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The 3-adic representations arising from elliptic curves over $\mathbb{Q}_3$ with potential good reduction

par Giovanni BOSCO

Résumé. Nous donnons une classification complète des représentations 3-adiques du groupe de Galois absolu de $\mathbb{Q}_3$ provenant des courbes elliptiques définies sur $\mathbb{Q}_3$ ayant potentiellement bonne réduction. Ces représentations sont décrites par leurs (φ, G)-modules filtrés associés. Les cas les plus intéressants sont ceux de potentielle bonne réduction sauvage.

Abstract. We give a complete classification of all the potentially crystalline 3-adic representations of the absolute Galois group of $\mathbb{Q}_3$ that are isomorphic to the Tate module of an elliptic curve defined over $\mathbb{Q}_3$. These representations are described in terms of their associated filtered (φ, G)-modules. The most interesting cases occur when the potential good reduction is wild.

1. Introduction

Fix a prime number p. The p-adic representations arising from elliptic curves over $\mathbb{Q}_p$ have been completely described for $p \geq 5$ in [13]. The goal of this paper is to treat the case of potential good reduction for $p = 3$. When $p \geq 5$, potential good reduction is necessarily tame with cyclic inertia. This is not the case for $p = 3$, where both wild potential good reduction and non abelian inertia do appear, sometimes simulteanously (see $e = 12$).

Let $\overline{\mathbb{Q}}_p$ be an algebraic closure of $\mathbb{Q}_p$ and $G = \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ be its absolute Galois group. Given an elliptic curve E defined over $\mathbb{Q}_p$, let $E[p^n]$ denote its group of p^n-torsion points with value in $\overline{\mathbb{Q}}_p$ and

$$T_p(E) = \varprojlim_{P \mapsto pP} E[p^n]$$

its p-adic Tate module. It is a free $\mathbb{Z}_p$-module of rank 2 with a continuous and linear action of G. The p-adic representation of G associated to E, also called Tate module, is

$$V_p(E) = \mathbb{Q}_p \otimes_{\mathbb{Z}_p} T_p(E).$$

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A p-adic representation V of G arises from an elliptic curve over $\mathbb{Q}_p$ if there exists $E/\mathbb{Q}_p$ such that $V \simeq V_p(E)$. We wish to classify all p-adic representations arising from elliptic curves over $\mathbb{Q}_p$ up to isomorphism for $p = 3$ with an additional condition: the considered elliptic curve has potential good reduction, that is, it acquires good reduction over a finite extension of $\mathbb{Q}_p$. Such curves have nice geometric properties which are carried over the representations. Indeed, it is well known that the Tate module of an elliptic curve with potential good reduction is potentially crystalline. Such representations are completely determined — via the contravariant functor $\mathbf{D}^*_{\mathrm{pcris}}$ — by their associated filtered (φ, G)-module, a purely semilinear object.

Let $E/\mathbb{Q}_p$ be an elliptic curve acquiring good reduction over a finite Galois extension $K/\mathbb{Q}_p$ with maximal unramified subfield K_0 such that its ramification index $e = e(K/\mathbb{Q}_p)$ is minimal. Let $\mathbf{D} = (D, \mathrm{Fil})$ be its associated filtered $(\varphi, \mathrm{Gal}(K/\mathbb{Q}_p))$-module, D_0 the subspace of elements fixed by $\mathrm{Gal}(K_0/\mathbb{Q}_p)$ (seen as a subgroup of $\mathrm{Gal}(K/\mathbb{Q}_p)$, see Section 2.2) and $\varphi_0 = \varphi\restriction_{D_0}$ the $\mathbb{Q}_p$-linear restriction of φ. We denote by $\mathbf{W}(\mathbf{D})$ the Weil representation associated to $\mathbf{D}$. It is known that $\mathbf{D}$ satisfies the following properties:

(1) $P_{\mathrm{char}}(\varphi_0)(X) = X^2 + a_pX + p$, with $|a_p|_\infty \leq 2\sqrt{p}$,
(2) $\mathbf{W}(\mathbf{D})$ is defined over $\mathbb{Q}$,
(3) $\bigwedge^2_{K_0} \mathbf{D} = K_0\{-1\}$ (i.e. $\bigwedge^2_{\mathbb{Q}_p} V_p(E) = \mathbb{Q}_p(1)$),
(4) $\mathbf{D}$ is of Hodge–Tate type $(0, 1)$.

These conditions alone are sufficient to guarantee that a 2-dimensional p-adic representation of G comes from an elliptic curve over $\mathbb{Q}_p$ in the case of tame potential good reduction (see for instance [13, Thm. 5.1] or [14, §5.4]). It is not known yet if these are sufficient in the presence of wild potential good reduction as well, however they are still necessary. Starting from these conditions, imposing geometric descent datum and a minimal field of good reduction K, we provide a list of isomorphism classes of possible filtered $(\varphi, \mathrm{Gal}(K/\mathbb{Q}_3))$-modules. Then we show that every object in the list arises from an elliptic curve over $\mathbb{Q}_3$.

Some of the classes described in this paper can directly be deduced from the $p \geq 5$ case when $(e, p) = 1$ (see [14]). To the best of our knowledge, a complete classification of ℓ-adic representations ($\ell \neq 3$) — which is encoded in terms of unfiltered $(\varphi, \mathrm{Gal}(K/\mathbb{Q}_3))$-modules — does not appear in the litterature. However, some particular cases can be found (see [4] for $e = 12$). Our genuine new results are the cases of wild potential good reduction ($e = 3, 6$ and 12) with $e = 12$ being the first case of non abelian inertia. We provide proofs in the tame case for the sake of completeness. The classification is synthetized in Table 1.1. Notations for the filtered

TABLE 1.1. Isomorphism classes of filtered $(\varphi, \mathrm{Gal}(K/\mathbb{Q}_3))$-modules arising from elliptic curves over $\mathbb{Q}_3$ with potential good reduction.

e	Red. type	K	Frob.	$\mathbf{D}$	#Classes
1	Supersing.	$\mathbb{Q}_3$	-3	$\mathbf{D}_{\mathrm{c}}(1;-3;0)$	1
			0	$\mathbf{D}_{\mathrm{c}}(1;0;0)$	1
			3	$\mathbf{D}_{\mathrm{c}}(1;3;0)$	1
	Ord.	$\mathbb{Q}_3$	-2	$\mathbf{D}_{\mathrm{c}}(1;-2;\alpha)$	2
			-1	$\mathbf{D}_{\mathrm{c}}(1;-1;\alpha)$	2
			1	$\mathbf{D}_{\mathrm{c}}(1;1;\alpha)$	2
			2	$\mathbf{D}_{\mathrm{c}}(1;2;\alpha)$	2
2	Supersing.	$\mathbb{Q}_3(\sqrt{3})$	-3	$\mathbf{D}_{\mathrm{c}}(2;-3;0)$	1
			0	$\mathbf{D}_{\mathrm{c}}(2;0;0)$	1
			3	$\mathbf{D}_{\mathrm{c}}(2;3;0)$	1
	Ord.	$\mathbb{Q}_3(\sqrt{3})$	-2	$\mathbf{D}_{\mathrm{c}}(2;-2;\alpha)$	2
			-1	$\mathbf{D}_{\mathrm{c}}(2;-1;\alpha)$	2
			1	$\mathbf{D}_{\mathrm{c}}(2;1;\alpha)$	2
			2	$\mathbf{D}_{\mathrm{c}}(2;2;\alpha)$	2
4	Supersing.	$\mathbb{Q}_3(\sqrt[4]{3})$	0	$\mathbf{D}_{\mathrm{pc}}(4;0;\alpha)$	$\mathbb{P}^1(\mathbb{Q}_3)$
3	Supersing.	$L^{\mathrm{na}}(\zeta_4)$	0	$\mathbf{D}_{\mathrm{pc}}^{\mathrm{na}}(3;0;\alpha)$	$\mathbb{P}^1(\mathbb{Q}_3)$
		$L^{\mathrm{a}} = \mathbb{Q}_3(\pi)$	-3	$\mathbf{D}_{\mathrm{pc}}^{\mathrm{a}}(3;-3,\mu;\pi)$	2
			0	$\mathbf{D}_{\mathrm{pc}}^{\mathrm{a}}(3;0,\mu;\pi)$	2
			3	$\mathbf{D}_{\mathrm{pc}}^{\mathrm{a}}(3;3,\mu;\pi)$	2
6	Supersing.	$L^{\mathrm{na}}(\zeta_4,\sqrt{3})$	0	$\mathbf{D}_{\mathrm{pc}}^{\mathrm{na}}(6;0;\alpha)$	$\mathbb{P}^1(\mathbb{Q}_3)$
		$L^{\mathrm{a}}(\sqrt{3})$	-3	$\mathbf{D}_{\mathrm{pc}}^{\mathrm{a}}(6;-3,\mu;\pi)$	2
			0	$\mathbf{D}_{\mathrm{pc}}^{\mathrm{a}}(6;0,\mu;\pi)$	2
			3	$\mathbf{D}_{\mathrm{pc}}^{\mathrm{a}}(6;3,\mu;\pi)$	2
12	Supersing.	K_1	0	$\mathbf{D}_{\mathrm{pc}}(12;0;1;\epsilon;\alpha)$	$\mathbb{P}^1(\mathbb{Q}_3)$
		K_2	0	$\mathbf{D}_{\mathrm{pc}}(12;0;2;\epsilon;\alpha)$	$\mathbb{P}^1(\mathbb{Q}_3)$
		K_3	0	$\mathbf{D}_{\mathrm{pc}}(12;0;3;\epsilon;\alpha)$	$\mathbb{P}^1(\mathbb{Q}_3)$
		K_4	0	$\mathbf{D}_{\mathrm{pc}}(12;0;4;\epsilon;\alpha)$	$\mathbb{P}^1(\mathbb{Q}_3)$
		K_5	0	$\mathbf{D}_{\mathrm{pc}}(12;0;5;\epsilon;\alpha)$	$\mathbb{P}^1(\mathbb{Q}_3)$

$(\varphi, \mathrm{Gal}(K/\mathbb{Q}_3))$-modules and their set of parameters are detailed in Section 4.

Note that the supersingular traces $a_3 = \pm 3$ occur, which is specific to the $p = 3$ case (compared to $p \geq 5$). One may expect that they should appear every time the reduction is supersingular, and yet this is not the case. The reason behind this absence lies in the structure of the automorphism group of the special fibre, which controls the possible descents. Furthermore, we need to deal with several different fields of good reduction. Indeed, wild finite extensions of $\mathbb{Q}_p^{\mathrm{un}}$ aren't unique as opposed to the tame ones. This leads to interesting new phenomena. The case $e = 12$ is uniform, the five fields are almost indistinguishable. When $e = 3$ the situation is different between the two possible fields. The non abelian extension occurs for only one possible Frobenius trace and has an infinity of isomorphism classes. The abelian extension, on the other hand, occurs for every supersingular traces value but has only two classes for each. Let us finally mention that the ordinary cases have simply disappeared when $e > 2$, again, a specific feature of elliptic curves over $\mathbb{F}_3$.

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2. Theoretical background

We denote by $\mathbb{Q}_p^{\mathrm{un}}$ the maximal unramified extension of $\mathbb{Q}_p$ inside $\overline{\mathbb{Q}}_p$ and $I = \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p^{\mathrm{un}})$ the inertia subgroup of G. For a finite extension K of $\mathbb{Q}_p$, we let $G_K = \mathrm{Gal}(\overline{\mathbb{Q}}_p/K)$ and $I_K = I \cap G_K$ be its absolute Galois group and inertia subgroup respectively.

2.1. Elliptic curves. Let $E/\mathbb{Q}_p$ be an elliptic curve (we refer to [11] for the arithmetic of elliptic curves). One may assume, after a suitable change of coordinates, that the coefficients of a Weierstrass equation of E are in $\mathbb{Z}_p$ and that the valuation of its discriminant is minimal. A Weierstrass equation satisfying these two properties is called minimal. Suppose that E is given by a minimal Weierstrass equation, after reducing each coefficient we obtain a curve $\widetilde{E}/\mathbb{F}_p$. The reduced curve need not be an elliptic curve itself. In fact, it will be if and only if $v_p(\Delta) = 0$ (i.e. $\Delta(\widetilde{E}) = \Delta \bmod p\mathbb{Z}_p \neq 0$). When the reduced curve is an elliptic curve, we say that E has good reduction. Let $L/\mathbb{Q}_p$ be a finite extension of $\mathbb{Q}_p$ and consider $E_L = E \times_{\mathbb{Q}_p} L$ the extension of E to L. Allowing changes of coordinates defined over L may give us a minimal model of E_L with $v_p(\Delta_L) = 0$, so that E_L has good reduction. When there exists such an extension, we say that E has potential good reduction. This property only depends on the action of inertia, which means we can choose $L/\mathbb{Q}_p$ to be totally ramified so $\widetilde{E}_L$ is defined over $\mathbb{F}_p$. Denote by $a_p(E) = a_p(\widetilde{E}_L)$ the trace of the characteristic polynomial of the Frobenius endomorphism acting on $V_\ell(\widetilde{E}_L)$ for some $\ell \neq p$. It is known that $a_p(E)$ is an integer independent of ℓ satisfying $|a_p(E)|_\infty \leq 2\sqrt{p}$, as well as an invariant of the isogeny class of $\widetilde{E}_L$ over $\mathbb{F}_p$ (see [12]). Furthermore we have the following relation:

$$a_p(\widetilde{E}_L) = p + 1 - \#\widetilde{E}_L(\mathbb{F}_p).$$

We say that $\widetilde{E}_L$ is ordinary when $a_p(\widetilde{E}_L)$ is coprime to p, and supersingular when p divides $a_p(\widetilde{E}_L)$.

2.2. ℓ-adic Galois representations. Let p, ℓ be distinct prime numbers. An ℓ-adic representation of G (or $\mathbb{Q}_\ell[G]$-module) is a finite dimensional $\mathbb{Q}_\ell$-vector space with a linear and continuous action of G. We denote such an object by (V, ρ_ℓ) where V is a $\mathbb{Q}_\ell$-vector space and $\rho_\ell \colon G \to \mathrm{Aut}_{\mathbb{Q}_\ell}(V)$ the group homomorphism defining the action. If the inertia subgroup I of G acts trivially on V, we say that the representation has good reduction. In this case, it factors into a representation of the absolute Galois group $G_{\mathbb{F}_p}$ of $\mathbb{F}_p$ and is completely determined by it. When there exists a finite extension $L/\mathbb{Q}_p$ such that I_L acts trivially on V, we say that the representation has potential good reduction. One easily checks that having potential good reduction is equivalent to $\rho_\ell(I)$ being finite. Let $E/\mathbb{Q}_p$ be an elliptic curve. The group G acts on $E(\overline{\mathbb{Q}}_p)$ by acting on the coefficients of its points. Since addition is G-equivariant, the group of ℓ^n-torsion points $E[\ell^n]$ of $E(\overline{\mathbb{Q}}_p)$ is stable by action of G and we define the ℓ-adic Tate module associated to E by

$$T_\ell(E) = \varprojlim_{P \mapsto \ell P} E[\ell^n].$$

It is a free $\mathbb{Z}_\ell$-module of rank 2 equipped with a continuous and $\mathbb{Z}_\ell$-linear action of G. Tensoring by $\mathbb{Q}_\ell$, we get an ℓ-adic representation $V_\ell(E) = \mathbb{Q}_\ell \otimes T_\ell(E)$ of G. It is well known that $V_\ell(E)$ has (potential) good reduction if and only if E has (potential) good reduction. If $E/\mathbb{Q}_p$ is an elliptic curve with potential good reduction, there exists a unique finite extension $M_E/\mathbb{Q}_p^{\mathrm{un}}$ of minimal degree over which E acquires good reduction. We call that minimal degree the semi-stability defect of E, denoted by $\mathbf{dst}(E)$. Consider $V_\ell(E)$ the ℓ-adic representation associated to E. Since E has potential good reduction, there exists $L/\mathbb{Q}_p$ finite of minimal ramification index satisfying $\rho_{E,\ell}(I_L) = 0$. It is then easy to see that

$$M_E = \overline{\mathbb{Q}}_p^{\ker(\rho_{E,\ell}\restriction_{I_L})}.$$

If $L/\mathbb{Q}_p$ is a finite extension with $L^{\mathrm{un}} = M_E$, then E acquires good reduction over L and $\mathbf{dst}(E) = e(L/\mathbb{Q}_p)$, it is the minimal ramification index among all fields of good reduction of E. It is also worth noticing that if L, $L'/\mathbb{Q}_p$ satisfy $L^{\mathrm{un}} = (L')^{\mathrm{un}}$, then they are interchangeable in the sense that $E/\mathbb{Q}_p$ acquires good reduction over L if and only if it acquires it over L'. We can always assume that $L/\mathbb{Q}_p$ is totally ramified, we denote by K its Galois closure, K_0 the maximal unramified extension of $\mathbb{Q}_p$ inside K and k the residue field of K. Then, K/L is unramified, $K\mathbb{Q}_p^{\mathrm{un}} = M_E$ and $\mathrm{Gal}(K/\mathbb{Q}_p) = I(K/\mathbb{Q}_p) \rtimes \mathrm{Gal}(K/L)$ with $\mathrm{Gal}(K/L) \simeq \mathrm{Gal}(K/\mathbb{F}_p) \simeq \mathrm{Gal}(K_0/\mathbb{Q}_p)$. Furthermore, when $p = 3$, we know that $\mathbf{dst}(E) \in \{1, 2, 3, 4, 6, 12\}$ and $\rho_{E,\ell}(I)$ is either a cyclic group of order $1, 2, 3, 4, 6$ or the non Abelian semi-direct product of a cyclic group of order 4 by a group of order 3 (see [10, §5.6]). The degree of a minimal field of good reduction is bounded by the image of inertia and the structure of its inertia subgroup is known.

2.3. Filtered $(\varphi, \mathrm{Gal}(K/\mathbb{Q}_p))$-modules. Let $K/\mathbb{Q}_p$ be a finite Galois extension and K_0 the maximal unramified extension of $\mathbb{Q}_p$ inside K. Denote by σ the absolute Frobenius on K_0. A filtered $(\varphi, \mathrm{Gal}(K/\mathbb{Q}_p))$-module $\mathbf{D} = (D, \mathrm{Fil})$ is a finite dimensional K_0-vector space D together with:

(i) a semilinear action of $\mathrm{Gal}(K/\mathbb{Q}_p)$,

(ii) a σ-semilinear, $\mathrm{Gal}(K/\mathbb{Q}_p)$-equivariant and bijective Frobenius

$$\varphi\colon D \xrightarrow{\sim} D,$$

(iii) a filtration $\mathrm{Fil} = (\mathrm{Fil}^i D_K)_{i\in\mathbb{Z}}$ on $D_K = K \otimes_{K_0} D$ by $\mathrm{Gal}(K/\mathbb{Q}_p)$-stable subspaces such that $\mathrm{Fil}^i D_K = D_K$ for $i \ll 0$ and $\mathrm{Fil}^i D_K = 0$ for $i \gg 0$.

Such objects form a category that we will denote by $\mathbf{MF}_\varphi(\mathrm{Gal}(K/\mathbb{Q}_p))$. The morphisms are the K_0-linear maps f commuting to the Frobenius and the action of $\mathrm{Gal}(K/\mathbb{Q}_p)$ as well as preserving the filtration (i.e. $f_K(\mathrm{Fil}^i D_K) \subseteq \mathrm{Fil}^i D'_K$). The Tate twist $\mathbf{D}\{-1\}$ of $\mathbf{D}$ is the K_0-vector space D with the same action of $\mathrm{Gal}(K/\mathbb{Q}_p)$, $\varphi\{-1\} = p\varphi$ and $\mathrm{Fil}^i (D\{-1\})_K = \mathrm{Fil}^{i-1} D_K$. We say that $\mathbf{D}$ is of Hodge–Tate type $(0, 1)$ if $\mathrm{Fil}^i D_K = D_K$ for $i \leq 0$, $\mathrm{Fil}^i D_K = 0$ for $i \geq 2$ and $\mathrm{Fil}^1 D_K$ is a non trivial subspace of D_K. We associate to $\mathbf{D}$ the following quantities:

(1) $t_N(\mathbf{D}) = v_p(\det \varphi)$,

(2) $t_H(\mathbf{D}) = \sum_{i\in\mathbb{Z}} i \dim_K(\mathrm{Fil}^i D_K / \mathrm{Fil}^{i+1} D_K)$,

where $\det \varphi$ is the determinant of a matrix representing φ. We say that $\mathbf{D}$ is admissible if $t_H(\mathbf{D}) = t_N(\mathbf{D})$ and $t_H(\mathbf{D}') \leq t_N(\mathbf{D}')$ for every subobject $\mathbf{D}'$ of $\mathbf{D}$. Given a p-adic representation V of G, one can associate to it a filtered $(\varphi, \mathrm{Gal}(K/\mathbb{Q}_p))$-module via the contravariant functor:

$$\mathbf{D}^*_{\mathrm{cris},K} : \mathbf{Rep}_{\mathbb{Q}_p}(G) \longrightarrow \mathbf{MF}_\varphi(\mathrm{Gal}(K/\mathbb{Q}_p)) : V \longmapsto \mathrm{Hom}_{\mathbb{Q}_p[G_K]}(V, \mathbf{B}_{\mathrm{cris}}).$$

Where $\mathbf{B}_{\mathrm{cris}}$ is the crystalline period ring (see [6]). The inequality

$$\dim_{K_0} \mathbf{D}^*_{\mathrm{cris},K}(V) \leq \dim_{\mathbb{Q}_p} V$$

is always satisfied, and we say that a representation V of G is crystalline over K if the equality holds. Viewing V as a representation of G_K by restriction, then V is potentially crystalline over K as a representation of

G if and only if it is crystalline as a representation of G_K. This functor establishes an anti-equivalence of categories between the category of p-adic representations of G crystalline over K and the category of admissible filtered $(\varphi, \mathrm{Gal}(K/\mathbb{Q}_p))$-modules (see [7] and [3]). The p-adic Tate module of an elliptic curve over $\mathbb{Q}_p$ with potential good reduction gives rise to such a representation. In fact, the following holds:

Theorem ([2, Thm. 4.7])**.** *Let $\mathcal{A}/\mathbb{Q}_p$ be an abelian variety, the p-adic representation $V_p(\mathcal{A})$ is (potentially) crystalline if and only if $\mathcal{A}$ has (potential) good reduction.*

Each filtered $(\varphi, \mathrm{Gal}(K/\mathbb{Q}_p))$-module has a linear object naturally attached to it, namely its Weil representation. Recall that the Weil group of $\overline{\mathbb{Q}}_p$ is defined by the short exact sequence

$$1 \longrightarrow I \longrightarrow W \xrightarrow{\ \nu\ } \mathbb{Z} \longrightarrow 1$$

where ν sends a lifting of the arithmetic Frobenius to 1, and we let $W_K = W \cap G_K$. To every $(\varphi, \mathrm{Gal}(K/\mathbb{Q}_p))$-module $\mathbf{D}$ we can associate a K_0-vector space Δ with a continuous K_0-linear action of W in the following way:

$$\rho\colon W \longrightarrow \mathrm{Aut}_{K_0}(\Delta)\colon \omega \longmapsto (\omega \bmod W_K)\varphi^{-\nu(\omega)}$$

where Δ is the underlying K_0-vector space of $\mathbf{D}$. The pair $\mathbf{W}(\mathbf{D}) = (\Delta, \rho)$ is called a Weil representation. It is defined over $\mathbb{Q}$ if $\mathrm{Tr}(\rho(w)) \in \mathbb{Q}$ for every $w \in W$.

3. Strategy

We begin by fixing a semi-stability defect $e \in \{1, 2, 3, 4, 6, 12\}$. The first step is to determine every finite Galois extension with ramification index e that arises as a field of good reduction for some elliptic curve defined over $\mathbb{Q}_3$. Cases $e = 1, 2$ and 4 are tame, hence necessarily given by $\mathbb{Q}_3(\sqrt[e]{3})$. For $e = 3$, we use Local Class Field Theory and the local fields database in [9]; $e = 6$ is then obtained by a ramified quadratic twist. Finally, $e = 12$ is treated using [9] since the structure of the Galois group and inertia subgroup are well known. We then fix $K/\mathbb{Q}_3$ to be one such extension. The next step is to describe the list of the 2-dimensional filtered $(\varphi, \mathrm{Gal}(K/\mathbb{Q}_3))$-modules $\mathbf{D}$ satisfying properties (1)–(4). We then show that given an elliptic curve $E/\mathbb{Q}_3$ with potential good reduction over K, its associated filtered $(\varphi, \mathrm{Gal}(K/\mathbb{Q}_3))$-module $\mathbf{D}^*_{\mathrm{cris},K}(V_3(E))$ is necessarily isomorphic to one object $\mathbf{D}$ of our list. Finally, given an object $\mathbf{D}$ in the list, we need to find an elliptic curve $E/\mathbb{Q}_3$ such that

$$\mathbf{D}^*_{\mathrm{cris},K}(V_3(E)) \simeq \mathbf{D}$$

and this is done in Section 5.

One last point require some discussion. Given an unfiltered 2-dimensional $(\varphi, \mathrm{Gal}(K/\mathbb{Q}_3))$-module $\mathbf{D}$, the set of Hodge–Tate type $(0,1)$ filtrations on $\mathbf{D}$ is in bijection with $\mathbb{P}^1(\mathbb{Q}_3)$. Indeed, by Galois Descent, it is easy to check that the $\mathrm{Gal}(K/\mathbb{Q}_3)$-stable lines in $D_K = K \otimes_{K_0} D$ are in bijection with the lines in

$$D_K^{\mathrm{Gal}(K/\mathbb{Q}_3)} = \{x \in D_K \mid \forall\, g \in \mathrm{Gal}(K/\mathbb{Q}_3), g.x = x\}$$

a 2-dimensional $\mathbb{Q}_3$-vector space. This means that if φ has only trivial stable subspaces, there are infinitely many admissible filtrations on $\mathbf{D}$. In the following, we will define sets that parameter our filtrations. This fact ensures that these sets will always be non empty, even though it could not be clear at first glance.

4. Classification

We provide the list of admissible filtered $(\varphi, \mathrm{Gal}(K/\mathbb{Q}_3))$-modules satisfying our geometric conditions:

(1) $P_{\mathrm{char}}(\varphi_0)(X) = X^2 + a_3X + 3$, with $|a_3|_\infty \leq 2\sqrt{3}$,
(2) $\mathbf{W}(\mathbf{D})$ is defined over $\mathbb{Q}$,
(3) $\bigwedge_{K_0}^2 \mathbf{D} = K_0\{-1\}$ (i.e. $\bigwedge_{\mathbb{Q}_3}^2 V_3(E) = \mathbb{Q}_3(1)$),
(4) $\mathbf{D}$ is of Hodge–Tate type $(0,1)$

with $K/\mathbb{Q}_3$ a minimal Galois extension of good reduction. We then show that every elliptic curve defined over $\mathbb{Q}_3$ with potential good reduction has its associated filtered $(\varphi, \mathrm{Gal}(K/\mathbb{Q}_3))$-module isomorphic to an object of the list.

4.1. The crystalline case. We start our classification with the representations coming from elliptic curves $E/\mathbb{Q}_3$ with good reduction ($K = \mathbb{Q}_3$). There are two distinct cases behaving differently depending on the trace $a_3(\widetilde{E})$ of the Frobenius of $\widetilde{E}/\mathbb{F}_3$.

4.1.1. *The supersingular case.* Let $a \in \{-3, 0, 3\}$ and $\alpha \in \mathbb{P}^1(\mathbb{Q}_3)$. We denote by $\mathbf{D}_{\mathrm{c}}(1; a; \alpha)$ the filtered φ-module (of Hodge–Tate type $(0,1)$) defined by:

- $D = \mathbb{Q}_3 e_1 \oplus \mathbb{Q}_3 e_2$,
- $M_B(\varphi) = \left(\begin{smallmatrix} 0 & -3 \\ 1 & -a \end{smallmatrix}\right)$, where $B = (e_1, e_2)$,
- $\mathrm{Fil}^1 D = (\alpha e_1 + e_2)\mathbb{Q}_3$.

Identifying $\mathbb{P}^1(\mathbb{Q}_3)$ with $\mathbb{Q}_3 \sqcup \{\infty\}$, we let $\alpha e_1 + e_2 = e_1$ when $\alpha = \infty$. For each $a \in \{-3, 0, 3\}$ and $\alpha \in \mathbb{P}^1(\mathbb{Q}_3)$, the filtered φ-module $\mathbf{D}_{\mathrm{c}}(1; a; \alpha)$ satisfies conditions (1)–(4) and is admissible. Condition (1) is obvious and (4) is satisfied by definition. Conditions (2) and (3) as well as admissibility are easily checked by computation.

Proposition 4.1. *Let $E/\mathbb{Q}_3$ be an elliptic curve with good reduction such that $a_3 = a_3(\widetilde{E}) \in \{-3, 0, 3\}$ and $\mathbf{D} = \mathbf{D}^*_{\mathrm{cris},\mathbb{Q}_3}(V_3(E))$. There exists an isomorphism of filtered φ-modules between $\mathbf{D}$ and $\mathbf{D}_{\mathrm{c}}(1; a_3; 0)$. Moreover, if $a, b \in \{-3, 0, 3\}$, then $\mathbf{D}_{\mathrm{c}}(1; a; 0)$ and $\mathbf{D}_{\mathrm{c}}(1; b; 0)$ are isomorphic if and only if $a = b$.*

Proof. Let D (resp. D') be the $\mathbb{Q}_3$-vector space associated to $\mathbf{D}$ (resp. $\mathbf{D}_{\mathrm{c}}(1; a_3; 0)$). Let $B = (e_1, e_2)$ and $B' = (e'_1, e'_2)$ be basis for D and D' respectively such that

$$M_B(\varphi) = \begin{pmatrix} 0 & -3 \\ 1 & -a_3 \end{pmatrix} = M_{B'}(\varphi').$$

Such a basis of D always exists since $P_{\mathrm{char}}(\varphi)(X) = X^2 + a_3 X + 3$ as $\mathbf{D}$ satisfies the condition (1). A $\mathbb{Q}_3$-isomorphism η between D and D' is φ-equivariant if and only if

$$M_{B,B'}(\eta) \in C(M_B(\varphi)),$$

where $C(M_B(\varphi))$ denotes the centralizer of $M_B(\varphi)$ in $\mathrm{M}_2(\mathbb{Q}_3)$. Notice that since

$$C(M_B(\varphi)) = C(\mathbb{Q}_3[M_B(\varphi)])$$

and $P_{\mathrm{char}}(\varphi)(X)$ is irreducible, the Double Centralizer Theorem implies

$$C(M_B(\varphi)) = \mathbb{Q}_3[M_B(\varphi)] = \mathbb{Q}_3(M_B(\varphi)).$$

In particular, every nonzero element of $\mathbb{Q}_3(M_B(\varphi))$ is an isomorphism of φ-modules between (D, φ) and (D', φ'). Consider $\mathrm{Fil}^1 D = (\alpha e_1 + \beta e_2)\mathbb{Q}_3$, $(\alpha, \beta) \neq (0, 0)$. The matrix

$$\begin{pmatrix} \alpha & -3\beta \\ \beta & \alpha - a_3\beta \end{pmatrix}$$

is invertible because the homogenous polynomial $X^2 - a_3 XY + 3Y^2$ only has trivial roots in $(\mathbb{Q}_3)^2$. Let $(\lambda, \mu) \in \mathbb{Q}_3^2$ be the unique solution to the system of equations

$$\begin{pmatrix} \alpha & -3\beta \\ \beta & \alpha - a_3\beta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

It follows that $(\lambda, \mu) \neq (0, 0)$ and

$$\begin{pmatrix} \lambda & -3\mu \\ \mu & \lambda - a_3\mu \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Therefore, the map $\lambda \,\mathrm{Id} + \mu M_B(\varphi) \in \mathbb{Q}_3[M_B(\varphi)]$ defines an isomorphism of filtered φ-modules between $\mathbf{D}$ and $\mathbf{D}_{\mathrm{c}}(1; a_3; 0)$. One checks the last assertion by a simple computation. □

Remark 4.2. There are 3 isomorphism classes of filtered φ-modules in the supersingular case, one for each value taken by a.

4.1.2. *The ordinary case.* Let $a \in \{\pm 1, \pm 2\}$ and $\alpha \in \mathbb{P}^1(\mathbb{Q}_3)$. We denote by $\mathbf{D}_{\mathrm{c}}(1; a; \alpha)$ the filtered φ-module defined by:

- $D = \mathbb{Q}_3 e_1 \oplus \mathbb{Q}_3 e_2$,
- $M_B(\varphi) = \left(\begin{smallmatrix} u & 0 \\ 0 & u^{-1}3 \end{smallmatrix}\right)$, where $u \in \mathbb{Z}_3^\times$ such that $u + u^{-1}3 = -a$,
- $\mathrm{Fil}^1 D = (\alpha e_1 + e_2)\mathbb{Q}_3$.

For each $a \in \{\pm 1, \pm 2\}$ and $\alpha \in \mathbb{P}^1(\mathbb{Q}_3)$, the filtered φ-module $\mathbf{D}_{\mathrm{c}}(1; a; \alpha)$ satisfies conditions (1)–(4) and is admissible for $\alpha \neq \infty$.

Proposition 4.3. *Let $E/\mathbb{Q}_3$ be an elliptic curve with good reduction such that $a_3 = a_3(\widetilde{E}) \in \{\pm 1, \pm 2\}$ and $\mathbf{D} = \mathbf{D}^*_{\mathrm{cris},\mathbb{Q}_3}(V_3(E))$. There exists an isomorphism of filtered φ-modules between $\mathbf{D}$ and $\mathbf{D}_{\mathrm{c}}(1; a_3; \alpha)$ for some $\alpha \in \{0, 1\}$. Moreover, if (α, a), $(\beta, b) \in \{0, 1\} \times \{\pm 1, \pm 2\}$, then $\mathbf{D}_{\mathrm{c}}(1; a; \alpha)$ and $\mathbf{D}_{\mathrm{c}}(1; b; \beta)$ are isomorphic if and only if $(\alpha, a) = (\beta, b)$.*

Proof. Since $\mathbf{D}$ is admissible, the only possible filtrations are defined by a $\mathbb{Q}_3$-line of the form $\mathrm{Fil}^1 D = (\beta e_1 + e_2)\mathbb{Q}_3$ for some $\beta \in \mathbb{Q}_3$. Let $\alpha \in \{0, 1\}$ and D' be the $\mathbb{Q}_3$-vector space associated to $\mathbf{D}_{\mathrm{c}}(1; a_3; \alpha)$. Let $B = (e_1, e_2)$, $B' = (e'_1, e'_2)$ be basis of D and D' respectively, such that

$$M_B(\varphi) = \begin{pmatrix} u & 0 \\ 0 & u^{-1}3 \end{pmatrix} = M_{B'}(\varphi'),\ u \in \mathbb{Z}_3^\times,\ u + u^{-1}3 = -a_3.$$

Such a basis exists because $\mathbf{D}$ satisfies (1) and $(a_3, 3) = 1$ and thus we have

$$P_{\mathrm{char}}(\varphi)(X) = X^2 + a_3 X + 3 = (X - u)(X - u^{-1}3)$$

for some $u \in \mathbb{Z}_3^\times$. A $\mathbb{Q}_3$-isomorphism η between D and D' is φ-equivariant if and only if

$$M_{B,B'}(\eta) \in C(M_B(\varphi)).$$

This time, since $P_{\mathrm{char}}(\varphi)(X)$ is a product of distinct linear factors

$$C(\mathbb{Q}_3[M_B(\varphi)]) = \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} :\ \lambda, \mu \in \mathbb{Q}_3 \right\}.$$

If $\beta = 0$, then every invertible element of $C(M_B(\varphi))$ defines an isomorphism of filtered φ-modules between $\mathbf{D}$ and $\mathbf{D}_{\mathrm{c}}(1; a_3; 0)$. If $\beta \neq 0$, then taking $\lambda = \beta^{-1}$ and $\mu = 1$ gives an isomorphism of filtered φ-modules between $\mathbf{D}$ and $\mathbf{D}_{\mathrm{c}}(1; a_3; 1)$. $\square$

Remark 4.4. There are 8 isomorphism classes of filtered φ-modules in the ordinary case, two for each possible value taken by a.

Remark 4.5. The elliptic curves $E/\mathbb{Q}_3$ with ordinary good reduction and $\alpha = 0$ are canonical lifts of their corresponding reduced curve $\widetilde{E}/\mathbb{F}_3$.

4.2. The quadratic case. Let $E/\mathbb{Q}_3$ with semi-stability defect $\mathbf{dst}(E) = 2$. Since 2 and 3 are coprime, the only quadratic extension of $\mathbb{Q}_3^{\mathrm{un}}$ is $\mathbb{Q}_3^{\mathrm{un}}(\sqrt{3})$. Let $K = \mathbb{Q}_3(\sqrt{3})$, it is a Galois extension of degree 2 with Galois group $\langle \tau_2 \rangle$ over which E acquire good reduction. As usual we denote by $a_3 = a_3(\widetilde{E})$ the trace of the Frobenius of $\widetilde{E}/\mathbb{F}_3$.

4.2.1. ***The supersingular case.*** Let $a \in \{-3, 0, 3\}$ and $\alpha \in \mathbb{P}^1(K)$. We denote by $\mathbf{D}_{\mathrm{c}}(2; a; \alpha)$ the filtered $(\varphi, \mathrm{Gal}(K/\mathbb{Q}_3))$-module defined by:

- $D = \mathbb{Q}_3 e_1 \oplus \mathbb{Q}_3 e_2$,
- $M_B(\varphi) = \begin{pmatrix} 0 & -3 \\ 1 & -a \end{pmatrix}$,
- $M_B(\tau_2) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$,
- $\mathrm{Fil}^1 D_K = (\alpha \otimes e_1 + 1 \otimes e_2)K$, where $D_K = K \otimes_{K_0} D$.

For each $a \in \{-3, 0, 3\}$ and $\alpha \in \mathbb{P}^1(\mathbb{Q}_3)$, the filtered $(\varphi, \mathrm{Gal}(K/\mathbb{Q}_3))$-module $\mathbf{D}_{\mathrm{c}}(2; a; \alpha)$ satisfies conditions (1)–(4) and is admissible.

Proposition 4.6. *Let $E/\mathbb{Q}_3$ be an elliptic curve with $\mathbf{dst}(E) = 2$ such that $a_3 = a_3(\widetilde{E}) \in \{-3, 0, 3\}$ and $\mathbf{D} = \mathbf{D}^*_{\mathrm{cris},K}(V_3(E))$. There exists an isomorphism of filtered $(\varphi, \mathrm{Gal}(K/\mathbb{Q}_3))$-modules between $\mathbf{D}$ and $\mathbf{D}_{\mathrm{c}}(2; a_3; 0)$. Moreover, if a, $b \in \{-3, 0, 3\}$, then $\mathbf{D}_{\mathrm{c}}(2; a; 0)$ and $\mathbf{D}_{\mathrm{c}}(2; b; 0)$ are isomorphic if and only if $a = b$.*

Proof. Let D be the underlying $\mathbb{Q}_3$-vector space associated to $\mathbf{D}$ and $B = (e_1, e_2)$ a basis of D such that

$$M_B(\varphi) = \begin{pmatrix} 0 & -3 \\ 1 & -a_3 \end{pmatrix}.$$

The element τ_2 is seen as a $\mathbb{Q}_3$-automorphism of D and is of order 2. Since $\mathbf{D}$ satisfies conditions (2)–(3), we have $P_{\mathrm{char}}(\tau_2)(X) \in \mathbb{Q}[X]$ and $\det(\tau_2) = 1$, so that

$$P_{\mathrm{char}}(\tau_2)(X) = (X + 1)^2$$

thus $\tau_2 = -\,\mathrm{Id}$. The K-line $(1 \otimes e_1)K$ is stable by τ_2 and if $\alpha \in K$, the K-line $(\alpha \otimes e_1 + 1 \otimes e_2)K$ is stable by τ_2 if and only if $\alpha \in \mathbb{Q}_3$. Let $\alpha \in \mathbb{P}^1(\mathbb{Q}_3)$ such that $\mathrm{Fil}^1 D_K = (\alpha \otimes e_1 + 1 \otimes e_2)K$ is the K-line defining the filtration of $\mathbf{D}$. The isomorphism between $\mathbf{D}$ and $\mathbf{D}_{\mathrm{c}}(2; a_3; 0)$ is given by $e_1 \mapsto (3/\alpha)e_1 + e_2$ and $e_2 \mapsto -3e_1 + (3/\alpha - a_3)e_2$. ∎

4.2.2. ***The ordinary case.*** Let $a \in \{\pm 1, \pm 2\}$ and $\alpha \in \mathbb{P}^1(\mathbb{Q}_3)$. We denote by $\mathbf{D}_{\mathrm{c}}(2; a; \alpha)$ the filtered $(\varphi, \mathrm{Gal}(K/\mathbb{Q}_3))$-module defined by:

- $D = \mathbb{Q}_3 e_1 \oplus \mathbb{Q}_3 e_2$,
- $M_B(\varphi) = \begin{pmatrix} u & 0 \\ 0 & u^{-1}3 \end{pmatrix}$ where $u \in \mathbb{Z}_3^\times$ such that $u + u^{-1}3 = -a$,
- $M_B(\tau_2) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$,
- $\mathrm{Fil}^1 D_K = (\alpha \otimes e_1 + 1 \otimes e_2)K$.

For each $a \in \{\pm 1, \pm 2\}$ and $\alpha \in \mathbb{P}^1(\mathbb{Q}_3)$, the filtered $(\varphi, \mathrm{Gal}(K/\mathbb{Q}_3))$-module $\mathbf{D}_{\mathrm{c}}(2; a; \alpha)$ satisfies conditions (1)–(4) and is admissible for $\alpha \neq \infty$.

Proposition 4.7. *Let $E/\mathbb{Q}_3$ be an elliptic curve with $\mathbf{dst}(E) = 2$ such that $a_3 = a_3(\widetilde{E}) \in \{\pm 1, \pm 2\}$ and $\mathbf{D} = \mathbf{D}^*_{\mathrm{cris},K}(V_3(E))$. There exists an isomorphism of filtered $(\varphi, \mathrm{Gal}(K/\mathbb{Q}_3))$-modules between $\mathbf{D}$ and $\mathbf{D}_{\mathrm{c}}(2; a_3; \alpha)$ for some $\alpha \in \{0, 1\}$. Moreover, if (α, a), $(\beta, b) \in \{0, 1\} \times \{\pm 1, \pm 2\}$, then $\mathbf{D}_{\mathrm{c}}(2; a; \alpha)$ and $\mathbf{D}_{\mathrm{c}}(2; b; \beta)$ are isomorphic if and only if $(\alpha, a) = (\beta, b)$.*

Proof. See the ordinary crystalline case for the description of φ and the supersingular quadratic case for the description of τ_2 and the filtration. □

Remark 4.8. These are exactly the twists by the ramified quadratic character associated to $\mathbb{Q}_3(\sqrt{3})/\mathbb{Q}_3$ of the corresponding crystalline cases.

Remark 4.9. As in the crystalline case, the elliptic curves $E/\mathbb{Q}_3$ with ordinary potential good reduction and $\alpha = 0$ are canonical lifts of their corresponding reduced curve $\widetilde{E}/\mathbb{F}_3$.

4.3. The quartic case. Let $E/\mathbb{Q}_3$ with semi-stability defect $\mathbf{dst}(E) = 4$. Again, since 4 and 3 are coprime, the only quartic extension of $\mathbb{Q}_3^{\mathrm{un}}$ is $\mathbb{Q}_3^{\mathrm{un}}(\sqrt[4]{3})$. Let us fix ζ_4 a primitive fourth root of unity and π_4 a root of $X^4 - 3$ in $\overline{\mathbb{Q}}_3$. Consider $L = \mathbb{Q}_3(\pi_4)$, $K = L(\zeta_4)$ its Galois closure and $K_0 = \mathbb{Q}_3(\zeta_4)$ the maximal unramified extension of $K/\mathbb{Q}_3$. Our curve necessarily acquires good reduction over L. Let $\sigma \in \mathrm{Gal}(K_0/\mathbb{Q}_3)$ be the absolute Frobenius on K_0, $\omega \in \mathrm{Gal}(K/\mathbb{Q}_3)$ a lifting of σ fixing L and τ_4 a generator of $\mathrm{Gal}(K/K_0) = I(K/\mathbb{Q}_3)$. Then $\mathrm{Gal}(K/\mathbb{Q}_3) = \langle \tau_4 \rangle \rtimes \langle \omega \rangle$ with $\tau_4 \omega = \omega \tau_4^{-1}$.

Let $\alpha \in \mathbb{P}^1(\mathbb{Q}_3)$. We denote by $\mathbf{D}_{\mathrm{pc}}(4; 0; \alpha)$ the filtered $(\varphi, \mathrm{Gal}(K/\mathbb{Q}_3))$-module defined by:

- $D = K_0 e_1 \oplus K_0 e_2$,
- $\varphi(e_1) = e_2$, $\varphi(e_2) = -3e_1$,
- $M_B(\tau_4) = \begin{pmatrix} \zeta_4 & 0 \\ 0 & \zeta_4^{-1} \end{pmatrix}$,
- $\omega(e_1) = e_1$, $\omega(e_2) = e_2$,
- $\mathrm{Fil}^1 D_K = (\alpha \pi_4^{-1} \otimes e_1 + \pi_4 \otimes e_2)K$.

For each $\alpha \in \mathbb{P}^1(\mathbb{Q}_3)$, the filtered $(\varphi, \mathrm{Gal}(K/\mathbb{Q}_3))$-module $\mathbf{D}_{\mathrm{pc}}(4; 0; \alpha)$ satisfies conditions (1)–(4) and is admissible.

Proposition 4.10. *Let $E/\mathbb{Q}_3$ be an elliptic curve with $\mathbf{dst}(E) = 4$ and $\mathbf{D} = \mathbf{D}^*_{\mathrm{cris},K}(V_3(E))$. There exists an isomorphism of filtered $(\varphi, \mathrm{Gal}(K/\mathbb{Q}_3))$-modules between $\mathbf{D}$ and $\mathbf{D}_{\mathrm{pc}}(4; 0; \alpha)$. Moreover, if α, $\beta \in \mathbb{P}^1(\mathbb{Q}_3)$, then $\mathbf{D}_{\mathrm{pc}}(4; 0; \alpha)$ and $\mathbf{D}_{\mathrm{pc}}(4; 0; \beta)$ are isomorphic if and only if $\alpha = \beta$.*

Proof. Let D be the underlying K_0-vector space associated to $\mathbf{D}$, the element τ_4 acts K_0-linearly over D and the morphism

$$I(K/\mathbb{Q}_3) \longrightarrow \mathrm{Aut}_{K_0}(D)$$

is injective by minimality of $e(K/\mathbb{Q}_3)$. We identify τ_4 to its image in $\mathrm{Aut}_{K_0}(D)$, it is an element of order 4. Again, because $\mathbf{D}$ satisfies (2)–(3) we have $\det(\tau_4) = 1$ and $P_{\mathrm{char}}(\tau_4)(X) \in \mathbb{Q}[X]$ so that

$$P_{\mathrm{char}}(\tau_4)(X) = P_{\min}(\tau_4)(X) = X^2 + 1 = (X - \zeta_4)(X - \zeta_4^{-1}).$$

In particular, τ_4 is diagonalizable in K_0 and has distinct eigenvalues. Let (e_1, e_2) be a diagonalization basis of τ_4 over K_0. The relation $\tau_4\omega = \omega\tau_4^{-1}$ implies that $\omega(e_i) \in K_0 e_i$, $i = 1, 2$. Denote by $\omega_i = \omega\restriction_{K_0 e_i}$, the group $\langle\omega_i\rangle$ acts semi-linearly over $K_0 e_i$. Descent theory tells us that $(K_0 e_i)^{\langle\omega_i\rangle} \neq \{0\}$. We can then find a basis (e_1, e_2) of D over K_0 which is fixed by ω and such that $\tau_4(e_1) = \zeta_4 e_1$, $\tau_4(e_2) = \zeta_4^{-1} e_2$. Since φ is $\mathrm{Gal}(K/\mathbb{Q}_3)$-equivariant, it commutes to τ_4 and ω. A simple calculation shows that $\varphi(e_1) \in \mathbb{Q}_3 e_2$ and $\varphi(e_2) \in \mathbb{Q}_3 e_1$. Since $\det(\varphi) = 3$, $\varphi(e_1) = a e_2$ and $\varphi(e_2) = -3a^{-1} e_1$, necessarily $a \in \mathbb{Q}_3^\times$. That way we show that there exists a K_0-basis of D such that

$$\begin{cases} \varphi(e_1) = e_2 \wedge \varphi(e_2) = -3e_1 \\ \tau_4(e_1) = \zeta_4 e_1 \wedge \tau_4(e_2) = \zeta_4^{-1} e_2 \\ \omega(e_1) = e_1 \wedge \omega(e_2) = e_2. \end{cases}$$

We have now described the $(\varphi, \mathrm{Gal}(K/\mathbb{Q}_3))$-module structure on D. In particular, we see that $a_3 = 0$, i.e. $\widetilde{E}_L$ is supersingular, but the two other supersingular values 3 and -3 cannot appear. What is left is to determine the K-line $\mathrm{Fil}^1 D_K$ which defines the filtration; it needs to satisfy the weak admissibility condition and be $\mathrm{Gal}(K/\mathbb{Q}_3)$-stable. Since φ does not have any stable subspaces, it is immediate. The K-line $(1 \otimes e_1)K$ is stable by action of $\mathrm{Gal}(K/\mathbb{Q}_3)$. Let $\beta \in \mathbb{Q}_3$ and $\mathrm{Fil}^1 D_K = (\beta \otimes e_1 + 1 \otimes e_2)K$. One easily shows that $\mathrm{Fil}^1 D_K$ is stable by ω if and only if $\beta \in L$ and by τ_4 if and only if $\pi_4^2\beta \in K_0$. Then, $\mathrm{Fil}^1 D_K$ is stable by the action of $\mathrm{Gal}(K/\mathbb{Q}_3)$ if and only if $\pi_4^2\beta \in L \cap K_0 = \mathbb{Q}_3$. Let $\alpha = \pi_4^2\beta \in \mathbb{Q}_3$, we can rewrite our K-line defining the filtration as

$$\mathrm{Fil}^1 D_K = (\alpha\pi_4^{-1} \otimes e_1 + \pi_4 \otimes e_2)K$$

it is then clear that $\mathbf{D} \simeq \mathbf{D}_{\mathrm{pc}}(4; 0; \alpha)$. Let $\alpha, \beta \in \mathbb{P}^1(\mathbb{Q}_3)$, consider the following filtered $(\varphi, \mathrm{Gal}(K/\mathbb{Q}_3))$-modules: $\mathbf{D} = \mathbf{D}_{\mathrm{pc}}(4; 0; \alpha)$ and $\mathbf{D}' = \mathbf{D}_{\mathrm{pc}}(4; 0; \beta)$. Let $B = (e_1, e_2)$ and $B' = (e'_1, e'_2)$ be K_0-basis of D and D' their respective underlying K_0-vector spaces. Let $\psi\colon \mathbf{D} \to \mathbf{D}'$ be a nonzero morphism of filtered $(\varphi, \mathrm{Gal}(K/\mathbb{Q}_3))$-modules. Let $D_0 = D^{\langle\omega\rangle}$ and $D'_0 = (D')^{\langle\omega'\rangle}$. The relation $\psi \circ \omega = \omega' \circ \psi$ implies $\psi(D_0) \subseteq D'_0 = \mathbb{Q}_3 e'_1 \oplus \mathbb{Q}_3 e'_2$. Moreover, $\psi \circ \tau_4 = \tau'_4 \circ \psi$ implies $\psi(e_i) \in K_0 e'_i$, $i = 1, 2$. Then there exists a, $d \in \mathbb{Q}_3$ such that $\psi(e_1) = a e'_1$ and $\psi(e_2) = d e'_2$. Finally, $\psi \circ \varphi = \varphi' \circ \psi$ leads to $a = d$. Denoting by ψ_K the K-linear extension of ψ, we see that $\psi_K(\mathrm{Fil}^1 D_K) \subseteq \mathrm{Fil}^1 D'_K$ if and only if $\alpha = \beta$. □

4.4. The cubic case. Let $E/\mathbb{Q}_3$ be an elliptic curve with semi-stability defect $\mathbf{dst}(E) = 3$. There are exactly 9 totally ramified extensions of degree 3 of $\mathbb{Q}_3$ (see [9]). Since E acquires good reduction over a degree 3 Galois extension of $\mathbb{Q}_3^{\mathrm{un}}$, we are interested in the ones that keep their ramification index after Galois closure. Indeed, if $e(F/\mathbb{Q}_3) = 3$ but $e(F^{\mathrm{Gal}}/\mathbb{Q}_3) > 3$, then $[(F^{\mathrm{Gal}})^{\mathrm{un}} : \mathbb{Q}_3^{\mathrm{un}}] > 3$ is not minimal. There are only 4 such extensions; among these, 3 are Abelian and the last one has a Galois closure of degree 6 with Galois group isomorphic to S_3. One easily shows (using the Kronecker–Weber Theorem) that the three considered Abelian extensions are exactly the degree 3 totally ramified subextensions of $\mathbb{Q}_3(\zeta_{13}, \zeta_9 + \zeta_9^{-1})$, so their compositum with $\mathbb{Q}_3^{\mathrm{un}}$ is $\mathbb{Q}_3^{\mathrm{un}}(\zeta_9 + \zeta_9^{-1})$, and they are therefore interchangeable. This is not the case of the non Abelian extension. We denote by $L^{\mathrm{a}} = \mathbb{Q}_3(\zeta_9 + \zeta_9^{-1})$ (resp. $L^{\mathrm{na}} = \mathbb{Q}_3(X^3 - 3X^2 + 6)$) a minimal field of good reduction for $E/\mathbb{Q}_3$ in the Abelian (resp. non Abelian) case.

Given an elliptic curve $E/\mathbb{Q}_3$ and $\ell \neq 3$ a prime, we consider:

$$\tau_E \colon I(\overline{\mathbb{Q}}_3/\mathbb{Q}_3) \xrightarrow{\rho_{E,\ell}} \mathrm{GL}_2(\mathbb{Q}_\ell) \hookrightarrow \mathrm{GL}_2(\mathbb{C}).$$

Proposition 4.11. *Let E, $E'/\mathbb{Q}_3$ be elliptic curves with semi-stability defects $\mathbf{dst}(E) = \mathbf{dst}(E') = 3$. We have the following equivalence:*

$$\tau_E \simeq_{\mathbb{C}} \tau_{E'} \iff M_E = M_{E'}.$$

Proof. The left to right implication is obvious since $M_E = (\mathbb{Q}_3^{\mathrm{un}})^{\ker(\tau_E)}$, $M_{E'} = (\mathbb{Q}_3^{\mathrm{un}})^{\ker(\tau_{E'})}$ and two isomorphic representations share the same kernel. If $M_E = M'_E$ then $\ker(\tau_E) = \ker(\tau_{E'})$ and both types factors into faithful irreducible representations of $\mathrm{Gal}(M_E/\mathbb{Q}_3^{\mathrm{un}}) \simeq \mathbb{Z}/3\mathbb{Z}$ defined over $\mathbb{Q}$, which are necessarily isomorphic. □

Using [5, Table 1] we see that there are only two isomorphism classes of such objects for $p = e = 3$ so that $L^{\mathrm{a}}\mathbb{Q}_3^{\mathrm{un}}$ and $L^{\mathrm{na}}\mathbb{Q}_3^{\mathrm{un}}$ are indeed distinct.

4.4.1. *The non Abelian case.* Let $E/\mathbb{Q}_3$ with $\mathbf{dst}(E) = 3$ acquiring good reduction over L^{na} with $K = L^{\mathrm{na}}(\zeta_4)$ its Galois closure. Denote by K_0 the maximal unramified extension of $K/\mathbb{Q}_3$ and $\sigma \in \mathrm{Gal}(K_0/\mathbb{Q}_3)$ the absolute Frobenius. Let $\omega \in \mathrm{Gal}(K/\mathbb{Q}_3)$ be a lifting of σ fixing L^{na} and τ_3 a generator of $\mathrm{Gal}(K/K_0) = I(K/\mathbb{Q}_3)$. Then, $\mathrm{Gal}(K/\mathbb{Q}_3) = \langle\tau_3\rangle \rtimes \langle\omega\rangle$ with $\tau_3\omega = \omega\tau_3^{-1}$ (the unique non trivial semi-direct product).

Let $\alpha \in \mathcal{M}_3^{\mathrm{na}} = \{x \in L^{\mathrm{na}} \mid \tau_3(x) = (3\zeta_4 + x)/(1 + \zeta_4 x)\}$. We denote by $\mathbf{D}_{\mathrm{pc}}^{\mathrm{na}}(3; 0; \alpha)$ the filtered $(\varphi, \mathrm{Gal}(K/\mathbb{Q}_3))$-module defined by:

- $D = K_0 e_1 \oplus K_0 e_2$,
- $\varphi(e_1) = e_2,\ \varphi(e_2) = -3e_1$,
- $M_B(\tau_3) = \begin{pmatrix} -\frac{1}{2} & \frac{3}{2}\zeta_4 \\ \frac{1}{2}\zeta_4 & -\frac{1}{2} \end{pmatrix}$,
- $\omega(e_1) = e_1,\ \omega(e_2) = e_2$,

- $\mathrm{Fil}^1 D_K = (\alpha \otimes e_1 + 1 \otimes e_2)K$.

For each $\alpha \in \mathcal{M}_3^{\mathrm{na}}$, the filtered $(\varphi, \mathrm{Gal}(K/\mathbb{Q}_3))$-module $\mathbf{D}_{\mathrm{pc}}^{\mathrm{na}}(3;0;\alpha)$ satisfies conditions (1)–(4) and is admissible.

Proposition 4.12. *Let $E/\mathbb{Q}_3$ be an elliptic curve with $\mathbf{dst}(E) = 3$ acquiring good reduction over L^{na} and $\mathbf{D} = \mathbf{D}^*_{\mathrm{cris},K}(V_3(E))$. There exists $\alpha \in \mathcal{M}_3^{\mathrm{na}}$ such that $\mathbf{D}$ and $\mathbf{D}_{\mathrm{pc}}^{\mathrm{na}}(3;0;\alpha)$ are isomorphic as filtered $(\varphi, \mathrm{Gal}(K/\mathbb{Q}_3))$-modules. Moreover, if α, $\beta \in \mathcal{M}_3^{\mathrm{na}}$, then $\mathbf{D}_{\mathrm{pc}}^{\mathrm{na}}(3;0;\alpha)$ and $\mathbf{D}_{\mathrm{pc}}^{\mathrm{na}}(3;0;\beta)$ are isomorphic if and only if $\alpha = \beta$.*

Proof. Denote by D the underlying K_0-vector space associated to $\mathbf{D}$, the element τ_3 acts K_0-linearly over D and the morphism

$$I(K/\mathbb{Q}_3) \longrightarrow \mathrm{Aut}_{K_0}(D)$$

is injective by minimality of $e(K/\mathbb{Q}_3)$. We identify τ_3 to its image in $\mathrm{Aut}_{K_0}(D)$, it is an element of order 3. Since $\zeta_3 \notin K_0$,

$$P_{\mathrm{char}}(\tau_3)(X) = P_{\min}(\tau_3)(X) = X^2 + X + 1.$$

Let $B = (e_1, e_2)$ be a K_0-basis of D fixed by ω such that $\varphi(e_1) = e_2$ and $\varphi(e_2) = -3e_1 - a_3 e_2$. Such a basis always exists since ω acts semi-linearly over D and $\varphi\omega = \omega\varphi$. Let λ, μ, λ', $\mu' \in K_0$ such that

$$M_B(\tau_3) = \begin{pmatrix} \lambda & \mu' \\ \mu & \lambda' \end{pmatrix}.$$

We already know that $\lambda' = -\lambda - 1$ and $\mu' = P(\lambda)/-\mu$ where $P = P_{\mathrm{char}}(\tau_3)$. The relations $\tau\omega = \omega\tau^{-1}$ and $\tau\varphi = \varphi\tau$ imply that $a_3 = 0$, $\sigma(\lambda) = -\lambda - 1$, $\sigma(\mu) = -\mu$ and $P(\lambda)/-\mu = 3\mu$. In conclusion:

$$\begin{cases} \varphi(e_1) = e_2 \wedge \varphi(e_2) = -3e_1 \\ \omega(e_1) = e_1 \wedge \omega(e_2) = e_2 \\ M_B(\tau_3) = \begin{pmatrix} \lambda & 3\mu \\ \mu & -\lambda-1 \end{pmatrix} \end{cases}$$

with $\lambda \in -\frac{1}{2} + \mathbb{Q}_3\zeta_4$, $\mu \in \mathbb{Q}_3^\times \zeta_4$ and $P(\lambda) + 3\mu^2 = 0$. Let

$$M = \begin{pmatrix} \lambda + \frac{1}{2} & -3\mu - \frac{3}{2}\zeta_4 \\ \mu - \frac{1}{2}\zeta_4 & \lambda + \frac{1}{2} \end{pmatrix}$$

clearly $\det(M) = 0$. Let $(a, b) \in \ker(M)^{\mathrm{Gal}(K/\mathbb{Q}_3)} \subseteq K_0^2$ be a nonzero element. Then

$$B' = (e_1', e_2') = (ae_1 + be_2, -3be_1 + ae_2)$$

is a K_0-basis of D such that

$$\begin{cases} \varphi(e_1') = e_2' \wedge \varphi(e_2') = -3e_1' \\ \omega(e_1') = e_1' \wedge \omega(e_2') = e_2' \\ M_{B'}(\tau_3) = \begin{pmatrix} -\frac{1}{2} & \frac{3}{2}\zeta_4 \\ \frac{1}{2}\zeta_4 & -\frac{1}{2} \end{pmatrix}. \end{cases}$$

Again, we denote by (e_1, e_2) such a basis. One easily checks that $(1 \otimes e_1)K$ and $(1 \otimes e_2)K$ are not stable by τ_3. Let $\alpha \in K^\times$ and $\mathrm{Fil}^1 D_K = (\alpha \otimes e_1 + 1 \otimes e_2)$. A simple calculation shows that such a K-line is stable by the action of $\mathrm{Gal}(K/\mathbb{Q}_3)$ if and only if $\alpha \in L^{\mathrm{na}}$ and $\tau_3(\alpha) = (3\zeta_4 + \alpha)/(1 + \zeta_4\alpha)$. Let B, B' be K_0-basis of $\mathbf{D} = \mathbf{D}_{\mathrm{pc}}^{\mathrm{na}}(3; 0; \alpha)$ and $\mathbf{D}' = \mathbf{D}_{\mathrm{pc}}^{\mathrm{na}}(3; 0; \beta)$ respectively. One easily shows that an isomorphism η of $(\varphi, \mathrm{Gal}(K/\mathbb{Q}_3))$-modules between $\mathbf{D}$ and $\mathbf{D}'$ is of the form

$$M_{B,B'}(\eta) = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}, \ a \in \mathbb{Q}_3^\times.$$

Denoting by $\eta_K \colon D_K \to D_K'$ the K-linear extension of η, it is then clear that $\eta_K((\alpha \otimes e_1 + 1 \otimes e_2)K) \subseteq (\beta \otimes e_1 + 1 \otimes e_2)K$ if and only if $\alpha = \beta$. $\square$

4.4.2. *The Abelian case.* Let $E/\mathbb{Q}_3$ with $\mathbf{dst}(E)$ acquiring good reduction over $K = L^{\mathrm{a}}$. Its Galois group $\mathrm{Gal}(K/\mathbb{Q}_3) = I(K/\mathbb{Q}_3) = \langle \tau_3 \rangle$ is cyclic of order 3.

Let $\alpha \in \mathcal{M}_3^{\mathrm{a}} = \{x \in L^{\mathrm{a}} \mid \tau_3(x) = (x-1)/x\}$ and

$$(a, \mu) \in S = (\{-3\} \times \{1, 2\}) \sqcup (\{0\} \times \{-1, 1\}) \sqcup (\{3\} \times \{-2, -1\}).$$

We denote by $\mathbf{D}_{\mathrm{pc}}^{\mathrm{a}}(3; a, \mu; \alpha)$ the filtered $(\varphi, \mathrm{Gal}(K/\mathbb{Q}_3))$-module defined by:

- $D = \mathbb{Q}_3 e_1 \oplus \mathbb{Q}_3 e_2$,
- $M_B(\tau_3) = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$,
- $M_B(\varphi) = \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}, \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}$, $a = -3$, $\mu = 1, 2$;
- $M_B(\varphi) = \begin{pmatrix} -2 & 1 \\ -1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & -2 \\ 2 & -1 \end{pmatrix}$, $a = 0$, $\mu = -1, 1$;
- $M_B(\varphi) = \begin{pmatrix} -1 & -1 \\ 1 & -2 \end{pmatrix}, \begin{pmatrix} -2 & 1 \\ -1 & -1 \end{pmatrix}$, $a = 3$, $\mu = -2, -1$;
- $\mathrm{Fil}^1 D_K = (\alpha \otimes e_1 + 1 \otimes e_2)K$.

For each $\alpha \in \mathcal{M}_3^{\mathrm{a}}$, the filtered $(\varphi, \mathrm{Gal}(K/\mathbb{Q}_3))$-module $\mathbf{D}_{\mathrm{pc}}^{\mathrm{a}}(3; a, \mu; \alpha)$ satisfies conditions (1)–(4) and is admissible.

Proposition 4.13. *Let $E/\mathbb{Q}_3$ be an elliptic curve with $\mathbf{dst}(E) = 3$ acquiring good reduction over K and $\mathbf{D} = \mathbf{D}_{\mathrm{cris},K}^*(V_3(E))$. There exists $\alpha \in \mathcal{M}_3^{\mathrm{a}}$ and $(a, \mu) \in S$ such that $\mathbf{D}$ and $\mathbf{D}_{\mathrm{pc}}^{\mathrm{a}}(3; a, \mu; \alpha)$ are isomorphic as filtered $(\varphi, \mathrm{Gal}(K/\mathbb{Q}_3))$-modules. Moreover, if α, $\beta \in \mathcal{M}_3^{\mathrm{a}}$ and (a, μ), $(b, \nu) \in S$, then $\mathbf{D}_{\mathrm{pc}}^{\mathrm{a}}(3; a, \mu; \alpha)$ and $\mathbf{D}_{\mathrm{pc}}^{\mathrm{a}}(3; b, \nu; \beta)$ are isomorphic if and only if $(a, \mu) = (b, \nu)$.*

Proof. Denote by D the underlying $\mathbb{Q}_3$-vector space associated to $\mathbf{D}$, the element τ_3 acts $\mathbb{Q}_3$-linearly over D and the natural morphism

$$I(K/\mathbb{Q}_3) \longrightarrow \mathrm{Aut}_{\mathbb{Q}_3}(D)$$

is injective by minimality of $e(K/\mathbb{Q}_3)$. We identify τ_3 to its image in $\mathrm{Aut}_{\mathbb{Q}_3}(D)$, it is an element of order 3. Since $\zeta_3 \notin \mathbb{Q}_3$,

$$P_{\mathrm{char}}(\tau_3)(X) = P_{\min}(\tau_3)(X) = X^2 + X + 1.$$

Let $B = (e_1, e_2)$ be a $\mathbb{Q}_3$-basis of D such that:

$$M_B(\tau_3) = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}.$$

Since $P_{\mathrm{char}}(\varphi)(X) = X^2 + a_3X + 3$ and $\varphi\tau_3 = \tau_3\varphi$, we have

$$M_B(\varphi) = \begin{pmatrix} \lambda & -2\lambda - a_3 \\ 2\lambda + a_3 & -\lambda - a_3 \end{pmatrix}, \ \lambda \in \mathbb{Q}_3$$

with $\det(\varphi) = 3\lambda^2 + 3\lambda a_3 + a_3^2 = 3$, i.e. λ is a root of $3X^2 + 3a_3X + a_3^2 - 3$. But this polynomial has roots in $\mathbb{Q}_3$ if and only if $3 \mid a_3$, so $a_3 \in \{-3, 0, 3\}$. Considering every possible value of a_3 we obtain:

- if $a_3 = -3$, λ is a root of $X^2 - 3X + 2$ i.e. $\lambda \in \{1, 2\}$,
- if $a_3 = 0$, λ is a root of $X^2 - 1$ i.e. $\lambda \in \{-1, 1\}$,
- if $a_3 = 3$, λ is a root of $X^2 + 3X + 2$ i.e. $\lambda \in \{-2, -1\}$.

Observe that $(1 \otimes e_1)K$ and $(1 \otimes e_2)K$ are not stable by action of $\mathrm{Gal}(K/\mathbb{Q}_3)$. Let $\alpha \in K^\times$, the K-line $(\alpha \otimes e_1 + 1 \otimes e_2)K$ is stable by τ_3 if and only if $\tau_3(\alpha) = (\alpha - 1)/\alpha$. So that $\mathbf{D} \simeq \mathbf{D}^{\mathrm{a}}_{\mathrm{pc}}(3; a_3, \lambda; \alpha)$ with α and (a_3, λ) satisfying the desired conditions. Let α, $\beta \in \mathcal{M}^{\mathrm{a}}_3$ and (a, μ), $(b, \nu) \in S$. Consider $\mathbf{D} = \mathbf{D}^{\mathrm{a}}_{\mathrm{pc}}(3; a, \mu; \alpha)$ and $\mathbf{D}' = \mathbf{D}^{\mathrm{a}}_{\mathrm{pc}}(3; b, \nu; \beta)$, we will first show that their underlying $(\varphi, \mathrm{Gal}(K/\mathbb{Q}_3))$-modules are not isomorphic, except in the obvious case. Let B and B' be $\mathbb{Q}_3$-basis of D and D' respectively. A morphism $\eta\colon D \to D'$ commuting to τ_3 and φ must be of the form

$$M_{B,B'}(\eta) = \begin{pmatrix} c & -d \\ d & c - d \end{pmatrix}$$

where $(c, d) \in \mathbb{Q}_3^2$ is in the kernel of the following linear map

$$\begin{pmatrix} \mu - \nu & 2(\nu - \mu) + b - a \\ \nu - \mu + b - a & \nu - \mu \end{pmatrix}.$$

The determinant of this matrix is $-(3(\nu - \mu)^2 + 3(\nu - \mu)(b - a) + (b - a)^2)$. There exists $(c, d) \neq (0, 0)$ in the kernel if and only if $(\nu - \mu)$ is a root of

$$3X^2 + 3(b - a)X + (b - a)^2.$$

But such a polynomial has roots in $\mathbb{Q}_3$ if and only if $a = b$, in which case its only root is 0. This shows that if $\mathbf{D}$ and $\mathbf{D}'$ are isomorphic as $(\varphi, \mathrm{Gal}(K/\mathbb{Q}_3))$-modules then $\mu = \nu$ and $a = b$. Now suppose that $(a, \mu) =$

(b,ν). Let $\mathrm{Fil}^1 D_K = (\alpha \otimes e_1 + 1 \otimes e_2)K$ and $\mathrm{Fil}^1 D'_K = (\beta \otimes e_1 + 1 \otimes e_2)K$, the K-lines defining the filtrations on $\mathbf{D}$ and $\mathbf{D}'$. If $\alpha = \beta$ taking $c = 1$ and $d = 0$ gives us an obvious isomorphism. In the other case, we see that $(\alpha\beta - \beta + 1)/(\alpha - \beta) \in \mathbb{Q}_3$ and taking some $d \neq 0$ and $c = d(\alpha\beta - \beta + 1)/(\alpha - \beta)$ gives us the desired isomorphism. □

Remark 4.14. We observe two differences with the non Abelian case: the supersingular traces 3 and -3 do appear and for each trace value there are two isomorphism classes of $(\varphi, \mathrm{Gal}(K/\mathbb{Q}_3))$-modules (not considering filtration). We will explain the absence of these traces in Section 5. These two isomorphism classes are unramified quadratic twists of each other.

4.5. The sextic case. This section can be summarized by the following result: if $E/\mathbb{Q}_3$ has a semi-stability defect of 3 then its quadratic twist $E'/\mathbb{Q}_3$ by the character associated to $\sqrt{3}$ has a semi-stability defect of 6, and vice versa. Consequently, if $F/\mathbb{Q}_3$ is a field of good reduction for E, then $F(\sqrt{3})$ is a field of good reduction for E'.

4.5.1. ***The non Abelian case.*** Let $E/\mathbb{Q}_3$ with $\mathbf{dst}(E) = 6$ acquiring good reduction over $L^{\mathrm{na}}(\sqrt{3})$ and let $K = L^{\mathrm{na}}(\sqrt{3}, \zeta_4)$ be its Galois closure. We have $\mathrm{Gal}(K/\mathbb{Q}_3) = (\langle\tau_3\rangle \times \langle\tau_2\rangle) \rtimes \langle\omega\rangle$ and $I(K/\mathbb{Q}_3) = \langle\tau_3\rangle \times \langle\tau_2\rangle$ is cyclic of order 6.

Let $\alpha \in \mathcal{M}_6^{\mathrm{na}} = \{x \in L^{\mathrm{na}}(\sqrt{3}) \mid \tau_3(x) = (3\zeta_4 + x)/(1 + \zeta_4 x)\}$. We denote by $\mathbf{D}_{\mathrm{pc}}^{\mathrm{na}}(6; 0; \alpha)$ the filtered $(\varphi, \mathrm{Gal}(K/\mathbb{Q}_3))$-module defined by:

- $D = K_0 e_1 \oplus K_0 e_2$,
- $\varphi(e_1) = e_2,\ \varphi(e_2) = -3e_1$,
- $M_B(\tau_3) = \begin{pmatrix} -\frac{1}{2} & \frac{3}{2}\zeta_4 \\ \frac{1}{2}\zeta_4 & -\frac{1}{2} \end{pmatrix}$,
- $M_B(\tau_2) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$,
- $\omega(e_1) = e_1,\ \omega(e_2) = e_2$,
- $\mathrm{Fil}^1 D_K = (\alpha \otimes e_1 + 1 \otimes e_2)K$.

For each $\alpha \in \mathcal{M}_6^{\mathrm{na}}$, the filtered $(\varphi, \mathrm{Gal}(K/\mathbb{Q}_3))$-module $\mathbf{D}_{\mathrm{pc}}^{\mathrm{na}}(6; 0; \alpha)$ satisfies conditions (1)–(4) and is admissible.

Proposition 4.15. *Let $E/\mathbb{Q}_3$ be an elliptic curve with $\mathbf{dst}(E) = 6$ acquiring good reduction over $L^{\mathrm{na}}(\sqrt{3})$ and $\mathbf{D} = \mathbf{D}^*_{\mathrm{cris},K}(V_3(E))$. There exists $\alpha \in \mathcal{M}_6^{\mathrm{na}}$ such that $\mathbf{D}$ and $\mathbf{D}_{\mathrm{pc}}^{\mathrm{na}}(6; 0; \alpha)$ are isomorphic as filtered $(\varphi, \mathrm{Gal}(K/\mathbb{Q}_3))$-modules. Moreover, if α, $\beta \in \mathcal{M}_6^{\mathrm{na}}$, then $\mathbf{D}_{\mathrm{pc}}^{\mathrm{na}}(6; 0; \alpha)$ and $\mathbf{D}_{\mathrm{pc}}^{\mathrm{na}}(6; 0; \beta)$ are isomorphic if and only if $\alpha = \beta$.*

Proof. Similar to the cubic non Abelian case using the natural injection

$$I(K/\mathbb{Q}_3) = \langle\tau_2\rangle \times \langle\tau_3\rangle \hookrightarrow \mathrm{Aut}_{K_0}(D).$$

□

4.5.2. ***The Abelian case.*** Let $E/\mathbb{Q}_3$ with $\mathbf{dst}(E) = 6$ acquiring good reduction over $K = L^{\mathrm{a}}(\sqrt{3})$. Then $\mathrm{Gal}(K/\mathbb{Q}_3) = I(K/\mathbb{Q}_3) = \langle\tau_3\rangle \times \langle\tau_2\rangle$ is cyclic of order 6.

Let $\alpha \in \mathcal{M}_6^{\mathrm{a}} = \{x \in L \mid \tau_3(x) = (x-1)/x\}$ and

$$(a, \mu) \in S = (\{-3\} \times \{1, 2\}) \sqcup (\{0\} \times \{-1, 1\}) \sqcup (\{3\} \times \{-2, -1\}).$$

We denote by $\mathbf{D}^{\mathrm{a}}_{\mathrm{pc}}(6; a, \mu; \alpha)$ the filtered $(\varphi, \mathrm{Gal}(K/\mathbb{Q}_3))$-module defined by:

- $D = \mathbb{Q}_3 e_1 \oplus \mathbb{Q}_3 e_2$,
- $M_B(\tau_3) = \left(\begin{smallmatrix} 0 & -1 \\ 1 & -1 \end{smallmatrix}\right)$,
- $M_B(\tau_2) = \left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$,
- $M_B(\varphi) = \left(\begin{smallmatrix} 1 & 1 \\ -1 & 2 \end{smallmatrix}\right), \left(\begin{smallmatrix} 2 & -1 \\ 1 & 1 \end{smallmatrix}\right)$, $a = -3$, $\mu = 1, 2$;
- $M_B(\varphi) = \left(\begin{smallmatrix} -2 & 1 \\ -1 & -1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & -2 \\ 2 & -1 \end{smallmatrix}\right)$, $a = 0$, $\mu = -1, 1$;
- $M_B(\varphi) = \left(\begin{smallmatrix} -1 & -1 \\ 1 & -2 \end{smallmatrix}\right), \left(\begin{smallmatrix} -2 & 1 \\ -1 & -1 \end{smallmatrix}\right)$, $a = 3$, $\mu = -2, -1$;
- $\mathrm{Fil}^1 D_K = (\alpha \otimes e_1 + 1 \otimes e_2)K$.

For each $\alpha \in \mathcal{M}_6^{\mathrm{a}}$, the filtered $(\varphi, \mathrm{Gal}(K/\mathbb{Q}_3))$-module $\mathbf{D}^{\mathrm{a}}_{\mathrm{pc}}(6; a, \mu; \alpha)$ satisfies conditions (1)–(4) and is admissible.

Proposition 4.16. *Let $E/\mathbb{Q}_3$ be an elliptic curve with $\mathbf{dst}(E) = 6$ acquiring good reduction over K and $\mathbf{D} = \mathbf{D}^*_{\mathrm{cris},K}(V_3(E))$. There exists $\alpha \in \mathcal{M}_6^{\mathrm{a}}$ and $(a, \mu) \in S$ such that $\mathbf{D}$ and $\mathbf{D}^{\mathrm{a}}_{\mathrm{pc}}(6; a, \mu; \alpha)$ are isomorphic as filtered $(\varphi, \mathrm{Gal}(K/\mathbb{Q}_3))$-modules. Moreover, if α, $\beta \in \mathcal{M}_6^{\mathrm{a}}$ and (a, μ), $(b, \nu) \in S$, then $\mathbf{D}^{\mathrm{a}}_{\mathrm{pc}}(6; a, \mu; \alpha)$ and $\mathbf{D}^{\mathrm{a}}_{\mathrm{pc}}(6; b, \nu; \beta)$ are isomorphic if and only if $(a, \mu) = (b, \nu)$.*

Proof. Similar to the cubic Abelian case using the following injection

$$I(K/\mathbb{Q}_3) = \langle\tau_2\rangle \times \langle\tau_3\rangle \hookrightarrow \mathrm{Aut}_{\mathbb{Q}_3}(D). \qquad \square$$

4.6. The dodecic case. If an elliptic curve $E/\mathbb{Q}_3$ has a semi-stability defect $\mathbf{dst}(E) = 12$, then its minimal field of good reduction has Galois closure $K/\mathbb{Q}_3$ satisfying:

$$\begin{cases} \mathrm{Gal}(K/\mathbb{Q}_3) \simeq \mathbb{Z}/3\mathbb{Z} \rtimes D_4 \simeq (\mathbb{Z}/3\mathbb{Z} \rtimes \mathbb{Z}/4\mathbb{Z}) \rtimes \mathbb{Z}/2\mathbb{Z} \\ I(K/\mathbb{Q}_3) \simeq \mathbb{Z}/3\mathbb{Z} \rtimes \mathbb{Z}/4\mathbb{Z}. \end{cases}$$

More precisely:

$$\begin{cases} \mathrm{Gal}(K/\mathbb{Q}_3) = (\langle\tau_3\rangle \rtimes \langle\tau_4\rangle) \rtimes \langle\omega\rangle \\ I(K/\mathbb{Q}_3) = \mathrm{Gal}(K/K_0) = \langle\tau_3\rangle \rtimes \langle\tau_4\rangle \end{cases}$$

with relations:

$$\begin{cases} \mathrm{ord}(\tau_4) = 4 \wedge \mathrm{ord}(\tau_3) = 3 \wedge \mathrm{ord}(\omega) = 2 \\ \tau_4\tau_3\tau_4^{-1} = \tau_3^2 = \tau_3^{-1} \\ \omega\tau_4\omega = \tau_4^{-1} \\ \tau_3\omega = \omega\tau_3. \end{cases}$$

This follows from the structure of $\mathrm{Aut}_{\mathbb{F}_9}(\widetilde{E})$, where $\widetilde{E}$ is the special fibre of $E_K = E \times_{\mathbb{Q}_3} K$. Looking at [9] we see that there are exactly 10 such fields, namely:

- $K_1 = \mathbb{Q}_3(X^{12} + 3X^4 + 3)$,
- $K_2 = \mathbb{Q}_3(X^{12} - 3X^{11} - 3X^9 + 3X^7 - 3X^4 - 3)$,
- $K_3 = \mathbb{Q}_3(X^{12} + 3)$,
- $K_4 = \mathbb{Q}_3(X^{12} + 9X^{10} + 9X^9 - 9X^8 + 6X^6 + 9X^5 - 9X^4 - 3X^3 + 9X^2 - 9X - 12)$,
- $K_5 = \mathbb{Q}_3(X^{12} + 9X^{11} + 9X^{10} + 9X^9 + 9X^8 - 9X^7 - 12X^6 - 9X^2 - 3)$,
- $K_6 = \mathbb{Q}_3(X^{12} + 3X^{10} - 3X^9 - 3X^7 + 3X^6 + 3X^5 + 3X^4 + 3X^3 - 3)$,
- $K_7 = \mathbb{Q}_3(X^{12} - 3X^{11} - 3X^{10} + 3X^9 + 3X^5 - 3X^4 + 3X^3 + 3)$,
- $K_8 = \mathbb{Q}_3(X^{12} - 9X^{11} + 9X^9 - 9X^8 + 9X^7 - 12X^6 + 3X^3 + 9X^2 + 9X - 12)$,
- $K_9 = \mathbb{Q}_3(X^{12} + 9X^{11} + 9X^{10} - 3X^9 - 9X^8 - 9X^7 + 3X^6 + 9X^5 - 9X^4 + 6X^3 - 9X^2 - 9X + 12)$,
- $K_{10} = \mathbb{Q}_3(X^{12} - 9X^{11} + 6X^9 + 9X^8 + 3X^6 + 9X^5 + 9X^4 + 3X^3 - 9X^2 + 9X + 3)$.

Looking at the lattice of each such field extension over $\mathbb{Q}_3$, we observe that $K_i^{\mathrm{un}} = K_j^{\mathrm{un}}$ if and only if $i \equiv j \bmod 5\mathbb{Z}$, so that there are in fact 5 fields of good reduction. Furthermore, every one of these 5 fields appears as the reduction field of some elliptic curve (see [8, Thm. 17(7)]). For $i \in \{1, \ldots, 5\}$, we let L_i be the maximal totally ramified sub-extension of K_i, so that $K_i = L_i(\zeta_4)$.

Let $\alpha \in \mathcal{M}_{12}^{i,\epsilon} = \{x \in L_i \mid \tau_3(x) = (x + 3(-1)^{\epsilon+1})/(1 + x(-1)^{\epsilon}) \wedge \tau_4(x) = -x\}$ for $i \in \{1, \ldots, 5\}$ and $\epsilon \in \{0, 1\}$. We let $K_0 = \mathbb{Q}_3(\zeta_4)$ be the maximal unramified extension of $\mathbb{Q}_3$ in K_i which is independant of i. We denote by $\mathbf{D}_{\mathrm{pc}}(12; 0; i; \epsilon; \alpha)$ the filtered $(\varphi, \mathrm{Gal}(K_i/\mathbb{Q}_3))$-module defined by:

- $D = K_0e_1 \oplus K_0e_2$,
- $M_B(\tau_4) = \begin{pmatrix} \zeta_4 & 0 \\ 0 & \zeta_4^{-1} \end{pmatrix}$,
- $M_B(\tau_3) = \begin{pmatrix} -\frac{1}{2} & \frac{(-1)^{\epsilon+1}3}{2} \\ \frac{(-1)^{\epsilon}}{2} & -\frac{1}{2} \end{pmatrix}$,
- $\varphi(e_1) = e_2;\ \varphi(e_2) = -3e_1$,
- $\omega(e_1) = e_1;\ \omega(e_2) = e_2$,
- $\mathrm{Fil}^1 D_{K_i} = (\alpha \otimes e_1 + 1 \otimes e_2)K_i$.

The filtered $(\varphi, \mathrm{Gal}(K_i/\mathbb{Q}_3))$-module $\mathbf{D}_{\mathrm{pc}}(12; 0; i; \epsilon; \alpha)$ satisfies conditions (1)–(4) and is admissible for each $i \in \{1, \dots, 5\}$, $\epsilon \in \{0, 1\}$ and $\alpha \in \mathcal{M}_{12}^{i,\epsilon}$.

Proposition 4.17. *Let $E/\mathbb{Q}_3$ be an elliptic curve with $\mathbf{dst}(E) = 12$ acquiring good reduction over K_i for some $i \in \{1, \dots, 5\}$ and $\mathbf{D} = \mathbf{D}^*_{\mathrm{cris},K_i}(V_3(E))$. There exists $\epsilon \in \{0, 1\}$ and $\alpha \in \mathcal{M}_{12}^{i,\epsilon}$ such that $\mathbf{D}$ and $\mathbf{D}_{\mathrm{pc}}(12; 0; i; \epsilon; \alpha)$ are isomorphic as filtered $(\varphi, \mathrm{Gal}(K_i/\mathbb{Q}_3))$-modules. Moreover, if ϵ, $\epsilon' \in \{0, 1\}$ and $\alpha \in \mathcal{M}_{12}^{i,\epsilon}$, $\beta \in \mathcal{M}_{12}^{i,\epsilon'}$, then $\mathbf{D}_{\mathrm{pc}}(12; 0; i; \epsilon; \alpha)$ and $\mathbf{D}_{\mathrm{pc}}(12; i; \epsilon'; \beta)$ are isomorphic if and only if $(\alpha, \epsilon) = (\beta, \epsilon')$.*

Proof. Let D be the underlying K_0-vector space associated to $\mathbf{D}$. As usual, the inertia subgroup of $K_i/\mathbb{Q}_3$ injects in $\mathrm{Aut}_{K_0}(D)$ and we identify τ_4 and τ_3 to their respective image. As in the quartic case we show that there is a K_0-basis $B = (e_1, e_2)$ of D such that:

$$\begin{cases} M_B(\tau_4) = \begin{pmatrix} \zeta_4 & 0 \\ 0 & \zeta_4^{-1} \end{pmatrix} \\ \omega(e_1) = e_1 \wedge \omega(e_2) = e_2 \\ \varphi(e_1) = e_2 \wedge \varphi(e_2) = -3e_1. \end{cases}$$

The relations between τ_3 and τ_4 as well as ω and φ implies that there is some $\epsilon' \in \{0, 1\}$ such that

$$M_B(\tau_3) = \begin{pmatrix} -\frac{1}{2} & \frac{3(-1)^{\epsilon'+1}}{2} \\ \frac{(-1)^{\epsilon'}}{2} & -\frac{1}{2} \end{pmatrix}.$$

A simple calculation shows that the K_i-lines of $D_{K_i} = K_i \otimes_{K_0} D$ stable by the action of $\mathrm{Gal}(K_i/\mathbb{Q}_3)$ are of the form

$$(\alpha \otimes e_1 + 1 \otimes e_2)K_i$$

with $\alpha \in L_i$ satisfying the desired conditions. Let ϵ, $\epsilon' \in \{0, 1\}$ and $\alpha \in \mathcal{M}_{12}^{i,\epsilon}$, $\beta \in \mathcal{M}_{12}^{i,\epsilon'}$. Looking only at their underlying $(\varphi, \mathrm{Gal}(K_i/\mathbb{Q}_3))$-modules, we see that $\mathbf{D}_{\mathrm{pc}}(12; 0; i; \epsilon; \alpha)$ and $\mathbf{D}_{\mathrm{pc}}(12; 0; i; \epsilon'; \beta)$ are isomorphic if and only if $\epsilon = \epsilon'$. Now supposing $\epsilon = \epsilon'$ and adding the filtration, we check that a morphism between $\mathbf{D}_{\mathrm{pc}}(12; 0; i; \epsilon; \alpha)$ and $\mathbf{D}_{\mathrm{pc}}(12; 0; i; \epsilon; \beta)$ must be of the form $\lambda\,\mathrm{Id}$ with $\lambda \in \mathbb{Q}_3^\times$, so that necessarily $\alpha = \beta$. ☐

Remark 4.18. As in the cubic Abelian case, observe that $\mathbf{D}_{\mathrm{pc}}(12; 0; i; 0; \alpha)$ and $\mathbf{D}_{\mathrm{pc}}(12; 0; i; 1; \alpha)$ are unramified quadratic twists of each other as unfiltered $(\varphi, \mathrm{Gal}(K_i/\mathbb{Q}_3))$-modules.

5. Elliptic curves with given Tate module

5.1. Minimal Galois pairs. Let $K/\mathbb{Q}_p$ be a finite Galois extension with maximal unramified extension K_0 and residue field $\mathbb{F}_{p^s}$. For an elliptic curve $\widetilde{E}_0/\mathbb{F}_p$, we let $\widetilde{E} = \widetilde{E}_0 \times_{\mathbb{F}_p} \mathbb{F}_{p^s}$ be its scalar extension to $\mathbb{F}_{p^s}$. Let σ be the absolute Frobenius generating $\mathrm{Gal}(\mathbb{F}_{p^s}/\mathbb{F}_p)$ and consider the $\mathbb{F}_{p^s}$-isomorphism $f_\sigma = \mathrm{Id}_{\tilde{E}_0} \otimes \sigma \colon \widetilde{E} \to \widetilde{E}^\sigma$. The Galois group $\mathrm{Gal}(\mathbb{F}_{p^s}/\mathbb{F}_p)$ acts on $\mathrm{Aut}_{\mathbb{F}_{p^s}}(\widetilde{E})$ via $\sigma.u = (f_\sigma \circ u \circ f_\sigma^{-1})^{\sigma^{-1}}$. This action defines a semidirect product that sits in the following split exact sequence:

$$1 \longrightarrow \mathrm{Aut}_{\mathbb{F}_{p^s}}(\widetilde{E}) \longrightarrow \mathrm{Aut}_{\mathbb{F}_{p^s}}(\widetilde{E}) \rtimes \mathrm{Gal}(\mathbb{F}_{p^s}/\mathbb{F}_p) \xrightarrow{\mathrm{pr}} \mathrm{Gal}(\mathbb{F}_{p^s}/\mathbb{F}_p) \longrightarrow 1.$$

A Galois pair for $K/\mathbb{Q}_p$ is a triple $(\widetilde{E}_0, \Gamma, \nu)$ where $\widetilde{E}_0/\mathbb{F}_p$ is an elliptic curve, Γ a subgroup of $\mathrm{Aut}_{\mathbb{F}_{p^s}}(\widetilde{E})$, and

$$\nu \colon \mathrm{Gal}(K/\mathbb{Q}_p) \longrightarrow \mathrm{Aut}_{\mathbb{F}_{p^s}}(\widetilde{E}) \rtimes \mathrm{Gal}(\mathbb{F}_{p^s}/\mathbb{F}_p)$$

is an antimorphism satisfying:

(1) $(\mathrm{pr} \circ \nu)(g) = g \bmod I(K/\mathbb{Q}_p)$ for all $g \in \mathrm{Gal}(K/\mathbb{Q}_p)$
(2) $\mathrm{Im}(\nu) = \Gamma \rtimes \langle \sigma \rangle$.

The Galois pair $(\widetilde{E}_0, \Gamma, \nu)$ is minimal if ν is injective and $\mathbb{F}_{p^s}$ is minimal with respect to Γ. We refer to [14, §3] for the properties of Galois pairs. A Galois pair functorially defines a $(\varphi, \mathrm{Gal}(K/\mathbb{Q}_p))$-module. Let us denote by D_0 (resp. D) the Dieudonné module of the p-divisible group associated to $\widetilde{E}_0$ (resp. $\widetilde{E}$). It is well known that D_0 is a 2-dimensional $\mathbb{Q}_p$-vector space with linear Frobenius φ_0 and $D = K_0 \otimes_{\mathbb{Q}_p} D_0$ is a 2-dimensional K_0-vector space with σ-semilinear Frobenius $\varphi = \sigma \otimes \varphi_0$. The following diagram

$$\begin{array}{ccc} \mathrm{Gal}(K/\mathbb{Q}_p) & \xrightarrow{\ \nu\ } & \mathrm{Aut}_{\mathbb{F}_{p^s}}(\widetilde{E}) \rtimes \mathrm{Gal}(\mathbb{F}_{p^s}/\mathbb{F}_p) \\ & \searrow^{\nu'} & \big\downarrow \text{Dieudonné functor} \\ & & \mathrm{Aut}_\varphi(D) \rtimes \mathrm{Gal}(K_0/\mathbb{Q}_p) \end{array}$$

defines, thanks to contravariancy of the Dieudonné functor, a group morphism ν' that induces the desired semilinear action.

Proposition 5.1. *Every (unfiltered) $(\varphi, \mathrm{Gal}(K/\mathbb{Q}_3))$-module appearing in Table 1.1 comes from a minimal Galois pair for $K/\mathbb{Q}_3$.*

Proof. We only treat the wild cases that are not quadratic twists, i.e. the cubic and dodecic ones. Let us denote $\widetilde{E} = \widetilde{E}_0 \times_{\mathbb{F}_3} \mathbb{F}_9$. The minimal Galois pairs are given in Table 5.1 below.

It is not hard to see that ν is injective and that the field of definition of Γ is minimal. Each of these objects gives rise to a $(\varphi, \mathrm{Gal}(K/\mathbb{Q}_3))$-module which is necessarily in our list by construction (they have the right Frobenius and Galois action). Except for the non abelian cubic case, there are

TABLE 5.1. Minimal Galois pairs for $e = 3$ and $e = 12$.

e	K	Tr.	Min. Gal. pair	(φ, G)-mod.
3	$L^{\mathrm{na}}(\zeta_4)$	0	$\widetilde{E}_0 : y^2 = x^3 + x$ $\Gamma = \langle \tau \rangle$, 3-Syl. of $\mathrm{Aut}_{\mathbb{F}_9}(\widetilde{E})$ $\nu : \tau_3 \mapsto \tau, \omega \mapsto f_\sigma$	$\mathbf{D}^{\mathrm{na}}_{\mathrm{pc}}(3; 0)$
	L^{a}	-3	$\widetilde{E}_0 : y^2 = x^3 - x + 1$ $\Gamma = \langle \tau \rangle$, 3-Syl. of $\mathrm{Aut}_{\mathbb{F}_3}(\widetilde{E}_0)$ $\nu : \tau_3 \mapsto \tau$	$\mathbf{D}^{\mathrm{a}}_{\mathrm{pc}}(3; -3)$
		0	$\widetilde{E}_0 : y^2 = x^3 - x$ $\Gamma = \langle \tau \rangle$, 3-Syl. of $\mathrm{Aut}_{\mathbb{F}_3}(\widetilde{E}_0)$ $\nu : \tau_3 \mapsto \tau$	$\mathbf{D}^{\mathrm{a}}_{\mathrm{pc}}(3; 0)$
		3	$\widetilde{E}_0 : y^2 = x^3 - x - 1$ $\Gamma = \langle \tau \rangle$, 3-Syl. of $\mathrm{Aut}_{\mathbb{F}_3}(\widetilde{E}_0)$ $\nu : \tau_3 \mapsto \tau$	$\mathbf{D}^{\mathrm{a}}_{\mathrm{pc}}(3; 3)$
12	K_i	0	$\widetilde{E}_0 : y^2 = x^3 + x$ $\Gamma = \mathrm{Aut}_{\mathbb{F}_9}(\widetilde{E})$ $\nu : \omega \mapsto f_\sigma$	$\mathbf{D}_{\mathrm{pc}}(12; 0; i; 1)$

always two isomorphisms classes of $(\varphi, \mathrm{Gal}(K/\mathbb{Q}_3))$-modules in our list (see Section 4). We have only checked that one of them comes from a Galois pair, but in fact both do since they are unramified quadratic twists of each other. As an illustration, we give an explicit construction of the obtained $(\varphi, \mathrm{Gal}(K/\mathbb{Q}_3))$-module for $e = 12$ and $K = K_i$ for some $i \in \{1, \ldots, 5\}$. Denote by D_0 (resp. D) the Dieudonné module of the 3-divisible group associated to $\widetilde{E}_0$ (resp. $\widetilde{E}$). Since $P_{\mathrm{char}}(\varphi_0) = P_{\mathrm{char}}(\mathrm{Frob}_{A_0})$, our φ-module D has the right trace $a_3 = 0$. The action of $\mathrm{Gal}(K/\mathbb{Q}_3)$ on D is obtained via the antimorphism ν:

$$\begin{array}{ccc} \mathrm{Gal}(K/\mathbb{Q}_3) & \overset{\nu}{\hookrightarrow} & \mathrm{Aut}_{\mathbb{F}_9}(\widetilde{E}) \rtimes \mathrm{Gal}(\mathbb{F}_9/\mathbb{F}_3) \\ & \underset{\nu'}{\searrow} & \big\downarrow \text{Dieudonné functor} \\ & & \mathrm{Aut}_\varphi(D) \rtimes \mathrm{Gal}(K_0/\mathbb{Q}_3) \end{array}$$

We have $\nu'(I(K/\mathbb{Q}_3)) \subseteq \mathrm{Aut}_\varphi(D)$, so that the action of $I(K/\mathbb{Q}_3) = \langle \tau_3 \rangle \rtimes \langle \tau_4 \rangle$ is K_0-linear, while the action of ω is σ-semilinear. It is then obvious to see that D is isomorphic to $\mathbf{D}_{\mathrm{pc}}(12; 0; i; \epsilon)$ as a $(\varphi, \mathrm{Gal}(K/\mathbb{Q}_3))$-module for some $\epsilon \in \{0, 1\}$. □

Remark 5.2. When $a_3(\widetilde{E}_0) = \pm 3$, a Galois pair for $K = L^{\mathrm{na}}(\zeta_4)/\mathbb{Q}_3$ is never minimal because $\mathrm{Aut}_{\mathbb{F}_9}(\widetilde{E})$ is too small compared to $\mathrm{Gal}(K/\mathbb{Q}_3)$. It is another way to see why those traces are absent from our list in that case.

5.2. A complete classification. To every 3-adic potentially crystalline representation V of $G_{\mathbb{Q}_3}$ corresponds an admissible filtered $(\varphi, \mathrm{Gal}(K/\mathbb{Q}_3))$-module $\mathbf{D}^*_{\mathrm{cris},K}(V)$. This association is functorial in a fully faithful way. In this section, we will show that every object described in Table 1.1 comes from an elliptic curve over $\mathbb{Q}_3$ with potential good reduction. It turns out that we can use the same tools and ingredients as M. Volkov in her treatment of the tame case (see [14]).

Theorem. *Let* $\mathbf{D}$ *be one of the filtered* $(\varphi, \mathrm{Gal}(K/\mathbb{Q}_3))$*-module in Table 1.1. There exists an elliptic curve* $E/\mathbb{Q}_3$ *such that* $\mathbf{D} \simeq \mathbf{D}^*_{\mathrm{cris},K}(V_3(E))$.

Proof. We sketch the proof, following the arguments of [14, Thm. 5.7]. The φ_0-module D_0 comes from an elliptic curve $\widetilde{E}_0/\mathbb{F}_3$ with the right Frobenius (via the Dieudonné module of its 3-divisible group). Let $\widetilde{E} = \widetilde{E}_0 \times_{\mathbb{F}_3} k$. Since $\mathbf{D} = \mathbf{D}^*_{\mathrm{cris},K}(V)$ for some crystalline representation V of $\mathrm{Gal}(\overline{\mathbb{Q}}_3/K)$ with Hodge–Tate weights $(0, 1)$, there exists a 3-divisible group $\mathcal{G}/\mathcal{O}_K$ lifting $\widetilde{E}(p)/k$ with Tate module isomorphic to V (see [1, Thm. 5.3.2]). By the Serre–Tate Theorem, the triple $(\mathcal{G}, \widetilde{E}, \widetilde{\mathcal{G}} \xrightarrow{\sim} \widetilde{E}(p))$ determines an elliptic curve E/K with good reduction (i.e. an elliptic scheme over $\mathcal{O}_K$) such that $V_3(E) \simeq V$. Finally, a minimal Galois pair $(\widetilde{E}_0, \Gamma, \nu)$ for $K/\mathbb{Q}_3$ (which always exists in the tame case by [14, Thm. 4.11] and in the wild case by Proposition 5.1) furnishes the necessary descent datum to obtain an elliptic curve $E_0/\mathbb{Q}_3$ such that $E = E_0 \times_{\mathbb{Q}_3} K$ and $V \simeq V_3(E_0)$. $\square$

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