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A mod- p metaplectic Montréal functor

par ROBIN WITTHAUS

RÉSUMÉ. Nous étendons le foncteur de Colmez, défini pour $\mathrm{GL}_2(\mathbf{Q}_p)$, à la catégorie des représentations modulo p admissibles, lisses de type fini du revêtement double métaplectique de $\mathrm{GL}_2(\mathbf{Q}_p)$ — une extension centrale par le groupe des racines de l'unité μ_2 de \mathbf{Q} . Nous calculons les images des objets absolument irréductibles, qui sont propres, i.e. sur lesquels le sous-groupe central μ_2 agit via le caractère non-trivial, et nous obtenons une bijection entre les représentations supersingulières propres et les représentations galoisiennes irréductibles de dimensions quatre invariantes par torsion par tous les caractères d'ordre deux. Restreint aux représentations propres, le foncteur étendu est naturellement à valeurs dans la catégorie des représentations galoisiennes munies d'une certaine structure supplémentaire qui encode l'invariance par torsion, que nous appelons représentations galoisiennes métaplectiques.

ABSTRACT. We extend Colmez's functor defined for $\mathrm{GL}_2(\mathbf{Q}_p)$ to the category of finitely generated smooth admissible mod- p representations of the two-fold metaplectic cover of $\mathrm{GL}_2(\mathbf{Q}_p)$ – a central extension by the roots of unity μ_2 in \mathbf{Q} . We compute the images of the absolutely irreducible objects, which are genuine, i.e. on which the central subgroup μ_2 acts via the non-trivial character, and obtain a bijection between genuine supersingular representations and four-dimensional irreducible Galois representations invariant under twist by all characters of order two. Restricted to genuine representations, the extended functor naturally takes values in the category of what we call metaplectic Galois representations – Galois representations with a certain extra structure encoding the aforementioned twist-invariance.

1. Introduction

1.1. The p -adic and mod- p local Langlands correspondence for $\mathrm{GL}_2(\mathbf{Q}_p)$. Integral weight modular forms may be viewed as automorphic representations of $\mathrm{GL}_2(\mathbf{A}_{\mathbf{Q}})$ and, after fixing an auxiliary prime p , they have a Galois representation of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ with coefficients in an algebraic closure of \mathbf{Q}_p attached to them satisfying local-global compatibility at all finite primes; that is to say, for each finite place ℓ , the smooth complex

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representation of $\mathrm{GL}_2(\mathbf{Q}_\ell)$ appearing as the local component of the automorphic representation is determined by the restriction of the Galois representation to the decomposition group $\mathrm{Gal}(\overline{\mathbf{Q}}_\ell/\mathbf{Q}_\ell)$ at ℓ through the local Langlands correspondence after passage to the corresponding Weil–Deligne representation. If $\ell \neq p$, then the local Galois representation at ℓ can essentially be recovered from the corresponding representation of $\mathrm{GL}_2(\mathbf{Q}_\ell)$. At $\ell = p$ however, passage to the Weil–Deligne representation is not fully faithful and hence the global correspondence does not restrict to a bijective correspondence locally at p . As explained in [8], the search for a p -adic local Langlands correspondence grew out of the desire to fix this issue. It roughly speaking aims to relate irreducible admissible unitary Banach space representation of $\mathrm{GL}_2(\mathbf{Q}_p)$ defined over a finite extension of \mathbf{Q}_p to Galois representations of $\mathcal{G}_{\mathbf{Q}_p} = \mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ defined over the same finite extension.

The starting point of the p -adic and mod- p local Langlands correspondence for $\mathrm{GL}_2(\mathbf{Q}_p)$ was the classification of the smooth irreducible (admissible) $\overline{\mathbf{F}}_p$ -linear representation by Barthel–Livné [2] (working more generally for finite extensions of \mathbf{Q}_p) and Breuil [6]. Namely, Barthel and Livné were able to describe all the irreducible representations occurring as a subquotient of a parabolic induction, and, building on their approach via universal spherical Hecke modules, Breuil managed to parametrize all the other (so-called *supersingular*) representations. This parametrization allowed him to write down a bijection between the set of isomorphism classes of supersingular representations and the set of isomorphism classes of two-dimensional continuous irreducible $\overline{\mathbf{F}}_p$ -linear representations of $\mathcal{G}_{\mathbf{Q}_p}$, which already suggested that there should be a mod- p local Langlands correspondence. Motivated by the possible existence of a p -adic local Langlands correspondence, Breuil [7] computed the reduction of (the completion of) locally algebraic unramified principal series depending on small weights, which led him to the formulation of the semi-simple mod- p local Langlands correspondence as it can be found at the beginning of *loc. cit.* In [14], Colmez observes a close relation between (φ, Γ) -modules and smooth representations of the mirabolic subgroup of $\mathrm{GL}_2(\mathbf{Q}_p)$ (and thus of the standard Borel if a central character is fixed). Inspired by the work [12] of Colmez, Berger and Breuil [5] compute the unitary completion of locally algebraic principal series representations and relate it to two-dimensional crystalline Galois representations via (φ, Γ) -modules; they also prove compatibility with Breuil’s semi-simple mod- p correspondence. Colmez [13] has turned his previously mentioned observation into an exact functor from the category of finite length $\mathrm{GL}_2(\mathbf{Q}_p)$ -representations (admitting a central character) to the category of (φ, Γ) -modules, which is equivalent to the category of Galois representations of $\mathcal{G}_{\mathbf{Q}_p}$ via Fontaine’s equivalence. This functor is not only

defined for mod- p coefficients, in which case it recovers Breuil’s correspondence after semi-simplification, but also for torsion coefficients and thus for Banach space representations yielding a *functorial* approach to a p -adic Langlands correspondence. This functor has been studied in more detail ever since, see for example the works [15] and [23] of Colmez–Dospinescu–Paškūnas and Paškūnas, respectively.

1.2. Motivation and goal. Half-integral weight modular forms, on the other hand, may be viewed as genuine automorphic representations of the so-called *metaplectic cover* $\widetilde{\mathrm{GL}}_2(\mathbf{A}_{\mathbf{Q}})$ of $\mathrm{GL}_2(\mathbf{A}_{\mathbf{Q}})$ – a central extension by the roots of unity μ_2 in \mathbf{Q} , where “genuine” means that μ_2 acts non-trivially. It is natural to ask whether these analytic objects also have a Galois representation attached to them in a reasonable way. One possible answer to this question is to make use of Shimura’s correspondence ([19, 21, 26]), which assigns an integral weight modular form to one of half-integral weight in a suitable Hecke-equivariant fashion, and then pass to the corresponding global Galois representation. This is unsatisfying as it does not give any new information about Galois representations, so it seems worthwhile to look for a direct way of realizing such a metaplectic correspondence.

We would like to investigate this question from a p -adic point of view locally at p , i.e. letting $\widetilde{\mathrm{GL}}_2(\mathbf{Q}_p)$ denote the pullback of the global covering to the local group at p , we are interested in a metaplectic version of the p -adic local Langlands correspondence relating irreducible admissible unitary genuine p -adic Banach space representations of $\widetilde{\mathrm{GL}}_2(\mathbf{Q}_p)$ to certain p -adic Galois representations of $\mathcal{G}_{\mathbf{Q}_p}$. Both, p -adic Banach space representations residually of finite length and p -adic Galois representations, can be reduced mod- p (and semi-simplified to obtain something well-defined), so any p -adic correspondence will give rise to a semi-simple mod- p metaplectic correspondence (at least if $p > 2$ so that there is a non-trivial character $\mu_2 \rightarrow \mathbf{F}_p^\times$). In fact, it seems easier to first study the mod- p situation. A complete classification of the smooth irreducible genuine $\overline{\mathbf{F}}_p$ -representations of $\widetilde{\mathrm{GL}}_2(\mathbf{Q}_p)$ is given in [28] and the goal of this paper is to establish a relation to mod- p Galois representations by extending Colmez’s functor.

1.3. Results. We fix an odd prime number p and a finite field k of characteristic p . We let $G = \mathrm{GL}_2(\mathbf{Q}_p)$. The metaplectic covering

$$1 \longrightarrow \mu_2 = \{\pm 1\} \longrightarrow \widetilde{G} \longrightarrow G \longrightarrow 1$$

is described by an explicit 2-cocycle relying on the Hilbert symbol, and it splits over the maximal compact open subgroup. Let $\iota: \mu_2 \hookrightarrow k^\times$ be the non-trivial group homomorphism. We let $\mathrm{Mod}_{\widetilde{G}}^{\mathrm{sm}}(k)$ denote the category of smooth k -linear representations of \widetilde{G} . Of particular interest for us is the

full subcategory $\text{Mod}_{\tilde{G},\iota}^{\text{sm}}(k)$ of *genuine* representations, i.e. those on which the central subgroup μ_2 acts via ι .

The category $\Phi\Gamma_{k((X))}^{\text{ét}}$ of étale (φ, Γ) -modules over $k((X))$ is equivalent to the category $\text{Mod}_{\mathcal{G}_{\mathbf{Q}_p}}^{\text{fin}}(k)$ of finite dimensional continuous k -linear representations of $\mathcal{G}_{\mathbf{Q}_p}$ via Fontaine’s equivalence. Colmez’s functor

$$\mathbf{D}: \text{Mod}_G^{\text{fg.adm}}(k) \longrightarrow \Phi\Gamma_{k((X))}^{\text{ét}}$$

is defined on the category of finitely generated admissible k -linear representations of G and has the property that the image of an object only depends on the restriction of that object to the mirabolic subgroup $P = \begin{pmatrix} \mathbf{Q}_p^\times & \mathbf{Q}_p \\ 0 & 1 \end{pmatrix}$. More formally, using Breuil’s generalization [10], one may view the functor as defined on the category of smooth P -representations at the cost of replacing the target category by its ind-category: $\mathbf{D}: \text{Mod}_P^{\text{sm}}(k) \rightarrow \text{Ind}(\Phi\Gamma_{k((X))}^{\text{ét}})$. When composed with the restriction functor $\text{Mod}_G^{\text{fg.adm}}(k) \rightarrow \text{Mod}_P^{\text{sm}}(k)$, this more general functor takes values in the subcategory of compact objects and recovers Colmez’s functor.

What allows us to extend Colmez’s functor is the observation that the pullback \tilde{P} of the metaplectic covering to P is simply equal to the product of groups $P \times \mu_2$. We may therefore consider the restriction functor $\text{Mod}_G^{\text{fg.adm}}(k) \rightarrow \text{Mod}_P^{\text{sm}}(k)$ and Corollary 4.14 establishes:

Proposition 1.1. *The composition*

$$\text{Mod}_G^{\text{fg.adm}}(k) \longrightarrow \text{Mod}_P^{\text{sm}}(k) \xrightarrow{\mathbf{D}} \text{Ind}(\Phi\Gamma_{k((X))}^{\text{ét}})$$

takes values in the subcategory of compact objects, yielding the exact functor

$$\mathbf{D}: \text{Mod}_G^{\text{fg.adm}}(k) \longrightarrow \Phi\Gamma_{k((X))}^{\text{ét}},$$

whose restriction to the full subcategory $\text{Mod}_G^{\text{fg.adm}}(k)$ recovers Colmez’s functor.

Putting $P^+ = \begin{pmatrix} \mathbf{Z}_p \setminus \{0\} & \mathbf{Z}_p \\ 0 & 1 \end{pmatrix}$ and letting Z denote the center of G , the key for establishing this result is to show the following: Given a finite dimensional $\tilde{K}\tilde{Z}$ -stable subspace M of $\pi \in \text{Mod}_G^{\text{fg.adm}}(k)$ generating π as a G -representation, the P^+ -subrepresentation of π generated by M is admissible in the sense that its subspace of $\begin{pmatrix} 1 & \mathbf{Z}_p \\ 0 & 1 \end{pmatrix}$ -invariants is finite dimensional. This can either be proved by making sure that an appropriate version of Emerton’s formalism of ordinary parts also works for $\text{Mod}_{\tilde{G},\iota}^{\text{fg.adm}}(k)$ to establish an analogue of [18, Theorem 3.4.7], or it can be deduced from known properties of the irreducible objects in $\text{Mod}_G^{\text{fg.adm}}(k)$. We choose to take the latter approach.

Denote by $\mathcal{V}: \Phi\Gamma_{k((X))}^{\text{ét}} \simeq \text{Mod}_{\mathcal{G}_{\mathbf{Q}_p}}^{\text{fin}}(k)$ Fontaine’s equivalence and put $\mathbf{V} = \mathcal{V} \circ \mathbf{D}$. Evaluating the extended functor at the genuine supersingular representations, we obtain our main result (Theorem 5.23):

Theorem 1.2. *The functor $\mathbf{V}: \text{Mod}_{\tilde{G}}^{\text{fg.adm}}(k) \rightarrow \text{Mod}_{\mathcal{G}_{\mathbf{Q}_p}}^{\text{fin}}(k)$ induces a bijection*

$$\left\{ \begin{array}{l} \text{Absolutely irreducible} \\ \text{genuine supersingular} \\ k\text{-representations of } \tilde{G} \end{array} \right\} \cong \left\{ \begin{array}{l} \text{Absolutely irreducible continuous} \\ \rho: \mathcal{G}_{\mathbf{Q}_p} \rightarrow \text{GL}_4(k) \text{ s.t. } \rho \cong \rho \otimes \chi \\ \text{for all characters } \chi \text{ with } \chi^2 = \mathbf{1} \end{array} \right\},$$

both sides considered up to isomorphism.

This theorem consists of two parts, namely surjectivity and injectivity. It is not difficult to write down what the Galois representations appearing as the target of the bijection look like. The hard part for showing surjectivity is to compute the image of a genuine supersingular representation explicitly, see Proposition 5.20 for a description. The detailed study of the supersingular objects in [28] allows us to put ourselves into a setup to which we can apply a general result of Florian Herzig on computing Galois representations attached to certain “nice” smooth k -representations of P^+ , see Section 4.2. Injectivity comes down to the fact, proved in [28], that two genuine supersingular representations are isomorphic if and only if their restrictions to P are.

There is a conceptual explanation for the twist-invariance under all characters of order two. Namely, consider the pullback $\widetilde{\mathbf{Q}_p^\times} \twoheadrightarrow \mathbf{Q}_p^\times$ of the metaplectic cover to the center $Z \cong \mathbf{Q}_p^\times$ of G , and for $\tilde{z} \in \widetilde{\mathbf{Q}_p^\times}$ with image $z \in \mathbf{Q}_p^\times$, define $\chi_{\tilde{z}} = (\mu_{-1}^{v(z)} \omega^{v(z)} \mu_{\omega(z)})^{\frac{p-1}{2}}$, where $v: \mathbf{Q}_p^\times \rightarrow \mathbf{Z}$ is the p -adic valuation, $\mu_?$ is the unramified character mapping p to $?$ and ω is the mod- p cyclotomic character. The conjugation action of an element $\tilde{z} \in \widetilde{\mathbf{Q}_p^\times}$ is given by the $\tilde{G} \supset \mu_2$ -valued character $\chi_{\tilde{z}} \circ \det$. In particular, the action of \tilde{z} on a genuine representation π defines a \tilde{G} -equivariant isomorphism between π and its twist by $\chi_{\tilde{z}} \circ \det$. Now, the functor \mathbf{D} is compatible with twist by a character χ in the sense that $\mathbf{D}(\pi \otimes \chi \circ \det) = \mathbf{D}(\pi) \otimes \chi$, meaning that the underlying vector space remains the same but the (φ, Γ) -action is twisted according to χ , see Definition 4.7. Thus, any Galois representation arising as the image of a genuine representation under \mathbf{V} is invariant under twist by the characters $\chi_{\tilde{z}}$, for $\tilde{z} \in \widetilde{\mathbf{Q}_p^\times}$, which assemble into a *surjective* homomorphism $\widetilde{\mathbf{Q}_p^\times} \twoheadrightarrow \text{Hom}^{\text{cts}}(\mathbf{Q}_p^\times, \mu_2)$. This more conceptual point of view leads to the following notion.

Definition 1.3. A *metaplectic* (φ, Γ) -module over $k((X))$ is a pair (D, θ) consisting of $D \in \Phi\Gamma_{k((X))}^{\text{ét}}$ and $\theta: \widetilde{\mathbf{Q}}_p^\times \rightarrow \text{Aut}_{k((X))}(D)$ a smooth and genuine $k((X))$ -linear action such that $\theta(\tilde{z}): D \cong D \otimes \chi_{\tilde{z}}$ is (φ, Γ) -equivariant. Denote the resulting category by $\widetilde{\Phi}\Gamma_{k((X)), \iota}^{\text{ét}}$.

The extended functor then naturally induces the *metaplectic Montréal functor*

$$\widetilde{\mathbf{D}}: \text{Mod}_{\widetilde{G}, \iota}^{\text{fg.adm}}(k) \longrightarrow \widetilde{\Phi}\Gamma_{k((X)), \iota}^{\text{ét}}.$$

Using Fontaine’s equivalence, one may translate the additional structure θ on a metaplectic (φ, Γ) -module (D, θ) into additional structure on the Galois representation $\mathcal{V}(D)$, and call the resulting object a metaplectic Galois representation. We note however that the additional structure on such a metaplectic Galois representation is *not* simply given by a (smooth and genuine) action on the underlying k -vector space defining appropriate isomorphisms, see Section 5.1.2.

If we let $S \subset \mathbf{Q}_p^\times$ denote the subgroup of squares, then the characters $\chi_{\tilde{z}}$ are trivial for all $\tilde{z} \in \widetilde{S}$, hence we have a restriction functor $\widetilde{\Phi}\Gamma_{k((X)), \iota}^{\text{ét}} \rightarrow \text{Mod}_{\widetilde{S}, \iota}^{\text{sm}}(\Phi\Gamma_{k((X))}^{\text{ét}}) \simeq \text{Mod}_S^{\text{sm}}(\Phi\Gamma_{k((X))}^{\text{ét}})$, where the equivalence follows from the equality of groups $\widetilde{S} = S \times \mu_2$ and is given by $-\boxtimes \iota$. This functor admits an explicit left adjoint denoted by $\text{Ind}_{\widetilde{S}}^{\mathbf{Q}_p^\times}(-\boxtimes \iota)$, see Section 5.1.1, nicely interacting with the functor $\widetilde{\mathbf{D}}$ as explained in Proposition 5.8.

We further utilize Colmez’s construction [14] from (φ, Γ) -modules to smooth representations of the mirabolic subgroup, to obtain a functor $\widetilde{\mathbf{N}}: \widetilde{\Phi}\Gamma_{k((X)), \iota}^{\text{ét}} \rightarrow \text{Mod}_{\widetilde{B}, \iota}^{\text{sm}}(k)$ from metaplectic (φ, Γ) -modules to smooth genuine representations of the metaplectic Borel. Moreover, there is a natural transformation $\widetilde{\mathbf{N}} \circ \widetilde{\mathbf{D}} \Rightarrow (-)|_{\widetilde{B}}$ of functors $\text{Mod}_{\widetilde{G}, \iota}^{\text{fg.adm}}(k) \rightarrow \text{Mod}_{\widetilde{B}, \iota}^{\text{sm}}(k)$, see Proposition 5.13.

We also compute the images of the absolutely irreducible representations in $\text{Mod}_{\widetilde{G}, \iota}^{\text{sm}}(k)$. As we will recall in more detail in the main text below, these are either supersingular or principal series representations. Putting $B_S = \begin{pmatrix} \mathbf{Q}_p^\times & \mathbf{Q}_p \\ 0 & S \end{pmatrix}$, we have $\widetilde{B}_S = B_S \times \mu_2$, and any absolutely irreducible genuine principal series is of the form $\widetilde{\pi}(\chi_1, \chi_2) = \text{Ind}_{\widetilde{B}_S}^{\widetilde{G}}(\chi_1 \otimes \chi_2 \boxtimes \iota)$ for smooth characters χ_1, χ_2 of \mathbf{Q}_p^\times and S , respectively. The isomorphism class of $\widetilde{\pi}(\chi_1, \chi_2)$ is uniquely determined by the pair $(\chi_1|_S, \chi_2|_S)$. Denoting by $\widetilde{\pi}(\chi_1, \chi_2)^{\text{bc}} \subset \widetilde{\pi}(\chi_1, \chi_2) \cong \text{Ind}_{\widetilde{B}}^{\widetilde{G}} \text{Ind}_{\widetilde{T}_S}^{\widetilde{T}}(\chi_1 \otimes \chi_2 \boxtimes \iota)$ the \widetilde{B} -invariant subspace of functions whose support is contained in the preimage of the big

cell $B \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} B$ under the projection $\tilde{G} \twoheadrightarrow G$, we obtain the following result (Proposition 5.16).

Proposition 1.4. *Let $\chi_1, \chi_2: \mathbf{Q}_p^\times \rightarrow k^\times$ be smooth characters, and let $\mathcal{D}(\chi_2)$ be the (φ, Γ) -module corresponding to χ_2 viewed as a character of $\mathcal{G}_{\mathbf{Q}_p}$ via local class field theory.*

- (i) *The object $\text{Ind}_{\tilde{S}}^{\tilde{\mathbf{Q}}_p^\times} ((\chi_1 \chi_2)|_S \boxtimes \mathcal{D}(\chi_2) \boxtimes \iota)$ is irreducible in $\tilde{\Phi}\Gamma_{k((X)), \iota}^{\text{ét}}$.*
- (ii) *There is an isomorphism*

$$\tilde{\mathbf{D}}(\tilde{\pi}(\chi_1, \chi_2)) \cong \text{Ind}_{\tilde{S}}^{\tilde{\mathbf{Q}}_p^\times} ((\chi_1 \chi_2)|_S \boxtimes \mathcal{D}(\chi_2) \boxtimes \iota)$$

in the category $\tilde{\Phi}\Gamma_{k((X)), \iota}^{\text{ét}}$.

- (iii) *The canonical \tilde{B} -equivariant map $\tilde{\Omega}\tilde{\mathbf{D}}(\tilde{\pi}(\chi_1, \chi_2)) \rightarrow \tilde{\pi}(\chi_1, \chi_2)$ has image $\tilde{\pi}(\chi_1, \chi_2)^{\text{bc}}$ and induces a non-split short exact sequence*

$$0 \longrightarrow \text{Ind}_{\tilde{T}_S}^{\tilde{T}} (\chi_2 \omega \otimes (\chi_1 \omega^{-1})|_S \boxtimes \iota) \longrightarrow \tilde{\Omega}\tilde{\mathbf{D}}(\tilde{\pi}(\chi_1, \chi_2)) \longrightarrow \tilde{\pi}(\chi_1, \chi_2)^{\text{bc}} \longrightarrow 0.$$

Interestingly, these statements can essentially be reduced to the corresponding known results for principal series of $\text{GL}_2(\mathbf{Q}_p)$.

Concerning absolutely irreducible genuine supersingular representations, we have already explained above that their image under \mathbf{V} , or equivalently \mathbf{D} , is explicitly described and absolutely irreducible. Parts (iii) and (iv) of Proposition 5.20 further establish:

Proposition 1.5. *Let π be an absolutely irreducible genuine supersingular k -representation of \tilde{G} , and let $\zeta \boxtimes \iota: S \times \mu_2 = \tilde{S} \cong Z(\tilde{G}) \rightarrow k^\times$ be its central character.*

- (i) *The canonical \tilde{B} -equivariant map $\tilde{\Omega}\tilde{\mathbf{D}}(\pi) \xrightarrow{\cong} \pi$ is an isomorphism.*
- (ii) *The metaplectic (φ, Γ) -module $\tilde{\mathbf{D}}(\pi)$ is the unique object in $\tilde{\Phi}\Gamma_{k((X)), \iota}^{\text{ét}}$ whose restriction to S is isomorphic to $\zeta \boxtimes \mathbf{D}(\pi) \in \text{Mod}_S^{\text{sm}}(\tilde{\Phi}\Gamma_{k((X))}^{\text{ét}})$ such that the \tilde{B} -action on $\tilde{\Omega}\tilde{\mathbf{D}}(\pi)$ extends to a smooth \tilde{G} -action.*

In the final section, we outline how the functors \mathbf{D} and $\tilde{\mathbf{D}}$ lift to p -adic coefficients, using that Breuil’s construction [10] works more generally for mod- p^n coefficients, for all $n \geq 1$. Concretely, let L be a finite extension of \mathbf{Q}_p with ring of integers \mathcal{O}_L and residue field k , and fix a uniformizer ϖ . Denote by $\text{Ban}_{\tilde{G}}^{\text{fg.adm}}(L)$ the category of unitary L -Banach space representations of \tilde{G} , which are residually finitely generated and admissible: for one, or equivalently any, \tilde{G} -invariant norm, the mod- ϖ reduction of the corresponding unit ball is finitely generated and admissible. Via Fontaine’s equivalence,

the category of continuous Galois representations of $\mathcal{G}_{\mathbf{Q}_p}$ over L is equivalent to the category $\Phi\Gamma_{\mathcal{E}}^{\text{ét}}$ of (φ, Γ) -modules over a certain topological field \mathcal{E} . From Proposition 1.1 we deduce that an analogous statement holds true for $\mathcal{O}_L/\varpi^n \mathcal{O}_L$ -coefficients, for all $n \geq 1$, see Proposition 6.2. Passing to the limit and inverting p , we thus obtain a functor $\mathbf{D}: \text{Ban}_{\widetilde{G}}^{\text{fg.adm}}(L) \rightarrow \Phi\Gamma_{\mathcal{E}}^{\text{ét}}$ extending Colmez's p -adic functor. It is compatible with semi-simplified mod- ϖ reduction in the sense that $\mathbf{D}(\overline{\Pi}^{\text{ss}}) = \overline{\mathbf{D}(\Pi)^{\text{ss}}}$, for $\Pi \in \text{Ban}_{\widetilde{G}}^{\text{fg.adm}}(L)$.

Viewing the character ι as valued in \mathcal{O}_L^\times via the Teichmüller lift, we may restrict to the category of genuine representation to obtain the p -adic metaplectic Montréal functor $\widetilde{\mathbf{D}}: \text{Ban}_{\widetilde{G}, \iota}^{\text{fg.adm}}(L) \rightarrow \widetilde{\Phi}\Gamma_{\mathcal{E}, \iota}^{\text{ét}}$, taking values in the category of metaplectic (φ, Γ) -modules over \mathcal{E} , which are defined as in the mod- p setting (Definition 1.3).

1.4. Outlook. We hope that this work serves as the starting point of a possible metaplectic p -adic local Langlands correspondence for $\text{GL}_2(\mathbf{Q}_p)$. Building on the computations of mod- p reductions of locally algebraic unramified genuine principal series appearing in [28], one may write down a speculative semi-simple mod- p metaplectic local Langlands correspondence relating the mod- p automorphic side to metaplectic Galois representations of dimension $2n^2 = 8$, where $n^2 = [L : \mathbf{Q}_p]$ with L being the compositum of all degree-two extensions of \mathbf{Q}_p . However, real evidence for such a correspondence can only come from the p -adic picture. This makes a closer investigation of the relation between the completions of the aforementioned locally algebraic representations and certain (crystalline) Galois representations, as in the work of Breuil–Berger [5] for $\text{GL}_2(\mathbf{Q}_p)$, inevitable.

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1.5. Conventions and notations. Throughout this work, we fix an odd prime number p and a finite field k of characteristic p with an algebraic closure \bar{k} .

We put $G = \text{GL}_2(\mathbf{Q}_p)$ and use the standard notations $B \supset T \supset Z$ to denote the subgroup of upper triangular matrices, the subgroup of diagonal matrices and the center of G , respectively, and we let $K = \text{GL}_2(\mathbf{Z}_p)$ be the

maximal compact open subgroup. We identify $Z \cong \mathbf{Q}_p^\times$ via the diagonal embedding and let $S \subset \mathbf{Q}_p^\times$ denote the subgroup of squares.

Given a topological group \mathcal{G} , the category of smooth k -linear representation of \mathcal{G} will be denoted by $\text{Mod}_{\mathcal{G}}^{\text{sm}}(k)$. Here, smooth means that the stabilizer of each vector is open.

For a closed subgroup $\mathcal{H} \subset \mathcal{G}$, we let $\text{Ind}_{\mathcal{H}}^{\mathcal{G}}(-): \text{Mod}_{\mathcal{H}}^{\text{sm}}(k) \rightarrow \text{Mod}_{\mathcal{G}}^{\text{sm}}(k)$ be the smooth induction defining a left adjoint of the restriction functor $(-)|_{\mathcal{H}}$. If \mathcal{H} is in fact open, we similarly denote by $\text{c-Ind}_{\mathcal{H}}^{\mathcal{G}}(-)$ the compact induction defining a right adjoint of the restriction functor.

Finally, if \mathcal{G} is abelian, we let \mathcal{G}^{sq} be its subgroup of squares.

2. Genuine representation theory

In this section, we remind the reader of the classification of the smooth absolutely irreducible genuine mod- p representations of the two-fold metaplectic cover of G in terms of universal spherical Hecke modules, and then prove a finiteness result, which will play an important role for extending Colmez’s functor.

2.1. Recollection. We recollect some facts about the metaplectic extension of G , for details we refer the reader to [28, Section 2]. The metaplectic cover of G is the topological central extension

$$1 \longrightarrow \mu_2 = \{\pm 1\} \longrightarrow \tilde{G} \longrightarrow G \longrightarrow 1$$

defined by the 2-cocycle $\sigma: G \times G \rightarrow \mu_2$,

$$(2.1) \quad \sigma(g_1, g_2) = \left(\frac{\mathfrak{c}(g_1 g_2)}{\mathfrak{c}(g_1)}, \frac{\mathfrak{c}(g_1 g_2)}{\mathfrak{c}(g_2)} \det(g_1) \right) \text{ for all } (g_1, g_2) \in G \times G,$$

where $\mathfrak{c}\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{cases} c & \text{if } c \neq 0 \\ d & \text{if } c = 0, \end{cases}$ and $(-, -): \mathbf{Q}_p^\times \times \mathbf{Q}_p^\times \rightarrow \mu_2$ is the qua-

dratic Hilbert symbol, explicitly given by $(a, b) = \omega\left(\left(-1\right)^{v(a)v(b)} \frac{b^{v(a)}}{a^{v(b)}}\right)^{\frac{p-1}{2}}$ for all $a, b \in \mathbf{Q}_p^\times$, where ω is the mod- p cyclotomic character.

Notation 2.1.

- (a) As a set, $\tilde{G} = G \times \mu_2$ and we write elements as pairs (g, ζ) for $g \in G$ and $\zeta \in \mu_2$.
- (b) The preimage of a subset $C \subset G$ in \tilde{G} will be denoted by \tilde{C} .

The map

$$K \times \mu_2 \xrightarrow{\cong} \tilde{K}$$

$$(g, \zeta) \mapsto \begin{cases} (g, \zeta(c, d \det(g)^{-1})) & \text{if } c \in p\mathbf{Z}_p \setminus \{0\} \\ (g, \zeta) & \text{otherwise} \end{cases}$$

defines a group isomorphism, which we fix throughout and use to view subgroups of K as subgroups of \tilde{K} .

As a group, the center $Z(\tilde{G}) = Z^{\text{sq}} \times \mu_2 \cong S \times \mu_2$ is equal to the product of the subgroup of squares in the center Z of G and μ_2 .

Let $\iota: \mu_2 \hookrightarrow k^\times$ be the non-trivial character. Given a subgroup $C \subset G$, one has the product decomposition

$$\text{Mod}_{\tilde{C}}^{\text{sm}}(k) = \text{Mod}_{\tilde{C},\iota}^{\text{sm}}(k) \times \text{Mod}_C^{\text{sm}}(k),$$

where the first factor is the full subcategory consisting of the *genuine* representations, i.e. those on which μ_2 acts via ι , and the second factor is viewed as a full subcategory via inflation.

Notation 2.2. Given a subgroup $C \subset G$ together with a group isomorphism $\tilde{C} \cong C \times \mu_2$, we denote the induced equivalence of categories $\text{Mod}_{\tilde{C}}^{\text{sm}}(k) \simeq \text{Mod}_{\tilde{C},\iota}^{\text{sm}}(k)$ by $\pi \mapsto \pi \boxtimes \iota$.

We will be interested in $\text{Mod}_{\tilde{G},\iota}^{\text{sm}}(k)$, the absolutely irreducible objects of which have been described in [28]: they are either genuine principal series or genuine supersingular representations. By definition, a genuine principal series on a k -vector space is of the form $\text{Ind}_{\tilde{B}}^{\tilde{G}}(\tau)$ for some smooth irreducible genuine k -representation τ of \tilde{T} , while a genuine supersingular representation is a smooth irreducible k -representation of \tilde{G} not occurring as a subquotient of, or equivalently being isomorphic to, a principal series.

Proposition 2.3. Let $B_S = \begin{pmatrix} \mathbf{Q}_p^\times & \mathbf{Q}_p \\ 0 & S \end{pmatrix} \supset \begin{pmatrix} \mathbf{Q}_p^\times & 0 \\ 0 & S \end{pmatrix} = T_S$.

- (i) As groups, $\tilde{B}_S = B_S \times \mu_2 \supset T_S \times \mu_2 = \tilde{T}_S$;
- (ii) The following three sets, considered up to isomorphism, are in bijection:
 - (a) $\{\text{Smooth characters } T^{\text{sq}} \rightarrow k^\times\}$;
 - (b) $\left\{ \begin{array}{l} \text{Smooth abs. irred. genuine representations of } \tilde{T} \text{ on} \\ k\text{-vector spaces} \end{array} \right\}$;
 - (c) $\{\text{Abs. irred. genuine principal series of } \tilde{G} \text{ on } k\text{-vector spaces}\}$.

The bijection from (a) to (b) is given by $\chi \mapsto \text{Ind}_{T_S}^{\tilde{T}}(\chi' \boxtimes \iota)$ for any extension χ' of χ to T_S , while the bijection between (b) and (c) is induced by the functor $\text{Ind}_{\tilde{B}}^{\tilde{G}}(-)$ composed with inflation along $\tilde{B} \twoheadrightarrow \tilde{T}$.

Proof. Part (i) is Lemma 2.4(ii), while part (ii) is the content of Remark 4.17, Definition-Proposition 4.16 and Proposition 4.20 in [28]. \square

The absolutely irreducible genuine principal series representations may also be described in terms of the universal module of the spherical Hecke algebras, which is also suitable to parametrize the supersingular objects:

For $0 \leq r \leq p - 1$, consider the irreducible $\mathrm{GL}_2(\mathbf{F}_p)$ -representation $\mathrm{Sym}^r(k^2)$, where k^2 denotes the standard representation (base changed to k), viewed as a smooth irreducible representation of K via inflation. We extend the action to KZ^{sq} by letting the central matrix p^2 act trivially. Via the identification $\tilde{K}Z(\tilde{G}) \cong KZ^{\mathrm{sq}} \times \mu_2$, we may thus form the smooth irreducible genuine k -representation $\mathrm{Sym}^r(k^2) \boxtimes \iota$ of $\tilde{K}Z(\tilde{G})$. According to [28, Proposition 5.2], the spherical Hecke algebra

$$\mathrm{End}_{\tilde{G}}(\mathrm{c}\text{-Ind}_{\tilde{K}Z(\tilde{G})}^{\tilde{G}}(\mathrm{Sym}^r(k^2) \boxtimes \iota)) = k[\tilde{T}]$$

is a polynomial ring in a single Hecke operator \tilde{T} .

Proposition 2.4 ([28, Theorem 5.13]). *Let $0 \leq r \leq p - 1$ and $\lambda \in k$.*

(i) *If $\lambda \neq 0$, then*

$$\mathrm{c}\text{-Ind}_{\tilde{K}Z(\tilde{G})}^{\tilde{G}}(\mathrm{Sym}^r(k^2) \boxtimes \iota) / (\tilde{T} - \lambda) \cong \mathrm{Ind}_{B_S}^{\tilde{G}}(\psi' \boxtimes \iota),$$

where $\psi': T_S \rightarrow k^\times$ is any character with $\psi'|_{K \cap T^{\mathrm{sq}}} = (\mathbf{1} \otimes \omega^r)|_{K \cap T^{\mathrm{sq}}}$, $\psi' \left(\begin{pmatrix} p^{-2} & 0 \\ 0 & 1 \end{pmatrix} \right) = \lambda$ and $\psi' \left(\begin{pmatrix} p^{-2} & 0 \\ 0 & p^{-2} \end{pmatrix} \right) = 1$.

(ii) *If $\lambda = 0$, then $\mathrm{c}\text{-Ind}_{\tilde{K}Z(\tilde{G})}^{\tilde{G}}(\mathrm{Sym}^r(k^2) \boxtimes \iota) / (\tilde{T})$ is an extension of two non-isomorphic absolutely irreducible genuine supersingular representations. The extension splits if and only if $r = \frac{p-1}{2}$.*

Definition 2.5. For $0 \leq r \leq p - 1$, $r \neq \frac{p-1}{2}$, $\lambda \in k$ and a smooth character $\eta: \mathbf{Q}_p^\times \rightarrow k^\times$, we define $\tilde{\pi}(r, \lambda, \eta)$ to be the unique irreducible quotient of $\mathrm{c}\text{-Ind}_{\tilde{K}Z(\tilde{G})}^{\tilde{G}}(\mathrm{Sym}^r(k^2) \boxtimes \iota) / (\tilde{T} - \lambda) \otimes \eta \circ \det$. If $\eta = \mathbf{1}$ is the trivial character, we simply write $\tilde{\pi}(r, \lambda)$.

Remark 2.6. The extension in Proposition 2.4(ii) is an extension of $\tilde{\pi}(r, 0)$ by $\tilde{\pi}(p-1-r, 0, \omega^r)$, if $r \neq \frac{p-1}{2}$, while it is the direct sum of $\tilde{\pi}(0)$ and $\tilde{\pi}(p-1)$ for $r = \frac{p-1}{2}$.

Proposition 2.7. *Every smooth absolutely irreducible genuine representation of \tilde{G} on a k -vector space is of the form $\tilde{\pi}(r, \lambda, \eta)$ for some $0 \leq r \leq p - 1$, $r \neq \frac{p-1}{2}$, $\lambda \in k$ and a smooth character $\eta: \mathbf{Q}_p^\times \rightarrow k^\times$.*

(i) *If $\lambda \neq 0$, then $\tilde{\pi}(r, \lambda, \eta)$ is a principal series.*

Moreover, $\tilde{\pi}(r, \lambda, \eta) \cong \tilde{\pi}(r', \lambda')$ if and only if $r \equiv r' \pmod{\frac{p-1}{2}}$, $\eta|_{\mathbf{Z}_p^{\times, \mathrm{sq}}} = \mathbf{1}$, $\eta(p^4) = 1$ and $\lambda' = \lambda\eta(p^{-2})$.

(ii) *If $\lambda = 0$, then $\tilde{\pi}(r, 0, \eta)$ is supersingular.*

Moreover, $\tilde{\pi}(r, 0, \eta) \cong \tilde{\pi}(r', \lambda')$ if and only if $\lambda' = 0$, $\eta(p^4) = 1$ and either of

- $r' = r$ and $\eta|_{\mathbf{Z}_p^{\times, \mathrm{sq}}} = \mathbf{1}$

$$\bullet r = \begin{cases} \frac{p-1}{2} - r' & \text{if } 0 < r' < \frac{p-1}{2} \\ \frac{3(p-1)}{2} - r' & \text{if } \frac{p-1}{2} < r' < p-1 \end{cases} \text{ and } \eta|_{\mathbf{Z}_p^{\times, \text{sq}}} = \omega^{r'}|_{\mathbf{Z}_p^{\times, \text{sq}}}.$$

Proof. This follows from Propositions 2.4, 2.3 and [28, Corollary 5.16]. \square

2.2. Some finiteness result. Recall that a smooth representation is admissible if the subspace of C -fixed vectors is finite dimensional for every open subgroup C . The domain of the extension of Colmez’s functor will be the category $\text{Mod}_{\tilde{G}}^{\text{fg.adm}}(k)$ of finitely generated admissible k -representations of \tilde{G} . We need to establish a certain finiteness result for this category in order to define the desired functor. The discussion for the full subcategory $\text{Mod}_{\tilde{G}}^{\text{fg.adm}}(k)$ by Emerton [16] will serve as a guideline.

By [17, Proposition 2.2.10], the category $\text{Mod}_{\tilde{G}}^{\text{adm}}(k)$ of smooth admissible k -representations of \tilde{G} is abelian.

Lemma 2.8. *An admissible object in $\text{Mod}_{\tilde{G}}^{\text{sm}}(k)$ is finitely generated if and only if it is of finite length. In particular, the category $\text{Mod}_{\tilde{G}}^{\text{fg.adm}}(k)$ is a Serre subcategory of $\text{Mod}_{\tilde{G}}^{\text{adm}}(k)$ and hence is abelian.*

Proof. We only need to prove that an object of $\text{Mod}_{\tilde{G}}^{\text{fg.adm}}(k)$ is of finite length. By [28, Theorem 5.18], this holds true if k is replaced by an algebraic closure \bar{k} . Let now $\pi \in \text{Mod}_{\tilde{G}}^{\text{fg.adm}}(k)$. We claim that every subobject $\tau \subset \pi$ is again finitely generated (and admissible). Indeed, $\pi \otimes_k \bar{k}$ is finitely generated and admissible, hence of finite length and so must be the subobject $\tau \otimes_k \bar{k}$, which is therefore finitely generated over the group ring $\bar{k}[\tilde{G}]$. By faithful flatness, we deduce that τ is finitely generated over $k[\tilde{G}]$ proving the claim. We may now choose a (possibly infinite) composition series of π , which must however be finite by the result over \bar{k} and faithfulness of the functor $- \otimes_k \bar{k}$, i.e. π is of finite length. \square

Denote by $P = \begin{pmatrix} \mathbf{Q}_p^\times & \mathbf{Q}_p \\ 0 & 1 \end{pmatrix}$ the mirabolic subgroup, which we view as a subgroup of \tilde{G} via Proposition 2.3 (i). The submonoid $P^+ = \begin{pmatrix} \mathbf{Z}_p \setminus \{0\} & \mathbf{Z}_p \\ 0 & 1 \end{pmatrix}$ is generated by the element $F = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$ and the submonoids $\Gamma = \begin{pmatrix} \mathbf{Z}_p^\times & 0 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & \mathbf{Z}_p \\ 0 & 1 \end{pmatrix}$.

Identifying the completed group ring of $\begin{pmatrix} 1 & \mathbf{Z}_p \\ 0 & 1 \end{pmatrix}$ with the power series ring $k[[X]]$ in the variable $X = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} - 1$, any smooth P^+ -representation on a k -vector space is naturally a left module over the ring $k[[X]][[F, \Gamma]]$ defined by the relations $Ff(X) = f(X^p)F$, $\gamma f(X) = f((1 + X)^\gamma - 1)\gamma$ and $F\gamma = \gamma F$ for all $f(X) \in k[[X]]$ and $\gamma \in \Gamma \cong \mathbf{Z}_p^\times$.

Lemma 2.9. *For $\pi \in \text{Mod}_{\tilde{G}}^{\text{adm}}(k)$, the $k[F]$ -module $\pi/X\pi$ is torsion, i.e. each element in it is killed by a polynomial in F with k -coefficients.*

Proof. Writing an admissible representation as the filtered colimit of its finitely generated subrepresentations, we may assume $\pi \in \text{Mod}_{\tilde{G}}^{\text{fg.adm}}(k)$. Any polynomial in $\bar{k}[F]$ is a factor of a polynomial in $k[F]$, hence we may replace k by \bar{k} . By [28, Theorem 5.18], an object in $\text{Mod}_{\tilde{G}}^{\text{fg.adm}}(\bar{k})$ is of finite length, and we are reduced to proving the statement for smooth irreducible \bar{k} -representation π of \tilde{G} . If μ_2 acts trivially, then this is a special case of [16, Proposition 4.5]. In the genuine case, this is Corollary 4.56 (supersingular) and Corollary 4.32 (principal series) in [28]. \square

Definition 2.10. Given a smooth k -representation π of P and a Γ -stable subspace $M \subset \pi$, define $\pi_M \subset \pi$ to be the $k[[X]][F]$ -submodule of π generated by M (which is in fact a $k[[X]][F, \Gamma]$ -submodule or equivalently a P^+ -subrepresentation).

Utilizing the previous lemma, one proves the following finiteness result just as Emerton [16, Theorem 4.7] does for GL_2 over an unramified extension of \mathbf{Q}_p .

Proposition 2.11. *Let $\pi \in \text{Mod}_{\tilde{G}}^{\text{adm}}(k)$ and let $M \subset \pi$ be a finite dimensional Γ -stable subspace. Then π_M is finitely presented and of finite length over $k[[X]][F]$ and the k -subspace of X -torsion elements $\pi_M[X]$ is finite dimensional.*

Definition 2.12. For $\pi \in \text{Mod}_{\tilde{G}}^{\text{fg.adm}}(k)$, define $\mathcal{G}(\pi)$ to be the directed set of finite dimensional $\tilde{K}\tilde{Z}$ -stable subspaces of π generating the latter as a \tilde{G} -representation, endowed with the partial order given by inclusion.

Remark 2.13. For $\pi \in \text{Mod}_{\tilde{G}}^{\text{fg.adm}}(k)$, the set $\mathcal{G}(\pi)$ is non-empty: a finite number of generators will be fixed by a small enough congruence subgroup $K_n = 1 + p^n M_{2 \times 2}(\mathbf{Z}_p)$, for some $n \geq 1$, so that π is generated by π^{K_n} , which is finite dimensional, by admissibility, and stable under the action of $\tilde{K}\tilde{Z}$.

3. Galois representations and (φ, Γ) -modules

One of the key ingredients for relating the smooth mod- p representation theory of \tilde{G} to Galois representations, as we will do in the next section, is Fontaine’s equivalence giving a semi-linear algebraic description of Galois representations in terms of (φ, Γ) -modules. The purpose of this section is to remind the reader of this categorical equivalence as well as to recollect the known description of the (φ, Γ) -module corresponding to an absolutely irreducible Galois representation.

3.1. Notations. For a field L with separable closure L^{sep} , denote its absolute Galois group by $\mathcal{G}_L = \text{Gal}(L^{\text{sep}}/L)$ and let $\text{Mod}_{\mathcal{G}_L}^{\text{fin}}(k)$ be the category of continuous \mathcal{G}_L -representations on finite dimensional (discrete) k -vector spaces.

We fix an algebraic closure $\overline{\mathbf{Q}}_p$ of \mathbf{Q}_p , and for $n \geq 1$ we let \mathbf{Q}_{p^n} denote the unique degree- n unramified extension of \mathbf{Q}_p (inside $\overline{\mathbf{Q}}_p$). We define

$$\omega_n: \mathcal{G}_{\mathbf{Q}_{p^n}} \twoheadrightarrow \mathbf{F}_{p^n}^\times, \quad g(\zeta^{p^n-1}\sqrt{-p}) = \omega_n(g) \zeta^{p^n-1}\sqrt{-p} \quad \text{for all } g \in \mathcal{G}_{\mathbf{Q}_{p^n}},$$

to be Serre's fundamental character of level n corresponding to the uniformizer $-p$ of \mathbf{Q}_p . In case $n = 1$, $\omega_1 = \omega$ is simply the mod- p cyclotomic character. For $\lambda \in k^\times$, let $\mu_\lambda: \mathcal{G}_{\mathbf{Q}_p} \rightarrow k^\times$ be the unique unramified character mapping the geometric Frobenius to λ .

Finally, normalizing local class field theory so that p maps to a geometric Frobenius, we will view continuous characters of $\mathcal{G}_{\mathbf{Q}_p}$ as continuous, or equivalently smooth, characters of \mathbf{Q}_p^\times , and vice versa.

3.2. Absolutely irreducible Galois representations. Following Berger [3], we say that an integer $1 \leq h \leq p^n - 2$ is *primitive* (relative to $n \geq 1$) if h is not divisible by $(p^n - 1)/(p^d - 1)$ for any $d < n$ dividing n . In this case, Mackey's decomposition implies that the Galois representation $\text{Ind}_{\mathcal{G}_{\mathbf{Q}_{p^n}}}^{\mathcal{G}_{\mathbf{Q}_p}}(\omega_n^h)$ is absolutely irreducible. Note that it is not only defined over \mathbf{F}_{p^n} but already over \mathbf{F}_p .

Proposition 3.1 ([3, Corollary 2.1.5]). *Every n -dimensional absolutely irreducible object in $\text{Mod}_{\mathcal{G}_{\mathbf{Q}_p}}^{\text{fin}}(k)$ is isomorphic to $\text{Ind}_{\mathcal{G}_{\mathbf{Q}_{p^n}}}^{\mathcal{G}_{\mathbf{Q}_p}}(\omega_n^h) \otimes \mu_\lambda$ for some primitive $1 \leq h \leq p^n - 2$ and $\lambda \in \bar{k}^\times$ with $\lambda^n \in k^\times$.*

Remark 3.2. For a field extension ℓ/k and $V_1, V_2 \in \text{Mod}_{\mathcal{G}_{\mathbf{Q}_p}}^{\text{fin}}(k)$, one has $\text{Hom}_{k[\mathcal{G}_{\mathbf{Q}_p}]}(V_1, V_2) \otimes_k \ell = \text{Hom}_{\ell[\mathcal{G}_{\mathbf{Q}_p}]}(V_1 \otimes_k \ell, V_2 \otimes_k \ell)$. In particular, there is no harm in extending scalars to a finite extension of k when dealing with absolutely irreducible Galois representations.

3.3. Fontaine's equivalence. Let $\mathbf{Q}_p(\mu_{p^\infty})$ be the subfield of $\overline{\mathbf{Q}}_p$ obtained by adjoining all p -power roots of unity to \mathbf{Q}_p . Consider the short exact sequence

$$0 \longrightarrow \mathcal{G}_{\mathbf{Q}_p(\mu_{p^\infty})} \longrightarrow \mathcal{G}_{\mathbf{Q}_p} \longrightarrow \Gamma := \text{Gal}(\mathbf{Q}_p(\mu_{p^\infty})/\mathbf{Q}_p) \longrightarrow 0.$$

The cyclotomic character $\chi_{\text{cyc}}: \mathcal{G}_{\mathbf{Q}_p} \twoheadrightarrow \mathbf{Z}_p^\times$ induces an isomorphism of topological groups $\chi_{\text{cyc}}: \Gamma = \text{Gal}(\mathbf{Q}_p(\mu_{p^\infty})/\mathbf{Q}_p) \cong \mathbf{Z}_p^\times$. For the kernel of the sequence, note that completing $\mathbf{Q}_p(\mu_{p^\infty})$ p -adically results in a perfectoid field $\mathbf{Q}_p(\mu_{p^\infty})^\wedge$, whose tilt (as defined in [25, Lemma 3.4(ii)]) is computed

to be

$$\begin{aligned} (\mathbf{Q}_p(\mu_{p^\infty})^\wedge)^\flat &= \left(\varprojlim_{x \mapsto x^p} \mathbf{Z}_p[\mu_{p^\infty}]/p \right) [1/X] \\ &\cong \mathbf{F}_p\llbracket X^{1/p^\infty} \rrbracket [1/X] = \mathbf{F}_p((X^{1/p^\infty})), \end{aligned}$$

the completion of the perfection $\mathbf{F}_p((X))^\text{perf}$, where $X = (\zeta_{p^n} - 1)_{n \geq 1}$, for a compatible system $\{\zeta_{p^n}\}_{n \geq 1}$ of p -power roots of unity. From now on we fix the choice of a compatible system of p -power roots and thus the above isomorphism. By the tilting equivalence [25, Theorem 3.7], we have identifications

$$\mathcal{G}_{\mathbf{Q}_p(\mu_{p^\infty})} \cong \mathcal{G}_{\mathbf{Q}_p(\mu_{p^\infty})^\wedge} \cong \mathcal{G}_{\mathbf{F}_p((X^{1/p^\infty}))} \cong \mathcal{G}_{\mathbf{F}_p((X))^\text{perf}} \cong \mathcal{G}_{\mathbf{F}_p((X))},$$

where the first and second-to-last isomorphism are induced by taking completions (Krasner’s Lemma).

Via functoriality of tilting and continuity, the Galois group $\mathcal{G}_{\mathbf{Q}_p}$ acts on $(\overline{\mathbf{Q}_p}^\wedge)^\flat$ and it preserves the subfield $\mathbf{F}_p((X))$: $g(X) = (X + 1)^{\chi_\text{cyc}(g)} - 1$, for $g \in \mathcal{G}_{\mathbf{Q}_p}$.

Remark 3.3. In particular, there is no (serious) clash of notation for Γ and X appearing in this section and Section 2.2: in both cases Γ is identified with \mathbf{Z}_p^\times and the two actions on X agree. In fact, this compatibility is what allowed Colmez to define his functor as we will see below.

Since the completed algebraic closure is algebraically closed, we obtain that $\mathbf{F}_p((X))^\text{sep} \subset (\overline{\mathbf{Q}_p}^\wedge)^\flat$, which is then also $\mathcal{G}_{\mathbf{Q}_p}$ -stable. We will let φ denote the Frobenius endomorphism $a \mapsto a^p$ on $\mathbf{F}_p((X))^\text{sep}$ and $\mathbf{F}_p((X))$. In particular, we have the commuting actions of φ and Γ on $\mathbf{F}_p((X))$.

Applying the functor $k \otimes_{\mathbf{F}_p} -$, we obtain similar actions of (φ, Γ) on $k((X))$.

Definition 3.4 ([20, 3.3]). An étale (φ, Γ) -module over $k((X))$ is a finite dimensional $k((X))$ -module D endowed with a φ -semilinear endomorphism φ_D and a continuous¹ semilinear Γ -action commuting with φ_D , such that the linearized map $\varphi_D: \varphi^* D \rightarrow D$ is an isomorphism.

We denote the resulting abelian category by $\Phi\Gamma_{k((X))}^\text{ét}$. It admits tensor products $- \otimes_{k((X))} -$, given as the tensor product of the underlying $k((X))$ -vector spaces with diagonal (φ, Γ) -action, as well as duals $(-)'$, uniquely characterized by $D' = \text{Hom}_{k((X))}(D, k((X)))$ on underlying $k((X))$ -vector spaces and requiring the evaluation map $D' \otimes_{k((X))} D \rightarrow k((X))$ to be (φ, Γ) -equivariant.

¹After fixing a basis of D , we can identify $D \cong k((X))^n$ for some $n \geq 0$ and we take the topology for which the $k[[X]]$ -submodule $k[[X]]^n$ endowed with the X -adic topology is an open subgroup.

Definition-Proposition 3.5 ([20, 3.4.3]). *The functors*

$$\mathcal{V} : \Phi\Gamma_{k((X))}^{\text{ét}} \rightleftarrows \text{Mod}_{\mathcal{G}_{\mathbf{Q}_p}}^{\text{fin}}(k) : \mathcal{D}$$

defined by $\mathcal{V}(D) = (D \otimes_{\mathbf{F}_p((X))} \mathbf{F}_p((X))^{\text{sep}})^{\varphi_D \otimes \varphi=1}$, for $D \in \Phi\Gamma_{k((X))}^{\text{ét}}$, and $\mathcal{D}(V) = (V \otimes_{\mathbf{F}_p} \mathbf{F}_p((X))^{\text{sep}})^{\mathcal{G}_{\mathbf{Q}_p}(\mu_{p^\infty})}$, for $V \in \text{Mod}_{\mathcal{G}_{\mathbf{Q}_p}}^{\text{fin}}(k)$, respectively, are quasi-inverse to each other and respect tensor products as well as duals.

Here, the $\mathcal{G}_{\mathbf{Q}_p}$ -action on $\mathcal{V}(D)$ is induced by the diagonal action². In the definition of $\mathcal{D}(V)$, the $\mathcal{G}_{\mathbf{Q}_p}(\mu_{p^\infty}) \cong \mathcal{G}_{\mathbf{F}_p((X))}$ -invariants are taken with respect to the diagonal action, so one obtains a natural action of the quotient Γ ; the φ -action on $\mathcal{D}(V)$ is induced by acting on $\mathbf{F}_p((X))^{\text{sep}}$. The action of k on either side is induced by the action of it on D and V , respectively.

Example 3.6 ([9, Corollaire 3.7]). Fix integers $n \geq 1$, $h \geq 0$ and let $\chi : \mathbf{Q}_p^\times \rightarrow \bar{k}^\times$ be a character with $\chi(p)^n \in k^\times$, so that the Galois representation $V = \text{Ind}_{\mathcal{G}_{\mathbf{Q}_p^n}}^{\mathcal{G}_{\mathbf{Q}_p}}(\omega_n^h) \otimes \chi$ is defined over k . The (φ, Γ) -module $\mathcal{D}(V)$ attached to V is the n -dimensional vector space over $k((X))$ with basis e_1, \dots, e_n and

$$\begin{aligned} \varphi(e_i) &= e_{i+1} \text{ for } 1 \leq i \leq n-1 \text{ and } \varphi(e_n) = \chi(p)^n X^{-h(p-1)} e_1 \\ \gamma(e_i) &= \chi(\gamma) \left(\frac{\omega(\gamma)X}{\gamma(X)} \right)^{h \frac{p^{i-1}(p-1)}{p^n-1}} e_i \text{ for all } 1 \leq i \leq n. \end{aligned}$$

In case $1 \leq h \leq p^n - 2$ is primitive relative to n (so that V is absolutely irreducible), this was also computed by Berger [3, Theorem 2.1.6].

3.4. (ψ, Γ) -modules. We finish this section with a brief discussion of (ψ, Γ) -modules following [14, II.].

The field $k((X))$ has basis $\{(X+1)^i\}_{i=0, \dots, p-1}$ over its subfield $k((X^p)) = \varphi(k((X)))$. Thus, if D is an étale (φ, Γ) -module over $k((X))$, then every element $v \in D$ is uniquely of the form $v = \sum_{i=0}^{p-1} (X+1)^i \varphi(v_i)$ for some elements $v_i \in D$.

Definition 3.7 ([14, II.3]). Let $D \in \Phi\Gamma_{k((X))}^{\text{ét}}$. For an element $v = \sum_{i=0}^{p-1} (X+1)^i \varphi(v_i) \in D$ with $v_i \in D$ for $i = 0, \dots, p-1$, we define $\psi(v) = v_0$, resulting in an additive map $\psi : D \rightarrow D$ commuting with the Γ -action and satisfying $\psi \circ \varphi = \text{id}_D$.

This definition applies in particular to the trivial (φ, Γ) -module $k((X))$ itself. For $D \in \Phi\Gamma_{k((X))}^{\text{ét}}$, one then has the identities $\psi(f(X)\varphi(v)) = \psi(f(X))v$ and $\psi(\varphi(f(X)v)) = f(X)\psi(v)$ for all $f(X) \in k((X))$ and $v \in D$.

Moreover, the operator ψ is functorial in the sense that a morphism in $\Phi\Gamma_{k((X))}^{\text{ét}}$ is ψ -equivariant.

² $\mathcal{G}_{\mathbf{Q}_p}$ acts on D through the quotient Γ and on $\mathbf{F}_p((X))^{\text{sep}}$ as explained prior to Definition 3.4.

Definition-Lemma 3.8 ([14, Proposition II.4.2, Corollaire II.5.12]). *Let $D \in \Phi\Gamma_{k((X))}^{\text{ét}}$.*

- (i) *There exists a unique lattice³ $D^\sharp \subset D$, which is stable under the action of ψ such that ψ is surjective on it and for each $v \in D$, there is some $n \geq 1$ so that $\psi^n(v) \in D^\sharp$.
Moreover, if $N \subset D$ is any ψ -stable lattice on which ψ is surjective, then $N \subset D^\sharp \subset X^{-1}N$.*
- (ii) *There exists a unique minimal ψ -stable lattice D^\natural of D . Moreover, ψ is surjective on it.*

Remark 3.9 ([14, Proposition II.4.6, Proposition II.5.17]). Note that the definitions of ψ , D^\natural and D^\sharp do not use the Γ -action, i.e. they are already defined for étale φ -modules and on this category the assignments $D \mapsto D^\sharp$ and $D \mapsto D^\natural$ are functorial. In particular, if D is a (φ, Γ) -module over $k((X))$, then the subspaces D^\natural and D^\sharp are Γ -stable (since the φ -action commutes with Γ). The resulting objects are instances of (ψ, Γ) -modules over $k[[X]]$, the definition of which the reader can guess but we will not need.

Example 3.10 ([4, Lemme 1.1.2]).

- (i) Let $D = k((X))e$ be the (φ, Γ) -module attached to a smooth character $\chi: \mathbf{Q}_p^\times \rightarrow k^\times$ in the sense that $\varphi(e) = \chi(p)e$ and $\gamma(e) = \chi(\gamma)e$ for all $\gamma \in \Gamma \cong \mathbf{Z}_p^\times$, i.e. $D = \mathcal{D}(\chi)$ via Fontaine’s equivalence 3.5 (and local class field theory). Then $D^\natural = k[[X]]e \subset X^{-1}k[[X]]e = D^\sharp$.
- (ii) Let $n \geq 2$ be an integer. If V is an n -dimensional absolutely irreducible Galois representation of $\mathcal{G}_{\mathbf{Q}_p}$ on a k -vector space, then $\mathcal{D}(V)^\natural = \mathcal{D}(V)^\sharp \subset \mathcal{D}(V)$. In fact, any non-trivial ψ -stable $k[[X]]$ -submodule of $\mathcal{D}(V)^\sharp$ must already be equal to $\mathcal{D}(V)^\sharp$.

4. An extension of Colmez’s functor

In [13], Colmez defines a functor from the category of finitely generated smooth admissible k -representations of G to the category of Galois representations of $\mathcal{G}_{\mathbf{Q}_p}$ on k -vector spaces. This functor has the property that the image of an object only depends on the restriction of that object to the mirabolic subgroup P . More formally, using Breuil’s generalization [10], one may view Colmez’s functor as defined on the category of smooth P -representations at the cost of replacing the target category of Galois representations by its ind-category. When composed with the restriction functor $\text{Mod}_G^{\text{fg.adm}}(k) \rightarrow \text{Mod}_P^{\text{sm}}(k)$, this more general functor takes values in the subcategory of compact objects, i.e. usual Galois representations, and recovers Colmez’s functor. By Proposition 2.3 (i), we have an equality

³This means a finitely generated $k[[X]]$ -stable subspace containing a basis of D over $k((X))$.

of groups $\tilde{P} = P \times \mu_2$, so we may also precompose with the restriction functor $\text{Mod}_{\tilde{G}}^{\text{fg.adm}}(k) \rightarrow \text{Mod}_P^{\text{sm}}(k)$ and the finiteness results of Section 2.2 will imply that the composition again takes values in the subcategory of compact objects, thus extending Colmez’s functor to the whole category $\text{Mod}_G^{\text{fg.adm}}(k)$.

4.1. From smooth P -representations to Galois representations.

We recollect the definition and basic properties of Breuil’s functor, as found in [10, Section 2], from the category of smooth k -representations of P to the ind-category of Galois representations of $\mathcal{G}_{\mathbb{Q}_p}$ on k -vector spaces or equivalently of (φ, Γ) -modules over $k((X))$.

Definition 4.1 ([10, p. 8]). Let π be a smooth k -representation of P . We let $\mathcal{M}(\pi)$ denote the set of finitely generated $k[[X]][[F, \Gamma]]$ -submodules τ of π such that the space $\tau[X]$ of X -torsion elements is finite dimensional. We turn $\mathcal{M}(\pi)$ into a filtered set by giving it the partial order induced by inclusion (cf. [10, Lemme 2.1]).

Although not necessary for constructing the Galois object associated to a smooth representation of P , the following observation seems to fit here just fine: As in [17, Definition 2.3.1], we say that $\pi \in \text{Mod}_B^{\text{sm}}(k)$ is *locally $Z(\tilde{G})$ -finite* if for each $v \in \pi$, the action map $k[Z(\tilde{G})] \rightarrow \text{End}_k(k[Z(\tilde{G})]v)$ of $Z(\tilde{G})$ on the $Z(\tilde{G})$ -subrepresentation of π generated by v has finite dimensional image. Denote the full abelian subcategory of $\text{Mod}_B^{\text{sm}}(k)$ consisting of locally $Z(\tilde{G})$ -finite representations by $\text{Mod}_B^{\text{sm}}(k)_{Z(\tilde{G})\text{-fin}}$.

Lemma 4.2. *For $\pi \in \text{Mod}_B^{\text{sm}}(k)_{Z(\tilde{G})\text{-fin}}$, the subset of $\mathcal{M}(\pi)$ consisting of those elements which are stable under the action of \tilde{Z} is cofinal.*

Proof. An element $\tau \in \mathcal{M}(\pi)$ is finitely generated over $k[[X]][[F]]$. Since $Z(\tilde{G}) \subset \tilde{Z}$ has finite index and π is locally $Z(\tilde{G})$ -finite, there exist $n \geq 1$ and $z_1, \dots, z_n \in \tilde{Z}$ such that $\sum_{z \in \tilde{Z}} z\tau = \sum_{i=1}^n z_i\tau$. Each of the summands lies in $\mathcal{M}(\pi)$ and, by [10, Lemme 2.1], so does the finite sum which contains τ , so we are done. □

As in Section 3.3, we let $\varphi = \text{id}_k \otimes (a \mapsto a^p)$ be the relative Frobenius endomorphism on $k \otimes_{\mathbf{F}_p} \mathbf{F}_p((X)) = k((X))$, and we consider the Γ -action on it defined by $\gamma f(X) = f((X + 1)^{\chi_{\text{cyc}}(\gamma)} - 1)$ for all $f(X) \in k((X))$ and $\gamma \in \Gamma$. Note that the subring $k[[X]] \subset k((X))$ is stable under the action of φ and Γ .

Let now π be a smooth k -representation of P and consider an element $\tau \in \mathcal{M}(\pi)$. Linearizing the action of F on τ , we obtain the $k[[X]]$ -linear map

$$(4.1) \quad F: \varphi^* \tau \longrightarrow \tau.$$

We wish to take the Pontryagin dual $(-)^{\vee} = \text{Hom}_k(-, k)$ of it: We turn τ^{\vee} into a profinite $k[[X]][\Gamma]$ -module by letting $k[[X]]$ and Γ act via $(g(X).f)(-) = f(g(X)(-))$ and $(\gamma.f)(-) = f(\gamma^{-1}(-))$ for all $f \in \tau^{\vee}$, $g(X) \in k[[X]]$ and $\gamma \in \Gamma$. By smoothness, we can write $\tau = \varinjlim_d \tau[X^{p^d}]$. Now, each $\tau[X^{p^d}]$ is a finite dimensional $k[[X]][\Gamma]$ -submodule of τ , so that $\tau^{\vee} = \varprojlim_d (\tau[X^{p^d}]^{\vee})$ is indeed a profinite $k[[X]][\Gamma]$ -module.

The following isomorphism is a corrected version of the one defined by Breuil, see also [11, Equation (14)] and the corresponding footnote there.

Lemma 4.3. *The map*

$$\begin{aligned}
 (\varphi^* \tau)^{\vee} &= (k[[X]] \otimes_{\varphi, k[[X]]} \tau)^{\vee} \xrightarrow{\cong} k[[X]] \otimes_{\varphi, k[[X]]} \tau^{\vee} = \varphi^*(\tau^{\vee}) \\
 f &\longmapsto \sum_{i=0}^{p-1} (1+X)^i \otimes f((1+X)^{-i} \otimes (-))
 \end{aligned}$$

is a $k[[X]]$ -linear isomorphism.

Proof. Since $k[[X]]$ has basis $\{(1+X)^j\}_{0 \leq j \leq p-1}$ over $k[[X^p]]$, it suffices to show that the map is $k[[X^p]]$ -linear and equivariant for the basis elements, which is straight forward. Similarly, $\{(1+X)^{-j}\}_{0 \leq j \leq p-1}$ is another basis, which implies that the map is injective and gives a recipe for constructing the inverse. \square

Since τ is finitely generated over $k[[X]][F]$ and torsion over $k[[X]]$ (by smoothness), the cokernel of (4.1) is finitely generated and torsion over $k[[X]]$, i.e. it is finite dimensional over k . Thus, taking Pontryagin duals and inverting X , we obtain a $k((X))$ -linear injection

$$(4.2) \quad F^{\vee}[1/X]: \tau^{\vee}[1/X] \hookrightarrow (\varphi^* \tau)^{\vee}[1/X] \cong \varphi^*(\tau^{\vee}[1/X]).$$

Since $\tau[X]$ is finite dimensional, we dually have that $\tau^{\vee}/X\tau^{\vee}$ is finite dimensional, i.e. the profinite $k[[X]]$ -module τ^{\vee} is finitely generated by the topological Nakayama Lemma [1, Corollary Section 3]. Thus, $\tau^{\vee}[1/X]$ is finite dimensional over $k((X))$. Hence, the above injection $F^{\vee}[1/X]$ between finite dimensional $k((X))$ -vector spaces of the same dimension must be an isomorphism. We now define $\varphi_{\tau^{\vee}[1/X]}$ to be the composition

$$\varphi_{\tau^{\vee}[1/X]}: \tau^{\vee}[1/X] \xrightarrow{1 \otimes \text{id}} \varphi^*(\tau^{\vee}[1/X]) \xrightarrow{(F^{\vee}[1/X])^{-1}} \tau^{\vee}[1/X].$$

Together with the action of Γ on $\tau^{\vee}[1/X]$ induced by the contragredient action on τ^{\vee} defined above, we then obtain the (φ, Γ) -module $\tau^{\vee}[1/X]$ over $k((X))$, which is étale by construction.

Remark 4.4. The following formula appearing in [11, Equation (158)] will come in handy:

$$\sum_{i=0}^{p-1} (1 + X)^i \varphi_{\tau^\vee[1/X]}(((1 + X)^{-i} f) \circ F) = f \text{ for all } f \in \tau^\vee[1/X].$$

It is enough to prove this equality after applying the injection (4.2), where it then follows from the definitions.

By taking duals $(-)'$ in the category of étale (φ, Γ) -modules, we obtain the assignment $\tau \mapsto (\tau^\vee[1/X])'$. After possibly applying Fontaine’s equivalence 3.5, which is compatible with duals, we may pass to the filtered colimit to arrive at the definition:

Definition 4.5 ([10, p. 9]). Given a smooth k -representation π of P , we put⁴

$$\mathbf{D}(\pi) := \varinjlim_{\tau \in \mathcal{M}(\pi)} (\tau^\vee[1/X])' \in \text{Ind}(\Phi\Gamma_{k((X))}^{\text{ét}})$$

and

$$\mathbf{V}(\pi) := \varinjlim_{\tau \in \mathcal{M}(\pi)} \mathcal{V}(\tau^\vee[1/X])^\vee \in \text{Ind}(\text{Mod}_{\mathcal{G}_{\mathbf{Q}_p}}^{\text{fin}}(k)),$$

where the filtered colimit is taken in the respective (abelian) ind-category.

Note that we have fully faithful embeddings $\Phi\Gamma_{k((X))}^{\text{ét}} \hookrightarrow \text{Ind}(\Phi\Gamma_{k((X))}^{\text{ét}})$ and $\text{Mod}_{\mathcal{G}_{\mathbf{Q}_p}}^{\text{fin}}(k) \hookrightarrow \text{Ind}(\text{Mod}_{\mathcal{G}_{\mathbf{Q}_p}}^{\text{fin}}(k))$ which define an equivalence onto the respective subcategory of compact objects.

Proposition 4.6 ([10, Proposition 2.7]).

(i) *The assignment $\pi \mapsto \mathbf{D}(\pi)$ induces a covariant functor*

$$\mathbf{D}: \text{Mod}_P^{\text{sm}}(k) \longrightarrow \text{Ind}(\Phi\Gamma_{k((X))}^{\text{ét}}).$$

(ii) *The functor \mathbf{D} is left-exact.*

(iii) *If $\pi \in \text{Mod}_P^{\text{sm}}(k)$ is $k[F]$ -torsion, then $\mathbf{D}(\pi) = 0$.*

(iv) *If $0 \rightarrow \pi_1 \rightarrow \pi_2 \rightarrow \pi_3 \rightarrow 0$ is an exact sequence in $\text{Mod}_P^{\text{sm}}(k)$ such that every finitely generated $k[[X]][F]$ -submodule of π_1 has finite dimensional X -torsion, then the sequence $0 \rightarrow \mathbf{D}(\pi_1) \rightarrow \mathbf{D}(\pi_2) \rightarrow \mathbf{D}(\pi_3) \rightarrow 0$ is exact.*

Definition 4.7. For a smooth character $\chi: \mathbf{Q}_p^\times \rightarrow k^\times$ and a (φ, Γ) -module $D \in \Phi\Gamma_{k((X))}^{\text{ét}}$, define $D \otimes \chi \in \Phi\Gamma_{k((X))}^{\text{ét}}$ by

- $D \otimes \chi = D$ on underlying $k((X))$ -vector spaces
- $\varphi_{D \otimes \chi} = \chi(p)\varphi_D$
- $\gamma_{D \otimes \chi} = \chi(\gamma)\gamma_D$ for all $\gamma \in \Gamma$,

⁴Breuil defines a pro-object. Since we dualized, we obtain an ind-object.

i.e. we twist the (φ, Γ) -action on D by the character χ .

Remark 4.8.

- (i) The twisting construction $D \mapsto D \otimes \chi$ defines an autoequivalence on $\Phi\Gamma_{k((X))}^{\acute{e}t}$ and hence induces one on the ind-category $\text{Ind}(\Phi\Gamma_{k((X))}^{\acute{e}t})$.
- (ii) The functor $\mathbf{D}: \text{Mod}_P^{\text{sm}}(k) \rightarrow \text{Ind}(\Phi\Gamma_{k((X))}^{\acute{e}t})$ is compatible with twists by characters: For $\pi \in \text{Mod}_P^{\text{sm}}(k)$, we have $\mathbf{D}(\pi \otimes \chi \circ \det) = \mathbf{D}(\pi) \otimes \chi$.

4.2. Calculating Galois representations. For specific $\pi \in \text{Mod}_P^{\text{sm}}(k)$, or rather a specific $\tau \in \mathcal{M}(\pi)$, the Galois representation $\mathcal{V}(\tau^\vee[1/X])$ can be computed explicitly, which will be relevant for us when computing the image of a genuine supersingular representation under the extension of Colmez’s functor. The results in this subsection are an elaborated version of parts of Florian Herzig’s lecture notes [22].

We will work in the following setup also appearing in [28, Section 4.4.1].

Setup. Let $\pi \in \text{Mod}_P^{\text{sm}}(k)$. Suppose that we are given k -linearly independent Γ -eigenvectors $v_1, \dots, v_n \in \pi[X]$ killed by X with Γ -eigencharacters χ_1, \dots, χ_n , satisfying

$$(4.3) \quad X^{s_i} F(v_i) = c_i v_{i+1} \text{ for some } s_i \in \mathbf{Z}_{\geq 0} \text{ and } c_i \in k^\times$$

for all $1 \leq i \leq n$. Here and in what follows we understand the subscript i to be the unique element in $\{1, \dots, n\}$ congruent to i modulo n . We define the Γ -stable subspace

$$M = \bigoplus_{i=1}^n kv_i$$

of π and let $\pi_M = k[[X]][F]M \subset \pi$ be the $k[[X]][F]$ -submodule generated by M as in Definition 2.10. Finally, for $1 \leq i \leq n$, we put $\pi_i = k[[X]][F^n]v_i \subset \pi_M$. The subspaces π_M and π_i , for $1 \leq i \leq n$, are stable under the action of Γ . We would like to understand the (φ, Γ) -action on $\pi_M^\vee[1/X]$. If $s_i = 0$ for all $1 \leq i \leq n$, then $\pi_M = M$ is finite dimensional over k and hence its localized Pontryagin dual vanishes, which is why we from now on assume that $s_j > 0$ for some $1 \leq j \leq n$.

Lemma 4.9 ([28, Lemma 4.41]).

- (i) As $k[[X]][F^n, \Gamma]$ -modules, one has the direct sum decomposition $\pi_M = \bigoplus_{i=1}^n \pi_i$.
- (ii) For each $1 \leq i \leq n$, the $k[[X]]$ -module π_i^\vee is free of rank one.
- (iii) Assume that the following condition is satisfied: If for some pair $1 \leq i \neq j \leq n$, the vectors v_i, v_j have the same Γ -eigencharacter, then $s_i \neq s_j$. Then π_M is an irreducible P^+ -representation.

By parts (i) and (ii) of the lemma, we may write $\pi_M^\vee = \bigoplus_{i=1}^n \pi_i^\vee$ as $k[[X]]$ -modules with each π_i^\vee being free of rank one: namely, a basis element of it is given by any k -linear function $\pi_i \rightarrow k$ not vanishing at v_i (by Nakayama's Lemma). In order to compute the action of φ and Γ , we would like to exhibit a particular basis element, for which we write $\pi_i = \varinjlim_m \pi_{i,m}$ as the increasing union of the subspaces $\pi_{i,m} = k[[X]]F^{nm}v_i$. Letting $c = \prod_{i=1}^n c_i$ be the product of the constants appearing in (4.3), $e(i) = \sum_{j=0}^{n-1} p^{n-1-j} s_{i+j}$ and defining $e(i)_m = e(i) \frac{1-p^{nm}}{1-p^n}$ for all $m \geq 1$, an iteration of the equation(s) just cited shows that the map

$$(4.4) \quad \begin{aligned} k[[X]]/(X^{e(i)_m+1}) &\xrightarrow{\cong} \pi_{i,m} \\ f(X) &\mapsto c^{-m} f(X)F^{nm}v_i \end{aligned}$$

is an isomorphism, see also [28, Lemma 4.43, Equation (23)]. We now define $f_i \in \pi_i^\vee = \varprojlim_m \pi_{i,m}^\vee$ by requiring

$$f_i|_{\pi_{i,m}} : \pi_{i,m} \cong k[[X]]/(X^{e(i)_m+1}) \twoheadrightarrow k,$$

to be the composition of the inverse of the isomorphism (4.4) with the projection $\sum_{d=0}^{e_i(m)} a_d X^d \mapsto a_{e(i)_m}$, for all $m \geq 1$. We have $f_i(v_i) = 1 \neq 0$ and so f_i indeed is a basis element of π_i^\vee . Thus, the set $\{f_1, \dots, f_n\}$ is a basis of the free $k[[X]]$ -module π_M^\vee .

Lemma 4.10. *For all $1 \leq i \leq n$,*

$$\varphi(f_i) \in c_i^{-1}(1 + Xk[[X]])X^{s_i-(p-1)}f_{i+1}$$

and

$$\gamma(f_i) \in \chi_i(\gamma)^{-1}(1 + Xk[[X]])f_i, \text{ for all } \gamma \in \Gamma,$$

where χ_i is the Γ -eigencharacter of v_i .

Proof. We follow the proof of [11, Proposition 3.2.4.2]. Let $1 \leq i \leq n$. We claim that for all $j \geq 0$,

$$F^\vee((X^{s_i+j}f_{i+1})) = (X^{s_i+j}f_{i+1}) \circ F \in \begin{cases} c_i(1 + Xk[[X]])f_i & \text{if } j = 0 \\ Xk[[X]]f_i & \text{otherwise.} \end{cases}$$

Indeed, the first equality is just the definition of F^\vee , and we have $F(\pi_i) \subset \pi_{i+1}$ (which follows by an iteration of (4.3), see [28, Lemma 4.43(i)]). In particular, $(X^{s_i+j}f_{i+1}) \circ F \in \pi_i^\vee$, i.e. there exists a unique $h(X) \in k[[X]]$ such that $(X^{s_i+j}f_{i+1}) \circ F = h(X)f_i$. Since v_i and v_{i+1} are killed by X , evaluation at v_i gives

$$h(0) = f_i(h(X)v_i) = (h(X)f_i)(v_i) = f_{i+1}(X^{s_i+j}F(v_i)) = \begin{cases} c_i & \text{if } j = 0 \\ 0 & \text{if } j > 0, \end{cases}$$

where the last equality uses the relation (4.3). This proves the claim. In particular, for all $j \geq 0$, $F^\vee((1+X)^{-j}X^{s_i}f_{i+1}) \in c_i(1+Xk[[X]])f_i$. Applying φ , we obtain

$$\varphi(F^\vee((1+X)^{-j}X^{s_i}f_{i+1})) \in c_i(1+X^pk[[X]])\varphi(f_i),$$

say it is equal to $c_i(1+X^ph_j(X))\varphi(f_i)$ for some $h_j(X) \in k[[X]]$. For $f = X^{s_i}f_{i+1}$, the formula in Remark 4.4 then reads

$$\left(\sum_{j=0}^{p-1} (1+X)^j(1+X^ph_j(X)) \right) c_i\varphi(f_i) = X^{s_i}f_{i+1}.$$

Putting $Y = X+1$, we have $\sum_{j=0}^{p-1}(1+X)^j = (Y^p-1)/(Y-1) = (Y-1)^{p-1} = X^{p-1}$. So the factor in front of $c_i\varphi(f_i)$ lies in $X^{p-1} + (X^p)$, which gives the required containment for $\varphi(f_i)$.

Let now $\gamma \in \Gamma$. We have $\gamma(f_i) \in \pi_i^\vee$, say $\gamma(f_i) = h(X)f_i$ for some $h(X) \in k[[X]]$. Evaluating at v_i shows that $\chi_i(\gamma^{-1}) = f_i(\gamma^{-1}(v_i)) = h(0)$, so we are done. \square

Proposition 4.11. *Let $D \in \Phi\Gamma_{k((X))}^{\text{ét}}$ with basis f_1, \dots, f_n such that for all $1 \leq i \leq n$,*

$$\begin{aligned} \varphi(f_i) &\in d_i(1+Xk[[X]])X^{t_i}f_{i+1} \\ \gamma(f_i) &\in \omega(\gamma)^{b_i}(1+Xk[[X]])f_i, \text{ for all } \gamma \in \Gamma, \end{aligned}$$

for some $b_i, t_i \in \mathbf{Z}$ and $d_i \in k^\times$. Then D is isomorphic to the (φ, Γ) -module with basis e_1, \dots, e_n such that

$$\begin{aligned} \varphi(e_i) &= e_{i+1} \text{ for } 1 \leq i \leq n-1 \text{ and } \varphi(e_n) = dX^{-t(n-1)}e_1 \\ \gamma(e_i) &= \omega(\gamma)^{b_1} \left(\frac{\omega(\gamma)X}{\gamma(X)} \right)^{t \frac{p^{i-1}(p-1)}{p^n-1}} e_i \text{ for all } 1 \leq i \leq n, \end{aligned}$$

where⁵ $t = -\frac{p^{n-1}t_1+p^{n-2}t_2+\dots+t_n}{p-1} \in \mathbf{Z}$ and $d = \prod_{i=1}^n d_i$. In particular, the associated Galois representation $\mathcal{V}(D)$ attached to D via Fontaine's equivalence 3.5 is isomorphic to

$$\mathcal{V}(D) \cong \omega^{b_1}\mu_\lambda \otimes \text{Ind}_{\mathcal{G}_{\mathbf{Q}_p^n}}^{\mathcal{G}_{\mathbf{Q}_p}}(\omega_n^t),$$

where $\lambda \in \bar{k}^\times$ with $\lambda^n = d$.

Proof. Write $\varphi(f_i) = d_i g_i(X)X^{t_i}f_{i+1}$ for some $g_i(X) \in 1+Xk[[X]]$. We want to get rid of the $g_i(X)$ by replacing f_i by $f'_i = h_i(X)f_i$ for suitable $h_i(X) \in 1+Xk[[X]]$. The condition $\varphi(f'_i) = d_i X^{t_i} f'_{i+1}$ is equivalent to $h_{i+1}(X) = h_i(X^p)g_i(X)$. This has the unique solution $h_i(X) = \prod_{j=1}^\infty g_{i-j}(X^{p^{j-1}})$ in

⁵We will see in the proof that indeed $t \in \mathbf{Z}$.

$1 + Xk[[X]]$, see also the end of the proof of [11, Proposition 3.2.4.2]. Since the f'_i still satisfy the same condition as the f_i , we may from now on assume that $\varphi(f_i) = d_i X^{t_i} f_{i+1}$ for all $1 \leq i \leq n$.

Put now $e_1 := f_1$ and $e_{i+1} := \varphi(e_i)$ for $1 \leq i \leq n - 1$. Then $\varphi(e_n) = dX^\Sigma e_1$, where $d = \prod_{i=1}^n d_i$ and $\Sigma = \sum_{i=1}^n p^{n-i} t_i$. We still have $\gamma(e_i) \in \omega(\gamma)^{b_i} (1 + Xk[[X]])e_i$ for all $1 \leq i \leq n$ and $\gamma \in \Gamma$. Fix $\gamma \in \Gamma$ and write $\gamma(e_1) = \omega(\gamma)^{b_1} h_1(X) e_1$ for some $h_1(X) \in 1 + Xk[[X]]$. Since φ and γ commute, one obtains inductively that

$$\omega(\gamma)^\Sigma \left(\frac{\gamma(X)}{\omega(\gamma)X} \right)^\Sigma = h_1(X^{p^n}) h_1(X)^{-1}.$$

Since $h_1(X)$ is a 1-unit, we deduce that $\omega(\gamma)^\Sigma = 1$. This holds for every $\gamma \in \Gamma$, so we may write $\Sigma = -t(p - 1)$ for some $t \in \mathbf{Z}$. Now $\left(\frac{\gamma(X)}{\omega(\gamma)X} \right)^\Sigma$ has a unique $(p^n - 1)$ -th root in $1 + X\mathbf{F}_p[[X]]$, say $g(X)$, and we deduce that $h(X^{p^n}) = h(X)$ for $h(X) = h_1(X)g(X)^{-1}$, i.e. $h(X)$ is constant equal to 1.

This proves that $h_1(X) = \left(\frac{\omega(\gamma)X}{\gamma(X)} \right)^{t \frac{(p-1)}{p^n-1}}$ and thus the correct formula for $\gamma(e_1)$, which also implies the correct formulas for $\gamma(e_i)$, for all $1 \leq i \leq n$.

The description of the Galois representation $\mathcal{V}(D)$ now follows from Example 3.6. Indeed, since $\omega_n^{p^n-1} = 1$, we may replace t by $t + p^n - 1$ on the Galois side, which amounts to replacing the basis $\{e_i\}$ by $\{X^{-p^{i-1}(p-1)} e_i\}$ on the (φ, Γ) -module side. This way we may assume that $t \geq 0$, in which case the cited example applies. \square

Corollary 4.12. *The Galois representation $\mathcal{V}(\pi_M^\vee[1/X])^\vee \in \text{Mod}_{\mathcal{G}_{\mathbf{Q}_p}}^{\text{fin}}(k)$ is isomorphic to*

$$\omega^{a_1-1} \mu_\lambda \otimes \text{Ind}_{\mathcal{G}_{\mathbf{Q}_{p^n}}}^{\mathcal{G}_{\mathbf{Q}_p}} (\omega_n^s)$$

where $\lambda \in \bar{k}$ such that $\lambda^n = \prod_{i=1}^n c_i$, $s = \frac{p^{n-1}s_1 + p^{n-2}s_2 + \dots + s_n}{p-1} \in \mathbf{Z}$ and $a_1 \in \mathbf{Z}$ is such that $\chi_1(\gamma) = \omega(\gamma)^{a_1}$ for all $\gamma \in \Gamma$. Moreover, if the condition formulated in part (iii) of Lemma 4.9 is satisfied, then this is an irreducible Galois representation.

Proof. Everything except the irreducibility statement follows from Lemma 4.10 and Proposition 4.11. Here note that we dualize the Galois representation and we have $\omega_n^{(p^n-1)/(p-1)} = \omega$. Assume now that the condition of Lemma 4.9 (iii) is satisfied, so that π_M is an irreducible P^+ -representation. Put $D = \pi_M^\vee[1/X]$. The canonical map $\pi_M^\vee \hookrightarrow D$ is compatible with the action of F^\vee on the source and of ψ on the target (as follows for example from Remark 4.4). We then have $D^\natural \subset \pi_M^\vee$, which must be an equality by the irreducibility of π_M as a P^+ -representation. If now $D_1 \subset D$ is a (φ, Γ) -submodule, then $D_1^\natural \subset D^\natural = \pi_M^\vee$ by functoriality of $(-)^{\natural}$, Remark 3.9, so

once again by irreducibility $D_1^{\natural} = D^{\natural}$ and thus, by inverting X , $D_1 = D$. This shows that the (φ, Γ) -module D is irreducible. \square

4.3. The extended functor. For $\pi \in \text{Mod}_{\tilde{G}}^{\text{fg.adm}}(k)$, we now check that the ind-object $\mathbf{D}(\pi) \in \text{Ind}(\Phi\Gamma_{k((X))}^{\text{ét}})$ is compact. Recall from Definition 2.12 that to π we have associated the filtered set $\mathcal{G}(\pi)$ consisting of the finite dimensional $\tilde{K}\tilde{Z}$ -stable subspaces of π generating the latter as a \tilde{G} -representation. On the other hand, we may consider the filtered set $\mathcal{M}(\pi)$, Definition 4.1. Given some element $M \in \mathcal{G}(\pi)$, Proposition 2.11 implies that the $k[[X]][F, \Gamma]$ -submodule $\pi_M \subset \pi$ of π generated by M lies in $\mathcal{M}(\pi)$. The resulting map $\mathcal{G}(\pi) \rightarrow \mathcal{M}(\pi)$ preserves the partial order and thus induces a morphism

$$(4.5) \quad \varinjlim_{M \in \mathcal{G}(\pi)} (\pi_M^{\vee}[1/X])' \longrightarrow \varinjlim_{\tau \in \mathcal{M}(\pi)} (\tau^{\vee}[1/X])' = \mathbf{D}(\pi)$$

in the category $\text{Ind}(\Phi\Gamma_{k((X))}^{\text{ét}})$.

Proposition 4.13. *Let $\tau_1, \tau_2 \in \mathcal{M}(\pi)$ with $\tau_1 \subset \tau_2$, and assume that $\tau_1[F^{-1}] = \pi$. The transition map $(\tau_1^{\vee}[1/X])' \xrightarrow{\cong} (\tau_2^{\vee}[1/X])'$ is an isomorphism. In particular, $\mathbf{D}(\pi) = (\tau_1^{\vee}[1/X])' \in \Phi\Gamma_{k((X))}^{\text{ét}}$ is a compact object of $\text{Ind}(\Phi\Gamma_{k((X))}^{\text{ét}})$. Moreover, any finitely generated $k[[X]][F]$ -submodule of π has finite dimensional X -torsion.*

Proof. This is similar to the proof of [16, Proposition 4.4]: If τ_2 is an arbitrary finitely generated $k[[X]][F]$ -submodule of $\pi = \tau_1[F^{-1}]$ (not necessarily an element of $\mathcal{M}(\pi)$), then we may choose $N \gg 0$ such that $F^N\tau_2 \subset \tau_1$. In particular, the injective operator F^N maps $\tau_2[X]$ into $\tau_1[X^{p^N}]$ (which is finite dimensional since $\tau_1 \in \mathcal{M}(\pi)$). This already proves the final assertion. For the first part, assume now that additionally $\tau_2 \in \mathcal{M}(\pi)$ and $\tau_1 \subset \tau_2$. Then the quotient τ_2/τ_1 is finitely generated over $k[[X]]$ (as τ_2 is finitely generated over $k[[X]][F]$ and the quotient is killed by F^N). But it is also torsion by smoothness, so it must be finite dimensional and is therefore killed when taking Pontryagin dual and inverting X , i.e. $\tau_1^{\vee}[1/X] = \tau_2^{\vee}[1/X]$, proving the first part. \square

Corollary 4.14. *The morphism (4.5) is an isomorphism, and all the transition maps in the inverse system $\{\pi_M^{\vee}[1/X]\}_{M \in \mathcal{G}(\pi)}$ are isomorphisms. In particular, $\mathbf{D}(\pi) \in \Phi\Gamma_{k((X))}^{\text{ét}}$ is a compact object of $\text{Ind}(\Phi\Gamma_{k((X))}^{\text{ét}})$. Moreover, any finitely generated $k[[X]][F]$ -submodule of π has finite dimensional X -torsion.*

Proof. The decomposition $\tilde{G} = P\tilde{K}\tilde{Z}$ implies that $\pi_M[F^{-1}] = \pi$ for any $M \in \mathcal{G}(\pi)$. \square

Remark 4.15. The composition

$$\mathrm{Mod}_G^{\mathrm{fg.adm}}(k) \xrightarrow{(-)|_P} \mathrm{Mod}_P^{\mathrm{sm}}(k) \xrightarrow{\mathbf{D}} \mathrm{Ind}(\Phi\Gamma_{k((X))}^{\mathrm{ét}})$$

may thus be viewed as valued in the category $\Phi\Gamma_{k((X))}^{\mathrm{ét}}$, and the resulting functor, still denoted by \mathbf{D} , recovers Colmez’s functor when restricted to the full subcategory $\mathrm{Mod}_G^{\mathrm{fg.adm}}(k)$, see [10, Proposition 3.2 (iii)].

5. Metaplectic Montréal functor

In this section, we study the restriction of the functor \mathbf{D} (or equivalently \mathbf{V}) to the full subcategory $\mathrm{Mod}_{G,t}^{\mathrm{fg.adm}}(k)$ of genuine objects. As we will soon see, the pullback of the metaplectic cover to the center $Z \cong \mathbf{Q}_p^\times$ of G defines an interesting action on this category. Translating this action over to the Galois side and passing to the category of equivariant objects, we obtain so-called metaplectic Galois representations and the previously extended functor induces the metaplectic Montréal functor.

5.1. Twist-equivariant (φ, Γ) -modules. Throughout this section we fix a group Ω . We remind the reader what it means for Ω to act on a category and what the corresponding category of Ω -equivariant objects is.

Definition 5.1 ([27, 2.1, 2.4]). Let \mathcal{C} be a category.

- (i) A *group action of Ω on \mathcal{C}* consists of the following data:
 - for each $g \in \Omega$, an autoequivalence $A_g : \mathcal{C} \xrightarrow{\sim} \mathcal{C}$
 - for each pair $g, h \in \Omega$, a natural isomorphism $\eta_{g,h} : A_g A_h \Rightarrow A_{gh}$
 such that the diagram

$$\begin{array}{ccc} A_g A_h A_k & \xrightarrow{A_g \eta_{h,k}} & A_g A_{hk} \\ \eta_{g,h} A_k \downarrow & & \downarrow \eta_{g,hk} \\ A_{gh} A_k & \xrightarrow{\eta_{gh,k}} & A_{ghk} \end{array}$$

commutes for all elements $g, h, k \in \Omega$.

- (ii) A *Ω -equivariant object* is a pair $(D, \{\theta_g : D \xrightarrow{\sim} A_g(D)\}_{g \in \Omega})$ consisting of an object D of \mathcal{C} and a collection of isomorphisms $\theta_g : D \xrightarrow{\sim} A_g(D)$, for $g \in \Omega$, such that the diagram

$$\begin{array}{ccc} D & \xrightarrow{\theta_g} & A_g(D) \\ \theta_{gh} \downarrow & & \downarrow A_g \theta_h \\ A_{gh}(D) & \xleftarrow{\eta_{g,h}(D)} & A_g(A_h(D)) \end{array}$$

commutes for all $g, h \in \Omega$.

The Ω -equivariant objects form in an evident way a category \mathcal{C}^Ω .

Recall from Definition 4.7 that for a (φ, Γ) -module D and a smooth character $\chi: \mathbf{Q}_p^\times \rightarrow k^\times$, we may form the twisted (φ, Γ) -module $D \otimes \chi$. This construction defines an action of the abelian group $\text{Hom}^{\text{cts}}(\mathbf{Q}_p^\times, k^\times)$ on the category $\Phi\Gamma_{k((X))}^{\text{ét}}$ by taking $A_\chi = - \otimes \chi$ and $\eta_{\chi, \psi} = \text{id}$ in Definition 5.1. In particular, given a group homomorphism

$$\Omega \longrightarrow \text{Hom}^{\text{cts}}(\mathbf{Q}_p^\times, k^\times), \quad g \longmapsto \chi_g,$$

we may consider the category $\left(\Phi\Gamma_{k((X))}^{\text{ét}}\right)^\Omega$ of Ω -equivariant (φ, Γ) -modules. Its objects are pairs (D, θ) consisting of a (φ, Γ) -module $D \in \Phi\Gamma_{k((X))}^{\text{ét}}$ and an action $\theta: \Omega \rightarrow \text{Aut}_{k((X))}(D)$ of Ω on the underlying $k((X))$ -vector space of D with $\theta(g): D \cong D \otimes \chi_g$ being an isomorphism of (φ, Γ) -modules.

5.1.1. Frobenius reciprocity. Assume now that $\Delta \subset \Omega$ is a subgroup of finite index, which we assume to be normal for convenience. Consider the restriction functor

$$(5.1) \quad (-)|_\Delta: \left(\Phi\Gamma_{k((X))}^{\text{ét}}\right)^\Omega \longrightarrow \left(\Phi\Gamma_{k((X))}^{\text{ét}}\right)^\Delta, \quad (D, \theta) \longmapsto (D, \theta|_\Delta).$$

We define a left adjoint by

$$\text{Ind}_\Delta^\Omega((D, \theta)) = \{f: \Omega \longrightarrow D : f(hg) = \theta(h)f(g) \text{ for all } g \in \Omega, h \in \Delta\}$$

with $k((X))$ -vector space structure induced by that on D and Ω -action defined by $(g.f)(g') = f(g'g)$, i.e. so far this is simply the usual induction of the representation $\theta: \Delta \rightarrow \text{Aut}_{k((X))}(D)$. We endow it with a (φ, Γ) -action by

- $(\varphi.f)(g) = \chi_g(p)\varphi(f(g))$ for all $g \in \Omega$
- $(\gamma.f)(g) = \chi_g(\gamma)\gamma(f(g))$ for all $g \in \Omega$, for all $\gamma \in \Gamma$.

Proposition 5.2.

- (i) *The (φ, Γ) -action on $\text{Ind}_\Delta^\Omega((D, \theta))$ is well-defined and yields a Ω -equivariant object in $\Phi\Gamma_{k((X))}^{\text{ét}}$.*
- (ii) *The resulting functor*

$$\text{Ind}_\Delta^\Omega(-): \left(\Phi\Gamma_{k((X))}^{\text{ét}}\right)^\Delta \longrightarrow \left(\Phi\Gamma_{k((X))}^{\text{ét}}\right)^\Omega$$

is a left adjoint of the restriction functor (5.1).

Proof. It is straight forward to check that the (φ, Γ) -action on $\text{Ind}_\Delta^\Omega((D, \theta))$ is well-defined. Since $\Delta \subset \Omega$ is normal, we have

$$(5.2) \quad \text{Ind}_\Delta^\Omega((D, \theta))|_\Delta \cong \bigoplus_{g \in \Delta \backslash \Omega} (D \otimes \chi_g, \theta^g),$$

where $\theta^g : \Delta \xrightarrow{g(-)g^{-1}} \Delta \xrightarrow{\theta} \text{Aut}_{k((X))}(D) = \text{Aut}_{k((X))}(D \otimes \chi_g)$. This space is finite dimensional over $k((X))$ since Δ has finite index in Ω . In particular, this is a (φ, Γ) -module, which finishes the proof of part (i). Part (ii) follows from the usual Frobenius reciprocity for representations of Δ and Ω , respectively. \square

Remark 5.3. The action of $\text{Hom}^{\text{cts}}(\mathbf{Q}_p^\times, k^\times)$ (hence of Ω) on $\Phi\Gamma_{k((X))}^{\text{ét}}$ extends to the ind-category $\text{Ind}(\Phi\Gamma_{k((X))}^{\text{ét}})$. The functors $\text{Ind}_\Delta^\Omega(-)$ and $(-)|_\Delta$ extend correspondingly and the adjunction of part (ii) in the previous proposition carries over.

5.1.2. Transferring twist-actions with a warning. Using Fontaine’s equivalence 3.5, we transfer the action of Ω on $\Phi\Gamma_{k((X))}^{\text{ét}}$ to the category $\text{Mod}_{\mathcal{G}_{\mathbf{Q}_p}}^{\text{fin}}(k)$, which then induces an equivalence

$$(5.3) \quad \left(\Phi\Gamma_{k((X))}^{\text{ét}}\right)^\Omega \simeq \left(\text{Mod}_{\mathcal{G}_{\mathbf{Q}_p}}^{\text{fin}}(k)\right)^\Omega$$

on Ω -equivariant objects.

We warn the reader that the transferred action of Ω on $\text{Mod}_{\mathcal{G}_{\mathbf{Q}_p}}^{\text{fin}}(k)$ is not simply given by twisting by the characters $\chi_g, g \in \Omega$, viewed as characters of $\mathcal{G}_{\mathbf{Q}_p}$ via local class field theory: For $D \in \Phi\Gamma_{k((X))}^{\text{ét}}$ and $\chi : \mathbf{Q}_p^\times \rightarrow k^\times$ is a smooth character, there is an isomorphism $\mathcal{V}(D \otimes \chi) \cong \mathcal{V}(D) \otimes \chi$, it however depends on the choice of a $(p - 1)$ -th root of $\chi(p)$ and is therefore not canonical. In general, one will not be able to choose the $(p - 1)$ -th roots of the elements $\chi_g(p)$ for $g \in \Omega$ in a multiplicatively compatible way.

5.2. Metaplectic Galois representations. We now take $\Omega = \widetilde{\mathbf{Q}}_p^\times \cong \widetilde{\mathbf{Q}}_p^\times$ to be the pullback of the metaplectic cover to the center $Z \cong \mathbf{Q}_p^\times$ of G :

$$1 \longrightarrow \mu_2 \longrightarrow \widetilde{\mathbf{Q}}_p^\times \longrightarrow \mathbf{Q}_p^\times \longrightarrow 1,$$

and we consider the surjective group homomorphism

$$\begin{aligned} \widetilde{\mathbf{Q}}_p^\times &\twoheadrightarrow \mathbf{Q}_p^\times \twoheadrightarrow \text{Hom}^{\text{cts}}(\mathbf{Q}_p^\times, \mu_2) \\ z &\longmapsto \chi_z = (\mu_{-1}^{v(z)} \omega^{v(z)} \mu_{\omega(z)})^{\frac{p-1}{2}}, \end{aligned}$$

where $v : \mathbf{Q}_p^\times \rightarrow \mathbf{Z}$ is the p -adic valuation. We write $\chi_{\tilde{z}} = \chi_z$ for $\tilde{z} \in \widetilde{\mathbf{Q}}_p^\times$ with image $z \in \mathbf{Q}_p^\times$. The definition of these characters comes from the following lemma, which is quickly verified using the explicit cocycle (2.1), see also [28, Lemma 4.4].

Lemma 5.4. *The conjugation action $\tilde{z}(-)(\tilde{z})^{-1}$ of an element $\tilde{z} \in \widetilde{Z} \cong \widetilde{\mathbf{Q}}_p^\times$ on \widetilde{G} is given by the $\widetilde{G} \supset \mu_2$ -valued homomorphism $\chi_{\tilde{z}} \circ \det$.*

In particular, for $\pi \in \text{Mod}_{\tilde{G},\iota}^{\text{sm}}(k)$ (resp. $\text{Mod}_{\tilde{B},\iota}^{\text{sm}}(k)$) and $\tilde{z} \in \widetilde{\mathbf{Q}_p^\times} \cong \tilde{Z}$, the action map $\tilde{z}: \pi \xrightarrow{\cong} \pi \otimes \chi_{\tilde{z}} \circ \det$ is \tilde{G} -equivariant (resp. \tilde{B} -equivariant).

Hence, if we let $\widetilde{\mathbf{Q}_p^\times}$ act on $\text{Mod}_{\tilde{B},\iota}^{\text{sm}}(k)$ via twisting by the characters $\chi_{\tilde{z}} \circ \det$, then we have the canonical functor

$$(5.4) \quad \text{Mod}_{\tilde{B},\iota}^{\text{sm}}(k) \longrightarrow \left(\text{Mod}_{\tilde{B},\iota}^{\text{sm}}(k) \right)^{\widetilde{\mathbf{Q}_p^\times}}, \quad \pi \longmapsto (\pi, \pi|_{\widetilde{\mathbf{Q}_p^\times}}),$$

the image of which lies in the full subcategory of $\widetilde{\mathbf{Q}_p^\times}$ -equivariant objects whose underlying $\widetilde{\mathbf{Q}_p^\times}$ -action is smooth and genuine. This motivates the following definition:

Definition 5.5.

- (i) A metaplectic (φ, Γ) -module over $k((X))$ is an étale $\widetilde{\mathbf{Q}_p^\times}$ -equivariant (φ, Γ) -module (D, θ) over $k((X))$ such that the action $\theta: \widetilde{\mathbf{Q}_p^\times} \rightarrow \text{Aut}_{k((X))}(D)$ is smooth and genuine. We denote the resulting category by $\widetilde{\Phi}\Gamma_{k((X)),\iota}^{\text{ét}}$.
- (ii) The category of metaplectic Galois representations of $\mathcal{G}_{\mathbf{Q}_p}$ on k -vector spaces is the category $\widetilde{\text{Mod}}_{\mathcal{G}_{\mathbf{Q}_p},\iota}^{\text{fin}}(k)$ equivalent to $\widetilde{\Phi}\Gamma_{k((X)),\iota}^{\text{ét}}$ under (5.3).

Now, as explained in Remark 4.8, the functor $\mathbf{D}: \text{Mod}_{\tilde{B},\iota}^{\text{sm}}(k) \xrightarrow{(-)|_P} \text{Mod}_P^{\text{sm}}(k) \rightarrow \text{Ind}(\Phi\Gamma_{k((X))}^{\text{ét}})$ is compatible with twists and thus induces a functor on $\widetilde{\mathbf{Q}_p^\times}$ -equivariant objects. Precomposing with (5.4) gives rise to the horizontal arrows in the left square of the diagram

$$(5.5) \quad \begin{array}{ccccc} & & \mathbf{D} & & \\ & & \curvearrowright & & \\ & \text{Mod}_{\tilde{B},\iota}^{\text{sm}}(k) & \xrightarrow{\tilde{\mathbf{D}}} & \text{Ind}(\Phi\Gamma_{k((X))}^{\text{ét}})^{\widetilde{\mathbf{Q}_p^\times}} & \xrightarrow{\text{forget}} & \text{Ind}(\Phi\Gamma_{k((X))}^{\text{ét}}) \\ & \uparrow & & \uparrow & & \parallel \\ & \text{Mod}_{\tilde{B},\iota}^{\text{sm}}(k)_{Z(\tilde{G})\text{-fin}} & \xrightarrow{\tilde{\mathbf{D}}} & \text{Ind}(\widetilde{\Phi}\Gamma_{k((X)),\iota}^{\text{ét}}) & \xrightarrow{\text{forget}} & \text{Ind}(\Phi\Gamma_{k((X))}^{\text{ét}}) \\ & \uparrow & & \uparrow & & \uparrow \\ & \text{Mod}_{\tilde{G},\iota}^{\text{fg.adm}}(k) & \xrightarrow{\tilde{\mathbf{D}}} & \widetilde{\Phi}\Gamma_{k((X)),\iota}^{\text{ét}} & \xrightarrow{\text{forget}} & \Phi\Gamma_{k((X))}^{\text{ét}}, \end{array}$$

where for the middle horizontal functor we implicitly made use of Lemma 4.2.

Definition 5.6. The *metaplectic Montréal functor* is the functor

$$\widetilde{\mathbf{D}}: \text{Mod}_{\widetilde{G},\iota}^{\text{fg.adm}}(k) \longrightarrow \widetilde{\Phi}\Gamma_{k((X)),\iota}^{\text{ét}}$$

appearing at the bottom of diagram (5.5). We further define

$$\widetilde{\mathbf{V}} = \mathcal{V} \circ \widetilde{\mathbf{D}}: \text{Mod}_{\widetilde{G},\iota}^{\text{fg.adm}}(k) \longrightarrow \widetilde{\text{Mod}}_{\mathcal{G}_{\mathbf{Q}_p},\iota}^{\text{fin}}(k)$$

to be the composition of $\widetilde{\mathbf{D}}$ with Fontaine’s equivalence, and we also refer to it as the metaplectic Montréal functor.

Remark 5.7. The functor $\widetilde{\mathbf{D}}: \text{Mod}_{\widetilde{B},\iota}^{\text{sm}}(k) \rightarrow \text{Ind}(\Phi\Gamma_{k((X))}^{\text{ét}})^{\widetilde{\mathbf{Q}}_p^\times}$ is compatible with twists: if $\widetilde{\mathbf{D}}(\pi) = (\mathbf{D}(\pi), \theta)$ and $\chi: \mathbf{Q}_p^\times \rightarrow k^\times$ is a smooth character, then

$$\widetilde{\mathbf{D}}(\pi \otimes \chi \circ \det) = (\mathbf{D}(\pi) \otimes \chi, \theta \otimes \chi^2),$$

where $(\theta \otimes \chi^2)_{\widetilde{z}} = \chi(z^2)\theta_{\widetilde{z}}$ for $\widetilde{z} \in \widetilde{\mathbf{Q}}_p^\times$ with image $z \in \mathbf{Q}_p^\times$.

Since the character $\chi_{\widetilde{z}}$ is trivial for $\widetilde{z} \in \widetilde{S} = S \times \mu_2$, Frobenius reciprocity for the inclusion $S \subset \widetilde{S} \subset \widetilde{\mathbf{Q}}_p^\times$ as presented in Section 5.1.1 (restricted to smooth genuine objects) may be interpreted as an adjoint pair

$$(5.6) \quad \text{Ind}_{\widetilde{S}}^{\widetilde{\mathbf{Q}}_p^\times}(- \boxtimes \iota): \text{Mod}_S^{\text{sm}}(\Phi\Gamma_{k((X))}^{\text{ét}}) \rightleftarrows \widetilde{\Phi}\Gamma_{k((X)),\iota}^{\text{ét}} : (-)|_S,$$

where the category on the left-hand side consists of group homomorphisms $S \rightarrow \text{Aut}_{\Phi\Gamma_{k((X))}^{\text{ét}}}(D)$, for $D \in \Phi\Gamma_{k((X))}^{\text{ét}}$, such that the underlying $k((X))$ -linear representation of S on D is smooth. A corresponding version exists on the level of ind-categories, cf. Remark 5.3.

Recall from Lemma 2.3(i) that $\widetilde{B}_S = B_S \times \mu_2$ as groups, so the functor $- \boxtimes \iota$ gives an equivalence of categories $\text{Mod}_{\widetilde{B}_S}^{\text{sm}}(k) \simeq \text{Mod}_{\widetilde{B}_S,\iota}^{\text{sm}}(k)$.

Proposition 5.8. *Let $\pi \in \text{Mod}_{\widetilde{B}}^{\text{sm}}(k)$ with central character $\chi: \mathbf{Q}_p^\times \rightarrow k^\times$. Then there is a canonical isomorphism*

$$\widetilde{\mathbf{D}}(\text{Ind}_{\widetilde{B}_S}^{\widetilde{B}}(\pi|_{B_S} \boxtimes \iota)) \cong \text{Ind}_{\widetilde{S}}^{\widetilde{\mathbf{Q}}_p^\times}(\chi|_S \boxtimes \mathbf{D}(\pi) \boxtimes \iota)$$

in the category $\text{Ind}(\widetilde{\Phi}\Gamma_{k((X)),\iota}^{\text{ét}})$, where $\chi|_S \boxtimes \mathbf{D}(\pi) \in \text{Mod}_S^{\text{sm}}(\Phi\Gamma_{k((X))}^{\text{ét}})$ is the action of S on $\mathbf{D}(\pi)$ via $\chi|_S$.

Proof. The inclusion $\widetilde{\mathbf{Q}}_p^\times \cong \widetilde{Z} \subset \widetilde{B}$ induces a bijection $\widetilde{S} \setminus \widetilde{\mathbf{Q}}_p^\times \cong \widetilde{B}_S \setminus \widetilde{B}$ and it follows from Lemma 5.4 that

$$\text{Ind}_{\widetilde{B}_S}^{\widetilde{B}}(\pi|_{B_S} \boxtimes \iota)|_{B_S} \cong \bigoplus_{\widetilde{z} \in \widetilde{S} \setminus \widetilde{\mathbf{Q}}_p^\times} (\pi \otimes \chi_{\widetilde{z}} \circ \det)|_{B_S}.$$

Since the definition of \mathbf{D} only depends on the restriction to P , we deduce that

$$(5.7) \quad \mathbf{D}(\mathrm{Ind}_{\tilde{B}_S}^{\tilde{B}}(\pi|_{B_S} \boxtimes \iota)) \cong \bigoplus_{\tilde{z} \in \tilde{S} \backslash \widehat{\mathbf{Q}}_p^\times} \mathbf{D}(\pi) \otimes \chi_{\tilde{z}}.$$

This contains $\mathbf{D}(\pi)$ and, since S acts on $\tilde{\mathbf{D}}(\mathrm{Ind}_{\tilde{B}_S}^{\tilde{B}}(\pi|_{B_S} \boxtimes \iota))$ via the character $\chi|_S$, we may apply (the ind-version of) Frobenius reciprocity (5.6) to obtain a non-trivial morphism

$$\mathrm{Ind}_{\tilde{S}}^{\widehat{\mathbf{Q}}_p^\times}(\chi|_S \boxtimes \mathbf{D}(\pi) \boxtimes \iota) \longrightarrow \tilde{\mathbf{D}}(\mathrm{Ind}_{\tilde{B}_S}^{\tilde{B}}(\pi|_{B_S} \boxtimes \iota))$$

in the category $\mathrm{Ind}(\widetilde{\Phi}\Gamma_{k((X)), \iota}^{\acute{e}t})$. That this is an isomorphism can be checked in the category $\mathrm{Ind}(\Phi\Gamma_{k((X))}^{\acute{e}t})$, where it follows from (5.7) and (5.2). \square

5.3. From metaplectic Galois to metaplectic Borel. Following the recipe [14, Chapter III] of Colmez, we explain how to attach a smooth genuine representation of \tilde{B} to a metaplectic (φ, Γ) -module.

For a (φ, Γ) -module $D \in \Phi\Gamma_{k((X))}^{\acute{e}t}$, we have the operator ψ as well as the largest (ψ, Γ) -stable lattice $D^\sharp \subset D$ as introduced in Section 3.4. The operator ψ on D^\sharp is in general not invertible, but this can be overcome by passing to the limit

$$\psi^{-\infty}(D) := \varprojlim_{\psi} D^\sharp = \{(v_n)_{n \geq 0} : \psi(v_{n+1}) = v_n, \forall n \geq 0\}.$$

Remark 5.9. The canonical map

$$\psi^{-\infty}(D) \xrightarrow{\cong} (\varprojlim_{\psi} D)_b,$$

where the right-hand side denotes bounded ψ -compatible sequences, is bijective. Indeed, D^\sharp is finitely generated over $k[[X]]$, hence bounded, i.e. the map is well-defined. Conversely, if $(v_n)_{n \geq 0}$ is a bounded ψ -compatible sequence, then there exists some $d \geq 0$ such that $v_n \in X^{-p^d} D^\sharp$. Since the inclusion $D^\sharp \subset X^{-p^d} D^\sharp$ has finite dimensional cokernel, the defining property of D^\sharp allows us to choose $N \geq 0$ with $\psi^N(X^{-p^d} D^\sharp) \subset D^\sharp$. In particular, $v_n = \psi^N(v_{n+N}) \in D^\sharp$ for all $n \geq 0$.

Proposition 5.10 ([14, Proposition III.1.1]). *The inverse limit $\psi^{-\infty}(D) = \varprojlim_{\psi} D^\sharp$ is a $k[[X]]$ -module via $f(X) \cdot (v_n)_n = (\varphi^n(f(X))v_n)_n$ and the following identities uniquely determine an action of the mirabolic subgroup P*

on it: For $v = (v_n)_{n \geq 0} \in \psi^{-\infty}(D)$,

$$\begin{aligned} \begin{pmatrix} p^m & 0 \\ 0 & 1 \end{pmatrix} v &= (v_{n+m})_{n \geq 0} \text{ for all } m \in \mathbf{Z}, \\ \begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} v &= (\gamma(v_n))_{n \geq 0} \text{ for all } \gamma \in \mathbf{Z}_p^\times = \Gamma, \\ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} v &= (X + 1)^{-b} v = ((X^{p^n} + 1)^{-b} v_n)_{n \geq 0} \text{ for all } b \in \mathbf{Z}_p, \end{aligned}$$

where we use the convention that $v_n = \psi^{-n}(v_0)$ if $n < 0$.

Remark 5.11. Our definition of the action of P on $\psi^{-\infty}(D)$ differs from the one of Colmez by conjugation by the element $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$.

Lemma 5.12. *The assignment $D \mapsto \psi^{-\infty}(D)$ defines an exact functor from $\Phi\Gamma_{k((X))}^{\text{ét}}$ to the category of k -linear representations of P , which is compatible with twists: If $\chi: \mathbf{Q}_p^\times \rightarrow k^\times$ is a smooth character, then the map*

$$\begin{aligned} \varprojlim_{\psi} (D \otimes \chi)^\sharp &\cong (\varprojlim_{\psi} D^\sharp) \otimes \chi \circ \det \\ (v_n)_{n \geq 0} &\longmapsto (\chi(p)^{-n} v_n)_{n \geq 0} \end{aligned}$$

is a P -equivariant isomorphism.

Proof. While functoriality follows from the assignment $D \mapsto D^\sharp$ being functorial, see Remark 3.9, exactness is the content of [14, Theorem III.3.5].

Let now $\chi: \mathbf{Q}_p^\times \rightarrow k^\times$ be a smooth character. It follows from the definition of $D \otimes \chi$, Definition 4.7, that $(D \otimes \chi)^\sharp = D^\sharp \otimes \chi$ with $\psi_{D \otimes \chi}(v) = \chi(p)^{-1} \psi_D(v)$ for all $v \in D$. From this one verifies that the described map is a well-defined P -equivariant isomorphism. \square

We endow D^\sharp with the X -adic topology and give $\psi^{-\infty}(D) = \varprojlim_{\psi} D^\sharp$ the inverse limit topology. Here note that the transition maps are $k[[X]]$ -linear if the n -th component is viewed as a $k[[X]]$ -module via the n -th power φ^n (this defines the same topology as the X -adic one). Hence, $\varprojlim_{\psi} D^\sharp$ is a profinite $k[[X]]$ -module. Continuity of the Γ -action on D implies that the action map $P \times \psi^{-\infty}(D) \rightarrow \psi^{-\infty}(D)$ is continuous (for the product topology on the source), hence the contragredient action of P on the Pontryagin dual $(\psi^{-\infty}(D))^\vee$ is smooth. Precomposing the functor $D \mapsto \psi^{-\infty}(D)$ of Lemma 5.12 with the dual $(-)'$ of (φ, Γ) -modules and postcomposing with the Pontryagin dual, we thus obtain the exact covariant functor

$$(5.8) \quad \Omega := (\psi^{-\infty} \circ (-)')^\vee: \Phi\Gamma_{k((X))}^{\text{ét}} \longrightarrow \text{Mod}_P^{\text{sm}}(k),$$

whose notation we adopt from [3]. By Lemma 5.4, we have the semi-direct product decomposition

$$(5.9) \quad \tilde{B} = \tilde{Z} \rtimes P,$$

where the conjugation action of P on \tilde{Z} is given by the group homomorphism $P \rightarrow \text{Aut}(\tilde{Z})$, $g \mapsto [\tilde{z} \mapsto \chi_{\tilde{z}}(\det(g))]$.

Let now $(D, \theta) \in \tilde{\Phi}\Gamma_{k((X)), \iota}^{\text{ét}}$ be a metaplectic (φ, Γ) -module. Besides the smooth k -representation $\Omega(D)$ of P , we then have, for $\tilde{z} \in \widetilde{\mathbf{Q}_p^\times} \cong \tilde{Z}$, the P -equivariant isomorphism $\Omega(\theta_{\tilde{z}}): \Omega(D) \xrightarrow{\cong} \Omega(D \otimes \chi_{\tilde{z}}) \cong \Omega(D) \otimes \chi_{\tilde{z}} \circ \det$ (implicitly making use of the isomorphism in Lemma 5.12), which assemble into a smooth and genuine action of \tilde{Z} on the underlying k -vector space of $\Omega(D)$. The semi-direct product decomposition (5.9) and the P -equivariance of the isomorphisms $\Omega(\theta_{\tilde{z}})$ allows us to uniquely glue the \tilde{Z} -action on $\Omega(D)$ to a smooth and genuine k -representation $\tilde{\Omega}(D)$ of \tilde{B} . This construction gives rise to an exact covariant functor

$$\tilde{\Omega}: \tilde{\Phi}\Gamma_{k((X)), \iota}^{\text{ét}} \longrightarrow \text{Mod}_{\tilde{B}, \iota}^{\text{sm}}$$

making the diagram

$$\begin{array}{ccc} \Phi\Gamma_{k((X))}^{\text{ét}} & \xrightarrow{\Omega} & \text{Mod}_P^{\text{sm}}(k) \\ \text{forget} \uparrow & & \uparrow (-)|_P \\ \tilde{\Phi}\Gamma_{k((X)), \iota}^{\text{ét}} & \xrightarrow{\tilde{\Omega}} & \text{Mod}_{\tilde{B}, \iota}^{\text{sm}}(k) \end{array}$$

commute.

Proposition 5.13. *For $\pi \in \text{Mod}_{\tilde{G}, \iota}^{\text{fg.adm}}(k)$, there is a natural \tilde{B} -equivariant map*

$$\tilde{\Omega}(\tilde{\mathbf{D}}(\pi)) \longrightarrow \pi.$$

If $\tilde{\mathbf{D}}(\pi) \neq 0$, then this map is non-zero.

Proof. We construct the Pontryagin dual of the desired map. Let $M \in \mathcal{G}(\pi)$, so that $\pi = \pi_M[F^{-1}]$ and $\mathbf{D}(\pi)' = \pi_M^\vee[1/X]$ by Corollary 4.14. The formula in Remark 4.4 implies that the canonical map $\pi^\vee \rightarrow \pi_M^\vee \rightarrow \pi_M^\vee[1/X] = \mathbf{D}(\pi)'$ is compatible with the map F^\vee on the source and ψ on the target. Since F is injective on π_M , the dual is surjective, hence the image of the bounded $k[[X]]$ -module π_M^\vee in $\pi_M^\vee[1/X]$ is contained in $(\pi_M^\vee[1/X])^\sharp$, so we may consider the map

$$\pi^\vee \cong \varprojlim_{F^\vee} \pi^\vee \longrightarrow \varprojlim_{\psi} (\pi_M^\vee[1/X])^\sharp = \psi^{-\infty}(\mathbf{D}(\pi)') = \tilde{\Omega}(\tilde{\mathbf{D}}(\pi))^\vee,$$

which is non-trivial if $\mathbf{D}(\pi)' = \pi_M^\vee[1/X] \neq 0$. Here, the first isomorphism is given by $f \mapsto \left(f \circ \begin{pmatrix} p^{-n} & 0 \\ 0 & 1 \end{pmatrix} \right)_{n \geq 0}$, from which one quickly verifies the P -equivariance. The \tilde{Z} -equivariance follows from the construction. \square

We finish this section with a useful compatibility of the functor $\tilde{\Omega}$ with the induction functor (5.6), which is a counterpart of Proposition 5.8.

Proposition 5.14. *For $D \in \Phi\Gamma_{k((X))}^{\text{ét}}$ and $\chi: S \rightarrow k^\times$ a smooth character, there is a canonical \tilde{B} -equivariant isomorphism*

$$\text{Ind}_{\tilde{B}_S}^{\tilde{B}}(\chi \boxtimes \Omega(D) \boxtimes \iota) \cong \tilde{\Omega} \left(\text{Ind}_{\tilde{S}}^{\tilde{\mathbf{Q}}_p^\times}(\chi \boxtimes D \boxtimes \iota) \right),$$

where $\chi \boxtimes \Omega(D)$ is viewed as a representation of B_S via the identification $B_S \cong S \times P$ (with S embedded diagonally) and $\chi \boxtimes D \in \text{Mod}_S^{\text{sm}}(\Phi\Gamma_{k((X))}^{\text{ét}})$ is given by the action of S on D via χ .

Proof. By equation (5.2) applied to $\tilde{S} \subset \tilde{\mathbf{Q}}_p^\times$ (which is a central subgroup), we have the isomorphism

$$\text{Ind}_{\tilde{S}}^{\tilde{\mathbf{Q}}_p^\times}(\chi \boxtimes D \boxtimes \iota)|_S \cong \bigoplus_{\tilde{z} \in \tilde{S} \backslash \tilde{\mathbf{Q}}_p^\times} \chi \boxtimes (D \otimes \chi_{\tilde{z}})$$

in the category $\text{Mod}_S^{\text{sm}}(\Phi\Gamma_{k((X))}^{\text{ét}})$. The right-hand side contains $\chi \boxtimes D$ as a subobject. Applying the functor Ω and using Frobenius reciprocity (S acts via the character χ on both sides), we obtain a \tilde{B} -equivariant map

$$\text{Ind}_{\tilde{B}_S}^{\tilde{B}}(\chi \boxtimes \Omega(D) \boxtimes \iota) \longrightarrow \tilde{\Omega} \left(\text{Ind}_{\tilde{S}}^{\tilde{\mathbf{Q}}_p^\times}(\chi \boxtimes D \boxtimes \iota) \right),$$

which is an isomorphism as can be seen after restricting to P , where both sides are identified with $\bigoplus_{\tilde{z} \in \tilde{S} \backslash \tilde{\mathbf{Q}}_p^\times} \Omega(D) \otimes \chi_{\tilde{z}}$, see also the proof of Proposition 5.8. □

5.4. Images of absolutely irreducible genuine representations. In this final section, we compute the images of the absolutely irreducible genuine representations under the functor $\tilde{\mathbf{D}}: \text{Mod}_{\tilde{G},\iota}^{\text{fg.adm}}(k) \rightarrow \tilde{\Phi}\Gamma_{k((X)),\iota}^{\text{ét}}$, or equivalently under $\tilde{\mathbf{V}}: \text{Mod}_{\tilde{G},\iota}^{\text{fg.adm}}(k) \rightarrow \widetilde{\text{Mod}}_{\mathcal{G}_{\mathbf{Q}_p},\iota}^{\text{fin}}(k)$. The results in Section 2.1 imply that an absolutely irreducible genuine k -linear representation of \tilde{G} is either a principal series or supersingular.

5.4.1. Genuine principal series. For a pair $\chi_1, \chi_2: \mathbf{Q}_p^\times \rightarrow k^\times$ of smooth characters, define

$$\tilde{\pi}(\chi_1, \chi_2) = \text{Ind}_{\tilde{B}_S}^{\tilde{G}}(\chi_1 \otimes \chi_2|_S \boxtimes \iota) = \text{Ind}_{\tilde{B}}^{\tilde{G}}(\text{Ind}_{\tilde{T}_S}^{\tilde{T}}(\chi_1 \otimes \chi_2|_S \boxtimes \iota)),$$

whose isomorphism class only depends on the restriction $(\chi_1|_S, \chi_2|_S)$, see Proposition 2.3. Evaluation at the identity yields the short exact sequence

$$(5.10) \quad 0 \longrightarrow \tilde{\pi}(\chi_1, \chi_2)^{\text{bc}} \longrightarrow \tilde{\pi}(\chi_1, \chi_2) \longrightarrow \text{Ind}_{\tilde{T}_S}^{\tilde{T}}(\chi_1 \otimes \chi_2|_S \boxtimes \iota) \longrightarrow 0$$

of \tilde{B} -representations, where the kernel is the subspace of functions whose support is contained in the big cell, i.e. in the preimage of $B \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} B$ under the projection $\tilde{G} \twoheadrightarrow G$. By the proposition just cited and [28, Proposition 4.28], both cokernel and kernel are irreducible. Note that any splitting of the above sequence would yield a finite dimensional \tilde{G} -stable subspace of the irreducible \tilde{G} -representation $\tilde{\pi}(\chi_1, \chi_2)$, which is infinite dimensional. Hence, the sequence is non-split.

We define $\text{Ind}_B^G(\chi_1 \otimes \chi_2) = \pi(\chi_1, \chi_2) \supset \pi(\chi_1, \chi_2)^{\text{bc}}$ similarly.

Lemma 5.15. *There is a \tilde{B} -equivariant isomorphism*

$$\tilde{\pi}(\chi_1, \chi_2)^{\text{bc}} \cong \text{Ind}_{\tilde{B}_S}^{\tilde{B}} (\pi(\chi_1, \chi_2)^{\text{bc}}|_{B_S} \boxtimes \iota).$$

Proof. The assignment $f \mapsto [x \mapsto f(\begin{pmatrix} 0 & 1 \\ 1 & x \end{pmatrix})]$ defines a B -equivariant isomorphism from $\pi(\chi_1, \chi_2)^{\text{bc}}$ onto $\mathcal{C}_c^\infty(\mathbf{Q}_p, k) \otimes (\chi_2 \otimes \chi_1)$ for an appropriate action of B on the space $\mathcal{C}_c^\infty(\mathbf{Q}_p, k)$ of locally constant compactly supported functions. Similarly, we may \tilde{B} -equivariantly identify $\tilde{\pi}(\chi_1, \chi_2)^{\text{bc}} \cong \mathcal{C}_c^\infty(\mathbf{Q}_p, k) \otimes \text{Ind}_{\tilde{T}_S}^{\tilde{T}}(\chi_2 \otimes \chi_1|_S \boxtimes \iota)$, see [28, Lemma 4.23] for more details. The projection formula now gives the assertion \square

Proposition 5.16. *Let $\chi_1, \chi_2: \mathbf{Q}_p^\times \rightarrow k^\times$ be smooth characters.*

- (i) *The object $\text{Ind}_S^{\tilde{\mathbf{Q}}_p^\times} ((\chi_1\chi_2)|_S \boxtimes \mathcal{D}(\chi_2) \boxtimes \iota)$ is irreducible in $\widetilde{\Phi}\Gamma_{k((X)), \iota}^{\text{ét}}$, where \mathcal{D} is Fontaine’s equivalence 3.5.*
- (ii) *There is an isomorphism*

$$\tilde{\mathbf{D}}(\tilde{\pi}(\chi_1, \chi_2)) \cong \text{Ind}_S^{\tilde{\mathbf{Q}}_p^\times} ((\chi_1\chi_2)|_S \boxtimes \mathcal{D}(\chi_2) \boxtimes \iota)$$

in the category $\widetilde{\Phi}\Gamma_{k((X)), \iota}^{\text{ét}}$.

- (iii) *The canonical \tilde{B} -equivariant map $\tilde{\mathbf{D}}(\tilde{\pi}(\chi_1, \chi_2)) \rightarrow \tilde{\pi}(\chi_1, \chi_2)$ of Proposition 5.13 has image $\tilde{\pi}(\chi_1, \chi_2)^{\text{bc}}$ and induces a non-split short exact sequence*

$$0 \longrightarrow \text{Ind}_{\tilde{T}_S}^{\tilde{T}}(\chi_2\omega \otimes (\chi_1\omega^{-1})|_S \boxtimes \iota) \longrightarrow \tilde{\mathbf{D}}(\tilde{\pi}(\chi_1, \chi_2)) \longrightarrow \tilde{\pi}(\chi_1, \chi_2)^{\text{bc}} \longrightarrow 0.$$

Proof. By equation (5.2), we have

$$(5.11) \quad \text{Ind}_S^{\tilde{\mathbf{Q}}_p^\times} ((\chi_1\chi_2)|_S \boxtimes \mathcal{D}(\chi_2) \boxtimes \iota)|_S = \bigoplus_{\tilde{z} \in \tilde{S} \setminus \tilde{\mathbf{Q}}_p^\times} (\chi_1\chi_2)|_S \boxtimes (\mathcal{D}(\chi_2) \otimes \chi_{\tilde{z}})$$

in the category $\text{Mod}_S^{\text{sm}}(\Phi\Gamma_{k((X))}^{\text{ét}})$. Since the summands are pairwise non-isomorphic, we deduce irreducibility proving part (i).

For (ii), note that the cokernel of (5.10) is finite dimensional and therefore is $k[F]$ -torsion. By Proposition 4.6(ii), we thus have the equality

$\tilde{\mathbf{D}}(\tilde{\pi}(\chi_1, \chi_2)) = \tilde{\mathbf{D}}(\tilde{\pi}(\chi_1, \chi_2)^{\text{bc}})$. Similarly, we have $\mathbf{D}(\pi(\chi_1, \chi_2)^{\text{bc}}) = \mathbf{D}(\pi(\chi_1, \chi_2))$, which is known to be equal to $\mathcal{D}(\chi_2)$, see [13, Théorème 0.10(iii)] (beware of Colmez’s normalization). Part (ii) now follows by applying the functor $\tilde{\mathbf{D}}$ to the isomorphism in Lemma 5.15 and utilizing Proposition 5.8.

For (iii), note that an analogue for $\pi(\chi_1, \chi_2)$ was proved by Berger [4, Proposition 1.2.4, Theoreme 2.1.1], namely there is a non-split short exact sequence

$$0 \longrightarrow \chi_2\omega \otimes \chi_1\omega^{-1} \longrightarrow \Omega(\mathcal{D}(\chi_2)) \longrightarrow \pi(\chi_1, \chi_2)^{\text{bc}} \longrightarrow 0$$

of P -representations, in fact of $B = Z \times P$ -representation if we let $Z \cong \mathbf{Q}_p^\times$ act on the middle term via $\chi_1\chi_2$. Applying $\text{Ind}_{B_S}^{\tilde{B}}((\chi_1\chi_2)|_S \boxtimes (-) \boxtimes \iota)$ to this sequence and using Proposition 5.14 together with part (ii), we obtain a short exact sequence

$$0 \longrightarrow \text{Ind}_{T_S}^{\tilde{T}}(\chi_2\omega \otimes (\chi_1\omega^{-1})|_S \boxtimes \iota) \longrightarrow \tilde{\Omega}\tilde{\mathbf{D}}(\tilde{\pi}(\chi_1, \chi_2)) \longrightarrow \tilde{\pi}(\chi_1, \chi_2)^{\text{bc}} \longrightarrow 0.$$

Comparing this to the non-split sequence (5.10) via the canonical map of interest, which is non-zero by Proposition 5.13, implies part (iii) except non-splitness, which however follows by observing that any splitting would dually give a non-zero map from the dual of the finite dimensional kernel to the X -torsionfree space $(\tilde{\Omega}\tilde{\mathbf{D}}(\tilde{\pi}(\chi_1, \chi_2)))^\vee$. \square

Remark 5.17. Let $L = \mathbf{Q}_{p^2}(\sqrt{p})$ be the abelian extension \mathbf{Q}_p with norm group S . By equation (5.11), the underlying Galois representation of $\tilde{\mathbf{V}}(\tilde{\pi}(\chi_1, \chi_2))$ is given by

$$\mathbf{V}(\tilde{\pi}(\chi_1, \chi_2)) \cong \text{Ind}_{\mathcal{G}_L}^{\mathcal{G}_{\mathbf{Q}_p}}(\chi_2|_{\mathcal{G}_L}),$$

which is the direct sum over all characters of $\mathcal{G}_{\mathbf{Q}_p}$ extending $\chi_2|_{\mathcal{G}_L}$.

5.4.2. Genuine supersingular. For computing the image of an absolutely irreducible genuine supersingular representation, we make use of Proposition 2.7, which allows us to write such a representation as $\tilde{\pi}(r, 0, \eta)$ for some $0 \leq r \leq p - 1$ with $r \neq \frac{p-1}{2}$ and a smooth character $\eta: \mathbf{Q}_p^\times \rightarrow k^\times$. Since the functor $\tilde{\mathbf{D}}$ is compatible with twist, see Remark 5.7, we may assume that η is the trivial character, i.e. we only need to treat the representation $\tilde{\pi}(r, 0)$.

Let now $I = \begin{pmatrix} \mathbf{Z}_p^\times & \mathbf{Z}_p \\ p\mathbf{Z}_p & \mathbf{Z}_p^\times \end{pmatrix}$ be the Iwahori subgroup. It admits the semi-direct product decomposition $I = I_1 \rtimes H$, where $I_1 = \begin{pmatrix} 1+p\mathbf{Z}_p & \mathbf{Z}_p \\ p\mathbf{Z}_p & 1+p\mathbf{Z}_p \end{pmatrix}$ is the pro- p Iwahori subgroup and H is the torus of $\text{GL}_2(\mathbf{F}_p)$ viewed as a subgroup of I via the Teichmüller lift. Since the order of the finite abelian group H is prime to p , every finite dimensional k -representation of H decomposes

as the direct sum of characters. Given a character $\chi: H \rightarrow \mathbf{F}_p^\times$, we let $e_\chi \in \mathbf{F}_p[H]$ denote the idempotent cutting out the χ -isotypic subspace.

Definition 5.18 ([28, Definition 3.8]). For a character $\chi: H \rightarrow \mathbf{F}_p^\times$ and $i, j \in \mathbf{Z}/2\mathbf{Z}$, define $\chi[i, j] = \chi \otimes (\omega^{i\frac{p-1}{2}} \otimes \omega^{j\frac{p-1}{2}}): H \rightarrow \mathbf{F}_p^\times$.

Given a character $\chi = \chi_1 \otimes \chi_2$ of H , we put $\chi^s = \chi_2 \otimes \chi_1$. Note that the element $\tilde{\Pi} = \left(\begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix}, 1\right) \in \tilde{G}$ normalizes I_1 and thus acts on the space of I_1 -invariants π^{I_1} of any \tilde{G} -representation π . In fact, if $v \in \pi^{I_1}e_\chi$, then $\tilde{\Pi}v \in \pi^{I_1}e_{\chi^s[1,0]}$.

By definition, $\tilde{\pi}(r, 0)$ is a quotient of $c\text{-Ind}_{KZ(\tilde{G})}^{\tilde{G}}(\text{Sym}^r(k^2) \boxtimes \iota)$ and thus contains the weight $\text{Sym}^r(k^2)$, whose I_1 -invariants are one-dimensional, equal to the character $\chi := \omega^r \otimes \mathbf{1}: H \rightarrow k^\times$. Let us fix a non-zero element $v_1 \in \text{Sym}^r(k^2)^{I_1} \subset \tilde{\pi}(r, 0)^{I_1}$, and define $v_i = (\tilde{\Pi})^{i-1}v_1 \in \tilde{\pi}(r, 0)^{I_1}$ for $1 \leq i \leq 4$. By [28, Corollary 4.52], we then have $\tilde{\pi}(r, 0)^{I_1} = \bigoplus_{i=1}^4 kv_i$. If we let $\chi_i: H \rightarrow k^\times$ denote the H -eigencharacter of v_i , then

$$(5.12) \quad \chi_1 = \chi, \quad \chi_2 = \chi^s[1, 0], \quad \chi_3 = \chi[1, 1], \quad \chi_4 = \chi^s[0, 1].$$

Moreover, Corollary 4.46 in *loc. cit.* tells us that each v_i generates a (smooth) irreducible representation of K , which we can then write as $\text{Sym}^{r_i}(k^2) \otimes \det^{b_i}$ for a unique $0 \leq r_i \leq p - 1$ and $b_i \in \mathbf{Z}/(p - 1)\mathbf{Z}$. In fact, if we put

$$r' = \begin{cases} \frac{p-1}{2} - r & \text{if } 0 \leq r < \frac{p-1}{2} \\ \frac{3(p-1)}{2} - r & \text{if } \frac{p-1}{2} < r \leq p - 1, \end{cases}$$

then

$$\begin{aligned} (r_1, b_1) &= (r, 0), & (r_2, b_2) &= (r', r), \\ (r_3, b_3) &= \left(r, \frac{p-1}{2}\right), & (r_4, b_4) &= \left(r', r + \frac{p-1}{2}\right). \end{aligned}$$

Finally, let us define $s_i = r_{i+1}$ for $1 \leq i \leq 4$ and

$$\begin{aligned} c_1 &= (-1)^r (r')!, & c_2 &= (-1)^{\frac{p-1}{2}} r!, \\ c_3 &= (-1)^{r+\frac{p-1}{2}} (r')!, & c_4 &= (-1)^{\frac{p-1}{2}} r!. \end{aligned}$$

Lemma 5.19 ([28, Corollary 4.48]). *For each $1 \leq i \leq 4$, the equality $X^{s_i}F(v_i) = c_i v_{i+1}$ holds.*

Recall that $(-)|_S: \tilde{\Phi}\Gamma_{\mathbf{k}((X)), \iota}^{\text{ét}} \rightarrow \text{Mod}_S^{\text{sm}}(\Phi\Gamma_{\mathbf{k}((X))}^{\text{ét}})$ denotes the restriction functor. In the language of Galois representations, we denote this more accurately by $(-)|_{S \times \mathcal{G}_{\mathbf{Q}_p}}: \text{Mod}_{\mathcal{G}_{\mathbf{Q}_p}, \iota}^{\text{fin}}(k) \rightarrow \text{Mod}_{S \times \mathcal{G}_{\mathbf{Q}_p}}^{\text{fin}}(k)$.

Proposition 5.20.

- (i) *The underlying Galois representation $\mathbf{V}(\tilde{\pi}(r, 0))$ of the metaplectic Galois representation $\tilde{\mathbf{V}}(\tilde{\pi}(r, 0))$ is irreducible.*
- (ii) *The restriction to $S \times \mathcal{G}_{\mathbf{Q}_p}$ is given by*

$$\tilde{\mathbf{V}}(\tilde{\pi}(r, 0))|_{S \times \mathcal{G}_{\mathbf{Q}_p}} \cong \omega^r|_S \boxtimes \omega^{r-1} \mu_\lambda \otimes \text{Ind}_{\mathcal{G}_{\mathbf{Q}_{p^4}}}^{\mathcal{G}_{\mathbf{Q}_p}} (\omega_4^{\frac{p^2+1}{2}s}),$$

where $\lambda^4 = (-1)^{\frac{p-1}{2}} (r!)^2 (r')^2$ and

$$s = \begin{cases} -2r + p & \text{if } r < \frac{p-1}{2} \\ -2r + 3p & \text{if } r > \frac{p-1}{2}. \end{cases}$$

- (iii) *The canonical \tilde{B} -equivariant map $\tilde{\Omega}\tilde{\mathbf{D}}(\tilde{\pi}(r, 0)) \xrightarrow{\cong} \tilde{\pi}(r, 0)$ of Proposition 5.13 is an isomorphism.*
- (iv) *The metaplectic (φ, Γ) -module $\tilde{\mathbf{D}}(\tilde{\pi}(r, 0))$ is the unique object in the category $\tilde{\Phi}\tilde{\Gamma}_{k((X)), \iota}^{\text{ét}}$ satisfying (ii) (after passing to Galois representations) such that the \tilde{B} -action on $\tilde{\Omega}\tilde{\mathbf{D}}(\tilde{\pi}(r, 0))$ extends to a smooth \tilde{G} -action.*

Proof. First, note that the center $Z(\tilde{G}) \cong S \times \mu_2$ acts on $\tilde{\pi}(r, 0)$ via the genuine character $\omega^r|_S \boxtimes \iota$, which trivially implies that the action of S on $\tilde{\mathbf{V}}(\tilde{\pi}(r, 0))$ is given by $\omega^r|_S$, so it is enough to determine the underlying Galois representation. The previous discussion places us in the setup of Section 4.12 with $M = \tilde{\pi}(r, 0)^{I_1}$. By [28, Proposition 4.49], we have $\tilde{\pi}(r, 0) = \pi_M[F^{-1}]$, where π_M is the P^+ -subrepresentation generated by M . By Lemma 4.9 (i) and (ii), the subspace of X -torsion elements in π_M is equal to M (which is finite dimensional), hence $\pi_M \in \mathcal{M}(\pi)$. We may therefore apply Proposition 4.13 to deduce that $\mathbf{D}(\tilde{\pi}(r, 0)) = (\pi_M^\vee[1/X])'$. Corollary 4.12, which we may apply by Lemma 5.19, now proves part (i) and (ii). The proof of it also shows that $\pi_M^\vee = (\pi_M^\vee[1/X])^\sharp$, which is equal to $(\pi_M^\vee[1/X])^\sharp$ by Example 3.10 (ii). The natural map $\tilde{\Omega}\tilde{\mathbf{D}}(\tilde{\pi}(r, 0)) \rightarrow \tilde{\pi}(r, 0)$ being an isomorphism is now just a reformulation of the equality $\tilde{\pi}(r, 0) = \pi_M[F^{-1}] = \varinjlim_F \pi_M$.

For part (iv), write $\tilde{\mathbf{D}}(\tilde{\pi}(r, 0)) = (D, \theta)$ and let $\lambda: \tilde{\mathbf{Q}}_p^\times \rightarrow \text{Aut}_{k((X))}(D)$ be another action such that (D, λ) is a metaplectic (φ, Γ) -module satisfying $\lambda|_{\tilde{\mathcal{S}}} = \theta|_{\tilde{\mathcal{S}}}$. Then the (φ, Γ) -equivariant action of $\tilde{\mathbf{Q}}_p^\times$ on D defined by $\lambda\theta^{-1}$ factors through $\tilde{S} \setminus \tilde{\mathbf{Q}}_p^\times \cong \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$. Since D is absolutely irreducible (e.g. by part (ii)), Schur's Lemma implies that this action is given by scalars. In conclusion, $\lambda = \theta \otimes \varepsilon$ for some $\varepsilon = \mu_{-1}^a \omega^{b\frac{p-1}{2}}$ for $a, b \in \mathbf{Z}/2\mathbf{Z}$.

In particular, there is a \tilde{B} -equivariant isomorphism

$$\tilde{\Omega}(D, \lambda) \cong \tilde{\Omega}(D, \theta) \otimes (\mathbf{1} \otimes \varepsilon),$$

where the character $\mathbf{1} \otimes \varepsilon$ of the torus T is viewed as a character of \tilde{B} via inflation. By part (iii), we are reduced to showing that the \tilde{B} -action on $\tilde{\pi}(r, 0) \otimes (\mathbf{1} \otimes \varepsilon)$ extends to a smooth \tilde{G} -action if and only if ε is trivial. Assume that the action does extend. It must then necessarily define a genuine supersingular representation (as its restriction to \tilde{B} is irreducible). By [28, Corollary 4.55], the isomorphism class of a genuine supersingular representation is determined by the restriction to P . In particular, we obtain a \tilde{B} -equivariant (in fact, \tilde{G} -equivariant) isomorphism $f: \tilde{\pi}(r, 0) \cong \tilde{\pi}(r, 0) \otimes (\mathbf{1} \otimes \varepsilon)$. We first check that ε is unramified. The argument at the beginning of the proof of the corollary just used goes through to show that f restricts to a Γ -equivariant isomorphism $M \cong M \otimes (\mathbf{1} \otimes \varepsilon)$. We repeat it for the convenience of the reader: Since $\tilde{\pi}(r, 0) = \pi_M[F^{-1}]$, we may choose $m \gg 0$ with $F^{4m}f(M) \subset \pi_M \otimes (\mathbf{1} \otimes \varepsilon)$. An iteration of the relations in Lemma 5.19 implies that $X^{e(i)_m}F^{4m}v_i = c^m v_i$ for $e(i)_m$ and c as defined prior to (4.4), and so $f(M) \subset \pi_M \otimes (\mathbf{1} \otimes \varepsilon)$. Comparing X -torsion shows that $f(M) \subset M \otimes (\mathbf{1} \otimes \varepsilon)$, which must be an equality for dimension reasons. Recall that χ_i denotes the H -eigencharacter of v_i . The H -equivariance of f forces $f(v_i)$ to lie in the subspace $(M \otimes (\mathbf{1} \otimes \varepsilon))e_{\chi_i} = Me_{\chi_i[0,b]}$. The explicit description of the characters (5.12) implies that, if $b \neq 0$, then this space can only possibly be non-zero if $\chi_i[0, b] = \chi_j$ with $j - i \equiv 1 \pmod{2}$, in which case $s_i \neq s_j$. Write $f(v_i) = av_j$ for some $a \in k^\times$. By symmetry, we may assume that $s_i > s_j$. Applying $X^{s_i}F$ and utilizing Lemma 5.19 once again, gives the contradiction

$$0 \neq c_i f(v_{i+1}) = f(X^{s_i}F(v_i)) = aX^{s_i}F(v_j) = ac_j X^{s_i-s_j}v_{j+1} = 0$$

since M is killed by X . This proves that $b = 0$, i.e. ε is unramified, and we must have $f(v_i) = a_i v_i$ for some $a_i \in k^\times$ for all $1 \leq i \leq 4$. Applying f to the relation $X^{s_i}F(v_i) = c_i v_{i+1}$ forces $a_i = a_{i+1}$ for all $1 \leq i \leq 4$. However, the $\tilde{\Pi}^2$ -equivariance of f implies that $a_3 = \varepsilon(p)a_1$ and thus $\varepsilon(p) = 1$. In conclusion, ε is the trivial character. \square

By definition, the underlying Galois representation of a metaplectic one is invariant under twist by any character of order two. The unramified part of such a character is trivial when restricted to $\mathcal{G}_{\mathbf{Q}_{p^2}}$ (hence also when restricted to $\mathcal{G}_{\mathbf{Q}_{p^4}}$), so the interesting part left when considering absolutely irreducible representation of dimension four is invariance under twist by the character $\omega^{\frac{p-1}{2}}$.

Lemma 5.21. *Up to twist by a character, the absolutely irreducible Galois representations of $\mathcal{G}_{\mathbf{Q}_p}$ on 4-dimensional k -vector spaces invariant under*

twist by $\omega^{\frac{p-1}{2}}$ are precisely those of the form $\text{Ind}_{\mathcal{G}_{\mathbf{Q}_{p^4}}}^{\mathcal{G}_{\mathbf{Q}_p}} \left(\omega_4^{\frac{p^2+1}{2}h} \right)$ for some odd integer $h \in \mathbf{Z}$ with $1 \leq h \leq 2(p^2 - 1)$

Proof. We first show that the representations of the form $\text{Ind}_{\mathcal{G}_{\mathbf{Q}_{p^4}}}^{\mathcal{G}_{\mathbf{Q}_p}} \left(\omega_4^{\frac{p^2+1}{2}h} \right)$, with $1 \leq h \leq 2(p^2 - 1)$ odd, are irreducible and invariant under twist by $\omega^{\frac{p-1}{2}}$. For irreducibility, it suffices to show, by Proposition 3.1, that $\frac{p^2+1}{2}h$ is primitive relative to 4, i.e. that $\frac{p^2+1}{2}h$ is not divisible by $\frac{p^d-1}{p^d-1}$ for $d = 1, 2$. This is clear since h is odd. The invariance under twist by $\omega^{\frac{p-1}{2}}$ is equivalent to $\omega^{\frac{p-1}{2}}$ being equal to $\omega_4^{(p^i-1)\frac{p^2+1}{2}h}$ for some $0 \leq i \leq 3$. Noting that $\omega^{\frac{p-1}{2}} = \omega_4^{\frac{p^4-1}{2}}$, it is enough to observe that, since h is odd, we have

$$\frac{p^4 - 1}{2} \equiv \frac{p^4 - 1}{2}h = (p^2 - 1)\frac{p^2 + 1}{2}h \pmod{p^4 - 1}.$$

Assume now that ρ is an arbitrary 4-dimensional absolutely irreducible Galois representation invariant under twist by $\omega^{\frac{p-1}{2}}$. By the proposition just used, we may, after possibly twisting, write $\rho \cong \text{Ind}_{\mathcal{G}_{\mathbf{Q}_{p^4}}}^{\mathcal{G}_{\mathbf{Q}_p}} (\omega_4^s)$ for some $1 \leq s \leq p^4 - 1$ primitive relative to 4. Invariance under twist by $\omega^{\frac{p-1}{2}}$ gives $\frac{p^4-1}{2} \equiv (p^i - 1)s \pmod{p^4 - 1}$ for some $0 \leq i \leq 3$. The case $i = 0$ is excluded for trivial reasons. If $i = 3$, then multiplying by the odd prime p gives

$$\frac{p^4 - 1}{2} \equiv p\frac{p^4 - 1}{2} \equiv (p^4 - p)s \equiv -(p - 1)s \pmod{p^4 - 1},$$

or equivalently $\frac{p^4-1}{2} \equiv (p - 1)s \pmod{p^4 - 1}$, which is the case $i = 1$. In the cases $i = 1, 2$, we obtain an equality of the form $s = (2a + 1)\frac{p^2-1}{p^i-1}\frac{p^2+1}{2}$, for some $a \in \mathbf{Z}$. If $i = 1$, then this implies that $\frac{p^4-1}{p^2-1} = p^2 + 1$ divides s , contradicting the irreducibility of ρ (or equivalently primitivity of s). In case $i = 2$, we obtain the desired result. \square

Lemma 5.22. *Given an odd integer $h \in \mathbf{Z}$, there exist integers $a, h' \in \mathbf{Z}$ with $3 \leq h' \leq 2p - 1$ odd such that*

$$\text{Ind}_{\mathcal{G}_{\mathbf{Q}_{p^4}}}^{\mathcal{G}_{\mathbf{Q}_p}} \left(\omega_4^{\frac{p^2+1}{2}h} \right) \cong \omega^a \otimes \text{Ind}_{\mathcal{G}_{\mathbf{Q}_{p^4}}}^{\mathcal{G}_{\mathbf{Q}_p}} \left(\omega_4^{\frac{p^2+1}{2}h'} \right).$$

Proof. We first show that we can choose $1 \leq h' \leq 2p - 1$ (i.e. including $h' = 1$) appropriately. Since ω_4 has order $p^4 - 1$, we may without loss of generality assume that $0 \leq h < 2(p^2 - 1)$. So, if $h = h_0 + ph_1 + p^2h_2$ denotes the base- p expansion with $0 \leq h_i \leq p - 1$, then $h_2 \in \{0, 1\}$. Next, note that replacing h by $h - 2a(p + 1)$ for some $a \in \mathbf{Z}$ amounts to twisting

the induced representation by ω^{-a} . If $h_2 = 1$, we take $a = \frac{p-1}{2}$ and may therefore assume that $0 \leq h < p^2 - 1$, whence $h_2 = 0$. Choose $a \in \mathbf{Z}$ such that $h'_1 := h_1 - 2a \in \{0, 1\}$ and put $h'_0 := h_0 - 2a$. We may replace h by $h' = h - 2a(p + 1) = h'_0 + ph'_1$. Note that $|h'_0| \leq p - 1$. If $h'_0 \geq 0$ or $h'_1 = 1$, then $1 \leq h' \leq 2p - 1$ as desired. Assume now that $h'_1 = 0$ and $h'_0 < 0$. Replace $h' = h'_0$ by $h'_0 + 2(p + 1) = 2p + (2 + h'_0)$. If $-h'_0 \geq 3$, then we are done; otherwise $h'_0 = -1$ (as it must be odd), in which case we use the intertwining

$$\text{Ind}_{\mathcal{G}_{\mathbf{Q}_{p^4}}}^{\mathcal{G}_{\mathbf{Q}_p}} \left(\omega_4^{\frac{p^2+1}{2}(2p+1)} \right) \cong \text{Ind}_{\mathcal{G}_{\mathbf{Q}_{p^4}}}^{\mathcal{G}_{\mathbf{Q}_p}} \left(\omega_4^{p\frac{p^2+1}{2}(2p+1)} \right) = \text{Ind}_{\mathcal{G}_{\mathbf{Q}_{p^4}}}^{\mathcal{G}_{\mathbf{Q}_p}} \left(\omega_4^{\frac{p^2+1}{2}(p+2)} \right).$$

This proves the statement with $1 \leq h' \leq 2p - 1$ odd. Similar to the first isomorphism in the above intertwining, we may replace h' by ph' and so replace $h' = 1$ by p . In particular, we may assume $3 \leq h' \leq 2p - 1$. \square

Theorem 5.23. *The functor $\mathbf{V}: \text{Mod}_{\tilde{G}}^{\text{fg.adm}}(k) \rightarrow \text{Mod}_{\mathcal{G}_{\mathbf{Q}_p}}^{\text{fin}}(k)$ induces a bijection*

$$\left\{ \begin{array}{l} \text{Absolutely irreducible} \\ \text{genuine supersingular} \\ \text{\(k\)-representations of } \tilde{G} \end{array} \right\} \cong \left\{ \begin{array}{l} \text{Absolutely irreducible continuous} \\ \rho: \mathcal{G}_{\mathbf{Q}_p} \rightarrow \text{GL}_4(k) \text{ s.t. } \rho \cong \rho \otimes \omega^{\frac{p-1}{2}} \end{array} \right\},$$

both sides considered up to isomorphism.

Proof. This follows from Proposition 5.20, Lemma 5.21, Lemma 5.22 and the fact, [28, Corollary 4.55], that the isomorphism class of a supersingular representation is determined by its restriction to the mirabolic subgroup P . Here, note that the restriction of the canonical map of part (iii) of the cited proposition to the mirabolic subgroup does not use the metaplectic structure. \square

Remark 5.24. Fix a square root $\varpi = \sqrt{-p}$, which serves as a uniformizer in the abelian extension $L = \mathbf{Q}_{p^2}(\sqrt{p})$ of \mathbf{Q}_p with norm group S . Consider the character

$$\omega_L: \mathcal{G}_L \longrightarrow \mathbf{F}_{p^2}^\times, \quad g(p^2\sqrt[2]{\varpi}) = \omega_L(g) p^2\sqrt[2]{\varpi}, \text{ for all } g \in \mathcal{G}_L;$$

since $\mu_{p^2-1}(\overline{\mathbf{Q}_p}) \subset \mathbf{Q}_{p^2} \subset L$, it is independent of the chosen $(p^2 - 1)$ -th root of ϖ . We then have $\omega_4^{(p^2+1)/2}|_{\mathcal{G}_L \cap \mathcal{G}_{\mathbf{Q}_{p^4}}} = \omega_L|_{\mathcal{G}_L \cap \mathcal{G}_{\mathbf{Q}_{p^4}}}$. In particular, for any odd integer $h \in \mathbf{Z}$, we have an isomorphism

$$\text{Ind}_{\mathcal{G}_{\mathbf{Q}_{p^4}}}^{\mathcal{G}_{\mathbf{Q}_p}} \left(\omega_4^{\frac{p^2+1}{2}h} \right) \cong \text{Ind}_{\mathcal{G}_L}^{\mathcal{G}_{\mathbf{Q}_p}} (\omega_L^h)$$

of irreducible $\mathcal{G}_{\mathbf{Q}_p}$ -representations. Conversely, the character ω_L^2 extends to $\mathcal{G}_{\mathbf{Q}_{p^2}}$, e.g. ω_2 , so the right-hand side will be reducible if h is even.

Remark 5.25. The results of this section imply that the functor $\tilde{\mathbf{D}}$ does not kill any non-zero representation and one may wonder whether it is in fact fully faithful. While we do not know if this is true, the canonical map of Proposition 5.13 and its behaviour for absolutely irreducible objects (Proposition 5.16 (iii), Proposition 5.20 (iii) and (iv)) suggest that the functor $\tilde{\Omega}$ seems to be a good starting point for defining a potential inverse functor (on the essential image). In fact, given an étale (φ, Γ) -module D and fixing a smooth character $\delta: \mathbf{Q}_p^\times \rightarrow k^\times$, Colmez defines a G -equivariant sheaf $U \mapsto D \boxtimes U$ on G/B , whose global sections carry an action of G with central character δ , see [13, Section II.1]. The functor Ω , cf. (5.8), is equal to the (bounded) \mathbf{Q}_p -sections of this sheaf. In order to realize $\tilde{\Omega}$ as the \mathbf{Q}_p -sections of a \tilde{G} -equivariant sheaf on $\tilde{G}/\tilde{B} = G/B$, one may try to adapt Colmez’s construction, see especially the paragraph preceding Lemme II.1.2 in *loc. cit.*, for which it seems natural to replace the character δ by the smooth and genuine $\widehat{\mathbf{Q}}_p^\times$ -action that a metaplectic (φ, Γ) -module comes equipped with.

Note that such a reverse functor would also give rise to a correspondence between smooth genuine representations of \tilde{G} and smooth representations of G : Given a metaplectic (φ, Γ) -module, one may consider the underlying (φ, Γ) -module along with the “square” of the underlying genuine $\widehat{\mathbf{Q}}_p^\times$ -representation (via $\mathbf{Q}_p^\times \xrightarrow{x \mapsto x^2} S \subset \widehat{\mathbf{Q}}_p^\times$). Assuming that the latter action of \mathbf{Q}_p^\times is given by a character, Colmez’s functor yields a smooth G -representation, while applying a hypothetical reverse functor in the metaplectic setting gives a smooth genuine \tilde{G} -representation. We do not a priori know whether this would turn out to be a meaningful correspondence.

6. The p -adic metaplectic Montréal functor

Let L be a finite extension of \mathbf{Q}_p with ring of integers \mathcal{O}_L and residue field k , and fix a uniformizer ϖ . The absolutely irreducible k -representations of \tilde{G} appear as constituents of the mod- ϖ reduction of a \tilde{G} -stable bounded lattice of (the unitary completion of) a locally algebraic unramified principal series over L , cf. [7] (for G) and Proposition 6.7, Theorem 5.13 and Corollary 5.16 in [28] (for genuine representations). In this context, it is natural to ask whether the extension $\mathbf{D}: \text{Mod}_{\tilde{G}}^{\text{fg.adm}}(k) \rightarrow \Phi\Gamma_{k((X))}^{\text{ét}}$ of Colmez’s functor, see Remark 4.15, and its metaplectic variant can be lifted to p -adic coefficients. In this section, we explain that this is rather a formal consequence of the results established for mod- ϖ representations and the fact that Breuil’s functor of Definition 4.5 is also defined for mod- ϖ^n coefficients, for all $n \geq 1$.

6.1. p -adic Banach space representations. In order to relate an L -Banach space representation of \tilde{G} to a smooth k -representation of \tilde{G} , we need it to be unitary. This means that it admits a \tilde{G} -invariant norm defining its underlying topology. Given such a representation Π , we may consider the unit ball Π^0 for a \tilde{G} -invariant norm, which is a ϖ -adically complete \mathcal{O}_L -representation of \tilde{G} . For $n \geq 1$, the mod- ϖ^n reduction $\Pi^0/\varpi^n\Pi^0$ is then a smooth $\mathcal{O}_L/\varpi^n\mathcal{O}_L$ -representation of \tilde{G} . The semi-simplification of $\Pi^0/\varpi\Pi^0$ in $\text{Mod}_{\tilde{G}}^{\text{sm}}(k)$ is independent of the chosen norm, provided it is of finite length.

Lemma 6.1. *Let Π be a unitary L -Banach representation of \tilde{G} , and let Π^0 be the unit ball for a \tilde{G} -invariant norm defining the topology of Π . Fix an integer $n \geq 1$.*

- (i) *If $\Pi^0/\varpi\Pi^0$ is finitely generated over $\mathcal{O}_L[\tilde{G}]$, then so is $\Pi^0/\varpi^n\Pi^0$.*
- (ii) *If $\Pi^0/\varpi\Pi^0$ is admissible as a representation of \tilde{G} , then so is $\Pi^0/\varpi^n\Pi^0$.*

Proof. This follows by induction from the short exact sequence

$$0 \longrightarrow \varpi^{n-1}\Pi^0/\varpi^n\Pi^0 \longrightarrow \Pi^0/\varpi^n\Pi^0 \longrightarrow \Pi^0/\varpi^{n-1}\Pi^0 \longrightarrow 0,$$

where the kernel is isomorphic to $\Pi^0/\varpi\Pi^0$. □

We denote by $\text{Ban}_{\tilde{G}}^{\text{fg.adm}}(L)$ the category of unitary L -Banach space representations of \tilde{G} , which are residually finitely generated and admissible: for one, or equivalently any, \tilde{G} -invariant norm, the mod- ϖ reduction of the corresponding unit ball is finitely generated and admissible⁶. By Lemma 2.8, a finitely generated admissible k -representation of \tilde{G} is of finite length. In particular, given $\Pi \in \text{Ban}_{\tilde{G}}^{\text{fg.adm}}(L)$, we obtain the semi-simplified mod- ϖ reduction $\bar{\Pi}^{\text{ss}} \in \text{Mod}_{\tilde{G}}^{\text{fg.adm}}(k)$.

Viewing $\iota: \mu_2 \hookrightarrow k^\times \subset \mathcal{O}_L^\times$ as valued in \mathcal{O}_L^\times via the Teichmüller lift, we may consider the full subcategory $\text{Ban}_{\tilde{G},\iota}^{\text{fg.adm}}(L) \subset \text{Ban}_{\tilde{G}}^{\text{fg.adm}}(L)$ of genuine objects; likewise for the category $\text{Mod}_{\tilde{G}}^{\text{fg.adm}}(\mathcal{O}_L/\varpi^n\mathcal{O}_L)$ of finitely generated smooth admissible $\mathcal{O}_L/\varpi^n\mathcal{O}_L$ -representations of \tilde{G} .

6.2. p -adic (φ, Γ) -modules. Following [13, Section I.1], we denote by $\mathcal{O}_{\mathcal{E}}$ the p -adic completion of $\mathcal{O}_L[[X]][1/X]$, endowed with the topology for which the subsets $p^k\mathcal{O}_{\mathcal{E}} + X^n\mathcal{O}_L[[X]]$, for integers $k, n \geq 0$, form an open neighbourhood basis of 0. We let $\mathcal{E} = \mathcal{O}_{\mathcal{E}}[1/p] = \bigcup_{n \geq 0} p^{-n}\mathcal{O}_{\mathcal{E}}$ be its fraction field, endowed with the inductive limit topology.

⁶By [24, Proposition 9.6], being residually admissible is equivalent to being admissible in the sense of Schneider–Teitelbaum.

Define the operator φ and the action of $\Gamma = \text{Gal}(\mathbf{Q}_p(\mu_{p^\infty})/\mathbf{Q}_p)$ on $\mathcal{O}_\mathcal{E}$ and \mathcal{E} by

$$\varphi(X) = (X + 1)^p - 1 \text{ and } \gamma(X) = (X + 1)^{\chi_{\text{cyc}}(\gamma)} - 1, \text{ for } \gamma \in \Gamma,$$

where $\chi_{\text{cyc}}: \Gamma = \text{Gal}(\mathbf{Q}_p(\mu_{p^\infty})/\mathbf{Q}_p) \cong \mathbf{Z}_p^\times$ is the cyclotomic character, as before. We then have the evident notion of a (φ, Γ) -module over any of the topological rings $\mathcal{O}_\mathcal{E}/\varpi^n \mathcal{O}_\mathcal{E}$, for $n \geq 1$, $\mathcal{O}_\mathcal{E}$ and \mathcal{E} : a finitely generated module D over the respective ring together with a continuous φ -semilinear operator φ_D on it as well as a continuous semilinear Γ -action commuting with φ_D . A (φ, Γ) -module D over $\mathcal{O}_\mathcal{E}/\varpi^n \mathcal{O}_\mathcal{E}$, for $n \geq 1$, resp. $\mathcal{O}_\mathcal{E}$, is étale if the linearization of φ_D is an isomorphism, while a (φ, Γ) -module over \mathcal{E} is étale if it is of the form $D \otimes_{\mathcal{O}_\mathcal{E}} \mathcal{E}$ for an étale (φ, Γ) -module D over $\mathcal{O}_\mathcal{E}$, cf. [13, Section I.2]. We will denote the corresponding categories of étale (φ, Γ) -modules by

$$\Phi\Gamma_{\mathcal{O}_\mathcal{E}/\varpi^n \mathcal{O}_\mathcal{E}}^{\text{ét}}, \Phi\Gamma_{\mathcal{O}_\mathcal{E}}^{\text{ét}} \text{ and } \Phi\Gamma_{\mathcal{E}}^{\text{ét}},$$

for $n \geq 1$, which are equivalent to the categories of Galois representations of $\mathcal{G}_{\mathbf{Q}_p}$ over $\mathcal{O}_L/\varpi^n \mathcal{O}_L$, \mathcal{O}_L and L , respectively, see [13, Théorème I.3.1].

Finally, as in Definition 5.5, we define the corresponding categories of metaplectic (φ, Γ) -modules:

$$\widetilde{\Phi}\Gamma_{\mathcal{O}_\mathcal{E}/\varpi^n \mathcal{O}_{\mathcal{E},\iota}}^{\text{ét}}, \widetilde{\Phi}\Gamma_{\mathcal{O}_{\mathcal{E},\iota}}^{\text{ét}} \text{ and } \widetilde{\Phi}\Gamma_{\mathcal{E},\iota}^{\text{ét}},$$

for $n \geq 1$. Concretely, its objects are pairs (D, θ) consisting of an étale (φ, Γ) -module D together with a smooth and genuine action θ of $\widetilde{\mathbf{Q}_p^\times}$ on the underlying module over the respective ring such that $\theta(\tilde{z}): D \cong D \otimes \chi_z$ is a (φ, Γ) -equivariant isomorphism, for $\tilde{z} \in \widetilde{\mathbf{Q}_p^\times}$ with image $z \in \mathbf{Q}_p^\times$, where $\chi_z = (\mu_{-1}^{v(z)} \omega^{v(z)} \mu_{\omega(z)})^{\frac{n-1}{2}}$ is viewed as valued in \mathcal{O}_L^\times via the Teichmüller lift.

6.3. Liftings. Fix an integer $n \geq 1$. In Definition 4.5, we stated Breuil’s functor with coefficients in k . However, Breuil works more generally with mod- ϖ^n coefficients, thus yielding the functor $\mathbf{D}: \text{Mod}_P^{\text{sm}}(\mathcal{O}_L/\varpi^n \mathcal{O}_L) \rightarrow \text{Ind}(\Phi\Gamma_{\mathcal{O}_\mathcal{E}/\varpi^n \mathcal{O}_\mathcal{E}}^{\text{ét}})$ on the category of smooth $\mathcal{O}_L/\varpi^n \mathcal{O}_L$ -representations of P , which is compatible with mod- ϖ^m reduction for $1 \leq m \leq n$.

Proposition 6.2. *For $\pi \in \text{Mod}_{\widetilde{G}}^{\text{fg.adm}}(\mathcal{O}_L/\varpi^n \mathcal{O}_L)$, the object $\mathbf{D}(\pi|_P)$ lies in the full subcategory $\Phi\Gamma_{\mathcal{O}_\mathcal{E}/\varpi^n \mathcal{O}_\mathcal{E}}^{\text{ét}} \hookrightarrow \text{Ind}(\Phi\Gamma_{\mathcal{O}_\mathcal{E}/\varpi^n \mathcal{O}_\mathcal{E}}^{\text{ét}})$ of compact objects.*

Proof. By Corollary 4.14, the statement holds true for $n = 1$. We proceed by induction and assume that it holds for $n - 1$. Consider the short exact sequence

$$0 \longrightarrow \varpi^{n-1} \pi \longrightarrow \pi \longrightarrow \pi/\varpi^{n-1} \pi \longrightarrow 0.$$

Noting that $\varpi^{n-1}\pi \in \text{Mod}_{\tilde{G}}^{\text{fg.adm}}(k)$, we may, by the final part of Proposition 4.13, apply [10, Proposition 2.7 (iv)] to obtain the short exact sequence

$$0 \longrightarrow \mathbf{D}(\varpi^{n-1}\pi) \longrightarrow \mathbf{D}(\pi) \longrightarrow \mathbf{D}(\pi/\varpi^{n-1}\pi) \longrightarrow 0,$$

where both kernel and cokernel lie in the subcategory $\Phi\Gamma_{\mathcal{O}_{\mathcal{E}}/\varpi^n\mathcal{O}_{\mathcal{E}}}^{\text{ét}}$, by the induction hypothesis. Write $\mathbf{D}(\pi) = \varinjlim_i D_i$ as the filtered colimit of compact objects. Now, a (φ, Γ) -module is in particular finitely generated over the underlying ring. Hence, there exists some i such that D_i surjects onto $\mathbf{D}(\pi/\varpi^{n-1}\pi)$. Using the compactness of $\mathbf{D}(\varpi^{n-1}\pi)$, we may increase i to also assume that $\mathbf{D}(\varpi^{n-1}\pi) \rightarrow \mathbf{D}(\pi)$ factors through D_i . We obtain the resulting short exact sequence

$$0 \longrightarrow \mathbf{D}(\varpi^{n-1}\pi) \longrightarrow D_i \longrightarrow \mathbf{D}(\pi/\varpi^{n-1}\pi) \longrightarrow 0.$$

Comparing this to the previous sequence and applying the snake lemma, we see that the canonical map $D_i \rightarrow \mathbf{D}(\pi)$ is an isomorphism. \square

Precomposing \mathbf{D} with the restriction functor $(-)|_P$ therefore yields the functor

$$\mathbf{D}: \text{Mod}_{\tilde{G}}^{\text{fg.adm}}(\mathcal{O}_L/\varpi^n\mathcal{O}_L) \longrightarrow \Phi\Gamma_{\mathcal{O}_{\mathcal{E}}/\varpi^n\mathcal{O}_{\mathcal{E}}}^{\text{ét}},$$

still denoted \mathbf{D} . Restricting to the subcategory of genuine representations, this functor may be promoted to its metaplectic version

$$\tilde{\mathbf{D}}: \text{Mod}_{\tilde{G},\iota}^{\text{fg.adm}}(\mathcal{O}_L/\varpi^n\mathcal{O}_L) \longrightarrow \tilde{\Phi}\Gamma_{\mathcal{O}_{\mathcal{E}}/\varpi^n\mathcal{O}_{\mathcal{E},\iota}}^{\text{ét}},$$

just as for $n = 1$ in Section 5.2.

Now, let $\Pi \in \text{Ban}_{\tilde{G}}^{\text{fg.adm}}(L)$ and choose a \tilde{G} -invariant norm defining its topology with corresponding unit ball Π^0 , so that in particular $\Pi = (\varprojlim_n \Pi^0/\varpi^n\Pi^0) \otimes_{\mathcal{O}_L} L$. By Lemma 6.1, we may form the étale (φ, Γ) -module $(\varprojlim_n \mathbf{D}(\Pi^0/\varpi^n\Pi^0)) \otimes_{\mathcal{O}_{\mathcal{E}}} \mathcal{E}$, which is independent of the chosen norm and will be denoted by $\mathbf{D}(\Pi)$. The resulting functor

$$\mathbf{D}: \text{Ban}_{\tilde{G}}^{\text{fg.adm}}(L) \longrightarrow \Phi\Gamma_{\mathcal{E}}^{\text{ét}}$$

is compatible with semi-simplified mod- ϖ reduction, i.e. $\mathbf{D}(\overline{\Pi}^{\text{ss}}) = \overline{\mathbf{D}(\Pi)}^{\text{ss}}$. When restricted to the subcategory $\text{Ban}_{\tilde{G}}^{\text{fg.adm}}(L)$, it agrees with Colmez’s functor, see [10, Proposition 3.2 (iii)]. On the other hand, restricting to the subcategory of genuine representations, it gives rise to a functor

$$\tilde{\mathbf{D}}: \text{Ban}_{\tilde{G},\iota}^{\text{fg.adm}}(L) \longrightarrow \tilde{\Phi}\Gamma_{\mathcal{E},\iota}^{\text{ét}},$$

which we call the p -adic metaplectic Montréal functor.

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