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Quantitative inverse Galois problem for semicommutative finite group schemes

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Résumé. Un schéma en groupes fini est dit semi-commutatif s'il peut être obtenu à partir de schémas en groupes finis commutatifs en effectuant de manière itérative des produits semi-directs avec des noyaux commutatifs et en prenant des quotients par des sous-groupes normaux. Dans cet article, pour un schéma en groupes fini semi-commutatif étale et modéré G, nous donnons une borne inférieure pour le nombre de G-torseurs connexes de hauteur bornée (tel que le discriminant).

Abstract. A semicommutative finite group scheme is a finite group scheme which can be obtained from commutative finite group schemes by iterated performing semidirect products with commutative kernels and taking quotients by normal subgroups. In this article, for an étale tame semicommutative finite group scheme G, we give a lower bound on the number of connected G-torsors of bounded height (such as discriminant).

1. Introduction

1.1. Inverse Galois problem for finite group schemes. One of the most famous questions in number theory is the *inverse Galois problem*, which asks whether every finite group G is realizable as the Galois group of a finite extension of the field $\mathbb{Q}$. This is a largely open question. It is classically known to admit an affirmative answer for G commutative, for $G = \mathfrak{S}_n$ symmetric, for $G = \mathfrak{A}_n$ alternating, etc. Without any modifications the question can be asked for other fields and in this paper we deal with the case of a global field F. In this context, one has (a generalization of) the celebrated Shafarevich theorem [21, Theorem 9.6.1] which states that every solvable G is a Galois group of a finite extension of F.

If K/F is an extension, then it is a Galois extension with the Galois group G if and only if $\operatorname{Spec}(K) \to \operatorname{Spec}(F)$ is a *connected* G-torsor. For non-constant finite group schemes G, one can thus ask:

Question 1.1.1. *Let F be a global field. Does every finite F-group scheme G admits a connected G-torsor?*

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Although the question is a very natural one, to our knowledge, in this form it was only asked in Section "The inverse problem of Galois theory for torsors" of [22] by Cassou-Noguès, Chinburg, Morin and Taylor, where an affirmative answer is provided for the case F is a number field and $G = \boldsymbol{\mu}_m$ is the finite group scheme of m-th roots of unity. In this article, we will always assume that, besides being finite, the group schemes are *étale* and *tame* (i.e. if the characteristic of F is positive, then the cardinality of G is coprime to the characteristic). In the literature one can find some other properties of a finite étale tame F-group scheme G which imply a positive solution to Question 1.1.1 such as "G satisfies *hyperweak approximation*" or "G is *supersolvable*" (for definitions see Appendix; where we have also added proofs that these properties indeed imply a positive solution to Question 1.1.1).

1.2. Quantitative aspect. A "quantitive" version of the inverse Galois problem is given by the *Malle conjecture.* We have a *height* function $H : BG(F) \to \mathbb{R}_{>0}$ and we count how many $X \in BG(F)$ satisfy $H(X) < B$, where $B > 0$.

Conjecture 1.2.1 (Malle [17])**.** *Let G be a non-trivial tame constant group which is embedded as a transitive subgroup of the group of permutations $\mathfrak{S}_n$ for some $n \geq 1$. Let $G_0 \subset G$ be a stabilizer of a point in $\{1, \dots, n\}$ for the action of G induced by the embedding. For a G-torsor X, we write $H(X) := \Delta(X/G_0)$, where Δ denotes the norm of the discriminant. One has that*

$$\#\{X \in BG(F) \,|\, X \textit{ is connected and } H(K) < B\} \underset{B\to\infty}{\asymp} B^{a(H)} \log(B)^{b(H)-1},$$

for some explicit invariants $a = a(H)$ and $b = b(H)$.

(This is a slightly different statement from the usual statement of the conjecture, and the equivalence with the usual one is explained in [10, Paragraph 1.1.2]). Over $\mathbb{Q}$, the conjecture is known to be true for G commutative embedded in its regular representation, $G = S_3, S_4, S_5$ embedded in its standard representations, etc (see [2, 3, 12, 24]). It is significantly harder than the inverse Galois problem: e.g. it is unknown for some usual groups such as $G = \mathfrak{A}_4$ embedded in its standard or regular representation. Conjecture 1.2.1 admits counterexamples as shown by Klüners [16]. For some G, only upper and lower bounds on the number of G-torsors (G-extensions) of bounded height are known. For the case of a number field, based on Shafarevich theorem, Alberts establishes in [1] a lower bound of the form $\gg B^a$, with $a > 0$ for every solvable group G.

In our previous article [10], we proposed a version of Malle conjecture for non-constant finite group schemes G.

Conjecture 1.2.2. *Let G be a non-trivial finite étale tame F-group scheme. Let $H : BG(F) \to \mathbb{R}_{>0}$ be a height. We define invariants $a(H)$ and $b(H)$ as in Definition 2.1.1. One has that*

$$\#\{x \in BG(F) \,|\, x \textit{ is secure and } H(x) \leq B\} \underset{B\to\infty}{\asymp} B^{a(H)} \log(B)^{b(H)-1}.$$

(A torsor x is secure if it lies outside of certain *thin* subset of rational points $BG(F)$, see [10, Definition 2.6.3]. The notion serves to avoid counterexamples. When G is commutative, every G-torsor is secure.) The conjecture is a special case of a *stacky Batyrev–Manin conjecture* [11, Conjecture 9.15] (see also [13] where only the constant a is predicted). Note that we have *not* imposed a connectivity condition. Conjecture 1.2.2 has been verified in [10, Theorem 1.3.2] for G commutative. However, it may happen that a positive proportion of G-torsors is not connected (e.g. this happens when $G = \boldsymbol{\mu}_m$ is the group scheme of m-th roots of unity, as remarked in [9, Remark 9.2.7.4]). Thus, a priori, Conjecture 1.2.2 does not imply the existence of a single connected G-torsor.

1.3. Content. The principal result of this article is a quantitative solution to the inverse Galois problem for *semicommutative* finite group schemes. These are the finite group schemes which can be obtained from finite commutative group schemes by iterated performing semidirect products with commutative kernels and taking quotients by normal subgroups (for details, see Definition 3.3.1). The constant semicommutative groups are precisely those which can be realized as Galois groups by successive solution to *split embedding problems with abelian kernels* and taking intermediate Galois extensions [18, Chapter IV, Section 2.2]. The methods of realization, however, do not work for the non-constant case.

Let us first suppose that G is commutative. Then, an assertion [10, Theorem 1.3.3], which is stronger than Conjecture 1.2.2, is valid: it allows to determine the asymptotic behaviour with prescribed certain local conditions. We will show that the stronger statement, together with Lemma 3.1.5 which gives local conditions which force torsors to be connected, implies the existence of (infinitely many) connected torsors. More precisely, we obtain that:

Theorem 1.3.1. *Let F be a global field. Suppose that G is a non-trivial commutative finite étale tame group scheme. One has that*

$$\#\{x \in BG(F) \,|\, x \textit{ is connected}, H(x) \leq B\} \underset{B\to\infty}{\asymp} B^{a(H)} \log(B)^{b(H)-1}.$$

We mention that in [9, Theorem 9.2.7.3] the precise asymptotic behaviour (with the leading constant) is established for the case F is a number field and $G = \boldsymbol{\mu}_m$ under additional assumption that $4 \nmid m$ or that $\sqrt{-1} \in F$.

Let us now treat the semicommutative case. A semicommutative finite étale group scheme G can be written as $G = \langle A, K\rangle$, where $\iota : A \hookrightarrow G$ is normal and commutative and $K \lneq G$ is semicommutative. We establish a similar bound to Alberts' bound for solvable constant groups:

Theorem 1.3.2. *Let F be a global field. Suppose that G is a non-trivial semicommutative finite étale and tame F-group scheme. Write $G = \langle A, K\rangle$ as above. There exists $C > 0$ such that*

$$\#\{x \in BG(F) \,|\, x \text{ is connected, } H(x) \leq B\} \geq CB^{a(\iota^* H)},$$

where $\iota^* H$ is the pullback height (defined precisely in Paragraph 2.1).

The obtained lower bounds may be as good as in the *weak Malle conjecture* (which predicts that the number grows at least as $CB^{a(H)}$ for some $C > 0$), as the following example shows. The constant alternating group $G = \mathfrak{A}_4$ is semicommutative (non-constant examples with $G(\overline{F}) = \mathfrak{A}_4$ do exist, as discussed in Example 3.3.4). Our result implies that if the characteristic of F is not 2 or 3, the number of $\mathfrak{A}_4$-fields of bounded discriminant is growing at least as $CB^{\frac{1}{2}}$. For the case F is a number field this was established in [1, Corollary 1.8].

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1.5. Notations. We will use notation F for a global field. We denote by M_F (respectively, by M_F^0 and by M_F^∞) the set of its places (respectively, of its finite and infinite places).

We fix algebraic closures of F and of F_v for $v \in M_F$ and embeddings of the algebraic closure of F in each of F_v. We denote by $\overline{F}$ (respectively, by $\overline{F}_v$) the separable closure of F in (respectively, of F_v for all $v \in M_F$) in the chosen algebraic closures. The notation Γ_F and Γ_v will be used to denote the absolute Galois group of F and F_v, respectively. For a finite place v, we denote by Γ_v^{un} the Galois group $\mathrm{Gal}(F_v^{\mathrm{un}}/F_v)$, where F_v^{un} is the maximal unramified extension of F_v and by q_v the cardinality of the residue field at v.

Let $f, g : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be two functions, such that for $B \gg 0$ one has that $g(B) \neq 0$. We write $f \asymp_{B\to\infty} g$ if there are constants $C_1, C_2 > 0$ such that for every B big enough one has that

$$C_1 g(B) \leq f(B) \leq C_2 g(B).$$

2. Notions

We recall some notions and results from [10, Section 2]. Let G be a non-trivial finite tame F-group scheme.

2.1. Heights. Let e be the exponent of $G(\overline{F})$ and let $\boldsymbol{\mu}_e$ be the group scheme of e-th roots of unity. The group $G(\overline{F})$ acts on the Γ_F-group $\operatorname{Hom}(\boldsymbol{\mu}_e, G(\overline{F}))$ by conjugation

$$h \cdot (x \mapsto g) := x \longmapsto (hgh^{-1}).$$

The action preserves Γ_F-orbits and the identity element. We let G_* be the finite pointed F-scheme given by the Γ_F-pointed set

$$\operatorname{Hom}(\boldsymbol{\mu}_e, G(\overline{F}))/G(\overline{F}).$$

For a closed immersion $G \hookrightarrow R$, we have a pointed morphism $G_* \to R_*$ of trivial kernel (but not necessarily injective). The following definitions are from [10, Paragraph 2.3.1].

Definition 2.1.1.

(1) *A counting function is a Γ_F-invariant function $c : G_*(\overline{F}) \to \mathbb{R}_{\geq 0}$, which satisfies that $c(x) = 0$ if and only if $x = 1$ is the distinguished element in $G_*(\overline{F})$.*

(2) *Let $c : G_*(\overline{F}) \to \mathbb{R}_{\geq 0}$ be a counting function. We define*

$$a(c) := \Big(\min_{x \in G_*(\overline{F}) - 1} c(x)\Big)^{-1} \in \mathbb{R}_{>0},$$
$$b(c) := \#\{x \in G_*(\overline{F}) \,|\, c(x) = a(c)^{-1}\}.$$

(3) *If $\iota : G' \hookrightarrow G$ is a closed immersion of a non-trivial subgroup scheme, and $c : G_*(\overline{F}) \to \mathbb{R}_{\geq 0}$ a counting function, then we set $\iota^* c := c \circ ((G')_* \to G_*)$ (it is a counting function).*

We have verified in [10, Lemma 2.2.4] that in the constant case the Galois orbits of the set $G_*(\overline{F})$ correspond to so-called F-*conjugacy classes* from the Malle conjecture, hence, our definition of a- and b-invariants coincides with the Malle's one. We denote by $BG(F)$ (respectively, for $v \in M_F$ by $BG(F_v)$) the pointed set of isomorphism classes of G-torsors over F (respectively, over F_v). The Γ_F-group $G(\overline{F})$ becomes, using inclusions $\Gamma_v \hookrightarrow \Gamma_F$, a Γ_v-group for $v \in M_F$. For $K \in \{\Gamma_F\} \cup \{\Gamma_v\}_{v \in M_F}$, we denote by $Z^1(K, G(\overline{F}))$ the set of continuous crossed homomorphisms $f : K \to G(\overline{F})$. There exist canonical pointed bijections

$$BG(F) = Z^1(\Gamma_F, G(\overline{F}))/\sim \;=: H^1(\Gamma_F, G(\overline{F})),$$
$$BG(F_v) = Z^1(\Gamma_v, G(\overline{F}))/\sim \;=: H^1(\Gamma_v, G(\overline{F})), \qquad (v \in M_F)$$

where $\sim$ is defined via

$$f \sim f' \iff \exists\, g \in G(\overline{F}) :\ \forall\, \gamma \in \Gamma_F : f'(\gamma) = g^{-1} f(\gamma)(\gamma \cdot g),$$

and analogously for $v \in M_F$. Let Σ_G be the finite set given by the places v such that $G(\overline{F})$ is ramified or not tame at v (that is, $\gcd(q_v, \#G(\overline{F})) > 1$). Whenever $v \in M_F - (\Sigma_G \cup M_F^\infty)$, we have defined in [10, Definition 2.4.5] a canonical map of pointed sets

$$\Psi_v^G : BG(F_v) \longrightarrow G_*(\overline{F}),$$

the kernel of which is

$$BG(\mathcal{O}_v) := H^1(\Gamma_v^{\mathrm{un}}, G(\overline{F})) \subset H^1(\Gamma_v, G(\overline{F})) = BG(F_v).$$

If $x \in BG(F)$, then for almost all finite v, the image of x for the map $BG(F) \to BG(F_v)$ lies in $BG(\mathcal{O}_v)$, hence, for almost all finite v, one has that x is in the kernel of the composite map

$$BG(F) = H^1(\Gamma_F, G(\overline{F})) \longrightarrow H^1(\Gamma_v, G(\overline{F})) = BG(F_v) \xrightarrow{\Psi_v^G} G_*(\overline{F}).$$

Definition 2.1.2. *Let $c : G_*(\overline{F}) \to \mathbb{R}_{\geq 0}$ be a counting function. Let $M_F^\infty \cup \Sigma_G \subset \Sigma \subset M_F$ be a finite set of places. For $v \in \Sigma$, we let $c_v : BG(F_v) \to \mathbb{R}_{\geq 0}$ be functions and for $v \in M_F - \Sigma$ let us set*

$$c_v = c \circ \Psi_v^G : BG(F_v) \longrightarrow \mathbb{R}_{\geq 0}.$$

For $v \in M_F$, we denote by H_v the function

$$H_v : BG(F_v) \longrightarrow \mathbb{R}_{>0} \qquad x \longmapsto q_v^{c_v(x)}.$$

The function

$$H = H((c_v)_v) : BG(F) \longrightarrow \mathbb{R}_{>0} \qquad x \longmapsto \prod_{v \in M_F} H_v(x_v),$$

where x_v is the image of x for the map $BG(F) \to BG(F_v)$, is called the height function defined by $(c_v)_v$ (sometimes simply the height). We say that c is the type of H. We set

$$a(H) := a(c)$$
$$b(H) := b(c).$$

The quotient of two heights is a function which is bounded from above and below by positive constants. If $\iota : G' \hookrightarrow G$ is a closed immersion of a non-trivial subgroup, then we define $\iota^* H$ to be the function $BG'(F) \to BG(F) \xrightarrow{H} \mathbb{R}_{>0}$, which turns out to be a height on $BG'(F)$.

2.2. Twists. The references for this paragraph are [10, Paragraph 2.2.2, Lemma 2.2.6, Lemma 2.5.4]. Let $\sigma \in Z^1(\Gamma_F, G(\overline{F}))$ be a cocycle. We define ${}_\sigma G$ to be the finite group scheme which corresponds to the Γ_F-action on $G(\overline{F})$ obtained by twisting by σ:

$$\gamma \cdot g := (\sigma(\gamma))(\gamma \cdot g)(\sigma(\gamma)^{-1}), \qquad \gamma \in \Gamma_F, g \in G(\overline{F}).$$

There exists a canonical bijection $\lambda_\sigma : B({}_\sigma G)(F) \to BG(F)$, induced by

$$\begin{aligned} \Lambda_\sigma : Z^1(\Gamma_F, {}_\sigma G(\overline{F})) &\longrightarrow Z^1(\Gamma_F, G(\overline{F})). \\ f &\longmapsto f \cdot \sigma. \end{aligned}$$

One has a canonical identification $({}_\sigma G)_* = G_*$. If $H : BG(F) \to \mathbb{R}_{>0}$ is a height, then $H \circ \lambda_\sigma : B({}_\sigma G)(F) \to \mathbb{R}_{>0}$ is a height. Moreover, one has that

$$\begin{aligned} a(H \circ \lambda_\sigma) &= a(H) \\ b(H \circ \lambda_\sigma) &= b(H). \end{aligned}$$

If R is another non-trivial finite étale tame F-group scheme and $\phi : G \hookrightarrow R$ a homomorphism which is a closed immersion, we may write ${}_\sigma R$ for ${}_{\phi(\overline{F})\circ\sigma} R$. We have a closed immersion ${}_\sigma G \to {}_\sigma R$, and the induced morphism $({}_\sigma G)_* \to ({}_\sigma R)_*$ coincides with the morphism $G_* \to R_*$.

3. Semicommutative groups

In this section we prove our principal results.

3.1. Commutative case. We prove our main result for the commutative case.

Definition 3.1.1. *Let J be a finite étale F-group scheme and let $X \in BJ(F)$. We say that X is connected if the underlying topological space is connected.*

In the situation of the definition, one has that X is connected if and only the canonical Γ_F-action on $X(\overline{F})$ is transitive. Let $x \in Z^1(\Gamma_F, X(\overline{F}))$ be a lift of X. We fix a bijection $X(\overline{F}) \xrightarrow{\sim} J(\overline{F})$ and identify the set $X(\overline{F})$ with $J(\overline{F})$ via this bijection. The action on $X(\overline{F})$ is given by

$$\gamma \cdot g = (x(\gamma))(\gamma(g)) \qquad \gamma \in \Gamma_F, g \in X(\overline{F}) = J(\overline{F}).$$

Remark 3.1.2. We clarify what happens in the situation when J is constant. Then $(\gamma \cdot g)(\gamma(g))^{-1} = (\gamma \cdot g)g^{-1} = x(\gamma)$ and we deduce that X is connected if and only if x is surjective.

Remark 3.1.3. We will compare the notion of a "connected torsor" with the notion of a "surjective element" introduced by Alberts in [1, p. 2518]. We will compare the two notions in Remark 3.2.2.

Lemma 3.1.4. *Let J be a finite étale F-group scheme. Let $X \in BJ(F)$ and let $x \in Z^1(\Gamma_F, J(\overline{F}))$ be a lift of X. Suppose that there exists a finite set of finite places $\{v_1, \dots, v_k\}$ of F such that for $1 \leq i \leq k$ one has that*

(1) *the finite group scheme $J_{F_{v_i}}$ is constant;*
(2) *one has that $J(\overline{F}) = \langle x(\Gamma_{v_i}) \rangle_{i=1}^k$.*

Then X is connected.

Proof. Let us show that $X(\overline{F}) = J(\overline{F})$ is a transitive Γ_F-set. Let $g_1, g_2 \in J(\overline{F})$ and let $g = (g_2)(g_1)^{-1}$. By the second assumption, there exists a finite product of $\prod x(\gamma_j)$, where $\gamma_j \in \Gamma_{v_1} \cup \dots \cup \Gamma_{v_k}$ such that $\prod x(\gamma_j) = g$. Note that for $\gamma_1, \gamma_2 \in \Gamma_{v_1} \cup \dots \cup \Gamma_{v_k}$ one has that

$$x(\gamma_1 \gamma_2) = (x(\gamma_1))(\gamma_1 \cdot x(\gamma_2)) = (x(\gamma_1))(x(\gamma_2)).$$

Hence, $x(\prod \gamma_j) = g$. We deduce that

$$\Big(\prod \gamma_j\Big) \cdot g_1 = \Big(x\Big(\prod \gamma_j\Big)\Big)\Big(\Big(\prod \gamma_j\Big)(g_2)\Big) = g g_1 = g_2.$$

The action is thus transitive and the statement follows. ☐

Lemma 3.1.5. *Let G be a finite étale F-group scheme and let $i : BG(F) \to \prod_{v \in M_F} BG(F_v)$ be the diagonal map. Let Σ be a finite set of places of F. There exists a finite subset $T \subset M_F^0 - (\Sigma \cup \Sigma_G)$, elements $y_v \in BG(\mathcal{O}_v)$ for $v \in T$, such that every $x \in BG(F)$, with*

$$i(x) \in \left(\prod_{v \in T} \{y_v\} \times \prod_{v \in M_F - T} BG(F_v)\right),$$

is connected.

Proof.

Step 1. First, we prove that for every $1 \neq g \in G(\overline{F})$ we can associate a finite place v_g of F such that the following conditions are verified:

- for every $g \in G(\overline{F}) - \{1\}$ one has that $v_g \notin \Sigma$;
- one has that $G_{F_{v_g}}$ are constant finite group schemes;
- one has that $v_g \neq v_{g'}$, whenever $g \neq g'$.

Indeed, there exists a finite Galois extension K/F contained in $\overline{F}$ such that Γ_F acts on $G(\overline{F})$ via the Galois group $\mathrm{Gal}(K/F)$. There exist infinitely many places v such that $K \subset F_v$. (We write $K = F(a)$ and let p_a be the minimal polynomial of a over F. There are infinitely many v such that $v(p_a(t)) > 0$ for some $t \in \mathcal{O}_F$, where $\mathcal{O}_F$ is the ring of integers of F. For any such v which satisfies that for every coefficient b_i of p_a one has $v(b_i) = 0$, by Hensel's lemma [20, Chapter II, Lemma 4.6], the polynomial p_a admits a root in F_v.) For such places v one has that G_{F_v} is constant. The claim follows.

Step 2. Now, for every $1 \neq g \in G(\overline{F})$, we fix a surjective homomorphism

$$\Gamma_{v_g} \longrightarrow \Gamma_{v_g}^{\mathrm{un}} = \widehat{\mathbb{Z}} \longrightarrow \langle g \rangle \subset G(\overline{F}).$$

This defines a $G_{F_{v_g}}$-torsor y_g such that $y_g \in BG(\mathcal{O}_v)$. Consider the open

$$U := \prod_{1 \neq g \in G(\overline{F})} \{y_g\} \times \prod_{v \in (M_F - \{v_g \mid 1 \neq g \in G(\overline{F})\})} BG(F_v) \subset \prod_{v \in M_F} BG(F_v),$$

which is also closed. By applying Lemma 3.1.4 to $J = G$ and to the set of places $T := \{v_g \mid g \in G(\overline{F}) - 1\}$, we have that if $i(x) \in U$, then x is connected. □

Theorem 3.1.6. *Let G be a commutative non-trivial finite étale and tame F-group scheme. Let H be a height on $BG(F)$. One has that*

$$\#\{x \in BG(F) \mid x \text{ is connected and } H(x) \leq B\} \underset{B \to \infty}{\asymp} B^{a(H)} \log(B)^{b(H)-1}.$$

Proof. Note that it suffices to assume that H is a normalized height, i.e. that $a(H) = 1$. Indeed, for every height H, one has that $H^{\frac{1}{a(H)}}$ is a normalized height and thus the claim for a normalized height then immediately implies the claim for a non-normalized height.

It follows from [21, Theorem 9.2.3 (vii)] that there exists a finite set of places Σ such that whenever $\Sigma' \subset M_F - \Sigma$ is finite, one has that the canonical map $BG(F) \to \prod_{v \in \Sigma'} BG(F_v)$ is surjective. Let T be and $(y_v)_{v \in T} \in \prod_{v \in T} BG(F_v)$ be given by applying Lemma 3.1.5 to the finite group scheme G and the set of places Σ. We set

$$U := \prod_{v \in T} \{y_v\} \times \prod_{v \in M_F - T} BG(F_v).$$

By construction one has that $i(BG(F)) \cap U \neq \emptyset$. In [10, Lemma 3.5.1], we have defined a Radon measure ω_H on the product space $\prod_{v \in M_F} BG(F_v)$. We have proven in [10, Lemma 3.5.2] that

$$\mathrm{supp}(\omega_H) = \overline{i(BG(F))}.$$

As U is an open neighbourhood of a point in $\mathrm{supp}(\omega_H)$, we deduce that $\omega_H(U) > 0$. Now, the statement follows by applying [10, Theorem 3.5.7] and [10, Theorem 3.5.8] to the characteristic function $\mathbf{1}_U$ of the open set with empty boundary having positive ω_H-volume U. □

3.2. Semidirect products. In this subsection, we will study torsors of semidirect products of finite étale tame F-group schemes.

Let A and K be finite étale tame F-group schemes. Suppose we are given an F-homomorphism $\phi : K \to \underline{\mathrm{Aut}}(A)$, where $\underline{\mathrm{Aut}}(A)$ is the finite étale F-group scheme given by the Γ_F-group $\mathrm{Aut}(A(\overline{F}))$ and the following action

$$\gamma \cdot t = \gamma \circ t \circ \gamma^{-1} \qquad (\gamma \in \Gamma_F, t \in \mathrm{Aut}(A(\overline{F}))).$$

We let $A \rtimes_\phi K$ be the group scheme given by the group $A(\overline{F}) \rtimes_{\phi(\overline{F})} K(\overline{F})$ which is endowed with the following Γ_F-action $\gamma \cdot (n_0, h_0) = (\gamma(n_0), \gamma(h_0))$.

Let $\theta \in Z^1(\Gamma_F, K(\overline{F}))$ be a crossed homomorphism and let Θ be the K-torsor defined by θ. The image of θ for the map $Z^1(\Gamma_F, K(\overline{F})) \to Z^1(\Gamma_F, A(\overline{F}) \rtimes_{\phi(\overline{F})} K(\overline{F}))$ induced by

$$K(\overline{F}) \longrightarrow A(\overline{F}) \rtimes_{\phi(\overline{F})} K(\overline{F}) \qquad h \longmapsto (1, h)$$

is the map $\sigma := \gamma \mapsto (1, \theta(\gamma))$. Let ${}_\sigma A$ be the group subscheme of ${}_\sigma(A \rtimes_\phi K)$ corresponding to the subgroup $A(\overline{F})$ which is Γ_F-invariant for the twisted action.

Lemma 3.2.1. *The canonical map*

$$u_\sigma^\phi : B({}_\sigma A)(F) \longrightarrow B({}_\sigma(A \rtimes_\phi K))(F) \xrightarrow{\lambda_\sigma} B(A \rtimes_\phi K)(F)$$

is given by $u_\sigma^\phi(X) = X \times_F \Theta$.

Proof. Let $X \in B({}_\sigma A)(F)$ and let $x \in Z^1(\Gamma_F, {}_\sigma A(\overline{F}))$ be a lift of X. The image of x under the canonical map

$$Z^1(\Gamma_F, {}_\sigma A(\overline{F})) \longrightarrow Z^1(\Gamma_F, {}_\sigma(A \rtimes_\phi K))$$

is given by $\gamma \mapsto (x(\gamma), 1)$. The map λ_σ is induced by the map

$$\Lambda_\sigma : Z^1(\Gamma_F, ({}_\sigma(A \rtimes_\phi K))(\overline{F})) \longrightarrow Z^1(\Gamma_F, (A \rtimes_\phi K)(\overline{F}))$$

which is given by $y \mapsto (\gamma \mapsto y(\gamma) \cdot (1, \theta(\gamma)))$. It follows that the image of X for the map u_σ^ϕ is the $A \rtimes_\phi K$-torsor induced by the crossed homomorphism $\gamma \mapsto (x(\gamma), \theta(\gamma))$. By [23, p. 47], the $A \rtimes_\phi K$-torsor induced by $\gamma \mapsto (x(\gamma), \theta(\gamma))$ is isomorphic to $A \rtimes_\phi K$-torsor given by the group $A(\overline{F}) \rtimes_{\phi(\overline{F})} K(\overline{F})$ and the following Γ_F-action:

$$\begin{aligned}\gamma \cdot (n_0, h_0) &= (x(\gamma), \theta(\gamma)) \cdot (\gamma(n_0), \gamma(h_0)) \\ &= \Big((x(\gamma))\Big(\phi(\theta(\gamma))(\gamma(n_0))\Big), \theta(\gamma)\gamma(h_0)\Big).\end{aligned}$$

The ${}_\sigma A$-torsor X is isomorphic to the ${}_\sigma A$-torsor given by the group $A(\overline{F})$ and the following Γ_F-action

$$\gamma \cdot n_0 = (x(\gamma)(\phi(\theta(\gamma)).\gamma(n_0))).$$

The K-torsor Θ is isomorphic to the K-torsor defined by the group $K(\overline{F})$ and the following Γ_F-action

$$\gamma \cdot h_0 = ((\theta(\gamma))(\gamma(h_0))).$$

By comparing the actions, we see immediately that $X \times_F \Theta = u_\sigma^\phi(X)$. The statement is proven. □

Remark 3.2.2. We now make a comparison of the Alberts' notion of "surjective element" with the notion of a connected torsor that we announced in Remark 3.1.3. Suppose that A, K, ϕ ar as above and suppose furthermore that A and K are constant and set $N = A \rtimes_\phi K$. We identify A and K with their images in N. Suppose that we are given a surjective homomorphism $\pi : \Gamma_F \to N(\overline{F})$ and set θ for its image for the map $Z^1(\Gamma_F, N(\overline{F})) \to Z^1(\Gamma_F, K(\overline{F}))$. Then for $\gamma \in \Gamma_F$ and $g \in {}_\sigma A(\overline{F})$, one has that $\gamma \cdot g = \pi(\gamma) g \pi(\gamma)^{-1}$ which is precisely the requirement of [1, Lemma 1.3]. In this lemma, Alberts establishes a bijection between the elements of $Z^1(\Gamma_F, {}_\sigma A(\overline{F}))$ and the group of homomorphisms $\pi' : \Gamma_F \to N(\overline{F})$ which satisfy that $\pi'(\gamma)({}_\sigma A(\overline{F})) = ({}_\sigma A(\overline{F}))\pi(\gamma)$. The bijection is given by $f \mapsto f \cdot \pi$. An element in $Z^1(\Gamma_F, A(\overline{F}))$ is said to be "surjective" if $x \cdot \pi : \Gamma_F \to N(\overline{F})$ is surjective, which by Remark 3.1.2 is equivalent to the claim that the class of $x \cdot \pi$ is connected. It can be seen from construction that the induced map from the map $f \mapsto f \cdot \pi$ on the cohomology classes is precisely the map u^ϕ_σ. Thus, we deduce that a $x \in Z^1(\Gamma_F, {}_\sigma A(\overline{F}))$ is surjective if and only if $u^\phi_\sigma([x])$ is connected, where $[x] \in B({}_\sigma A)(\overline{F})$ is the class of x. Note that this implies that $[x]$ is connected but not vice versa.

The following notion is a "quantitative" variant of the notion of weak weak approximation.

Definition 3.2.3. *Let G be a non-trivial finite étale tame F-group scheme and let $\alpha > 0$ and $\beta \geq 0$. Let $H : BG(F) \to \mathbb{R}_{>0}$ be a height. We say that G is (H, α, β)-saturated if the following condition is satisfied:*

- *there exists a finite subset $S \subset M_F$ such that for every finite $T \subset M_F - S$ and every $(z_v)_{v \in T} \in \prod_{v \in T} BG(F_v)$, one has that there exists $C > 0$ such that*

$$\#\{y \in BG(F) \,|\, y \text{ is connected}, \forall\, v \in T, y \otimes_F F_v \cong z_v, H(x) \leq B\} \geq C B^\alpha \log(B)^\beta$$

for $B \gg 0$.

Remark 3.2.4. We may drop the assumption that y is connected. Indeed, it follows from Lemma 3.1.5 that one can choose finitely many local conditions at places disjoint from S which will force every G-torsor satisfying them to be connected. We then add the corresponding places to S.

Clearly, for two heights H_1 and H_2 which have the same type, one has that G is (H_1, α, β)-saturated if and only if it is (H_2, α, β)-saturated. It is well known [21, Theorem 9.2.3 (vii)] that if G is commutative, then G satisfies the weak weak approximation. Moreover, Theorem 3.1.6 implies that G is $(H, \alpha(H), \beta(H) - 1)$-saturated.

Proposition 3.2.5. *Let G be a non-trivial finite étale tame F-group scheme. We suppose that $G = \langle A, K\rangle$, where $A \leq G$ and $K \lneq G$ are closed subgroups, such that A is normal in G and K admits a connected torsor Θ. Let $\phi : K \to \underline{\mathrm{Aut}}(A)$ be the homomorphism given by the conjugation. Let $\sigma_K \in Z^1(\Gamma_F, K(\overline{F}))$ be a lift of Θ and let σ be the image of σ_K for the map $Z^1(\Gamma_F, K(\overline{F})) \to Z^1(\Gamma_F, (A \rtimes_\phi K)(\overline{F}))$ induced by the map*

$$K \longrightarrow A \rtimes_\phi K, \qquad k \longmapsto (1, k).$$

Let $\alpha, \beta > 0$. Let $H : BG(F) \to \mathbb{R}_{>0}$ be a height. Suppose that ${}_\sigma A$ is $(H \circ u^\phi_\sigma, \alpha, \beta)$-saturated. There exists $C > 0$ such that

$$\#\{x \in BG(F) \,|\, x \text{ is connected, } H(x) \leq B\} \geq CB^\alpha \log(B)^\beta$$

for $B \gg 0$.

Proof. We split the proof in the several steps.

Step 1. We recall a known fact: if $G_1 \subset G_2$ is normal subgroup of a finite étale and tame F-group scheme G_2, the canonical map $B(G_2)(F) \to B(G_2/G_1)(F)$ is given by $x \mapsto (x/G_1)$. Indeed, let $\widetilde{x} \in Z^1(\Gamma_F, G_2(\overline{F}))$ be a lift of $X \in B(G_2)(F)$. Its image in $Z^1(\Gamma_F, (G_2/G_1)(\overline{F}))$ is $w \circ \widetilde{x}$, where $w : G_2(\overline{F}) \to (G_2/G_1)(\overline{F})$ is the quotient map. The element in $B(G_2/G_1)(F)$ associated to $w \circ \widetilde{x}$ is isomorphic to Γ_F-set given by $(G_2/G_1)(\overline{F})$ endowed with the following Γ_F-action, where $y \in G_2(\overline{F})$:

$$\gamma \cdot w(y) := (w(\widetilde{x}(\gamma)))\gamma(w(y))) = w(\widetilde{x}(\gamma)\gamma(y)).$$

On the other side, the quotient of x by G_2 is isomorphic to the Γ_F-set

$$\gamma \cdot w(y) = w(\gamma \cdot y) = w(\widetilde{x}(\gamma)\gamma(y)),$$

and the claim follows.

Step 2. We have a surjective homomorphism

$$A \rtimes_\phi K \longrightarrow \langle A, K\rangle = G \qquad (a, k) \longmapsto ak$$

and we denote by σ_G the image of σ for the induced map $Z^1(\Gamma_F, (A \rtimes_\phi K)(\overline{F})) \to Z^1(\Gamma_F, G(\overline{F}))$. The composite map $K \to A \rtimes_\phi K \to G$ is the inclusion $K \hookrightarrow G$, hence, one has that σ_G is precisely the image of σ_K for the map induced by the inclusion. It is immediate that ${}_{\sigma_G}A = {}_\sigma A$ and that the homomorphism ${}_{\sigma_G}A \to {}_{\sigma_G}G$ induced by σ_G is the homomorphism ${}_\sigma A \hookrightarrow {}_\sigma(A \rtimes_\phi K) \to {}_\sigma G = {}_{\sigma_G}G$ induced by σ. It follows that the map

$$u^\phi_\sigma : B({}_{\sigma_G}A)(F) \longrightarrow B({}_{\sigma_G}G)(F) = B({}_\sigma G)(F) \xrightarrow{\lambda_\sigma} BG(F),$$

which by [10, Lemma 2.6.1] has all fibers of cardinality at most $\#G(\overline{F})$, coincides with the map

$$B({}_\sigma A)(F) \longrightarrow B({}_\sigma(A \rtimes_\phi K))(F) \longrightarrow B({}_\sigma G)(F) \xrightarrow{\lambda_\sigma} BG(F).$$

Now, [10, Lemma 2.2.1, Part (5)] gives that the maps coincide with the map

$$u_\sigma^\phi : B({}_\sigma A)(F) \longrightarrow B({}_\sigma(A \rtimes_\phi K))(F) \xrightarrow{\lambda_\sigma} B(A \rtimes_\phi K)(F) \longrightarrow BG(F).$$

By combining Part (1) together with Lemma 3.2.1, we obtain that the composite map is given by $x \mapsto (x \times_F \Theta)/N$, where N is the kernel of $A \rtimes_\phi K \to G$. It follows, in particular, that the image $u_\sigma^\phi(x)$ is connected if $x \times_F \Theta$ is connected.

Step 3. By an abuse of notation, we may use the same letters for fields and corresponding spectra. Let $\widetilde{\Theta}/F$ be the Galois closure of Θ. Note that in order for $x \otimes_F \Theta$ is a field it suffices that x and $x \otimes_F \widetilde{\Theta}$ is a field. Let $\Theta_1, \ldots, \Theta_k$ be the minimal subextensions of $\widetilde{\Theta}/F$ which are strictly larger than F. By [4, Chapter V, §10, n° 8, Theorem 5], if x is a field, one has that $x \otimes_F \widetilde{\Theta}$ is a field if and only if x does not contain any of the subfields $\Theta_1, \ldots, \Theta_k$. If $\Theta_j \subset x$ then for every $v \in M_F^0$, one has that $\Theta_j \otimes_F F_v \subset x \otimes_F F_v$.

Step 4. For every $j = 1, \ldots, k$, it follows from Čebotarev theorem [21, Theorem 9.1.3] that there exist infinitely many places $v \in M_F^0$, such that Θ_j does not have a degree 1 place over it. (We recall the implication. Let $\widetilde{\Theta}_j$ be the Galois closure of Θ_j. By [20, Lemma 13.5], which is stated only for number fields, but the presented proof is valid for function fields as well, the Dirichlet density that v does admit a degree 1 place over it is equal to

$$\frac{\# \bigcup_{g \in \mathrm{Gal}(\widetilde{\Theta}_j/F)} g \,\mathrm{Gal}(\widetilde{\Theta}_j/\Theta_j) g^{-1}}{\# \,\mathrm{Gal}(\widetilde{\Theta}_j/F)}.$$

We verify that the last quotient is strictly less than 1. Indeed, there are at most $[\mathrm{Gal}(\widetilde{\Theta}_j/F) : \mathrm{Gal}(\widetilde{\Theta}_j/\Theta_j)]$ conjugates of the subgroup $\mathrm{Gal}(\widetilde{\Theta}_j/\Theta_j)$ and each of them contains the element $1 \in \mathrm{Gal}(\widetilde{\Theta}_j/F)$. Hence,

$$\begin{aligned} \# \bigcup_{g \in \mathrm{Gal}(\widetilde{\Theta}_j/F)} & g \,\mathrm{Gal}(\widetilde{\Theta}_j/\Theta_j) g^{-1} \\ & \leq (\# \,\mathrm{Gal}(\widetilde{\Theta}_j/\Theta_j) - 1) \cdot [\mathrm{Gal}(\widetilde{\Theta}_j/F) : \mathrm{Gal}(\widetilde{\Theta}_j/\Theta_j)] + 1 \\ & < \# \,\mathrm{Gal}(\widetilde{\Theta}_j/F), \end{aligned}$$

where the strict inequality is valid because $F \subsetneq \Theta_j \subset \widetilde{\Theta}_j$. The claim follows.) Let v_j be such a place not contained in the finite set $S \subset M_F$ which is as in the Definition 3.2.3 (recall that ${}_\sigma A$ is $(H \circ u_\sigma^\phi, \alpha, \beta)$-saturated.) One has that $\Theta_j \otimes_F F_{v_j}$ is a product of fields, none of which is isomorphic to F_{v_j}.

We set

$$U := \prod_{j=1}^{k}\{({}_{\sigma}A)_{F_{v_j}}\} \times \prod_{v\notin\{v_1,\dots,v_k\}} B({}_{\sigma}A)(F_v) \subset \prod_{v\in M_F} B({}_{\sigma}A)(F_v).$$

For $x \in B({}_{\sigma}A)(F)$ which is connected and such that $i(x) \in U$, where $i : B({}_{\sigma}A)(F) \to \prod_{v\in M_F} B({}_{\sigma}A)(F_v)$ is the diagonal map, we have that $x \not\supset \Theta_j$ because the trivial $({}_{\sigma}A)_{F_{v_j}}$-torsor has a component isomorphic to $\mathrm{Spec}(F_{v_j})$. Hence, for such x one has that $x \otimes_F \Theta_j$ is a field. Set $T = \{v_1, \dots, v_k\}$. It follows from [10, Lemma 2.6.1] and the assumption that ${}_{\sigma}A$ is $(H\circ u_{\sigma}^{\phi}, \alpha, \beta)$-saturated that for $B \gg 0$ one has

$$\begin{aligned}\#\{y \in BG(F)\,|\, y \text{ is connected and } H(x) \le B\}\\ &\ge \#G(F)\cdot \#u_{\sigma}^{\phi}\left(\left\{x \in B({}_{\sigma}A)(F)\,\middle|\, \begin{matrix} x \text{ is connected, } i(x)\in U,\\ H(u_{\sigma}^{\phi}(x)) \le B\end{matrix}\right\}\right)\\ &\ge B^{\alpha}\log(B)^{\beta},\end{aligned}$$

for some $C > 0$, where $i : B({}_{\sigma}A)(F) \to \prod_{v\in M_F} B({}_{\sigma}A)(F_v)$ is the diagonal map. The proposition has been proven. ☐

Remark 3.2.6. We note that connected ${}_{\sigma}A$-torsors have Θ for a *resolvent* (that is, the Galois closure of the corresponding extensions contain the extension corresponding to Θ). The question of counting extensions with a fixed resolvent has been studied in [5, 6, 7], etc.

3.3. Semi-commutative groups. We establish a lower bound on the number of connected torsors for *semicommutative* group schemes. A reference for the definition and basic properties for the constant case is [18, Chapter IV, Section 2.2].

Definition 3.3.1. *We say that a finite étale F-group scheme G is semicommutative if there exists a finite set of commutative subgroup schemes $\{A_i\}_{i=1}^m$ such that*

$$G = \langle A_i\rangle_{i=1}^m \text{ and } A_i \le \mathcal{N}_G(A_j) \text{ whenever } i \le j,$$

where $\langle K_i\rangle_{i=1}^m$ denotes the smallest closed subgroup scheme containing the subschemes K_i of G and $\mathcal{N}_G(K)$ denotes the normalizer of K, i.e. the largest closed subgroup scheme of G containing the closed subgroup scheme K as a normal subgroup.

The following characterization for the constant case is due to Dentzer.

Proposition 3.3.2. *Let G be a non-trivial finite étale F-group scheme. The following conditions are equivalent.*

(1) *G is semicommutative.*

(2) *There exists a commutative normal subgroup A of G and a semi-commutative closed subgroup $K \lneq G$, such that $G = \langle A, K\rangle$.*
(3) *There exist a sequence $(G_i)_{i=0}^{k}$ of finite étale F-group schemes with $G_0 = \{0\}$ and $G_k \cong G$, a sequence $(A_i)_{i=1}^{k-1}$ of commutative finite étale F-group schemes, a sequence of homomorphisms $(\phi_i : G_i \to \underline{\mathrm{Aut}}(A_i))_{i=0}^{k-1}$ of finite F-group schemes and a sequence of normal subgroup schemes $(N_i \subset (A_i \rtimes_{\phi_i} G_i))_{i=0,\dots,k-1}$ such that for every $i = 1, \dots, k$ one has that*

$$G_i = (A_{i-1} \rtimes_{\phi_{i-1}} G_{i-1})/N_{i-1}.$$

Proof. The proof is identical to the constant case [18, Chapter IV, Theorem 2.7]. □

Theorem 3.3.3. *Suppose that G is a semicommutative étale tame F-group scheme and write $G = \langle A, K\rangle$, with $\iota : A \hookrightarrow G$ commutative, $K \lneq G$ semicommutative. Let $c : G_*(\overline{F}) \to \mathbb{R}_{\geq 0}$ be a counting function and let $H : BG(F) \to \mathbb{R}_{>0}$ be a height of type c. There exists $C > 0$ such that*

$$\#\{x \in BG(F) \,|\, x \textit{ is connected, } H(x) \leq B\} \geq CB^{a(\iota^* c)}$$

for $B \gg 0$.

Proof. The proof is by induction on the cardinality of G. By induction, we can suppose that there exists at least one connected K-torsor Θ. Let σ be the image of a lift of Θ for the map $Z^1(\Gamma_F, K(\overline{F})) \to Z^1(\Gamma_F, (A \rtimes_\phi K)(\overline{F}))$. Consider the inclusion $\kappa : {}_\sigma A \to {}_\sigma G$. One has that $a(\iota^* c) = a(\kappa^* c)$, because the homomorphism $({}_\sigma A)(\overline{F}) \to ({}_\sigma G)(\overline{F})$ coincides with the homomorphism $A(\overline{F}) \to G(\overline{F})$, hence the map $({}_\sigma A)_*(\overline{F}) \to ({}_\sigma G)_*(\overline{F})$ coincides with the map $A_*(\overline{F}) \to G_*(\overline{F})$. It follows from Theorem 3.1.6, that the finite group scheme ${}_\sigma A$ is $(H \circ g, a(\iota^* c), 0)$-saturated, where g is the map

$$B({}_\sigma A)(\overline{F}) \longrightarrow B({}_\sigma G)(F) \xrightarrow{\lambda_\sigma} BG(F).$$

By Proposition 3.2.5, we have for $B \gg 0$ that

$$\#\{x \in BG(F) \,|\, x \text{ is connected}, H(x) \leq B\} \geq CB^{a(\iota^* c)}$$

for some $C > 0$. □

Example 3.3.4. Suppose that the characteristic of F is not 2 or 3. The alternating group $\mathfrak{A}_4$ of order 12 has a normal commutative subgroup of order 4 which is preserved by every automorphism of $\mathfrak{A}_4$. It follows from this fact and Proposition 3.3.2 that a finite étale group scheme G, for which $G(\overline{F}) = \mathfrak{A}_4$ as abstract groups, is semicommutative if and only if it contains a closed subgroup of order 3. This happens e.g. when $G = \mathfrak{A}_4$ is constant, but also for any (not necessarily constant) G of the form $G = {}_\sigma(\mathfrak{A})_4$ where $\sigma : \Gamma_F \to \mathfrak{S}_4$ is such that the induced action of Γ_F on $\{1, \dots, 4\}$ fixes an element. The natural representation $\mathfrak{A}_4 \subset \mathfrak{S}_4$ induces a counting function $c : (\mathfrak{A}_4)_* \to \mathbb{R}_{>0}$ given by $c(x) = 2$ if $x \neq 1$. We deduce, in particular, from Theorem 3.3.3 that the number of $\mathfrak{A}_4$-fields of bounded discriminant is growing as $CB^{\frac{1}{2}}$ for some $C > 0$.

Appendix A.

Let F be a global field. In this appendix, we will write down some examples of non-constant étale and tame finite F-group schemes for which Question 1.1.1 is known to have a positive answer, even though it is not explicitly written down in the literature. The implications are certainly well-known to experts.

A.1. Hyperweak approximation. Harari defines the following notion in [14, Section 4].

Definition A.1.1 (Harari)**.** *Let G be étale and tame finite F-group scheme. We say that G satisfies the hyperweak approximation, if there exists a finite set of places $S_0 \subset M_F$ such that for every finite $S \subset M_F - S_0$ one has that the image of the canonical map*

$$BG(F) \longrightarrow \prod_{v\in S} BG(F_v)$$

contains the subset $\prod_{v\in S} BG(\mathcal{O}_v)$.

This is a weaker property than the one given below

Definition A.1.2. *Denote by $BG(F)$ the set of isomorphism classes of G-torsors over F and for a place v of F, denote by $BG(F_v)$ the set of isomorphism classes of G-torsors over the completion F_v. We say that the weak weak approximation is valid for G, if there exists a finite set S_0 of places of F, such that for every finite subset S of places of F which is disjoint from S_0, we have that the canonical map*

$$BG(F) \longrightarrow \prod_{v\in S} BG(F_v) \tag{A.1.3}$$

is surjective.

He establishes in [14, Proposition 1] that if F is assumed to be a number field and G to be constant étale finite group scheme satisfying the hyperweak approximation, then the inverse Galois problem has an affirmative answer for G. As we have said in Section 1.1, the "constant" assumption is redunant. (We thank Lucchini Arteche for indicating us this).

Proposition A.1.4. *Let G be an étale tame finite F-group scheme. Suppose that G satisfies the hyperweak approximation. Then G admits a connected G-torsor.*

Proof. Let $S_0 \subset M_F$ be as in Definition A.1.1. By Lemma 3.1.5, it is possible to choose for $g \in G(\overline{F}) - \{1\}$ places $v_g \in M_F - S_0$ and elements $y_g \in BG(\mathcal{O}_v)$ such that $v_g \neq v_{g'}$ if $g \neq g'$ and such that whenever $x \in BG(F)$ satisfies that

$$i(x) \in \prod_{g \in G(\overline{F}) - \{1\}} \{y_g\} \times \prod_{v \in M_F - \{v_g | 1 \neq g \in G(\overline{F})\}} BG(F_v),$$

then x is connected. By the assumption that G satisfies the hyperweak approximation, the set of such x is non-empty. The claim follows. □

We list two results due to Harari. He proves them with the assumption that F is a number field, but the identical proofs work in the global field case with the tameness assumption.

Proposition A.1.5 ([14, Harari, Proposition 2 and 3])**.** *Let G be étale and tame finite F-group scheme which satisfies the hyperweak approximation.*

(1) *Let N be a subgroup scheme of G. The finite groups scheme G/N satisfies the hyperweak approximation.*
(2) *Let A be a commutative étale and tame finite F-group scheme and let $\phi : G \to \underline{\mathrm{Aut}}(A)$ be a homomorphsm. The semidirect product $A \rtimes_\phi G$ satisfies the hyperweak approximation.*

Thus, the results of Harari are sufficient to conclude the following fact.

Corollary A.1.6. *Let G be a semicommutative étale tame finite F-group scheme. Then G admits a connected G-torsor.*

A.2. Supersolvable groups. We now suppose that F is a number field.

Definition A.2.1. *Let G be an étale finite F-group scheme. We say that G is supersolvable if there exists composition series of finite normal subgroup schemes $\{1\} = G_0 \subset \cdots \subset G_k = G$ such that for every $i = 1, \dots, k$ one has that $(G_i/G_{i-1})(\overline{F}) \cong \mathbb{Z}/r_i\mathbb{Z}$ for some $r_i \in \mathbb{Z}_{>1}$.*

The following property is an immediate consequence of [15, Theorem B] due to Harpaz and Wittenberg. (In the article, they derived it only with the assumption G is constant.)

Proposition A.2.2. *Let G be an étale finite F-group scheme which is supersolvable. Then G satisfies the weak weak approximation.*

Proof. There exists a closed embedding $G \hookrightarrow \mathrm{SL}_n$ for some $n \geq 1$ (indeed, by [19, Corollary 4.10], one can embed G to GL_m for certain m and GL_m can be embedded in SL_{m+1} via $A \mapsto A \oplus (\det(A)^{-1})$). By Hironaka theorem, there exists a proper smooth geometrically integral variety X which is F-birational to SL_n / G. By [15, Theorem B], one has that the Brauer–Manin obstruction is the only one for the weak approximation for X (that is, the closure of the image of the canonical map

$X(F) \to \prod_{v \in M_F} X(F_v)$ coincides with the vanishing locus of the Brauer–Manin pairing). The variety X is unirational, thus by [8, p. 347, (6)] satisfies the conditions of [8, Lemma 13.3.13] and applying it, gives that the weak weak approximation is valid for X (that is, the closure of the image of the map $X(F) \to \prod_{v \in M_F} X(F_v)$ is of the form $Z \times \prod_{v \in M_F - T} X(F_v)$ for some finite set of places T and some closed subset $Z \subset \prod_{v \in T} X(F_v)$). Hence, by [8, Proposition 13.2.3], one has that the weak weak approximation is valid for SL_n / G and by the equivalence from [14, p. 551], we obtain that G satisfies the weak weak approximation. □

Hence, it is immediate from Proposition A.1.4 that:

Corollary A.2.3. *Let G be a supersolvable étale finite group scheme over the number field F. Then G admits a connected G-torsor.*

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