1(-r,r) &= \prod_{\substack{p\geqslant 3 \\ (-d/p)=1}} \frac{\Big(1 - 2p^{-1} + p^{-(1+2r)}\Big)\Big(1 - p^{-(1+2r)}\Big)}{(1-p^{-1})^2}, \\ A_{-1}(-r,r) &= \prod_{\substack{p\geqslant 3 \\ (-d/p)=-1}} \frac{1 - p^{-2(1+2r)}}{1 - p^{-2}}, \end{aligned}$$

we conclude that

$$\begin{aligned} J(r) &:= \frac{\partial}{\partial\alpha} \frac{1}{1-2\alpha} \left(\frac{2\pi}{KN\sqrt{|D|}}\right)^{2\alpha} G(-\alpha,\gamma)\Big|_{\alpha=\gamma=r} \\ &= -\frac{\zeta(1-2r)L(1+2r,\chi)}{L(1,\chi)} \frac{1}{1-2r} \left(\frac{2\pi}{KN\sqrt{|D|}}\right)^{2r} \\ &\qquad \times F_d(-r,r)H_2(-r,r)A_1(-r,r)A_{-1}(-r,r). \end{aligned}$$

We sum up this discussion in the following proposition.

Proposition 3.6. *The Ratios Conjecture implies that for any $r \in \mathbb{C}$ satisfying* (3.12)*, we have*

$$\begin{aligned} \frac{1}{K}\sum_{k=1}^{K} \frac{L_k'(1/2+r)}{L_k(1/2+r)} &= \frac{\zeta'(1+2r)}{\zeta(1+2r)} - \frac{L'(1+2r,\chi)}{L(1+2r,\chi)} - \frac{d^{r+1/2}\log d}{d^{2r+1}-1} + H_2'(r) \\ &\qquad - 2\sum_{\substack{p\geqslant 3 \\ (-d/p)=-1}} \frac{\log p}{p^{4r+2}-1} + J(r) + O_\varepsilon\Big(K^{-1/2+\varepsilon}\Big). \end{aligned}$$

We now let $f:\mathbb{R}\to\mathbb{R}$ be an even Schwartz function with supp $\widehat{f}$ compact. By the argument given in the beginning of [19, Section 4], the above result allows us to express the 1-level density $D(\mathcal{F}(K);f)$ (conditionally on the Ratios Conjecture) as

$$\frac{1}{2\pi i}\int_{(c)}\frac{1}{K}\left(\sum_{k=1}^{K}\left(2\cdot\frac{L_k'(1/2+r)}{L_k(1/2+r)}-\frac{X_k'(1/2+r)}{X_k(1/2+r)}\right)\right)f\left(\frac{ir\log K}{\pi}\right)\mathrm{d}r \\ = S_X+S_\zeta+S_L+S_{A'}+S_J+S_d+S_H+O_\varepsilon\left(K^{-1/2+\varepsilon}\right), \tag{3.13}$$

where c is any real number satisfying $1/\log K<c<1/4$, and

$$S_X:=-\frac{1}{2K\log K}\int_{(C)}\sum_{k=1}^{K}\frac{X_k'(1/2+\pi i\tau/\log K)}{X_k(1/2+\pi i\tau/\log K)}f(\tau)\,\mathrm{d}\tau, \tag{3.14}$$

$$S_\zeta:=\frac{1}{\log K}\int_{(C)}\frac{\zeta'(1+2\pi i\tau/\log K)}{\zeta(1+2\pi i\tau/\log K)}f(\tau)\,\mathrm{d}\tau, \tag{3.15}$$

$$S_L:=-\frac{1}{\log K}\int_{(C)}\frac{L'(1+2\pi i\tau/\log K,\chi)}{L(1+2\pi i\tau/\log K,\chi)}f(\tau)\,\mathrm{d}\tau, \tag{3.16}$$

$$S_{A'}:=-\frac{2}{\log K}\int_{(C)}\sum_{\substack{p\geqslant 3\\(-d/p)=-1}}\frac{\log p}{p^{4\pi i\tau/\log K+2}-1}f(\tau)\,\mathrm{d}\tau, \tag{3.17}$$

$$S_J:=\frac{1}{\log K}\int_{(C)}J\left(\frac{\pi i\tau}{\log K}\right)f(\tau)\,\mathrm{d}\tau, \tag{3.18}$$

$$S_d:=-\frac{\log d}{\log K}\int_{(C)}\frac{d^{\pi i\tau/\log K+1/2}}{d^{2\pi i\tau/\log K+1}-1}f(\tau)\,\mathrm{d}\tau, \tag{3.19}$$

$$S_H:=\frac{1}{\log K}\int_{(C)}H_2'\left(\frac{\pi i\tau}{\log K}\right)f(\tau)\,\mathrm{d}\tau, \tag{3.20}$$

where we denoted by (C) the set of all τ with $\mathrm{Im}(\tau)=-c\log K/\pi$.

In order to determine explicitly the prediction of the one-level density $D(\mathcal{F}(K);f)$ offered by the Ratios Conjecture, our next goal will be to provide estimates of each of these integrals in terms of f and the relevant parameters of our family $\mathcal{F}(K)$. This is the point of the next section.

4. Computations of the Integrals (3.14)–(3.20)

In the following we will allow all implicit constants to depend on the test function f. We start by recalling and adapting some results from [19].

Lemma 4.1. *Let B be a positive integer. As $K \to \infty$, we have*

$$S_\zeta = -\frac{f(0)}{2} - \sum_{j=1}^{B} \frac{c_j \widehat{f}^{(j-1)}(0)}{(\log K)^j} + O_B\bigg(\frac{1}{(\log K)^{B+1}}\bigg),$$

where $c_1, c_2, c_3, \dots$ are numbers defined in [19, eq. (5.4), (5.5)]. *In particular,*

$$c_1 = 1 + \int_1^\infty \frac{\psi(t) - t}{t^2}\,\mathrm{d}t = -\gamma,$$

where γ denotes the Euler–Mascheroni constant and $\psi(t) = \sum_{n \leqslant t} \Lambda(n)$ is the second Chebyshev function.

Proof. The asymptotic expression for S_ζ is simply [19, Lemma 5.2], so we only need to prove that

$$\int_1^\infty \frac{\psi(t) - t}{t^2}\,\mathrm{d}t = -\gamma - 1. \tag{4.1}$$

We begin by rewriting the integral as

$$\int_1^\infty \frac{\psi(t) - t}{t^2}\,\mathrm{d}t = \lim_{T\to\infty} \int_1^T \frac{\psi(t) - t}{t^2}\,\mathrm{d}t = \lim_{T\to\infty} \int_1^T \frac{\psi(t)}{t^2}\,\mathrm{d}t - \log T. \tag{4.2}$$

To evaluate the integral of $\psi(t)/t^2$, we let $\{q_k : k \geqslant 1\}$ denote the sequence of prime powers in increasing order, and we write $p_k := \exp(\Lambda(q_k))$ for the unique prime dividing q_k. It will also be convenient to write $\Pi_k := p_1 \cdot p_2 \cdots p_k$. Then, using that $\psi(t)$ is constantly equal to $\log \Pi_k = \psi(q_k)$ on the interval $[q_k, q_{k+1})$, we compute that for any large prime power q_M,

$$\begin{aligned}
\int_1^{q_M} \frac{\psi(t)}{t^2}\,\mathrm{d}t &= \sum_{k=1}^{M} \log \Pi_k \int_{q_k}^{q_{k+1}} \frac{1}{t^2}\,\mathrm{d}t = \sum_{k=1}^{M} \log \Pi_k \bigg(\frac{1}{q_k} - \frac{1}{q_{k+1}}\bigg) \\
&= \frac{\log \Pi_1}{q_1} - \frac{\log \Pi_M}{q_{M+1}} + \sum_{k=2}^{M-1} \frac{1}{q_k}\Big(\log \Pi_k - \log \Pi_{k-1}\Big) \\
&= -\frac{\log \Pi_M}{q_{M+1}} + \sum_{k=1}^{M-1} \frac{\log p_k}{q_k} = -\frac{\psi(q_M)}{q_{M+1}} + \sum_{n=1}^{q_{M-1}} \frac{\Lambda(n)}{n}.
\end{aligned}$$

Inserting this into (4.2) and replacing T by q_M, we have

$$\begin{aligned}
&\int_1^\infty \frac{\psi(t) - t}{t^2}\,\mathrm{d}t \\
&\quad = \lim_{M\to\infty} \bigg(-\frac{\psi(q_M)}{q_M} \cdot \frac{q_M}{q_{M+1}} + \sum_{n=1}^{q_{M-1}} \frac{\Lambda(n)}{n} - \log q_{M-1} + \log \frac{q_{M-1}}{q_M}\bigg).
\end{aligned} \tag{4.3}$$

Here we can absorb the $\log q_{M-1}$ term into the sum that precedes it by using the familiar asymptotic for the harmonic numbers in the form

$$\log k = -\gamma + 1 + \frac{1}{2} + \cdots + \frac{1}{k} + O\Big(\frac{1}{k}\Big).$$

Observing also that the prime number theorem implies $q_M/q_{M+1} \to 1$ and $\psi(q_M)/q_M \to 1$ as $M \to \infty$, we obtain from (4.3) that

$$\int_1^\infty \frac{\psi(t)-t}{t^2}\,\mathrm{d}t = -1+\gamma+\sum_{n=1}^{\infty}\frac{\Lambda(n)-1}{n}$$
$$= -1+\gamma+\lim_{s\to 1^+}\Big(-\frac{\zeta'(s)}{\zeta(s)}-\zeta(s)\Big) = -1-\gamma,$$

where we also used that $\lim_{s\to 1^+} -\zeta'(s)/\zeta(s)-\zeta(s) = -2\gamma$. □

Although our Dirichlet L-function $L(s,\chi)$ is defined in terms of the character $\chi(\,\cdot\,) = (-d/\cdot)$ and not the non-principal character modulo 4 as in [19, Lemma 5.4], we note that this result, including the statement about $S_{A'}$, remains valid in our case with exactly the same proof. We record these two results in the following two separate lemmas.

Lemma 4.2 ([19, Lemma 5.4])**.** *As $K \to \infty$, we have*

$$S_L = -\frac{\widehat{f}(0)}{\log K}\frac{L'(1,\chi)}{L(1,\chi)} + O\Big(\frac{1}{(\log K)^2}\Big).$$

Lemma 4.3 ([19, Lemma 5.4])**.** *As $K \to \infty$, we have*

$$S_{A'} = -\frac{2\widehat{f}(0)}{\log K}\sum_{\substack{p\geqslant 3\\(-d/p)=-1}}\frac{\log p}{p^2-1} + O\Big(\frac{1}{(\log K)^2}\Big).$$

Moreover, we can adapt [19, Lemma 5.1] to our situation and obtain the following result.

Lemma 4.4. *As $K \to \infty$, we have*

$$S_X = \widehat{f}(0)\left(1+\frac{\log\sqrt{|D|}-\log 2\pi+\log N-1}{\log K}\right) + O\Big(\frac{1}{K}\Big).$$

Proof. We have

$$\frac{X_k'(s)}{X_k(s)} = \frac{\mathrm{d}}{\mathrm{d}s}\log\Gamma(1-s+Nk) - \frac{\mathrm{d}}{\mathrm{d}s}\log\Gamma(s+Nk) - \log|D| + 2\log 2\pi.$$

The contribution of the constants to the integral (3.14) is

$$\frac{1}{\log K}\widehat{f}(0)\Big(\log\sqrt{|D|}-\log 2\pi\Big).$$

As for the contribution of the gamma functions, we recall that, in the notation of [19], we have $\mathbf{k} = 1/2+Nk$. It then follows immediately from [19, eq. (5.1)], that this contribution equals

$$\frac{1}{K\log K}\widehat{f}(0)\sum_{k=1}^{K}\log\mathbf{k} + O\Big(\frac{1}{K}\Big).$$

By arguing as in [19, eq. (5.2)], we find that

$$\begin{aligned}\sum_{k=1}^{K} \log \mathbf{k} = \sum_{k=1}^{K} \log(1/2 + Nk) &= K \log N + \log K! + O(\log K)\\ &= K \log N + K \log K - K + O(\log K)\\ &= K \log K + K(\log N - 1) + O(\log K).\end{aligned}$$

This proves the claim. □

We now turn to the integral (3.18). As Lemma 4.6 below will show, the asymptotic expression for this integral will involve a special value of the logarithmic derivative of $L(s,\chi)$. In anticipation of this, we will first compute this special value.

Lemma 4.5. *Let η denote Dedekind's eta function,*

$$\eta(\tau) = e^{\pi i\tau/12} \prod_{n\geqslant 1}\Big(1 - e^{2\pi i n\tau}\Big), \qquad \mathrm{Im}(\tau) > 0,$$

and let γ denote the Euler–Mascheroni constant. Then we have

$$\frac{L'(1,\chi)}{L(1,\chi)} = \gamma - \log 2 - (\log|D|)/2 - \log \mathrm{Im}(\tau_0) - 4\log|\eta(\tau_0)|$$

where $\tau_0 := i\sqrt{2}$ if $d = 2$, or otherwise $\tau_0 := (-1 + i\sqrt{d})/2$.

Proof. If $\zeta_{\mathbb{K}}(s)$ denotes the Dedekind zeta function of $\mathbb{K}$, [5, Proposition 10.5.5] gives the factorization $\zeta_{\mathbb{K}}(s) = \zeta(s)L(s,\chi)$. Indeed, this is clear from the discussion in Section 2.3. On the other hand, since any non-zero ideal $\mathfrak{m} \subseteq \mathcal{O}_{\mathbb{K}}$ has exactly $|\mathcal{O}_{\mathbb{K}}^{\times}|$ generators, we also have

$$\zeta_{\mathbb{K}}(s) = \frac{1}{|\mathcal{O}_{\mathbb{K}}^{\times}|} \sum_{N(a,b)\neq 0} \frac{1}{N(a,b)^s}, \qquad \mathrm{Re}(s) > 1,$$

where $N(a,b)$ denotes the norm of the element $j(a,b)$ as defined in Section 2.2. It now follows from this and [5, Corollary 10.4.8] that

$$\begin{aligned}\zeta(s)L(\chi,s) &= \frac{1}{|\mathcal{O}_{\mathbb{K}}^{\times}|} \sum_{N(a,b)\neq 0} \frac{1}{N(a,b)^s}\\ &= \frac{2}{|\mathcal{O}_{\mathbb{K}}^{\times}|}\frac{\pi}{\sqrt{|D|}}\Big(\frac{1}{s-1} + C(d) + O(s-1)\Big),\end{aligned} \tag{4.4}$$

where $C(d) := 2\gamma - \log 2 - (\log|D|)/2 - \log \mathrm{Im}(\tau_0) - 4\log|\eta(\tau_0)|$. Furthermore, by writing L and ζ as Laurent series around $s = 1$, we find that

$$\begin{aligned}L(s,\chi) &= L(1,\chi) + L'(1,\chi)(s-1) + O\Big((s-1)^2\Big),\\ \zeta(s) &= \frac{1}{s-1} + \gamma + O(s-1),\end{aligned}$$

and hence

$$\zeta(s)L(s,\chi) = \frac{L(1,\chi)}{s-1} + L'(1,\chi) + \gamma L(1,\chi) + O(s-1). \tag{4.5}$$

Together, the two different expressions (4.4) and (4.5) for $\zeta(s)L(s,\chi)$ now force an equality of coefficients, namely

$$L(1,\chi) = \frac{2\pi}{|\mathcal{O}_{\mathbb{K}}^{\times}|\sqrt{|D|}}, \qquad L'(1,\chi) = \frac{2\pi(C(d)-\gamma)}{|\mathcal{O}_{\mathbb{K}}^{\times}|\sqrt{|D|}}.$$

By computing the quotient $L'(1,\chi)/L(1,\chi)$, we obtain the claim. □

Lemma 4.6. *As $K \to \infty$, we have*

$$S_J = \frac{f(0)}{2} - \frac{1}{2}\int_{\mathbb{R}} \widehat{f}(\tau)\mathbb{1}_{[-1,1]}(\tau)\,\mathrm{d}\tau + \frac{\widehat{f}(1)}{\log K}\ell_1 + O_d\Big(\frac{1}{(\log K)^2}\Big),$$

where

$$\ell_1 = \frac{L'(1,\chi)}{L(1,\chi)} + 2\sum_{\substack{p\geqslant 3\\ (-d/p)=-1}} \frac{\log p}{p^2-1} + \log\frac{2\pi e}{N\sqrt{|D|}} + \frac{\sqrt{d}\log d}{d-1} - \gamma - \frac{2a\log 2}{3},$$

$a = -\mathbb{1}(d \neq 2, 7)$, and $L'(1,\chi)/L(1,\chi)$ is given in Lemma 4.5.

Proof. We proceed as in the proof of [19, Lemma 5.5], which relies on the methods of [9]. That is, we will replace the domain of integration (C) with the union of the compact and non-compact contours $C_1 \cup C_\eta$ and C_0, respectively, where

$$\begin{aligned} C_0 &:= \{\tau \in \mathbb{C} : \mathrm{Im}(\tau) = 0,\ |\mathrm{Re}(\tau)| \geqslant (\log K)^{\varepsilon}\}, \\ C_1 &:= \{\tau \in \mathbb{C} : \mathrm{Im}(\tau) = 0,\ \eta \leqslant |\mathrm{Re}(\tau)| \leqslant (\log K)^{\varepsilon}\}, \\ C_\eta &:= \Big\{\eta e^{i\theta} : -\pi \leqslant \theta \leqslant 0\Big\}. \end{aligned}$$

The utility of such a decomposition is two-fold: On the non-compact part C_0 we can bound the integrand using the rapid decay of our test function f. On the other hand, on the compact part we can estimate the individual factors in the integrand with the first few terms of their Taylor expansions. Moreover, the fact that $C_1 \cup C_\eta$ tends towards a symmetric subset of the real line when we let $\eta \to 0$ means that we do not have to take into account contributions from any odd, positive powers of τ in these expansions, as the integrals of $\tau f(\tau)$, $\tau^3 f(\tau)$, $\tau^5 f(\tau)$, … over this set vanish. We exploit this fact to get rid of any occurrences of ε in the error terms. We now proceed to the details.

Initially, we recall the following elementary estimates: If γ denotes the Euler–Mascheroni constant, then as $K \to \infty$,

$$\zeta\Big(1 - \frac{2\pi i\tau}{\log K}\Big) = -\frac{\log K}{2\pi i\tau} + \gamma + O\Big(\frac{|\tau|}{\log K}\Big), \tag{4.6}$$

$$\frac{L(1 + 2\pi i\tau/\log K, \chi)}{L(1,\chi)} = 1 + \frac{L'(1,\chi)}{L(1,\chi)}\frac{2\pi i\tau}{\log K} + O\bigg(\frac{|\tau|^2}{(\log K)^2}\bigg), \tag{4.7}$$

and

$$\begin{aligned} &A_1\Big(-\frac{\pi i\tau}{\log K}, \frac{\pi i\tau}{\log K}\Big)A_{-1}\Big(-\frac{\pi i\tau}{\log K}, \frac{\pi i\tau}{\log K}\Big) \\ &\qquad = 1 + \Bigg(2\sum_{\substack{p\geqslant 3\\ (-d/p)=-1}} \frac{\log p}{p^2-1}\Bigg)\frac{2\pi i\tau}{\log K} + O\bigg(\frac{|\tau|^2}{(\log K)^2}\bigg). \end{aligned} \tag{4.8}$$

We now obtain Taylor expansions of the other factors in J. First of all,

$$\begin{aligned} &\frac{1}{1-2r}\bigg(\frac{2\pi}{KN\sqrt{|D|}}\bigg)^{2r}\bigg|_{r=\frac{\pi i\tau}{\log K}} \\ &= \exp\bigg(\frac{2\pi i\tau}{\log K}\log\frac{2\pi}{N\sqrt{|D|}} - 2\pi i\tau\bigg)\bigg(1 + \frac{2\pi i\tau}{\log K} + O\bigg(\frac{|\tau|^2}{(\log K)^2}\bigg)\bigg) \\ &= e^{-2\pi i\tau}\bigg(1 + \frac{2\pi i\tau}{\log K}\log\frac{2\pi}{N\sqrt{|D|}} + O\bigg(\frac{|\tau|^2(\log|D|)^2}{(\log K)^2}\bigg)\bigg) \\ &\qquad \times \bigg(1 + \frac{2\pi i\tau}{\log K} + O\bigg(\frac{|\tau|^2}{(\log K)^2}\bigg)\bigg) \\ &= e^{-2\pi i\tau} + e^{-2\pi i\tau}\frac{2\pi i\tau}{\log K}\log\frac{2\pi e}{N\sqrt{|D|}} + O\bigg(\frac{|\tau|^2(\log|D|)^2}{(\log K)^2}\bigg), \end{aligned} \tag{4.9}$$

whenever K is so large that

$$\bigg|\frac{2\pi i\tau}{\log K}\bigg| < 1, \qquad \bigg|\frac{2\pi i\tau}{\log K}\log\frac{2\pi}{N\sqrt{|D|}}\bigg| < 1.$$

(Note that this is certainly satisfied for $\tau \in C_1 \cup C_\eta$.) Next, we see that

$$F_d\Big(-\frac{\pi i\tau}{\log K}, \frac{\pi i\tau}{\log K}\Big) = 1 + \frac{\sqrt{d}\log d}{d-1}\frac{2\pi i\tau}{\log K} + O_d\bigg(\frac{|\tau|^2}{(\log K)^2}\bigg). \tag{4.10}$$

Finally, we also record the bound

$$H_2\Big(-\frac{\pi i\tau}{\log K}, \frac{\pi i\tau}{\log K}\Big) = 1 - \frac{2a\log 2}{3}\frac{2\pi i\tau}{\log K} + O\bigg(\frac{|\tau|^2}{(\log K)^2}\bigg). \tag{4.11}$$

By taking the product of all the series expansions (4.6)–(4.11) and disregarding all those resulting terms whose order of magnitude is at least $\tau/\log K$, we obtain that

$$\begin{aligned}
J\Big(\frac{\pi i\tau}{\log K}\Big) &= \Big(\frac{1}{x}-\gamma+O(|x|)\Big)\Big(e^{-2\pi i\tau}+x\Big(e^{-2\pi i\tau}\frac{L'(1,\chi)}{L(1,\chi)} \\
&\qquad + 2e^{-2\pi i\tau}\sum_{\substack{p\geqslant 3\\(-d/p)=-1}}\frac{\log p}{p^2-1}+e^{-2\pi i\tau}\log\frac{2\pi e}{N\sqrt{|D|}} \\
&\qquad + e^{-2\pi i\tau}\frac{\sqrt{d}\log d}{d-1}-e^{-2\pi i\tau}\frac{2a\log 2}{3}\Big)+O_d\big(|x|^2\big)\Big) \\
&= \frac{e^{-2\pi i\tau}}{x}+e^{-2\pi i\tau}\Big(-\gamma+\frac{L'(1,\chi)}{L(1,\chi)}+2\sum_{\substack{p\geqslant 3\\(-d/p)=-1}}\frac{\log p}{p^2-1} \\
&\qquad +\log\frac{2\pi e}{N\sqrt{|D|}}+\frac{\sqrt{d}\log d}{d-1}-\frac{2a\log 2}{3}\Big)+O_d(|x|),
\end{aligned}$$

where we wrote $x=2\pi i\tau/\log K$ for simplicity. Now, as we described at the beginning of the proof, if we had retained even the linear terms in x and truncated the Laurent series expansion of $J(\pi i\tau/\log K)$ at its first order term, thus obtaining an error of size $O_d(|x|^2)$, the first order term in x would not contribute to the integral of $f(\tau)J(\pi i\tau/\log K)$ over $C_1\cup C_\eta$. Rather, we would obtain the same asymptotic expression for S_J as we do below, but with an error term of magnitude $O_d((\log K)^{-3+\varepsilon})=O_d((\log K)^{-2})$. Because of this trick, we therefore obtain that

$$\begin{aligned}
(4.12)\quad S_J &= \int_{C_1\cup C_\eta} f(\tau)\frac{e^{-2\pi i\tau}}{2\pi i\tau}\,\mathrm{d}\tau+\frac{1}{\log K}\Big(-\gamma+\frac{L'(1,\chi)}{L(1,\chi)} \\
&\quad +2\sum_{\substack{p\geqslant 3\\(-d/p)=-1}}\frac{\log p}{p^2-1}+\log\frac{2\pi e}{N\sqrt{|D|}}+\frac{\sqrt{d}\log d}{d-1}-\frac{2a\log 2}{3}\Big) \\
&\quad \times\int_{C_1\cup C_\eta} f(\tau)e^{-2\pi i\tau}\,\mathrm{d}\tau+\frac{1}{\log K}\int_{C_0} J\Big(\frac{\pi i\tau}{\log K}\Big)f(\tau)\,\mathrm{d}\tau \\
&\quad +O_d\Big(\frac{1}{(\log K)^2}\Big).
\end{aligned}$$

By arguing exactly as in [19, Lemma 5.5], we relate the integrals above to special values of the Fourier transform $\widehat{f}$, namely

$$\int_{C_1\cup C_\eta} f(\tau)e^{-2\pi i\tau}\,\mathrm{d}\tau = \widehat{f}(1) + O\Big(\frac{1}{(\log K)^3}\Big), \tag{4.13}$$

$$\int_{C_1\cup C_\eta} f(\tau)\frac{e^{-2\pi i\tau}}{2\pi i\tau}\,\mathrm{d}\tau = \frac{f(0)}{2} - \frac{1}{2}\int_{-1}^{1}\widehat{f}(\tau)\,\mathrm{d}\tau + O\Big(\frac{1}{(\log K)^3}\Big). \tag{4.14}$$

Since the rapid decay of f on $\mathbb{R}$ shows that the integral over C_0 in (4.12) is at most a constant times $(\log K)^{-2}$ (for example), it now follows from (4.13) and (4.14) that

$$S_J = \frac{f(0)}{2} - \frac{1}{2}\int_{-1}^{1}\widehat{f}(\tau)\,\mathrm{d}\tau + \frac{\widehat{f}(1)}{\log K}\Bigg(\frac{L'(1,\chi)}{L(1,\chi)} + 2\sum_{\substack{p\geqslant 3\\(-d/p)=-1}}\frac{\log p}{p^2-1} + \log\frac{2\pi e}{N\sqrt{|D|}} + \frac{\sqrt{d}\log d}{d-1} - \gamma - \frac{2a\log 2}{3}\Bigg) + O_d\Big(\frac{1}{(\log K)^2}\Big),$$

which completes the proof. □

We now turn to the final two integrals (3.19) and (3.20).

Lemma 4.7. *As $K\to\infty$, we have the estimate*

$$S_d = -\frac{\log d}{\log K}\frac{\sqrt{d}}{d-1}\widehat{f}(0) + O_d\Big(\frac{1}{(\log K)^2}\Big).$$

Proof. The integrand only has poles when $\mathrm{Im}(\tau) = (\log K)/2\pi > 0$, so analogously to the proof of Lemma 4.6, we use the Cauchy–Goursat theorem and the rapid decay of f to move the contour (C) to the real line without changing the value of the integral. As before, we partition this set into a compact and a non-compact part in order to, respectively, use the (even-indexed terms of the) Taylor expansion of the integrand and bound the integral using the decay of the test function. Concretely, we write $\mathbb{R} = C_0\cup C_1$ with

$$C_0 := \{\tau\in\mathbb{C} : \mathrm{Im}(\tau)=0,\ |\mathrm{Re}(\tau)| > (\log K)^{\varepsilon}\},$$

$$C_1 := \{\tau\in\mathbb{C} : \mathrm{Im}(\tau)=0,\ |\mathrm{Re}(\tau)| \leqslant (\log K)^{\varepsilon}\}.$$

As in the proof of Lemma 4.6, we note that

$$\int_{C_0}\frac{d^{\pi i\tau/\log K+1/2}}{d^{2\pi i\tau/\log K+1}-1}f(\tau)\,\mathrm{d}\tau \ll_d \frac{1}{(\log K)^2},$$

whereas for the integral over C_1, we use the Taylor expansion

$$\frac{d^{r+1/2}}{d^{2r+1}-1} = \frac{\sqrt{d}}{d-1} + r\cdot\frac{\mathrm{d}}{\mathrm{d}r}\Big(\frac{d^{r+1/2}}{d^{2r+1}-1}\Big)\Big|_{r=0} + O_d\big(r^2\big) \qquad (r\to 0)$$

and the fact that $\tau f(\tau)$ is odd to obtain

$$\begin{aligned}\int_{C_1} \frac{d^{\pi i\tau/\log K+1/2}}{d^{2\pi i\tau/\log K+1}-1} f(\tau)\,\mathrm{d}\tau &= \int_{C_1}\left(\frac{\sqrt{d}}{d-1}+O_d\left(\frac{\tau^2}{(\log K)^2}\right)\right) f(\tau)\,\mathrm{d}\tau\\ &= \left(\frac{\sqrt{d}}{d-1}+O_d\left(\frac{1}{(\log K)^{2-2\varepsilon}}\right)\right)\int_{C_1} f(\tau)\,\mathrm{d}\tau\\ &= \left(\frac{\sqrt{d}}{d-1}+O_d\left(\frac{1}{(\log K)^{2-2\varepsilon}}\right)\right)\\ &\quad\times\left(\int_{\mathbb{R}} f(\tau)\,\mathrm{d}\tau+O\left(\frac{1}{(\log K)^2}\right)\right),\end{aligned}$$

where we used the rapid decay of f in the last step. The claim now follows. □

Finally, we have the following asymptotic estimate. We assume that $d \neq 2, 7$ since otherwise $S_H = 0$.

Lemma 4.8. *Suppose that $d \neq 2$, 7. As $K \to \infty$, we have*

$$S_H = \frac{-2\log 2}{3\log K}\widehat{f}(0) + O\left(\frac{1}{(\log K)^2}\right).$$

Proof. The method of proof is identical to that of the previous lemma. Once again, we note that $H_2'(\pi i\tau/\log K)$ only has poles if $\mathrm{Im}(\tau) = (\log K)/2\pi > 0$, so that we are justified in moving the contour to the real line. As before, the integral over the non-compact part C_0 of our partition $C_0 \cup C_1$ of $\mathbb{R}$ simply contributes to the error term, while the integral over the compact part is

$$\int_{C_1}\left(\frac{-2\log 2}{3}+\frac{\pi i\tau}{\log K}\cdot\frac{\mathrm{d}}{\mathrm{d}r}H_2'(r)\bigg|_{r=0}+O\left(\frac{|\tau|^2}{(\log K)^2}\right)\right) f(\tau)\,\mathrm{d}\tau,$$

which follows from the Taylor expansion

$$H_2'(r) = \frac{-2\log 2}{3} + r\cdot\frac{\mathrm{d}}{\mathrm{d}r}H_2'(r)\bigg|_{r=0} + O\left(r^2\right).$$

We now proceed exactly as in the proof of the previous lemma. □

By combining the results from Lemma 4.1 to Lemma 4.8 with (3.13), we have completed the proof of Theorem 1.2.

5. An Unconditional Asymptotic for the One-Level Density

In this section, we use the following formula for logarithmic derivatives of the L-functions L_k in order to give an unconditional expression for the one-level density $D(\mathcal{F}(K); f)$.

Lemma 5.1. *For $k \geqslant 1$ and $\mathrm{Re}(s) > 1$, we have*

$$\frac{L_k'(s)}{L_k(s)} = -\sum_{n\geqslant 1} \frac{c_k(n)}{n^s}, \quad c_k(n) = \Lambda(n) \sum_{\mathbb{N}(\mathfrak{p}^m)=n} \Big(1 + \mathbb{1}(\mathfrak{p} = \langle p\rangle)\Big)\psi_k(\mathfrak{p}^m),$$

where we understand that the indicator function specifies whether or not $\mathfrak{p}$ lies over an inert rational prime.

Proof. By taking the logarithmic derivative of the (absolutely convergent) Euler product that defines $L_k(s)$ in the half-plane $\mathrm{Re}(s) > 1$, we get

$$\frac{L_k'(s)}{L_k(s)} = -\sum_{\mathfrak{p}} \log \mathbb{N}(\mathfrak{p}) \frac{\psi_k(\mathfrak{p})/\mathbb{N}(\mathfrak{p})^s}{1 - \psi_k(\mathfrak{p})/\mathbb{N}(\mathfrak{p})^s} = -\sum_{\mathfrak{p}} \log \mathbb{N}(\mathfrak{p}) \sum_{m\geqslant 1} \frac{\psi_k(\mathfrak{p}^m)}{\mathbb{N}(\mathfrak{p}^m)^s},$$

where we also used that ψ_k and the norm are completely multiplicative.

To finish the computation we note that, if $\mathfrak{p}$ lies over a ramified or split prime p, then for any $m \geqslant 1$,

$$\log \mathbb{N}(\mathfrak{p}) = \log p = \Lambda(\mathbb{N}(\mathfrak{p})) = \Lambda(\mathbb{N}(\mathfrak{p}^m)),$$

whereas if $\mathfrak{p} = \langle p\rangle$ lies over an inert prime,

$$\log \mathbb{N}(\mathfrak{p}) = 2\log p = 2\Lambda(\mathbb{N}(\mathfrak{p})) = 2\Lambda(\mathbb{N}(\mathfrak{p}^m))$$

for any $m \geqslant 1$. In light of this and the previous computation, we can rewrite the logarithmic derivative as

$$\frac{L_k'(s)}{L_k(s)} = -\sum_{\mathfrak{p}} \Big(1 + \mathbb{1}(\mathfrak{p} = \langle p\rangle)\Big) \sum_{m\geqslant 1} \frac{\psi_k(\mathfrak{p}^m)}{\mathbb{N}(\mathfrak{p}^m)^s} \Lambda(\mathbb{N}(\mathfrak{p}^m)).$$

We now obtain the claim by grouping together all $\mathfrak{p}^m$ with norm n, for all $n \geqslant 1$. □

Towards a computation of the 1-level density, we note that just as in [19, Section 6], we have

$$D(\mathcal{F}(K); f) = \frac{1}{2\pi i} \int_{(c')} \frac{1}{K} \sum_{k=1}^{K} \left(2\frac{L_k'(1/2+r)}{L_k(1/2+r)} - \frac{X_k'(1/2+r)}{X_k(1/2+r)}\right) \times f\left(\frac{ir\log K}{\pi}\right) \mathrm{d}r,$$

where now $c' \geqslant 1/2$. Using Lemma 5.1 and arguing as in [13, Theorem 5.12], we now obtain

$$\begin{aligned}
D(\mathcal{F}(K); f) &= S_X - \frac{1}{\pi i}\frac{1}{K}\int_{(c')}\sum_{k=1}^{K}\sum_{n\geqslant 1}\frac{c_k(n)}{n^{1/2+r}} f\left(\frac{ir\log K}{\pi}\right)\mathrm{d}r \\
&= S_X - \frac{1}{\pi i}\frac{1}{K}\sum_{k=1}^{K}\sum_{n\geqslant 1}\frac{c_k(n)}{\sqrt{n}}\int_{(c')} e^{-r\log n} f\left(\frac{ir\log K}{\pi}\right)\mathrm{d}r \\
&= S_X - \frac{1}{K\log K}\sum_{k=1}^{K}\sum_{n\geqslant 1}\frac{c_k(n)}{\sqrt{n}} \\
&\qquad \times \int_{\{\mathrm{Im}(r)=c'\log K/\pi\}} e^{2\pi i r\log n/2\log K} f(r)\,\mathrm{d}r \\
&= S_X - \frac{1}{K\log K}\sum_{k=1}^{K}\sum_{n\geqslant 1}\frac{c_k(n)}{\sqrt{n}}\widehat{f}\left(\frac{\log n}{2\log K}\right),
\end{aligned}$$

where we replaced $\{\mathrm{Im}(r) = c'\log K/\pi\}$ with $\mathbb{R}$ due to the rapid decay of f on horizontal strips.

The occurrence of the von Mangoldt function in $c_k(n)$ means that the only indices n contributing to the sum above are the prime powers. If we replace n with p^n (to save notation), we can then describe the resulting values of $c_k(p^n)$ depending on the splitting behavior of p. Thus, with the help of Lemma 2.5, we find that

$$c_k(p^n) = \begin{cases} (\psi_k(\mathfrak{q}_1^n) + \psi_k(\mathfrak{q}_2^n))\log p & \text{if } \langle p\rangle = \mathfrak{q}_1\mathfrak{q}_2,\ \mathfrak{q}_1 \neq \mathfrak{q}_2, \\ 2\log p & \text{if } p \text{ is inert and } n \text{ is even}, \\ 0 & \text{if } p \text{ is inert and } n \text{ is odd}, \\ \log p & \text{if } \langle p\rangle = \mathfrak{q}^2. \end{cases}$$

Considering these special values of $c_k(p^n)$, we now define

$$\begin{aligned}
S_{\mathrm{inert}} &= -\frac{2}{\log K}\sum_{p \text{ inert}}\sum_{n\geqslant 1}\frac{\log p}{p^n}\widehat{f}\left(\frac{n\log p}{\log K}\right), \\
S_{\mathrm{split}} &= -\frac{1}{K\log K}\sum_{p \text{ split}}\sum_{n\geqslant 1}\frac{\log p}{p^{n/2}}\widehat{f}\left(\frac{n\log p}{2\log K}\right)\sum_{k=1}^{K}(\psi_k(\mathfrak{q}_1^n) + \psi_k(\mathfrak{q}_2^n)), \\
S_{\mathrm{ram}} &= -\frac{1}{\log K}\sum_{n\geqslant 1}\frac{\log d}{d^{n/2}}\widehat{f}\left(\frac{n\log d}{2\log K}\right).
\end{aligned}$$

Thus,

$$D(\mathcal{F}(K); f) = S_X + S_{\mathrm{inert}} + S_{\mathrm{split}} + S_{\mathrm{ram}}. \tag{5.1}$$

We now want to compare the terms S_{inert} and S_{ram} with the terms (3.14)–(3.20) appearing in the expression of the one-level density conditional on the Ratios Conjecture. To facilitate this comparison, we begin by rewriting these terms as follows.

Lemma 5.2. *We have that*

$$S_\zeta = -\frac{1}{\log K}\sum_{n\geqslant 1}\frac{\Lambda(n)}{n}\widehat{f}\Big(\frac{\log n}{\log K}\Big), \tag{5.2}$$

$$S_{A'} = -\frac{2}{\log K}\sum_{\substack{p\geqslant 3\\ (-d/p)=-1}}\sum_{n\geqslant 1}\frac{\log p}{p^{2n}}\widehat{f}\Big(\frac{2n\log p}{\log K}\Big), \tag{5.3}$$

$$S_L = \frac{1}{\log K}\sum_{n\geqslant 1}\frac{\Lambda(n)\chi(n)}{n}\widehat{f}\Big(\frac{\log n}{\log K}\Big), \tag{5.4}$$

$$S_d = -\frac{\log d}{\sqrt{d}\log K}\sum_{n\geqslant 0}\frac{1}{d^n}\widehat{f}\Big(\frac{(1/2+n)\log d}{\log K}\Big), \tag{5.5}$$

$$S_H = \frac{a}{2}\frac{\log 2}{\log K}\sum_{n\geqslant 0}\frac{1}{4^n}\widehat{f}\Big(\frac{(2n+2)\log 2}{\log K}\Big). \tag{5.6}$$

Proof. The proofs of the equalities (5.2) and (5.3) follow immediately from the corresponding proofs in [19, Lemma 4.2], once we substitute our character χ for the character χ_1 in that paper.

As for the equality (5.4), we note the standard formula

$$\frac{L'(s,\chi)}{L(s,\chi)} = -\sum_{n\geqslant 1}\frac{\Lambda(n)\chi(n)}{n^s}, \qquad \mathrm{Re}(s) > 1.$$

Substituting this infinite sum for the logarithmic derivative appearing in the definition of S_L, we obtain the claim by changing variables and moving the contour of integration to the real line, which is justified due to the rapid decay of f in combination with the lack of poles of the integrand in the part of the complex plane enclosed by these contours.

Turning to S_d, we note that when τ has imaginary part $-c\log K/\pi$ where $c > 0$, the number $-2\pi i\tau/\log K - 1$ has real part $-2c-1 < 0$, and so we can write

$$\frac{d^{\pi i\tau/\log K+1/2}}{d^{2\pi i\tau/\log K+1}-1} = d^{-\pi i\tau/\log K-1/2}\sum_{n\geqslant 0}\Big(d^{-2\pi i\tau/\log K-1}\Big)^n.$$

Inserting this into the expression (3.19), we get that

$$S_d = -\frac{\log d}{\sqrt{d}\log K}\int_{(C)}\exp\Big(-\pi i\tau\frac{\log d}{\log K}\Big) \times \sum_{n\geqslant 0}\exp\Big(-n\log d\Big(\frac{2\pi i\tau}{\log K}+1\Big)\Big)f(\tau)\,\mathrm{d}\tau$$
$$= -\frac{\log d}{\sqrt{d}\log K}\sum_{n\geqslant 0}d^{-n}\int_{(C)}f(\tau)\exp\Big(-2\pi i\tau(1/2+n)\frac{\log d}{\log K}\Big)\,\mathrm{d}\tau,$$

which equals the claimed expression for S_d as we can move the contour (C) to the real line for the usual reasons.

Finally, we turn our attention to S_H. In the same way as before, we rewrite the function

$$H_2'(r) = \begin{cases} 0 & \text{if } d = 2, 7, \\ -2\log 2\Big(2^{2(2r+1)}-1\Big)^{-1} & \text{otherwise,} \end{cases}$$

as a geometric series. Letting $r = \pi i\tau/\log K$, we then find that

$$H_2'\Big(\frac{\pi i\tau}{\log K}\Big) = \frac{a\log 2}{2}\sum_{n\geqslant 0}4^{-n}\exp\Big(-2\pi i\tau\frac{(2n+2)\log 2}{\log K}\Big),$$

where $a = -\mathbb{1}(d \neq 2, 7)$. The equality (5.6) now follows once we substitute this expression for H_2' in (3.20) and shift the contour of integration. □

It now follows from (5.2) and (5.4) that

$$S_\zeta + S_L = \frac{1}{\log K}\sum_{n\geqslant 1}\frac{\Lambda(n)(\chi(n)-1)}{n}\widehat{f}\Big(\frac{\log n}{\log K}\Big).$$

Since those d we are interested in satisfy $d \not\equiv 1 \pmod 4$, we know that $\chi(n)$ is a quadratic Dirichlet character of modulus $4d = d^3$ (in case $d = 2$) or modulus d (otherwise). This fact means that for all prime powers n appearing in the above sum, $\chi(n) = 0$ if and only if n is a power of d. Moreover, since $\chi(n) - 1 = 0$ whenever n is the power of a split prime or an even power of an inert prime, the computation above shows that

$$S_\zeta + S_L = -\frac{\log d}{\log K}\sum_{n\geqslant 1}d^{-n}\widehat{f}\Big(\frac{n\log d}{\log K}\Big) - \frac{2}{\log K}\sum_{n\geqslant 0}\sum_{p\text{ inert}}\frac{\log p}{p^{2n+1}}\widehat{f}\Big(\frac{(2n+1)\log p}{\log K}\Big). \tag{5.7}$$

Regarding the first infinite sum, we note with the help of Lemma 5.2 that

$$(5.8)\quad -\frac{\log d}{\log K}\sum_{n\geqslant 1} d^{-n}\widehat{f}\Big(\frac{n\log d}{\log K}\Big) = S_{\mathrm{ram}} + \frac{\log d}{\log K}\sum_{n\geqslant 0} d^{-n-1/2}\widehat{f}\Big(\frac{(2n+1)\log d}{2\log K}\Big) = S_{\mathrm{ram}} - S_d.$$

Similarly, in the case of the second infinite sum, we see that

$$\begin{aligned}(5.9)\quad -\frac{2}{\log K}\sum_{n\geqslant 0}\sum_{p\ \mathrm{inert}}\frac{\log p}{p^{2n+1}}\widehat{f}\Big(\frac{(2n+1)\log p}{\log K}\Big) &= S_{\mathrm{inert}} + \frac{2}{\log K}\sum_{p\ \mathrm{inert}}\sum_{n\geqslant 1}\frac{\log p}{p^{2n}}\widehat{f}\Big(\frac{2n\log p}{\log K}\Big)\\ &= S_{\mathrm{inert}} + \frac{2}{\log K}\sum_{\substack{p\geqslant 3\\ p\ \mathrm{inert}}}\sum_{n\geqslant 1}\frac{\log p}{p^{2n}}\widehat{f}\Big(\frac{2n\log p}{\log K}\Big)\\ &\quad + \mathbb{1}(2\ \mathrm{inert})\frac{2\log 2}{\log K}\sum_{n\geqslant 1}2^{-2n}\widehat{f}\Big(\frac{2n\log 2}{\log K}\Big)\\ &= S_{\mathrm{inert}} - S_{A'} + \mathbb{1}(2\ \mathrm{inert})\frac{2\log 2}{\log K}\sum_{n\geqslant 1}2^{-2n}\widehat{f}\Big(\frac{2n\log 2}{\log K}\Big),\end{aligned}$$

again using Lemma 5.2. Since 2 is inert in $\mathbb{K}$ if and only if $d \neq 2$, 7, we see that when $d \neq 2$, 7, the last term above is

$$\begin{aligned}\frac{2\log 2}{\log K}\sum_{n\geqslant 1}2^{-2n}\widehat{f}\Big(\frac{2n\log 2}{\log K}\Big) &= \frac{4}{2}\frac{\log 2}{\log K}\sum_{n\geqslant 0}2^{-2n-2}\widehat{f}\Big(\frac{(2n+2)\log 2}{\log K}\Big)\\ &= \frac{1}{2}\frac{\log 2}{\log K}\sum_{n\geqslant 0}2^{-2n}\widehat{f}\Big(\frac{(2n+2)\log 2}{\log K}\Big)\\ &= -S_H.\end{aligned}$$

Since S_H furthermore vanishes if $d = 2$, 7, combining (5.7), (5.8), and (5.9), we therefore obtain that

$$(5.10)\qquad S_{\mathrm{inert}} + S_{\mathrm{ram}} = S_\zeta + S_L + S_{A'} + S_d + S_H.$$

It now follows from (5.1) and (3.13) that our unconditional expression for the 1-level density $D(\mathcal{F}(K); f)$ agrees with the expression conditional on the Ratios Conjecture, and hence with the Katz–Sarnak prediction (1.1), if

$$S_{\mathrm{split}} \approx S_J.$$

6. Comparison with the Katz–Sarnak Density Conjecture

The goal of this section is to unify our explicit computation with the prediction of the Katz–Sarnak Density Conjecture. As we described earlier, this goal amounts to verifying that the term S_{split} coming from the split rational primes is equal to the term S_J predicted by the Ratios Conjecture, at least up to some small error.

We now generalize the result [18, Lemma 2.1] which provides a useful relation between the angle θ_I and the norm $\mathbb{N}(I)$ of a non-zero ideal $I \subseteq \mathcal{O}_{\mathbb{K}}$.

Theorem 6.1. *Let $\Lambda = g\mathbb{Z}^2 \subseteq \mathbb{R}^2$ be a unimodular lattice with*

$$g = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}).$$

Let ℓ be a line through the origin such that the angle between the positive x-axis and ℓ is $\theta \in [0, \pi)$. If $\theta = \pi/2$ (in which case we let $q := 1$), or if $-n + a^2 \tan\theta$ is an algebraic number of degree $q \geqslant 1$, then there exists $C = C(\Lambda, \ell) > 0$ such that for every $\mathbf{v} \in \Lambda \setminus \ell$,

$$|\alpha(\mathbf{v})| \geqslant \frac{C}{\|\mathbf{v}\|^q},$$

where $\alpha(\mathbf{v})$ denotes the angle between $\mathbf{v}$ and ℓ.

Proof. Let us first note that if ℓ is the y-axis, then the claim follows easily: Indeed, the set of first coordinates of lattice points in Λ is discrete, as any $\mathbf{v} \in \Lambda$ has the form

$$\mathbf{v} = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix} \begin{pmatrix} s \\ t \end{pmatrix} = \begin{pmatrix} as \\ (ns+t)/a \end{pmatrix} \in \Lambda,$$

where $s, t \in \mathbb{Z}$. Therefore, if the angle $\alpha(\mathbf{v})$ between $\mathbf{v}$ and the y-axis is non-zero, but small, we have

$$\begin{aligned} 2|\alpha(\mathbf{v})| &\geqslant |\tan\alpha(\mathbf{v})| \\ &= \frac{|as|}{|(sn+t)/a|} \\ &\gg_a \frac{1}{|(sn+t)/a|} \\ &\geqslant \frac{1}{\sqrt{(sn+t)^2/a^2 + a^2 s^2}} \\ &= \frac{1}{\|\mathbf{v}\|}. \end{aligned}$$

We can therefore assume that $\theta \neq \pi/2$, so that $\cos\theta \neq 0$.

Similarly, if ℓ is the x-axis, Liouville's Theorem [1, Theorem 1.1] and the assumption that $-n$ is algebraic of degree q imply that $|sn+t| \gg |s|^{1-q}$ uniformly in $(s,t) \in \mathbb{Z}^2 \setminus \{\mathbf{0}\}$, and hence

$$\begin{aligned} 2|\alpha(\mathbf{v})| &\geqslant |\tan \alpha(\mathbf{v})| \\ &= \frac{|sn+t|}{a^2|s|} \\ &\gg_a \frac{1}{|a^q s^q|} \\ &\geqslant \frac{1}{\sqrt{a^{2q}s^{2q} + (sn+t)^{2q}/a^{2q}}} \\ &\geqslant \frac{1}{\|\mathbf{v}\|^q}, \end{aligned}$$

if $\alpha(\mathbf{v})$ is non-zero and small. (Note that, in particular, this implies that $s \neq 0$ and $sn+t \neq 0$.) We can therefore also assume that $\theta \neq 0$, so that $\sin\theta \neq 0$.

Let us now rotate ℓ and Λ clockwise by the angle θ, which transforms ℓ into the x-axis and Λ into the lattice

$$\Lambda' = \left\{ \begin{pmatrix} x \cdot \cos\theta + y \cdot \sin\theta \\ -x \cdot \sin\theta + y \cdot \cos\theta \end{pmatrix} : x = as,\ y = (sn+t)/a,\ s,t \in \mathbb{Z} \right\}.$$

Since the substance of the claim pertains to the situation where $\alpha(\mathbf{v}) \neq 0$ is very small, we assume that $\mathbf{v} \in \Lambda'$ is any non-zero lattice point with $\alpha(\mathbf{v})$ non-zero, but small. Under this assumption, we have the estimate

$$\begin{aligned} 2|\alpha(\mathbf{v})| &\geqslant |\tan \alpha(\mathbf{v})| \\ &= \frac{|-x \cdot \sin\theta + y \cdot \cos\theta|}{|x \cdot \cos\theta + y \cdot \sin\theta|} \\ &= \frac{|y - x \cdot \tan\theta|}{|x + y \cdot \tan\theta|} \\ &\gg_a \frac{|t - s \cdot (a^2 \tan\theta - n)|}{|x + y \cdot \tan\theta|}. \end{aligned} \tag{6.1}$$

If $s = 0$, we have $x = 0$ and $y = t/a \neq 0$, and the right-hand side of (6.1) is bounded from below (independently of $\mathbf{v}$) by $|a/\tan\theta| > 0$. We can therefore suppose that $s \neq 0$. In this case, the assumption about $a^2\tan\theta - n$ implies, by Liouville's Theorem, that there exists $C = C(\Lambda, \ell) > 0$ such that

$$\left| t - s \cdot (a^2 \tan\theta - n) \right| \geqslant \frac{C}{|s|^{q-1}}. \tag{6.2}$$

Since we have $|s|^{q-1} \ll_a \left(a^2s^2 + (sn+t)^2/a^2\right)^{(q-1)/2} = \|\mathbf{v}\|^{q-1}$ and

$$|x + y \cdot \tan\theta| \leqslant \sqrt{|x + y \cdot \tan\theta|^2 + |y - x \cdot \tan\theta|^2} = \frac{\|\mathbf{v}\|}{|\cos\theta|},$$

we obtain the claim from (6.1). □

Remark 6.2. In anticipation of the lemma below, we use Theorem 6.1 to define

$$Q := \max_{0\leqslant m\leqslant 2N-1} \left\{ q \geqslant 1 : \begin{array}{l} -n(\mathcal{O}_{\mathbb{K}}) + a(\mathcal{O}_{\mathbb{K}})^2 \tan(\pi m/(2N)) \\ \text{is algebraic of degree } q \end{array} \right\},$$

where $n(\mathcal{O}_{\mathbb{K}})$ and $a(\mathcal{O}_{\mathbb{K}})$ denote the parameters appearing in the Iwasawa decomposition of the lattice $\mathcal{O}_{\mathbb{K}}$ from Lemma 2.1. That is, Q is the largest of the degrees of all the algebraic numbers $-n(\mathcal{O}_{\mathbb{K}}) + a(\mathcal{O}_{\mathbb{K}})^2 \tan(\pi m/(2N))$, where $m = 0, \dots, 2N-1$. Note that these numbers are indeed algebraic: This follows from Lemma 2.1 and from the fact that $\tan(\pi m/2N)$ is algebraic, as $\tan(\pi m) = 0$ can be written as a quotient of two polynomials in $\tan(\pi m/2N)$ with integer coefficients.

In particular, there exists a constant $0 < c_0 < 1/4N$, which only depends on $\mathcal{O}_{\mathbb{K}}$ and N, such that for $m = 0, \dots, 2N-1$, we have $|\alpha(\mathbf{v})| \geqslant c_0/\|\mathbf{v}\|^Q$, where $\alpha(\mathbf{v})$ denotes the angle between $\ell_m := \left\{re^{i\theta} \in \mathbb{C} : \theta = \pi m/2N\right\}$ and $\mathbf{v} \in \mathcal{O}_{\mathbb{K}} \setminus \ell_m$.

We can now repeat the argument of Waxman in [19, Section 6] to prove that, at least when $\alpha := \sup\operatorname{supp}\widehat{f} < 1$, the unconditional asymptotic for the one-level density obtained above is in agreement with the prediction of the Katz–Sarnak Density Conjecture. Observe that when $\alpha < 1$, this is the case precisely if S_{split} is very small, cf. Lemma 4.6.

Lemma 6.3. *Suppose that* $\alpha < 1$. *Then* $S_{\text{split}} \ll_{\widehat{f},\varepsilon} K^{\alpha-1+\varepsilon}$.

Proof. Note that the character sum appearing in the definition of S_{split} satisfies

$$\left|\sum_{k=1}^{K} \psi_k(I)\right| = \left|\sum_{k=1}^{K} e^{2iNk\theta_I}\right| \leqslant \frac{2}{|e^{2iN\theta_I} - 1|} \tag{6.3}$$

whenever θ_I is not a multiple of π/N. Also, note that θ_I can't even be a multiple of $\pi/2N$, for in that case, if $I = \langle\beta\rangle$, we have $(\beta/\overline{\beta})^N = \pm 1$, and hence

$$\langle\beta\rangle^N = \langle\beta^N\rangle = \langle\overline{\beta}^N\rangle = \langle\overline{\beta}\rangle^N,$$

which forces $\beta \equiv \overline{\beta} \pmod{\mathcal{O}_{\mathbb{K}}^{\times}}$ due to the unique factorization of ideals in $\mathcal{O}_{\mathbb{K}}$. It therefore follows that $\langle\beta\rangle$ can't lie over a split prime, which is a contradiction.

Now, if for some $n = 0, \dots, 2N-1$,

$$\frac{n\pi}{2N} < \theta_I < \frac{(n+1)\pi}{2N}, \tag{6.4}$$

then we find that

$$\begin{aligned} \frac{2}{|e^{2iN\theta_I}-1|} &= \frac{1}{|\sin N\theta_I|} \\ &\leqslant \frac{\pi}{2N}\Big(\frac{1}{(n+1)\pi/(2N)-\theta_I} + \frac{1}{\theta_I - n\pi/(2N)}\Big). \end{aligned} \tag{6.5}$$

Indeed, by using the left- or rightmost inequality in (6.4) if n is even or odd, respectively, we can identify a representative in $(0, \pi/2)$ of $N\theta_I$ modulo the symmetries (evenness and $\frac{\pi}{N}$-periodicity) of $\theta_I \mapsto |\sin N\theta_I|$. Bounding the function $1/|\sin N\theta_I|$ independently of the parity of n, we arrive at (6.5) after applying the inequality $x/\sin x \leqslant \pi/2$ (for $0 < x \leqslant \pi/2$).

Since $\pm 1 \in \mathcal{O}_{\mathbb{K}}^\times$ for all possible values of d, we can always take $\theta_I \in (0,\pi)$ and thus define $n(\theta_I)$ to be the unique $n \in \{0, \dots, 2N-1\}$ such that (6.4) is satisfied. In combination with (6.3), ignoring the constant $\pi/2N$ in (6.5), we therefore obtain

$$\Big|\sum_{k=1}^{K} \psi_k(I)\Big| \ll \frac{1}{(n(\theta_I)+1)\pi/(2N) - \theta_I} + \frac{1}{\theta_I - n(\theta_I)\pi/(2N)}. \tag{6.6}$$

Now, from the definition of S_{split} and (6.6) it is clear that

$$\begin{aligned} S_{\text{split}} \ll \frac{1}{K\log K} \sum_{n=0}^{2N-1} \sum_{\substack{I \subseteq \mathcal{O}_{\mathbb{K}} \\ n(\theta_I)=n}} \frac{\Lambda(N(I))}{\sqrt{N(I)}} \cdot \Big|\widehat{f}\Big(\frac{\log\sqrt{N(I)}}{\log K}\Big)\Big| \\ \times \Big(\frac{1}{(n+1)\pi/(2N)-\theta_I} + \frac{1}{\theta_I - n\pi/(2N)}\Big) \end{aligned} \tag{6.7}$$

since all the powers p^m ($m \geqslant 1$) of a split prime will appear as norms of suitable ideals $I \subseteq \mathcal{O}_{\mathbb{K}}$ with $\theta_I \neq 0$.

Since $\mathcal{O}_{\mathbb{K}}$ is a two-dimensional lattice in $\mathbb{C}$, the basic idea is that the right-hand side of (6.7) can be estimated rather sharply by replacing it with an integral over certain parts of the ambient complex plane, where a change to polar coordinates will simplify the integrand greatly. Concretely, these domains of integration will be annulus sectors S_I indexed by $I \subseteq \mathcal{O}_{\mathbb{K}}$. As such, in order to describe them, we only need to specify for each I an angular interval and a radial interval.

To define the angular intervals, we take an arbitrary ideal $I \subseteq \mathcal{O}_{\mathbb{K}}$ and consider the interval

$$\mathcal{J}_I := \Big(\frac{n(\theta_I)\pi}{2N} + \frac{c_0}{\mathbb{N}(I)^{Q/2}}, \frac{(n(\theta_I)+1)\pi}{2N} - \frac{c_0}{\mathbb{N}(I)^{Q/2}}\Big).$$

By (6.4) and the remark following the proof of Theorem 6.1, we have $\theta_I \in \mathcal{J}_I$. As the length of $\mathcal{J}_I$ is asymptotically equal to $\pi/2N$ (as $\mathbb{N}(I) \to \infty$), there exists a constant $0 < c_1 < (\sqrt{3} - \sqrt{2})/2$, which is independent of I, such that for all $I \subseteq \mathcal{O}_{\mathbb{K}}$,

$$\frac{c_1}{\sqrt{\mathbb{N}(I)}} < \text{length}(\mathcal{J}_I).$$

For each I we may therefore choose an interval $(a_I, b_I) \subseteq \mathcal{J}_I$ that has length $b_I - a_I = c_1/\sqrt{\mathbb{N}(I)}$ and contains θ_I. The interval (a_I, b_I) is the angular interval of S_I.

To specify the radial interval, we choose a constant c_2 such that $0 < c_2 < (\sqrt{3} - \sqrt{2} - 2c_1)/4$. The radial interval of S_I will then be $(\sqrt{\mathbb{N}(I)} - c_2, \sqrt{\mathbb{N}(I)} + c_2)$. In conclusion, the annulus sector S_I is now given by

$$S_I = \left\{ re^{i\theta} \in \mathbb{C} : a_I < \theta < b_I, \text{ and } \sqrt{\mathbb{N}(I)} - c_2 < r < \sqrt{\mathbb{N}(I)} + c_2 \right\}.$$

By construction, the collection $\{S_I : I \subseteq \mathcal{O}_{\mathbb{K}} \text{ is an ideal}\}$ has the following properties:

(i) All the sets S_I have the same area.
(ii) For any two different ideals $I_1, I_2 \subseteq \mathcal{O}_{\mathbb{K}}$, we have $S_{I_1} \cap S_{I_2} = \varnothing$.

Indeed, the area of S_I is given by the integral

$$\int_{a_I}^{b_I} \int_{\sqrt{\mathbb{N}(I)}-c_2}^{\sqrt{\mathbb{N}(I)}+c_2} r \,\mathrm{d}r \,\mathrm{d}\theta = \frac{b_I - a_I}{2} \cdot 4c_2\sqrt{\mathbb{N}(I)} = 2c_1c_2.$$

Furthermore, in order to show pairwise disjointness, it is enough to show that each S_I is contained within the disk with center $\sqrt{\mathbb{N}(I)}e^{i\theta_I}$ and radius $(\sqrt{3} - \sqrt{2})/2$, since this number is a lower bound for the distance between generators of different prime ideals. On the other hand, to demonstrate this containment we observe that the "diagonal" of S_I has length

$$\left| \left(\sqrt{\mathbb{N}(I)} - c_2 \right) e^{ib_I} - \left(\sqrt{\mathbb{N}(I)} + c_2 \right) e^{ia_I} \right|$$
$$= \left| \left(\sqrt{\mathbb{N}(I)} - c_2 \right) e^{i(b_I - a_I)} - \sqrt{\mathbb{N}(I)} - c_2 \right|.$$

Using the triangle inequality and estimating the chord length $\left| e^{i(b_I - a_I)} - 1 \right|$ with the arc length $b_I - a_I$, we find that this is at most

$$\sqrt{\mathbb{N}(I)} \cdot \left| e^{i(b_I - a_I)} - 1 \right| + 2c_2$$
$$\leqslant c_1 + 2c_2 < c_1 + \frac{\sqrt{3} - \sqrt{2} - 2c_1}{2} = \frac{\sqrt{3} - \sqrt{2}}{2},$$

and our second claim follows as well.

Continuing our estimate of S_{split}, as all the sets S_I have the same area, we now bound the sum appearing in (6.7) by approximating each term with its average over the appropriate sector S_I. We therefore find that

$$\begin{aligned}
&\frac{1}{K\log K}\sum_{n=0}^{2N-1}\sum_{\substack{I\subseteq\mathcal{O}_{\mathbb{K}}\\ n(\theta_I)=n}}\frac{\Lambda(\mathbb{N}(I))}{\sqrt{\mathbb{N}(I)}}\left|\widehat{f}\left(\frac{\log\sqrt{\mathbb{N}(I)}}{\log K}\right)\right|\\
&\qquad\times\left(\frac{1}{(n+1)\pi/(2N)-\theta_I}+\frac{1}{\theta_I-n\pi/(2N)}\right)\\
&\quad\ll\frac{1}{K\log K}\sum_{n=0}^{2N-1}\sum_{\substack{I\subseteq\mathcal{O}_{\mathbb{K}}\\ n(\theta_I)=n}}\int_{S_I}\log r\left|\widehat{f}\left(\frac{\log r}{\log K}\right)\right|\cdot\frac{1}{(n+1)\pi/2N-\theta}\,\mathrm{d}r\,\mathrm{d}\theta\\
&\quad\ll\frac{1}{K\log K}\sum_{n=0}^{2N-1}\int_1^\infty\log r\left|\widehat{f}\left(\frac{\log r}{\log K}\right)\right|\\
&\qquad\times\int_{n\pi/2N+c_0/r^Q}^{(n+1)\pi/2N-c_0/r^Q}\frac{1}{(n+1)\pi/2N-\theta}\,\mathrm{d}\theta\,\mathrm{d}r\\
&\quad\ll\frac{1}{K\log K}\int_1^\infty\log r\left|\widehat{f}\left(\frac{\log r}{\log K}\right)\right|\int_{c_0/r^Q}^{\pi/2N-c_0/r^Q}\frac{1}{\theta}\,\mathrm{d}\theta\,\mathrm{d}r,
\end{aligned}$$

where we used Lemma 6.1 and the remark following it to bound the angular parameter θ in terms of r. Continuing, we obtain from (6.7) and this estimate that

$$\begin{aligned}
S_{\text{split}}&\ll_Q\frac{1}{K\log K}\int_1^\infty(\log r)^2\left|\widehat{f}\left(\frac{\log r}{\log K}\right)\right|\mathrm{d}r\\
&\ll_{\widehat{f}}\frac{1}{K\log K}\int_1^{K^\alpha}(\log r)^2\,\mathrm{d}r\ll K^{\alpha-1}\log K.
\end{aligned}$$

This concludes the proof. □

By the remark after (5.10), we have therefore proved Theorem 1.1.

It would be satisfactory to give a reason why S_{split} and S_J would have anything to do with each other, so that the approximate equality given by Lemma 6.3 is not just “accidental”. It seems difficult to give any such reason due to the arithmetic complexity of the term S_J. However, we can make the following observation, which at least serves to tidy up the S_{split} term to some extent. Namely, the character sum appearing in S_{split} is expressible (cf. Lemma 6.4 below) in terms of the *Dirichlet kernel*

$$D_K(x)=\frac{\sin(Kx+x/2)}{\sin x/2}=1+2\sum_{k=1}^{K}\cos(kx).$$

As a consequence, we can rewrite S_{split} as

$$S_{\text{split}} = \frac{1}{\log K} \sum_{(-d/p)=1} \sum_{n \geqslant 1} \frac{\log p}{p^{n/2}} \widehat{f}\left(\frac{n \log p}{2 \log K}\right) \frac{1 - D_K(2Nn\theta_p)}{K},$$

where $\theta_p \in (0, \pi)$ denotes the argument of one of the generators of either of the prime ideals lying over the split prime p. In light of this expression, it seems that a better understanding of the angles θ_p is crucial if one wishes to understand the relation $S_{\text{split}} \approx S_J$ on a deeper level.

We conclude this final section by justifying our claim above about the character sum.

Lemma 6.4. *Suppose $\mathfrak{q}_1$ and $\mathfrak{q}_2$ are different prime ideals in $\mathcal{O}_{\mathbb{K}}$ lying over a rational prime p, and let $n \geqslant 1$. Then we have*

$$\sum_{k=1}^{K} \Big(\psi_k(\mathfrak{q}_1^n) + \psi_k(\mathfrak{q}_2^n)\Big) = -1 + D_K(2Nn\theta_{\mathfrak{q}_1}),$$

where $\theta_{\mathfrak{q}_1} \in (0, \pi)$ denotes the argument of a generator of $\mathfrak{q}_1$ in the upper half-plane.

Proof. Suppose that $z = a + ib$ is any generator of $\mathfrak{q}_1$. Since $\bigcap_d \mathcal{O}_{\mathbb{K}}^{\times} = \mathbb{Z}/2\mathbb{Z}$, we can assume that $b > 0$. Moreover, since the generator of the ideal $\mathfrak{q}_1\mathfrak{q}_2$ is a rational prime, the conjugate $\bar{z} = a - ib$ must be a generator of $\mathfrak{q}_2$. Therefore,

$$\begin{aligned}
\psi_k(\mathfrak{q}_1^n) + \psi_k(\mathfrak{q}_2^n) &= (z/\bar{z})^{Nnk} + (\bar{z}/z)^{Nnk} \\
&= \exp\big(Nnk \log(z/\bar{z})\big) + \exp\big(-Nnk \log(z/\bar{z})\big) \\
&= 2\cos\big(i \cdot Nnk \log(z/\bar{z})\big) \\
&= 2\cos\left(i \cdot Nnk \log \frac{a/b + i}{a/b - i}\right) \\
&= 2\cos\left(2Nnk \cdot \frac{1}{2i} \log \frac{a/b - i}{a/b + i}\right) \\
&= 2\cos\big(2Nnk \cdot \arctan(b/a)\big),
\end{aligned}$$

where we used the identity $\arctan 1/x = i/2 \cdot \log((x - i)/(x + i))$ (for $x \neq 0$). Moreover, since the angle $\theta_{\mathfrak{q}_1}$ between 1 and $a + ib$ is given by

$$\theta_{\mathfrak{q}_1} = \begin{cases} \arctan(b/a) & \text{if } a > 0, \\ \arctan(b/a) + \pi & \text{if } a < 0, \end{cases}$$

the computation above and the periodicity of cosine show that $\psi_k(\mathfrak{q}_1^n) + \psi_k(\mathfrak{q}_2^n) = 2\cos(2Nnk\theta_{\mathfrak{q}_1})$. It now follows that

$$\sum_{k=1}^{K}(\psi_k(\mathfrak{q}_1^n) + \psi_k(\mathfrak{q}_2^n)) = 2\sum_{k=1}^{K}\cos(2Nnk\theta_{\mathfrak{q}_1}) = -1 + D_K(2Nn\theta_{\mathfrak{q}_1}),$$

which completes the proof. □

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