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On Periodicity of Continued Fractions with Partial Quotients in Quadratic Number Fields

par Zhaonan WANG et Yingpu DENG

Résumé. Dans cet article, nous fixons un corps quadratique réel K et considérons une fraction continue ultimement périodique à quotients partiels dans $\mathcal{O}_K$. Nous étudions le problème de convergence et examinons l'augmentation de la taille des numérateurs et dénominateurs partiels des fractions convergentes. En outre, nous établissons des conditions nécessaires et suffisantes pour qu'un nombre irrationnel quartique réel admette un développement en fraction continue ultimement périodique à quotients partiels dans $\mathcal{O}_K$. Enfin, nous analysons l'exemple du corps $K = \mathbb{Q}(\sqrt{5})$. À partir des résultats obtenus, nous proposons un algorithme de développement en fraction continue des irrationnels quartiques réels ξ appartenant à une extension quadratique de K dont les conjugués algébriques sont tous réels. Nous démontrons que la fraction continue obtenue à partir de l'algorithme est ultimement périodique et converge vers ξ.

Abstract. In this paper, we fix a real quadratic field K and take an ultimately periodic continued fraction with partial quotients in $\mathcal{O}_K$. We examine the convergence of the sequence and the increase in the sizes of both the numerators and denominators of the convergent fractions. Additionally, we establish necessary and sufficient conditions for a real quartic irrational to possess an ultimately periodic continued fraction that converges to it, with partial quotients belonging to $\mathcal{O}_K$. Finally, we analyze a specific example with $K = \mathbb{Q}(\sqrt{5})$. By the obtained results, we give a continued fraction expansion algorithm for those real quartic irrationals ξ belonging to a quadratic extension of K whose algebraic conjugates are all real. We prove that the expansion obtained from the algorithm is ultimately periodic and converges to the specified ξ.

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1. Introduction

The history of simple continued fractions traces back to ancient Greece, and their study persists today due to their remarkable properties and applications, such as their relationship with quadratic irrationals and Diophantine approximation. In 1770, Lagrange introduced a well-known theorem in [10], which states that the simple continued fraction expansion of a real number is ultimately periodic if and only if it is a quadratic irrational. Also, the simple continued fraction of a given positive real number ξ gives the best rational approximation of ξ [11]. Inspired by classical continued fractions, we consider some extended forms of continued fraction expansions and attempt to investigate whether these new forms of continued fractions still possess similar desirable properties.

In 1977, Rosen firstly raised a problem in [15]: Can one devise a continued fraction that represents uniquely all real numbers, so that the finite continued fractions represent the elements of an algebraic number field? Conversely, can every element of the number field be represented by a finite continued fraction? He defined the λ-continued fractions for $\lambda = 2\cos(\frac{\pi}{q})$ with partial quotients multiples of λ. He proved that when $\lambda = \frac{\sqrt{5}+1}{2}$ (i.e. $q = 5$), the λ-continued fraction will satisfy the desired properties.

The β-numeration introduced by [13] and [14] generates the integral-base numeration system with the non-integral β-base. Let $\beta > 1$, one can expand any positive real x as $x = \sum_{k=-\infty}^{N} x_k \beta^k$ where $x_k \in \{0, 1, \ldots, \lfloor \beta \rfloor\}$. Among all the expansions of x, the greatest sequence for the lexicographical order is defined to be the β-expansion of x. The sum consisting of non-negative powers of the expansion of x is called the β-integral part of x, denoted by $[x]_\beta$. Also they defined the (positive) β-integers by those positive elements x such that $x = [x]_\beta$, denoted by $\mathbb{Z}_\beta^+$.

Based on the above definitions, it is natural to consider continued fraction expansions with partial quotients being β-integers. In 2006, Bernat defined such β-continued fraction expansion algorithm with $\beta = \frac{\sqrt{5}+1}{2}$ being the golden ratio in [1], and he showed that any element in $\mathbb{Q}(\beta)$ can be represented with finite expansions. In [12], Zuzana Masáková et.al. used a different approach from Bernat to consider more general continued fractions with partial quotients belonging to some discrete subset of the ring of algebraic integers of a real quadratic field K, hence Bernat's result is a special case when the discrete subset is chosen to be $\mathbb{Z}_\beta^+$. The authors showed that for all quadratic Perron numbers β, every element of $\mathbb{Q}(\beta)$ has a periodic or finite β-continued fraction expansion, and there exist four quadratic Perron numbers including $\beta = \frac{\sqrt{5}+1}{2}$ such that every element of $\mathbb{Q}(\beta)$ has a finite β-continued fraction expansion. Moreover, they have also proved that, for any quadratic real field K, if an element $\alpha \in K$ has a continued fraction expansion with all partial quotients $a_i \in \mathcal{O}_K \cap \mathbb{R}_{\geq 1}$ such that $|a'_n| \leq a_n$, where a'_n is the conjugate of a_n in K, then this expansion is finite or ultimately periodic. However, the β-continued fraction expansion, as defined, fails to preserve the properties, such as the Lagrange theorem. Specifically, the expansion of quartic irrationals in some quadratic extension of $\mathbb{Q}(\beta)$ may not be ultimately periodic.

In this paper, we assume that K is an arbitrary quadratic real field with a ring of algebraic integers $\mathcal{O}_K$, and all partial quotients belong to $\mathcal{O}_K$. In contrast to classic continued fractions and β-continued fractions, we do not impose a constraint on a_n to be greater than 1. It is worth noting that a_n may also take negative values.

The paper is organized as follows: Section 2 provides an overview of the properties of continued fractions and multiplicative Weil heights. In Section 3, we present some results on convergence of an ultimately periodic continued fraction expansion with partial quotients in $\mathcal{O}_K$, we also establish the necessary and sufficient condition for the periodicity of a continued fraction expansion converging to some element ξ, especially when ξ is a real quartic number. Section 4 presents an explicit example with $K = \mathbb{Q}(\beta) = \mathbb{Q}(\sqrt{5})$.We present an algorithm designed to guarantee that any real quartic irrational within a quadratic extension of K, possessing all real conjugates, exhibits an ultimately periodic continued fraction expansion with partial quotients belonging to $\mathcal{O}_K$.

2. Preliminaries

2.1. Continued fractions. Let $\{a_n\}$ be a sequence of real numbers, $a_i \neq 0$ for all $i > 0$. Define sequences $\{P_n\}$ and $\{Q_n\}$ as the $\mathcal{Q}$-pair associated to $\{a_n\}$ recursively as follows:

$$P_{-1} = 1,\ P_0 = a_0,\ P_{n+1} = a_{n+1}P_n + P_{n-1},\ (n \geq 0)$$

$$Q_{-1} = 0,\ Q_0 = 1,\ Q_{n+1} = a_{n+1}Q_n + Q_{n-1}.\ (n \geq 0)$$

Then we have

$$[a_0, a_1, \ldots, a_n] := \frac{P_n}{Q_n} = a_0 + \frac{1}{a_1 + \frac{1}{\cdots + \frac{1}{a_n}}}.$$

Alternatively, P_n and Q_n can be called the partial numerator and partial denominator, respectively. If $\{a_n\}$ is an infinite sequence with $a_n > 0$ for all $n \geq 1$, then $\frac{P_n}{Q_n}$ converges if and only if $\sum_{i=0}^{\infty} a_i$ approaches infinity, which is equivalent to the condition $\lim_{n\to+\infty} Q_n = +\infty$ [9], and we can write $[a_0, a_1, a_2, \ldots]$ for the limit $\lim_{n\to+\infty} \frac{P_n}{Q_n}$. If the expansion is convergent and takes ξ as the limit, denote $\xi := [a_0, a_1, a_2, \ldots]$, $\{a_n\}$ is called the sequence of partial quotients of ξ, and a_i is the i-th partial quotients of ξ.

Define the complete quotient $\xi_n = [a_n, a_{n+1}, \dots]$, we have

$$\xi = a_0 + \cfrac{1}{a_1 + \cfrac{1}{\dots + \cfrac{1}{a_n + \cfrac{1}{\xi_{n+1}}}}} = \frac{\xi_{n+1} P_n + P_{n-1}}{\xi_{n+1} Q_n + Q_{n-1}}.$$

Hence we obtain

$$\xi_{n+1} = -\frac{\xi Q_{n-1} - P_{n-1}}{\xi Q_n - P_n},$$

$$\xi - \frac{P_n}{Q_n} = \frac{\xi_{n+1} P_n + P_{n-1}}{\xi_{n+1} P_n + P_{n-1}} - \frac{P_n}{Q_n} = \frac{(-1)^n}{Q_n(\xi_{n+1} Q_n + Q_{n-1})}\ .$$

Proposition 2.1. *For all $n \geq 0$ we have the following properties:*

(1) $P_{n-1} Q_n - P_n Q_{n-1} = (-1)^n$,
(2) $\left|\xi - \frac{P_n}{Q_n}\right| = \frac{1}{|Q_n(\xi_{n+1} Q_n + Q_{n-1})|}$,
(3) $|Q_n \xi - P_n| = |\xi_1 \dots \xi_{n+1}|^{-1}$.

For proof one can see the first chapter in [9]. Note that if all $a_n \geq 1$ for all $n > 0$, then $Q_{n+1} > Q_n > 0$ and $a_{n+1} + 1 \geq \xi_{n+1} \geq a_{n+1}$ for all $n \geq 0$. Specifically,

$$\frac{1}{Q_n\left(Q_{n+1} + Q_n\right)} \leq \left|\xi - \frac{P_n}{Q_n}\right| = \frac{1}{Q_n\left(\xi_{n+1} Q_n + Q_{n-1}\right)} \leq \frac{1}{Q_n Q_{n+1}} \leq \frac{1}{a_{n+1} Q_n^2}.$$

Now let us examine the $\mathcal{Q}$-pair P_n and Q_n from the perspective of so-called continuants.

Define the multivariate polynomials by the recurrence

$$K_{-1} = 0,\ K_0 = 1,$$
$$K_n(t_1, \dots, t_n) = t_n K_{n-1}(t_1, \dots, t_{n-1}) + K_{n-2}(t_1, \dots, t_{n-2}).$$

It is evident, based on the provided definition, that K_n is a polynomial in $t_1, \dots, t_n$ with positive integer coefficients. Furthermore, each t_i appears in at least one monomial with a non-zero coefficient.

Therefore, we can represent P_n and Q_n as $P_n = K_{n+1}(a_0, \dots, a_n)$ and $Q_n = K_n(a_1, \dots, a_n)$.

The continuants possess further remarkable properties, we suggest referring to [5].

Here we present a proposition that is relevant to this paper, which can be proven through induction.

Proposition 2.2. *The polynomials $K_n(t_1, \dots, t_n)$ defined above satisfy the following properties:*

(1) $K_n(t_1, \dots, t_n) = K_n(t_n, \dots, t_1)$.

(2) *For all $j, l \geq 1$, we have*

$$K_{j+l}(t_1, \ldots, t_{j+l}) = K_j(t_1, \ldots, t_j) K_l(t_{j+1}, \ldots, t_{j+l}) + K_{j-1}(t_1, \ldots, t_{j-1}) K_{l-1}(t_{j+2}, \ldots, t_{j+l}).$$

2.2. Weil heights of algebraic numbers. In this section, we provide definitions and properties of the Weil height that are relevant to our research.

Definition 2.3 (Weil Height)**.** The absolute value $|\cdot|_v$ is the unique absolute value on $\mathbb{Q}(x)$ which restricts to the usual v-adic absolute value over $\mathbb{Q}$. Let also n and n_v be respectively the global and local degrees of x. Then the Weil height of x is defined as

$$H(x) := \prod_v \max\{1, |x|_v^{n_v}\}^{1/n},$$

for v running over all places of $\mathbb{Q}(x)$.

Remark 2.4. There are only finitely many factors in the infinite product of $H(x)$ that do not equal 1. Also, the Weil height of x is independent from the field we choose that contains x.

Next we introduce some useful properties of the Weil height.

Proposition 2.5. *Let r/s be a rational number, and let $\alpha \neq 0$ and $\beta \neq 0$ be elements of $\overline{\mathbb{Q}}^{\times}$. Then*

(1) $H(\alpha \pm \beta) \leq 2H(\alpha)H(\beta)$,
(2) $H(\alpha\beta) \leq H(\alpha)H(\beta)$,
(3) $H(\alpha^{r/s}) = |r/s|_\infty H(\alpha)$,
(4) $H(\sigma(\alpha)) = H(\alpha)$ *for all* $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$,
(5) $H(\alpha) = 1$ *if and only if α is a root of unity.*

Proposition 2.6 (Northcott's theorem on heights)**.** *The set*

$$\{\alpha \in \overline{\mathbb{Q}} \mid H(\alpha) \leq C,\ [\mathbb{Q}(\alpha) : \mathbb{Q}] \leq d\}$$

is finite for any positive constants C, d.

Proposition 2.7. *Let $\alpha \in \overline{\mathbb{Q}}$, $n = [\mathbb{Q}(\alpha) : \mathbb{Q}]$ and d be the leading coefficient of the minimal polynomial of α over $\mathbb{Z}$. Then*

$$H(\alpha)^n = |d| \prod_\sigma \max\{1, |\sigma(\alpha)|\},$$

where σ runs through all embeddings of $\mathbb{Q}(\alpha) \hookrightarrow \mathbb{C}$.

For a comprehensive introduction to the theory of heights and the proof of the listed properties, we refer the reader to the first two chapters of [2].

3. Continued fractions with partial quotients in quadratic fields

In this section, we assume K is a real quadratic number field with the ring of algebraic integers $\mathcal{O}_K$, and $[a_0, \ldots, a_N, \overline{a_{N+1}, \ldots, a_{N+k}}]$ is an ultimately periodic continued fraction expansion with partial quotients in $\mathcal{O}_K$. We make a convention that $Q_n \neq 0$ for all $n \geq 0$.

3.1. Convergence and periodicity. We firstly provide some notation and definitions related to formal continued fraction expansions and examine the convergence of this expansion.

For any $\alpha \in \mathbb{C}$, define the matrix $D(\alpha) := \left[\begin{smallmatrix} \alpha & 1 \\ 1 & 0 \end{smallmatrix}\right]$, and $t := D(0)$. Note that $D(\alpha)^{-1} = tD(-\alpha)t = \left[\begin{smallmatrix} 0 & 1 \\ 1 & -\alpha \end{smallmatrix}\right]$. Let $F = [a_1, a_2, a_3, \ldots, a_n]$ be a finite continued fraction. Define

$$M(F) = \begin{bmatrix} P_n & P_{n-1} \\ Q_n & Q_{n-1} \end{bmatrix} = \prod_{i=1}^{n} D(a_i) = D(a_1) D(a_2) \ldots D(a_n).$$

It is worth noting that this is precisely the standard matrix used to represent a continued fraction. One can observe that

$$\begin{aligned} M(F)^{-1} &= \prod_{i=0}^{n-1} tD(-a_{n-i})\, t = t \left[\prod_{i=0}^{n-1} D(-a_{n-i})\right] t \\ &= M([0, -a_n, \ldots, -a_1, 0]), \end{aligned}$$

and $\det M(F) = \det M(F)^{-1} = (-1)^n$.

Note that the periodic continued fraction $[b_1, \ldots, b_N, \overline{a_1, \ldots, a_k}]$ can be expressed as a purely periodic continued fraction

$$[\overline{b_1, \ldots, b_N, a_1, \ldots, a_k, 0, -b_N, \ldots, -b_1, 0}],$$

as the expression $[\ldots a, 0, b, \ldots]$ is equivalent to $[\ldots, a+b, \ldots]$, hence for an ultimately periodic continued fraction $P = [b_1, \ldots, b_N, \overline{a_1, \ldots, a_k}]$, we define

$$\begin{aligned} E(P) = \begin{bmatrix} E_{11}(P) & E_{12}(P) \\ E_{21}(P) & E_{22}(P) \end{bmatrix} &:= M([b_1, \ldots, b_N, a_1, \ldots, a_k, 0, -b_N, \ldots, -b_1, 0]) \\ &= D(b_1) \ldots D(b_N) D(a_1) \ldots D(a_k)\, tD(-b_N) \ldots D(-b_1)\, t \\ &= M([b_1, \ldots, b_N])\, M([a_1, \ldots, a_k])\, M([b_1, \ldots, b_N])^{-1}, \end{aligned}$$

It should be noted that the determinant of the matrix $E(P)$ is given by $(-1)^k$, which is solely determined by the periodic part.

Finally, we define the polynomial $f(P) = E_{21}(P)x^2 + (E_{22}(P) - E_{11}(P))x - E_{12}(P)$, and call it the polynomial associated to the periodic continued fraction expansion P. In instances where clarity is retained, we denote E_{ij} without explicitly referencing P.

In [3], the authors describe an algorithm to determine if a periodic continued fraction expansion $[a_0, a_1, \dots, a_n, \overline{a_{n+1}, \dots, a_{n+k}}]$ with partial quotients in rings $\mathcal{O}$ of S-integers in a number field is convergent, hence it is also applicable to our case where all $a_i \in \mathcal{O}_K$. Now we describe the main theorem and algorithm in their paper.

Let $A = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ with $ad - bc = \varepsilon = \pm 1$. Let $\lambda_\pm$ be the eigenvalues of A chosen so that $|\lambda_+| \geq 1 \geq |\lambda_-|$. If $A \neq \pm\sqrt{\varepsilon} I$ let

$$\gamma_\pm = \gamma_\pm(A) = \frac{\lambda_\pm - d}{c} = \frac{b}{\lambda_\pm - a} \left(= \frac{a - \lambda_\mp}{c} = \frac{b}{d - \lambda_\mp} \right),$$

where we take whichever expression is not the indeterminate 0/0.

Let **INEQ** be:

$$M([a_{j+1}, \dots, a_{k+j}])_{21} = 0 \text{ and } |M([a_{j+1}, \dots, a_{k+j}])_{22}| > 1 \quad \text{for some } 0 \leq j \leq k-1.$$

Lemma 3.1 ([3, Theorem 4.3]). *Let $P = [b_1, \dots, b_N, \overline{a_1, \dots, a_k}]$ be a periodic continued fraction expansion. The expansion converges if and only if* none *of the following three conditions is satisfied:*

(1) $E(P) = \pm i^k I$.
(2) ***INEQ** holds.*
(3) $\mathrm{Tr}(E(P))^2 \in \mathbb{R}$ *and* $0 \leq (-1)^k \,\mathrm{Tr}(E(P))^2 < 4$.

If it converges, then the limit $\widehat{\gamma}(P) = \gamma_+(E(P))$.

The following algorithm can be derived directly from the above lemma.

Algorithm 3.2. The value or limit of a periodic continued fraction:
Input: A periodic continued fraction $P = [b_1, \dots, b_N, \overline{a_1, \dots, a_k}]$.
Output: Either its value (or limit) $\widehat{\gamma}(P)$ or "doesn't exist".

(1) compute $E = E(P)$ from the above definition.
(2) if E is a multiple of the identity then print "doesn't exist" and break;
(3) compute the multi-set of roots $\{\gamma, \gamma^*\}$ of

$$f(P) = E_{21}x^2 + (E_{22} - E_{11})x - E_{12};$$

(4) if $\gamma = \gamma^*$, then $\widehat{\gamma}(P) := \gamma = \gamma^*$, return $\widehat{\gamma}(P)$, break;
(5) Without loss of generality assume $\gamma \neq \infty$ and compute $|E_{21}\gamma + E_{22}|$;
(6) if $|E_{21}\gamma + E_{22}| = 1$ then print "doesn't exist"; break;
(7) if $|E_{21}\gamma + E_{22}| > 1$ then $\widehat{\gamma}(P) := \gamma$, else $\widehat{\gamma}(P) := \gamma^*$;
(8) Test if **INEQ** is satisfied: if so, then print "doesn't exist"; break; else return $\widehat{\gamma}(P)$.

Lemma 3.3. *Assume $P = [a_0, \dots, a_N, \overline{a_{N+1}, \dots, a_{N+k}}]$ is an ultimately periodic continued fraction expansion, where all partial quotients $a_i \in \mathcal{O}_K$. If $f(P) = E_{21}x^2 + (E_{22} - E_{11})x - E_{12}$ has a positive determinant which is* *not a square in K, then the expansion is convergent. If the determinant is negative, the expansion does not converge.*

Proof. The proof is straightforward and can be derived directly from the steps of the above algorithm. If the determinant $\Delta(f(P))$ is positive and not a square in K, then $\{\gamma, \gamma^*\}$ are real numbers that are quadratic over K (hence being quartic irrationals). Since $E_{21}, E_{22} \in \mathcal{O}_K$, the absolute value $|E_{21}\gamma + E_{22}|$ must not be equal to 1. Additionally, since we have assumed that $\Delta(f(P))$ is not a square, then **INEQ** cannot hold, since $M([a_{j+1}, \ldots, a_{k+j}])_{21} \neq 0$ for any j. Therefore, we can find the limit of the continued fraction expansion by applying Algorithm 3.2.

In the case where the determinant is negative, we will obtain two complex roots γ, γ^* with identical absolute values. We observe that $|E_{21}\gamma + E_{22}| = \left|E_{21} \cdot \frac{E_{11}-E_{22}+\sqrt{(E_{22}-E_{11})^2+4E_{12}E_{21}}}{2E_{21}} + E_{22}\right|$. As the determinant is negative, we can simplify this expression as follows:

$$\begin{aligned} |E_{21}\gamma + E_{22}|^2 &= \left| \left(\frac{E_{11}+E_{22}}{2}\right)^2 - \frac{(E_{22}-E_{11})^2 + 4E_{12}E_{21}}{4} \right| \\ &= |E_{11}E_{22} - E_{12}E_{21}| = 1. \end{aligned}$$

Therefore, the continued fraction expansion does not converge due to Step 6 of the algorithm. ☐

Lemma 3.4. *If the ultimately periodic continued fraction expansion*

$$[a_0, \ldots, a_N, \overline{a_{N+1}, \ldots, a_{N+k}}]$$

with partial quotients in $\mathcal{O}_K$ converges to a real number ξ, then $|P_n|$, $|Q_n|$ tend to infinity.

Proof. We prove that $|Q_n| \to \infty$. Since $\frac{P_n}{Q_n}$ is convergent, we could also derive that $|P_n| \to \infty$.

As stated in Proposition 2.1, we have $P_{n-1}Q_n - P_nQ_{n-1} = (-1)^n$, hence $\frac{P_n}{Q_n} - \frac{P_{n-1}}{Q_{n-1}} = \frac{(-1)^{n-1}}{Q_{n-1}Q_n}$. Therefore, if the continued fraction expansion is convergent, this difference should approach 0, thus $|Q_{n-1}Q_n| \to \infty$. Hence, there exists no infinite subsequence Q_{n_i} characterized by bounded absolute values, and simultaneously, the index sequence $\{n_i\}$ will not contain an infinite number of adjacent integers.

Since the lemma is in fact equivalent to the statement that there does not exist an infinite subsequence $\{Q_{n_i}\}$ with bounded absolute values, we will prove it by contradiction, assuming that $\{Q_{n_i}\}$ is such a sequence.

Since the continued fraction expansion of ξ is ultimately periodic, we may assume that the value-bounded subsequence $\{Q_{n_i}\}$ satisfies the condition that a_{n_i} are identical for all indices n_i appearing in the subsequence, otherwise we can extract a proper subsequence from $\{Q_{n_i}\}$ to meet the condition. Based on the proceeding discussion, it can be observed that, for the bounded subsequence $\{Q_{n_i}\}$ described above, the absolute values of the subsequence $\{Q_{n_i-1}\}$ will tend towards infinity, hence the absolute values of $\left\{\frac{Q_{n_i}}{Q_{n_i-1}}\right\}$ approaches 0.

We have $Q_{n_i} = a_{n_i}Q_{n_i-1} + Q_{n_i-2}$, thus the ratio $\frac{Q_{n_i}}{Q_{n_i-1}}$ can be formally expressed as

$$\begin{aligned}\frac{Q_{n_i}}{Q_{n_i-1}} &= a_{n_i} + \frac{Q_{n_i-2}}{Q_{n_i-1}} \\ &= a_{n_i} + \frac{Q_{n_i-2}}{a_{n_i-1}Q_{n_i-2} + Q_{n_i-3}} \\ &= a_{n_i} + \frac{1}{a_{n_i-1} + \frac{Q_{n_i-3}}{Q_{n_i-2}}} \\ &= \cdots \\ &= a_{n_i} + \frac{1}{a_{n_i-1} + \frac{1}{a_{n_i-2}+\frac{1}{\cdots+\frac{1}{a_1}}}}.\end{aligned}$$

We denote this continued fraction as $[a_{n_i}, \ldots, a_1]$, which converges to 0 as n_i increases.

An important observation lies in the fact that, if $[\overline{\alpha_0, \ldots, \alpha_m}]$ represents a purely periodic continued fraction expansion, taking a real root of the polynomial $ax^2 + bx + c$ as the limit, then $[\overline{\alpha_m, \ldots, \alpha_0}]$ also converges. This can be substantiated by confirming that this expansion satisfies all the convergence criteria delineated in the previously outlined algorithm. In fact, if we assume that the final two convergents of $[\alpha_0, \ldots, \alpha_m]$ are $\frac{p_{m-1}}{q_{m-1}}$ and $\frac{p_m}{q_m}$, and those of $[\alpha_m, \ldots, \alpha_0]$ are $\frac{p'_{m-1}}{q'_{m-1}}$ and $\frac{p'_m}{q'_m}$, it can be deduced from Proposition 2.2 that

$$\begin{aligned} p'_m &= p_m, \qquad q'_m = p_{m-1}, \\ p'_{m-1} &= q_m, \quad q'_{m-1} = q_{m-1}. \end{aligned}$$

We have $a = q_m$, $b = -(p_m - q_{m-1})$, $c = -p_{m-1}$, thus the quadratic equation corresponding to $[\overline{\alpha_m, \ldots, \alpha_0}]$ is $cx^2 - bx + a$, which has the same discriminant as ax^2+bx+c. Therefore, according to Lemma 3.3, $[\overline{\alpha_m, \ldots, \alpha_0}]$ is convergent to one root of $cx^2 - bx + a = 0$.

Based on our previous assumptions, we may assume the periodic part of the expansion of ξ ends with the a_{n_i} chosen above. I.e., if we denote $a_{n_i} := b_k$, then ξ can be expressed as $[a_0, \ldots, a_{N'}, \overline{b_1, \ldots, b_k}]$ for some integer N' with $(b_1, \ldots, b_k)$ obtained by appropriately cyclically shifting

$(a_{N+1},\dots,a_{N+k})$ to ensure the following holds:

$$\frac{Q_{n_i}}{Q_{n_i-1}}=[b_k,\dots,b_1,b_k,\dots,b_1,\dots,a_{N'},\dots,a_1].$$

Denote $\alpha:=[a_{N'},\dots,a_1]$, $l_i:=\frac{n_i-N'}{k}$ (by above discussion, $l_i\in\mathbb{Z}_+$), and the last two convergents of $[\underbrace{b_k,\dots,b_1}_{1},\underbrace{b_k,\dots,b_1}_{2},\dots,\underbrace{b_k,\dots,b_1}_{l_i}]$ are $\frac{\gamma_{i,1}}{\delta_{i,1}}$ and $\frac{\gamma_{i,2}}{\delta_{i,2}}$, both of which converges to $[\overline{b_k,\dots,b_1}]$. Then we have

$$\frac{Q_{n_i}}{Q_{n_i-1}}=\frac{\alpha\gamma_{i,1}+\gamma_{i,2}}{\alpha\delta_{i,1}+\delta_{i,2}},$$

hence

$$\lim_{n_i\to\infty}\frac{Q_{n_i}}{Q_{n_i-1}}=[\overline{b_k,\dots,b_1}],$$

which cannot be zero. This leads to a contradiction to the initial assumption. □

Through a proof similar to that of the theorem, we can establish the following corollary.

Corollary 3.5. *If the ultimately periodic continued fraction expansion*

$$[a_0,\dots,a_N,\overline{a_{N+1},\dots,a_{N+k}}]$$

with partial quotients in $\mathcal{O}_K$ converges to some real number ξ, then there does not exist a subsequence $\{Q_{n_i}\}$ or $\{P_{n_i}\}$ such that $\left|\frac{Q_{n_i}}{Q_{n_i-1}}\right|$ or $\left|\frac{P_{n_i}}{P_{n_i-1}}\right|$ tends towards either infinity or 0. Therefore, $\left|\frac{Q_n}{Q_{n-1}}\right|$ and $\left|\frac{P_n}{P_{n-1}}\right|$ are upper bounded and have a lower bound larger than 0.

Now we consider a continued fraction expansion

$$P=[a_0,\dots,a_N,\overline{a_{N+1},\dots,a_{N+k}}],$$

which is ultimately periodic with all partial quotients $a_i\in\mathcal{O}_K$. Assume the discriminant of $f(P)$ is positive, then by Lemma 3.3, the expansion converges to some real number ξ_P. We may assume $\Delta(f(P))$ is not a real square in K, i.e. ξ_P is a quartic irrational, otherwise ξ_P will be a quadratic irrational, and by the well-known Lagrange's theorem it can be expanded into an ultimately periodic continued fraction with $a_i\in\mathbb{Z}_+$ for all $i\geq 1$. We call it a continued fraction expansion of ξ_P in K.

Notice that all complete quotients ξ_n lie in the same quadratic extension L over K, where $L=K(\sqrt{\Delta(f(P))})$. Hence this is a necessary condition for an element ξ to admit a periodic expansion over K. Next we assume ξ is a real quartic irrational, lying in a quadratic extension of K.

Denote $\mathrm{Gal}(K/\mathbb{Q})=\{1,\sigma\}$, where for all $\alpha\in K$, σ maps α to its conjugate $\overline{\alpha}$, which is a real number. Assume a real quartic irrational ξ has minimal polynomial $f(x) = Ax^2 + Bx + C$ over K, where $A, B, C \in \mathcal{O}_K$, and the discriminant $B^2 - 4AC > 0$ is not a square in $\mathcal{O}_K$. We can describe the periodicity of a continued fraction expansion alternatively through the finite existence of the complete quotients ξ_n. Since all $\xi_n \in L \setminus K$, where $L = K(\sqrt{B^2 - 4AC})$, we can apply the Northcott theorem in Proposition 2.6. Therefore, we can deduce that if a continued fraction expansion $[a_0, \dots, a_n, \dots]$ converges to ξ, then it is ultimately periodic if and only if there exists an upper bound for all $H(\xi_{n+1})$. Here for convenience we use the shifted index $n+1$. With the formula mentioned in Proposition 2.7 we have

$$H(\xi_{n+1})^4 = d \prod_{\phi:\text{embeddings of } L \hookrightarrow \mathbb{C}} \max\{1, |\phi(\xi_{n+1})|\},$$

where d is the leading coefficient of the minimal polynomial of ξ_{n+1} over $\mathbb{Z}$.

Now we calculate the minimal polynomial f_{n+1} of all complete quotients ξ_{n+1} over $\mathcal{O}_K$. It is worth noting that the calculation is analogous to that in the classical continued fraction case, which can be found in [9]. Hence we have

$$f_{n+1}(x) = A_{n+1}x^2 + B_{n+1}x + C_{n+1},$$

where

$$\begin{cases} A_{n+1} = AP_n^2 + BP_nQ_n + CQ_n^2, \\ B_{n+1} = 2AP_nP_{n-1} + B(P_nQ_{n-1} + P_{n-1}Q_n) + 2CQ_nQ_{n-1}, \\ C_{n+1} = AP_{n-1}^2 + BP_{n-1}Q_{n-1} + CQ_{n-1}^2 = A_n. \end{cases}$$

Then the minimal polynomial of ξ_{n+1} over $\mathbb{Z}$ is $f_{n+1}(x) \cdot \sigma(f_{n+1}(x))$, and the leading coefficient of the minimal polynomial of ξ_n over $\mathbb{Z}$ is

$$A_{n+1} \cdot \sigma(A_{n+1}).$$

We can proceed to compute the Weil height of ξ_{n+1}:

$$\begin{aligned} H(\xi_{n+1})^4 &= |A_{n+1} \cdot \sigma(A_{n+1})| \prod_{\phi} \max\{1, |\phi(\xi_{n+1})|\} \\ &= |A_{n+1} \cdot \sigma(A_{n+1})| \prod_{\phi} \max\left\{1, \left|\phi\left(\frac{\xi Q_{n-1} - P_{n-1}}{\xi Q_n - P_n}\right)\right|\right\}. \end{aligned}$$

Multiply then divide the term $\prod_\phi |\phi(\xi Q_n - P_n)|$, and we obtain

$$H(\xi_{n+1})^4 = \frac{|A_{n+1} \cdot \sigma(A_{n+1})|}{\prod_\phi |\phi(\xi Q_n - P_n)|} \prod_{\phi} \max\{|\phi(\xi Q_{n-1} - P_{n-1})|, |\phi(\xi Q_n - P_n)|\}.$$

We define $S_n = \xi Q_n - P_n$. Since ξ satisfies the quadratic equation $Ax^2 + Bx + C = 0$ over $\mathcal{O}_K$, S_n is the root of

$$A\left(\frac{x + P_n}{Q_n}\right)^2 + B\left(\frac{x + P_n}{Q_n}\right) + C = 0.$$

Simplifying this expression yields $Ax^2 + (BQ_n + 2AP_n)x + A_{n+1} = 0$, thus we have

$$\prod_{\phi} |\phi(S_n)| = \frac{|A_{n+1} \cdot \sigma(A_{n+1})|}{|A \cdot \sigma(A)|},$$

and

$$H(\xi_{n+1})^4 = |A \cdot \sigma(A)| \prod_{\phi} \max\{|\phi(S_n)|, |\phi(S_{n-1})|\}.$$

Based on the previous discussion, we can explicitly express all ϕ. Since ϕ maps ξ to one of the roots of the equation

$$(Ax^2 + Bx + C) \cdot (\sigma(A)x^2 + \sigma(B)x + \sigma(C)) = 0,$$

we denote the maps

$$\begin{aligned} \tau_1 &: \sqrt{B^2 - 4AC} \longrightarrow -\sqrt{B^2 - 4AC}, \\ \tau_2 &: \sqrt{B^2 - 4AC} \longrightarrow \sqrt{\sigma(B^2 - 4AC)}, \\ \tau_3 &: \sqrt{B^2 - 4AC} \longrightarrow -\sqrt{\sigma(B^2 - 4AC)}, \end{aligned}$$

Then the embeddings of $L \hookrightarrow \mathbb{C}$ are $\{1, \tau_1, \sigma\tau_2, \sigma\tau_3\}$.

We give the following lemma to show that, for an ultimately periodic continued fraction expansion, both

$$\prod_{\phi \in \{1, \tau_1\}} \max\{|\phi(S_n)|, |\phi(S_{n-1})|\}$$

and

$$\prod_{\phi \in \{\tau_2\sigma, \tau_3\sigma\}} \max\{|\phi(S_n)|, |\phi(S_{n-1})|\}$$

should be bounded.

Lemma 3.6. *If an ultimately periodic continued fraction expansion* $P = [a_0, \ldots, a_N, \overline{a_{N+1}, \ldots, a_{N+k}}]$ *converges to a real quartic irrational with all* $a_i \in \mathcal{O}_K$*, then*

$$\prod_{\phi \in G_1} \max\{|\phi(S_n)|, |\phi(S_{n-1})|\} \text{ and } \prod_{\phi \in G_2} \max\{|\phi(S_n)|, |\phi(S_{n-1})|\}$$

are bounded for all n*, where* $G_1 = \{1, \tau_1\}$*,* $G_2 = \{\tau_2\sigma, \tau_3\sigma\}$*.*

Proof. For convenience we denote $F_i(n) = \prod_{\phi \in G_i} \max\{|\phi(S_n)|, |\phi(S_{n-1})|\}$. It can be seen that $F_1(n) \cdot F_2(n)$ are bounded for all n, since $F_1(n) \cdot F_2(n) = H(\xi_{n+1})^4$, and the assumption of periodicity results in only finitely many ξ_n, hence establishing the boundedness of the Weil height and consequently, the boundedness of $F_1(n)\cdot F_2(n)$. Therefore, to prove the theorem, it suffices to demonstrate that $F_1(n)$ is upper and lower bounded for all n.

$$F_1(n) = \max\{|\xi Q_n - P_n|, |\xi Q_{n-1} - P_{n-1}|\} \cdot \max\{|\tau_1(\xi) Q_n - P_n|, |\tau_1(\xi) Q_{n-1} - P_{n-1}|\}.$$

Again by Proposition 2.1 we have $|\xi Q_n - P_n| = \frac{1}{|\xi_{n+1} Q_n + Q_{n-1}|}$, hence

$$|Q_n| \cdot |\xi Q_n - P_n| = \frac{1}{|\xi_{n+1} + \frac{Q_{n-1}}{Q_n}|}.$$

Next we show that $\frac{1}{|\xi_{n+1} + \frac{Q_{n-1}}{Q_n}|}$ is bounded if the continued fraction expansion is periodic. As stated in Corollary 3.5, it is lower bounded, hence it suffices to show that there is no subsequence of $\{|\xi_{n+1} + \frac{Q_{n-1}}{Q_n}|\}$ that tends to 0. We still prove this by contradiction. Suppose there exists such a subsequence $\{|\xi_{n_i+1} + \frac{Q_{n_i-1}}{Q_{n_i}}|\}$, we may assume that all a_{n_i} are identical, and the period length of the expansion is k. From the proof of Lemma 3.4, we have

$$\lim_{n_i \to \infty} \frac{Q_{n_i}}{Q_{n_i-1}} = [\overline{a_{n_i}, a_{n_i-1}, \dots, a_{n_i-k+1}}],$$

and

$$\xi_{n_i+1} = [\overline{a_{n_i+1}, \dots, a_{n_i+k}}].$$

Assume the limit of $[\overline{a_{n_i}, a_{n_i-1}, \dots, a_{n_i-k+1}}]$ is γ, we firstly prove that, $-\frac{1}{\gamma} \neq \xi_{n_i+1}$.

Let

$$P_1 = [\overline{a_{n_i+1}, \dots, a_{n_i+k}}] = [\overline{a_{n_i+k-1}, \dots, a_{n_i}}],\ P_2 = [\overline{a_{n_i}, a_{n_i-1}, \dots, a_{n_i-k+1}}].$$

Since P_1 and P_2 are purely periodic, we have $E(P_i) = M(P_i)$. According to the discussion in the proof of Lemma 3.4, if we assume

$$E(P_1) = M(P_1) = \begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix},$$

then

$$E(P_2) = M(P_2) = E(P_1)^T = \begin{bmatrix} E_{11} & E_{21} \\ E_{12} & E_{22} \end{bmatrix}.$$

Since ξ_{n_i+1} is the limit of P_1, by Algorithm 3.2, it is a root of $f(P_1)$ which satisfies $|E_{21}\xi_{n_i+1} + E_{22}| > 1$. Denote the other root of $f(P_1)$ by $\xi^*_{n_i+1}$, then $|E_{21}\xi^*_{n_i+1} + E_{22}| < 1$. Without loss of generality, we may assume

$$\xi_{n_i+1} = \frac{E_{11} - E_{22} + \sqrt{(E_{11} - E_{22})^2 + 4E_{21}E_{12}}}{2E_{21}},$$

and the other case can be proven in a similar way. Then

$$-\frac{1}{\xi_{n_i+1}}=\frac{E_{11}-E_{22}-\sqrt{(E_{11}-E_{22})^2+4E_{21}E_{12}}}{2E_{12}},$$

and

$$\begin{aligned}\left|-\frac{1}{\xi_{n_i+1}}E_{12}+E_{22}\right|&=\left|\frac{E_{11}+E_{22}+\sqrt{(E_{11}-E_{22})^2+4E_{21}E_{12}}}{2}\right|\\&=|\xi^*_{n_i+1}E_{21}+E_{22}|<1.\end{aligned}$$

Therefore, $\gamma\neq-\frac{1}{\xi_{n_i+1}}$, since $|\gamma E_{12}+E_{22}|>1$.

This claim allows us to assume $\frac{1}{\left|\xi_{n+1}+\frac{Q_{n-1}}{Q_n}\right|}\in[t_1,t_2]$, and $\left|\frac{Q_n}{Q_{n-1}}\right|\in[t_3,t_4]$ by Corollary 3.5, where both $t_1,t_3\geq 0$. Therefore, for any $\epsilon>0$, there exists an $n_\epsilon\in\mathbb{N}_+$ such that for all $n\geq n_\epsilon$,

$$\begin{aligned}F_1(n)&\leq\max\{|\xi Q_n-P_n|,|\xi Q_{n-1}-P_{n-1}|\}\cdot\max\{|Q_n|,|Q_{n-1}|\}\\&\quad\cdot\max\left\{\left|\tau_1(\xi)-\frac{P_n}{Q_n}\right|,\left|\tau_1(\xi)-\frac{P_{n-1}}{Q_{n-1}}\right|\right\}\\&\leq t_2t_4(|\tau_1(\xi)-\xi|+\epsilon),\end{aligned}$$

and similarly

$$F_1(n)\geq t_1t_3(|\tau_1(\xi)-\xi|-\epsilon).$$

Hence, we can select ϵ to be sufficiently small, ensuring that $|F_1(n)|$ is bounded from above and below, consequently proving the lemma. □

The following theorem will give a sufficient and necessary condition for the periodicity of a continued fraction expansion over K.

Theorem 3.7. *Assume a continued fraction expansion $[a_0,\dots,a_n,\dots]$ with all partial quotients in $\mathcal{O}_K$ converges to a real quartic irrational ξ, which is quadratic over K, and the minimal polynomial of ξ over K is Ax^2+Bx+C, then it is ultimately periodic if and only if $|A_n|$ and $|\sigma(A_n)|$ are bounded for all n, where $A_n=AP_n^2+BP_nQ_n+CQ_n^2$ as defined before.*

Proof. The necessity of the proof is straightforward, since

$$F_1(n)\geq|(\xi Q_n-P_n)\cdot(\tau_1(\xi)Q_n-P_n)|=|A_n|,$$

and

$$F_2(n)\geq|(\tau_2\sigma(\xi)\sigma(Q_n)-\sigma(P_n))\cdot(\tau_3\sigma(\xi)\sigma(Q_n)-\sigma(P_n))|=|\sigma(A_n)|,$$

and $|F_i(n)|$ are bounded by the Lemma 3.6.

To prove the sufficiency, we recall the definition of naive height. Let α be an algebraic number of degree d, if the minimal polynomial of α has integer coefficients $c_0, \ldots, c_d$, then the naive height of α is defined as

$$h(\alpha) = \frac{\max\{|c_i| : i = 0, \ldots, d\}}{\gcd(c_0, \ldots, c_d)}.$$

The connection between naive height and multiplicative Weil height concerning algebraic integers is well-established. Specifically, the naive height and Weil height are equivalent for $\alpha \in \mathbb{Q}$. For $\alpha \in \overline{\mathbb{Q}}$ with degree d, it can be observed that:

$$H(\alpha) \leq (d+1)^{1/2} h(\alpha) \quad \text{and} \quad h(\alpha) \leq 2^d H(\alpha).$$

Details about the naive height can be found in [6]. As previously mentioned, periodicity implies the finite existence of ξ_n, initially established by the boundedness of the Weil height. This observation suggests an alternative proof of this finiteness through the boundedness on the naive height. Furthermore, one can demonstrate the finiteness by the boundedness of $\max\{|c_i| : i = 0, \ldots, d\}$.

Recall that all ξ_n are quartic irrationals with the minimal polynomial over $\mathbb{Z}$ $f_n(x) \cdot \sigma(f_n)(x)$, where

$$f_n(x) = A_n x^2 + B_n x + C_n,$$

and $C_n = A_{n-1}$. Here $\sigma(f)$ denotes the action of σ on every coefficient of the polynomial f. The discriminant of f_n is equal to

$$B_n^2 - 4A_nC_n = (P_{n-1}Q_{n-2} - Q_{n-1}P_{n-2})^2 (B^2 - 4AC) = B^2 - 4AC,$$

hence the discriminant of $\sigma(f_n)$ is $\sigma(B^2 - 4AC)$, both of which remain constant while verifying n. Hence assuming $|A_n|$ and $|\sigma(A_n)|$ are bounded, we can obtain

$$B_n^2 \leq (B^2 - 4AC) + 4|A_{n-1}A_n|$$

and

$$\sigma(B_n)^2 \leq |\sigma(B^2 - 4AC)| + 4|\sigma(A_{n-1})\sigma(A_n)|.$$

As a result, both polynomials $f_n(x)$ and $\sigma(f_n)(x)$ has bounded absolute values of the coefficients, which suggests that the absolute values of the coefficients of the polynomial product are bounded. □

Corollary 3.8. *With all the assumptions in Theorem 3.7, if* $\sigma(B^2 - 4AC) < 0$ *and the continued fraction expansion is ultimately periodic, then* $|\sigma(P_n)|$ *and* $|\sigma(Q_n)|$ *are bounded for all* n.

Proof. If $\sigma(B^2 - 4AC) < 0$, then

$$\sigma(A_{n+1}) = \sigma(A) \left[\sigma\left(P_n - \frac{B}{2A}Q_n\right)^2 + \sigma\left(AC - \frac{B^2}{4}\right) \cdot \sigma(Q_n)^2 \right],$$

and the absolute value is bounded if and only if $|\sigma(Q_n)|$, $|\sigma(P_n - \frac{B}{2A}Q_n))|$ are bounded. Consequently, if the expansion is ultimately periodic, both $|\sigma(P_n)|$ and $|\sigma(Q_n)|$ are bounded. □

Corollary 3.9. *Under all the assumptions specified in Corollary 3.8, assume* $\xi = [a_1, \dots, a_N, \overline{a_{N+1}, \dots, a_{N+k}}]$, *then the period length* k *is even, and* $|\sigma(B^2 - 4AC)| < 4$. *Conversely, if some* ξ *has the minimal polynomial* $Ax^2 + Bx + C = 0$ *over* $\mathcal{O}_K$ *with* $\sigma(B^2 - 4AC) < -4$, *then* ξ *does not have any periodic continued fraction expansions in* K.

Proof. Denote this periodic expansion converging to ξ by P. From Algorithm 3.2, ξ is the root of

$$f(P) = E_{21}(P)x^2 + (E_{22}(P) - E_{11}(P))x - E_{12}(P) = 0,$$

and the discriminant of $f(P)$ is

$$(E_{22}(P) - E_{11}(P))^2 + 4E_{21}(P)E_{12}(P) = (E_{22}(P) + E_{11}(P))^2 + 4(-1)^{k-1}.$$

Therefore, if k is odd, then $\sigma(\Delta(f(P))) = \sigma(B^2 - 4AC) > 0$, contradicting the assumption. Moreover, if $\sigma(B^2 - 4AC) < -4$, then $B^2 - 4AC$ cannot match any $\Delta(f(P))$. □

Next, we provide a necessary and sufficient condition for ξ to possess an ultimately periodic continued fraction expansion in K, where all the algebraic conjugates of ξ are real. This theorem essentially restates Theorem 3.7, shedding light on the property of continued fractions as a method for approximating real numbers.

Theorem 3.10. *With all the assumptions in Theorem 3.7, if* $\sigma(B^2 - 4AC) > 0$ *and the continued fraction expansion is ultimately periodic, then* $|\xi Q_n - P_n|$ *and* $|\xi' \sigma(Q_n) - \sigma(P_n)|$ *are bounded. Moreover, there exist positive constants* M_1, M_2, *and a (real) root* ξ' *of the equation* $\sigma(A)x^2 + \sigma(B)x + \sigma(C) = 0$, *such that*

$$|\xi Q_n - P_n| \le \frac{M_1}{|Q_n|},$$
$$|\xi' \sigma(Q_n) - \sigma(P_n)| \le \frac{M_2}{|\sigma(Q_n)|}.$$

The two conditions are also sufficient for a continued fraction expansion of such a ξ *to be ultimately periodic.*

Proof. Denote the ultimately periodic continued fraction $[a_0, a_1, \dots]$ which converges to ξ by P, then $\sigma(P) = [\sigma(a_0), \sigma(a_1), \dots]$ is also an ultimately periodic continued fraction. Since $\Delta(f(P)) = B^2 - 4AC$, it follows that $\Delta(f(\sigma(P))) = \Delta(\sigma(f(P))) = \sigma(B^2 - 4AC) > 0$. Lemma 3.3 implies that it converges to a root of $f(\sigma(P)) = \sigma(A)x^2 + \sigma(B)x + \sigma(C)$. According to the proof of Lemma 3.6, it is evident that for any periodic continued fraction converging to some α with corresponding $\mathcal{Q}$-pairs $\{p_n\}, \{q_n\}$,

$$|q_n| \cdot |\alpha q_n - p_n| = \frac{1}{\left|\alpha_{n+1} + \frac{q_{n-1}}{q_n}\right|}$$

is bounded, hence it holds for the expansion P and $\sigma(P)$.

To prove the sufficiency, we firstly prove the boundedness of $|A_n|$. By the assumption we have

$$|\xi Q_n - P_n| \leq \frac{M_1}{|Q_n|},$$

hence we may assume

$$P_n = \xi Q_n + \frac{M_1 \delta_n}{Q_n}$$

for some δ_n such that $|\delta_n| < 1$, and

$$\begin{aligned} A_{n+1} &= A\left(\xi Q_n + \frac{M_1\delta_n}{Q_n}\right)^2 + B \cdot \left(\xi Q_n + \frac{M_1\delta_n}{Q_n}\right) \cdot Q_n + C \cdot Q_n^2 \\ &= (A\xi^2 + B\xi + C) \cdot Q_n^2 + (2A\xi + B) \cdot M_1\delta_n + A\left(\frac{M_1\delta_n}{Q_n}\right)^2. \end{aligned}$$

Since ξ is the root of $Ax^2 + Bx + C = 0$, it follows that

$$A_{n+1} = (2A\xi + B) \cdot M_1\delta_n + A\left(\frac{M_1\delta_n}{Q_n}\right)^2.$$

We examine the two cases where $|Q_n| < 1$ and $|Q_n| \geq 1$ separately. Starting with $|Q_n| \geq 1$, we have

$$|A_{n+1}| \leq (|2A\xi| + |B|) \cdot M_1 + |A|M_1^2.$$

On the other hand, when $|Q_n| < 1$, we observe that the assumption of the boundedness of $|\xi Q_n - P_n|$ implies the boundedness of $|P_n|$. Consequently, $|A_{n+1}|$ is bounded in both cases. Similarly establishing the boundedness of $|\sigma(A_{n+1})|$, the condition stated in Theorem 3.7 is satisfied, thereby confirming the periodicity of the continued fraction expansion. $\square$

Remark 3.11. Indeed, we can obtain the stronger result that $|\xi Q_n - P_n|$ and $|\xi'\sigma(Q_n) - \sigma(P_n)|$ approach 0, given that $|Q_n|$ and $|\sigma(Q_n)|$ tend to infinity, assuming periodicity as per Lemma 3.4. Nonetheless, their boundedness suffices to demonstrate the periodicity of the continued fraction expansion.

3.2. Constraints on the partial quotients. In this subsection, we establish some conclusions about periodicity by imposing additional constraints on the partial quotients.

As demonstrated in previous discussions, the ambiguity introduced by the undetermined signs of the partial quotients significantly complicates the analysis of periodicity. However, if we enforce the condition that $a_n \in \mathcal{O}_K \cap \mathbb{R}_{\geq 1}$, as seen in certain existing algorithms (e.g. β-continued fractions for some $\beta > 1$), we can infer from the remark of Proposition 2.1 that

$$|S_{n-1}| = |\xi Q_{n-1} - P_{n-1}| > |\xi Q_n - P_n| = |S_n|,$$

and

$$\frac{1}{Q_n + Q_{n+1}} \leq |S_n| \leq \frac{1}{Q_{n+1}}.$$

Therefore,

$$\frac{1}{2Q_n} \leq \frac{1}{Q_n + Q_{n-1}} \leq |S_{n-1}| \leq \frac{1}{Q_n}.$$

For $\phi = \tau_1$, it can be seen that

$$|\phi(S_n)| = |S_n + (\tau_1(\xi) - \xi)\, Q_n|\,.$$

Since $Q_n \to \infty$ and $S_n \to 0$, for sufficiently large n, $|\tau_1(S_n)|$ and $|(\tau_1(\xi) - \xi)Q_n|$ are of similar magnitudes. Thus there exists an integer $N > 0$ such that for all $n \geq N$,

$$|\tau_1(S_n)| > |\tau_1(S_{n-1})|\,,$$

$$(\tau_1(\xi) - \xi)\, Q_n - \frac{1}{Q_n} \leq |\tau_1(S_n)| \leq (\tau_1(\xi) - \xi)\, Q_n + \frac{1}{Q_n}.$$

Now for $n \geq N$ with N chosen as above, it can be observed that

$$F_1(n) = \prod_{\phi \in \{1,\tau_1\}} \max\{|\phi(S_n)|\,, |\phi(S_{n-1})|\} = |S_{n-1}| \cdot |\tau_1(S_n)|$$

$$\in \left[\frac{\tau_1(\xi) - \xi}{2} - \frac{1}{Q_n^2}, \tau_1(\xi) - \xi + \frac{1}{Q_n^2}\right],$$

which has both upper and lower bounds. Consequently, the periodicity of the expansion is equivalent to the boundedness of $F_2(n)$.

Theorem 3.12. *Assume that a continued fraction expansion* $[a_0, a_1, \dots]$ *converges to a real quartic irrational* ξ*, where all partial quotients* a_n *are chosen such that* $a_n \geq 1, \sigma(a_n) > 0$ *for all* $n \geq 1$*, then:*

(1) *If all algebraic conjugates of* ξ *are real numbers, then the continued fraction expansion of* ξ *is ultimately periodic if and only if there exists a lower bound for all* $\sigma(a_n)$ *when* $n \geq 1$ *and* $\frac{\sigma(P_n)}{\sigma(Q_n)}$ *converges to one of the roots of the equation* $\sigma(A)x^2 + \sigma(B)x + \sigma(C) = 0$.

(2) *If* $\sigma(B^2 - 4AC) < 0$*, then* ξ *does not have periodic continued fraction expansion in* K.

Proof. The proof of this theorem is straightforward. As for the first claim, the "if" part follows naturally from Theorem 3.10. Now we assume that there exists $\alpha > 0$ such that $\sigma(a_n) \geq \alpha > 0$ for all $n \geq 1$, and $\frac{\sigma(P_n)}{\sigma(Q_n)}$ converges to one of the roots of $\sigma(A)x^2 + \sigma(B)x + \sigma(C) = 0$, denoted by ξ'.

By Proposition 2.1 we have

$$|\xi' \sigma(Q_n) - \sigma(P_n)| = \frac{1}{\xi'_{n+1}\sigma(Q_n) + \sigma(Q_{n-1})} \leq \frac{1}{\sigma(Q_{n+1})} \leq \frac{1}{\sigma(a_{n+1})\sigma(Q_n)}.$$

Therefore, we can set $M_2 = \frac{1}{\alpha}$ to demonstrate the boundedness of $F_2(n)$ and apply Theorem 3.10 again.

Regarding the second assertion, Khinchin demonstrated in [9] that, if a continued fraction expansion $[a_0, a_1, \dots]$ is convergent and all $a_i \geq 0$, then $\sum_{i=1}^{\infty} a_i$ diverges. Therefore, the ultimately periodic expansions $[a_0, a_1, \dots]$ with $\sigma(a_n) > 0$ will lead to the divergence of $\sum_{n=1}^{\infty} \sigma(a_n)$ by the convergence of $[\sigma(a_0), \sigma(a_1), \dots]$. Consequently, a subsequence in $\sigma(Q_n)$ with values approaching infinity will be obtained, which contradicts Corollary 3.8. □

Corollary 3.13. *Assume that the partial quotients are chosen such that* $\sigma(a_n) > 0$ *for all* $n \geq 1$*. If no roots of the equation* $\sigma(A)x^2+\sigma(B)x+\sigma(C) = 0$ *fall within the interval* $\left(\sigma(a_0), \sigma(a_0) + \frac{1}{\sigma(a_1)}\right)$*, then the continued fraction expansion of* ξ *over* K *starting with* $[\sigma(a_0), \sigma(a_1)]$ *will not be ultimately periodic.*

Proof. Given that all $\sigma(a_i)$ are positive, we can conclude that the limit of any convergent continued fraction expansion, $[\sigma(a_0), \sigma(a_1), \dots]$, falls within the interval $(\sigma(a_0), \sigma(a_0) + \frac{1}{\sigma(a_1)})$. Hence the statement can be proved directly using Theorem 3.12. □

4. K=$\mathbb{Q}(\sqrt{5})$: An Explicit Example

In this section, we take $\beta = \frac{\sqrt{5}+1}{2}$, $K = \mathbb{Q}(\beta) = \mathbb{Q}(\sqrt{5})$, then $\mathcal{O}_K = \mathbb{Z}[\beta]$. The Galois group $\mathrm{Gal}(K/\mathbb{Q}) = \{1, \sigma\}$, where σ maps β to its algebraic conjugate $-\frac{1}{\beta}$. We will present a continued fraction expansion algorithm such that for a given element ξ, which lies in the quadratic extension of K with four real algebraic conjugates, the algorithm generates a continued fraction expansion with partial quotients in $\mathcal{O}_K$ that converges to ξ. Moreover, this continued fraction expansion of ξ will be ultimately periodic.

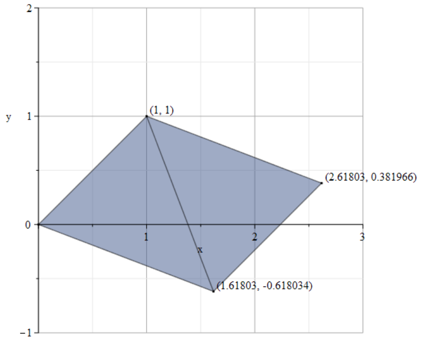

FIGURE 4.1. The fundamental parallelepiped of $v(\mathcal{O}_K)$

Now we consider the function $v : K \to \mathbb{R}^2$ defined by

$$v(x) = (x, \sigma(x)),$$

it is clear that v is an additive map. Recall that an abelian subgroup $\mathcal{L}$ of a real vector space V is called a lattice if $\mathcal{L} = \mathbf{Z}v_1 + \cdots + \mathbf{Z}v_r$ for some linearly independent vectors $v_1, \ldots, v_r$, and we say $\mathcal{L}$ is a full lattice if r is the dimension of V over $\mathbb{R}$. If $\mathcal{L}$ is such a full lattice with basis $v_1, \ldots, v_n$, then the set

$$T = \{r_1v_1 + \cdots + r_nv_n : 0 \le r_i < 1,\ 1 \le i \le n\}$$

is called a fundamental parallelepiped for $\mathcal{L}$. From the definition of the map v we know that $v(\mathcal{O}_K)$ is a full lattice in $\mathbb{R}^2$ with basis $v(1) = (1, 1)$ and $v(\beta) = \left(\beta, -\frac{1}{\beta}\right)$, and the fundamental parallelepiped of $v(\mathcal{O}_K)$ is shown in Figure 4.1. Further information regarding the lattice theory of algebraic number rings can be found in [7].

Consider the triangle formed by the points $(0, 0), (1, 1)$ and $\left(\beta, \frac{1}{\beta}\right)$. By calculating the coordinates of the circumcenter of this triangle, we find that it is located at $\left(\frac{3}{10} + \frac{2}{5}\beta, \frac{7}{10} - \frac{2}{5}\beta\right)$, while the radius of the circumcircle is $\sqrt{\frac{9}{10}}$, which is less than 1.

Let ξ be any real quartic irrational belonging to a quadratic extension of K whose conjugates are all real. Assume the minimal polynomial of ξ is

$$(Ax^2 + Bx + C) \cdot (\sigma(A)x^2 + \sigma(B)x + \sigma(C)),$$

we choose one of the roots of $\sigma(A)x^2 + \sigma(B)x + \sigma(C) = 0$ arbitrarily and denote it as ξ'. By defining the vector $\mathbf{\Xi} = (\xi, \xi')$, it becomes feasible to represent it as a sum of a lattice vector and another vector within the fundamental parallelepiped in polynomial time. Additionally, the latter vector can be modified to have length less than $\sqrt{\frac{9}{10}}$. The details about this process can be found in the algorithm below. In other words, we can express $\mathbf{\Xi} = (a_0, \sigma(a_0)) + \mathbf{\Xi'}$, where $(a_0, \sigma(a_0)) \in v(\mathcal{O}_K)$ and $|\Xi| < \sqrt{\frac{9}{10}} < 1$.

Based on the above discussion, for such a given ξ and ξ', we define the continued fraction expansion algorithm for ξ as follows.

Algorithm 4.1. The continued fraction expansion for pair (ξ, ξ'):

(1) $\xi_0 := \xi$, $\xi_0' := \xi'$, $\mathbf{\Xi_0} := (\xi_0, \xi_0')$.

(2) For all $n \ge 0$, find $\boldsymbol{a_n} := (a_n, \sigma(a_n)) \in v(\mathcal{O}_K)$ such that $|\mathbf{\Xi_n} - \boldsymbol{a_n}| < \sqrt{\frac{9}{10}}$. This can be achieved by performing the following calculation:

(a) Solve the equation

$$\begin{bmatrix} 1 & \beta \\ 1 & -\frac{1}{\beta} \end{bmatrix} \begin{bmatrix} \tilde{x_n} \\ \tilde{y_n} \end{bmatrix} = \begin{bmatrix} \xi_n \\ \xi_n' \end{bmatrix}$$

(b) Choose $x_n, y_n \in \mathbb{Z}$ such that

$$0 < |x_n - \tilde{x_n}| < 1,\ 0 < |y_n - \tilde{y_n}| < 1$$

and

$$|x_n + y_n\beta - \xi_n|^2 + \left|x_n - y_n\frac{1}{\beta} - \xi'_n\right|^2 < \frac{9}{10}.$$

One can find a satisfying (x_n, y_n) after at most four attempts. Set $a_n = x_n + y_n\beta$.

Note that the choices may not be unique, one can choose a solution arbitrarily.

(3) $\xi_{n+1} := \frac{1}{\xi_n - a_n}$, $\xi'_{n+1} := \frac{1}{\xi'_{n+1} - \sigma(a_n)}$, $\boldsymbol{\Xi_{n+1}} := (\xi_{n+1}, \xi'_{n+1})$.

Then we obtain an infinite continued fraction expansion of ξ by using the aforementioned algorithm, which generates the sequence $[a_0, a_1, \ldots]$, with all partial quotients belong to $\mathcal{O}_K$. It can be seen that both $|\xi_n|$ and $|\xi'_n|$ are larger than $\sqrt{\frac{10}{9}}$.

The subsequent lemma is derived from the discussion in Corollary 4.5 in [4], which addresses the characteristics of complex continued fractions, with the proof remaining essentially unchanged.

Lemma 4.2. *Suppose ξ as above is a real quartic irrational contained in some quadratic extension of K whose conjugate roots are all real, and the continued fraction expansion of ξ over K is given by our defined algorithm with respect to a chosen conjugate root ξ'. Apply Algorithm 4.1 on the pair (ξ, ξ'). Let $\gamma = \sqrt{\frac{10}{9}}$. Let N be the set $\{n \in \mathbb{N} \mid |Q_n| > |Q_{n-1}|\}$. If N is not empty, say the minimal element in N is n_0, then for all $n \geq n_0$, we have*

$$|Q_n\xi - P_n| < \frac{(\gamma - 1)^{-1}}{|Q_n|}.$$

Similarly for ξ', we can define $N' := \{n \in \mathbb{N} \mid |\sigma(Q_n)| > |\sigma(Q_{n-1})|\}$. If N' is not empty with minimal element n'_0, then for all $n \geq n'_0$, we have

$$|\sigma(Q_n)\xi' - \sigma(P_n)| < \frac{(\gamma - 1)^{-1}}{|\sigma(Q_n)|}.$$

Proof. We prove the claim only for ξ, since the proof for ξ' is essentially identical.

By the definition of our algorithm, we have $|\xi_n| > \gamma > 1$ for all n. Thus if $n_0 \in N \neq \emptyset$, then

$$\left|\xi_{n_0+1} + \frac{Q_{n_0}}{Q_{n_0-1}}\right| > |\xi_{n_0+1}| - \left|\frac{Q_{n_0}}{Q_{n_0-1}}\right| > (\gamma - 1)^{-1},$$

which implies

$$|Q_{n_0}\xi - P_{n_0}| = \left|\frac{1}{\xi_{n_0+1}Q_{n_0} + Q_{n_0-1}}\right| < \frac{(\gamma-1)^{-1}}{|Q_{n_0}|}.$$

If $n_0 + 1 \in N$, we can similarly deduce that $|Q_{n_0+1}\xi - P_{n_0+1}| < \frac{(\gamma-1)^{-1}}{|Q_{n_0+1}|}$. Otherwise if $n_0 + 1 \notin N$, then $|Q_{n_0+1}| \leq |Q_{n_0}|$. Hence

$$|Q_{n_0+1}\xi - P_{n_0+1}| = |\xi_1 \dots \xi_{n_0+2}|^{-1} < |Q_{n_0}\xi - P_{n_0}| < \frac{(\gamma-1)^{-1}}{|Q_{n_0}|} \leq \frac{(\gamma-1)^{-1}}{|Q_{n_0+1}|}.$$

Therefore, by induction, we can conclude that for all $n \geq n_0$, $|Q_n\xi - P_n| < \frac{(\gamma-1)^{-1}}{|Q_n|}$. □

Now we demonstrate that this algorithm will generate an ultimately periodic continued fraction expansion that converges to ξ.

Theorem 4.3. *With all the assumptions in Lemma 4.2, the continued fraction obtained by (ξ, ξ') using Algorithm 4.1 is ultimately periodic. Also, if we denote the expansion by P, then P takes ξ as the limit, while $\sigma(P)$ converges to ξ' as chosen before.*

Proof. If both sets N and N', as previously defined, are non-empty, then Lemma 4.2 holds. Utilizing Proposition 2.1, we derive $|\xi Q_n - P_n| = |\xi_1 \dots \xi_{n+1}|^{-1}$ and $|\xi'\sigma(Q_n) - \sigma(P_n)| = |\xi'_1 \dots \xi'_{n+1}|^{-1}$, both approaching 0 as n increases, provided that $|\xi_i|, |\xi'_i| > \sqrt{\frac{10}{9}}$. Consequently, in accordance with Theorem 3.10, the expansion eventually becomes periodic.

On the other hand, if N is empty, then $|Q_n|$ is bounded, hence $|P_n|$ is also bounded, as $|Q_n\xi - P_n| = |\xi_1 \dots \xi_n|^{-1} \to 0$. The similar result holds if N' is empty. Therefore, we always have bounded $|A_n| = |AP_n^2 + BP_n + C_n|$ and $|\sigma(A_n)| = |\sigma(AP_n^2 + BP_n + C_n)|$, and by Theorem 3.7 we conclude again that expansion is ultimately periodic.

Denote the obtained continued fraction expansion by P, then Lemma 3.3 guarantees the convergence of P and $\sigma(P)$, and by Lemma 3.4, both $|Q_n|, |\sigma(Q_n)| \to \infty$, hence by $|\xi Q_n - P_n|$, $|\xi'\sigma(Q_n) - \sigma(P_n)| \to 0$, we can establish the convergence of P to ξ and $\sigma(P)$ to ξ', as required in the theorem. □

Remark 4.4. Although we have discussed the cases whether the set N, N' is empty during the proof, it should be seen by Lemma 3.4 that, neither of them can be empty.

Our approach depends heavily on the property of the lattice $v(\mathcal{O}_K)$, considering the definition of the covering radius as the minimum radius that encompasses the closed circles centered at vertices covering the polygon. In fact, our method is only applicable to those $v(\mathcal{O}_K)$ with covering radius less than 1.

Using the method in [8], we can easily determine the covering radius r of $v(\mathcal{O}_K)$ for $K = \mathbb{Q}(\sqrt{D})$: When $D \equiv 2, 3 \mod 4$, $r = \sqrt{\frac{D+1}{2}}$, and for $D \equiv 1 \mod 4$, $r = \frac{1}{2\sqrt{2}}\left(\sqrt{D} + \frac{1}{\sqrt{D}}\right)$. One can check that for $D \neq 5$ the covering radius is always larger than 1.

Example 4.5. Let $\xi = 1 + \sqrt{\beta^2 + 1}$ and $\xi' = 1 + \sqrt{\frac{1}{\beta^2} + 1}$. Define $\mathbf{\Xi_0} = (\xi, \xi')$. Let $a_0 = 2$, and $\mathbf{\Xi_1} = \left(\frac{\sqrt{\beta^2+1}}{\beta^2}, \beta^2 + \sqrt{\beta^4 + \beta^2}\right)$. If we take $a_1 = 4 - 2\beta$, then $\mathbf{\Xi_2} = \mathbf{\Xi_0}$, hence we obtain a purely periodic continued fraction expansion of $\xi = \overline{[2, 4 - 2\beta]}$ with period length 2. Alternatively, take $\xi' = 1 - \sqrt{1 + \frac{1}{\beta^2}}$, we will obtain another expansion of $\xi = [\beta^2, \overline{2\beta}]$ that is ultimately periodic with period length 1.

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