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Approximation by non-convergents and second Lagrange spectrum

par Dmitry GAYFULIN

Résumé. Etant donné un nombre irrationnel α, on considère sa mesure d'irrationalité

$$\psi_\alpha(t) = \min_{1\le q\le t, q\in\mathbb{Z}} \|q\alpha\|.$$

L'ensemble $\mathbb{L}$ des valeurs de la fonction

$$\lambda(\alpha) = \limsup_{t\to\infty}((t\psi_\alpha(t))^{-1})$$

où α parcourt l'ensemble $\mathbb{R}\setminus\mathbb{Q}$, est appelé le spectre de Lagrange. Il est très bien étudié. Dans cet article, nous considérons une autre mesure d'irrationalité, $\psi_\alpha^{[2]}(t)$, qui traite l'approximation du nombre α par des rationnels non réduits. En remplaçant la fonction ψ_α par $\psi_\alpha^{[2]}$ dans la définition de $\mathbb{L}$, on obtient un ensemble $\mathbb{L}_2$ appelé le spectre de Lagrange d'ordre deux. Dans cet article, nous donnons la structure complète de la partie discrète initiale de $\mathbb{L}_2$.

Abstract. Given an irrational number α consider its irrationality measure function

$$\psi_\alpha(t) = \min_{1\le q\le t, q\in\mathbb{Z}} \|q\alpha\|.$$

The set of all values of

$$\lambda(\alpha) = \limsup_{t\to\infty}((t\psi_\alpha(t))^{-1})$$

where α runs through the set $\mathbb{R}\setminus\mathbb{Q}$ is known as the Lagrange spectrum $\mathbb{L}$. It is very well studied. In the present paper, we consider another irrationality measure function $\psi_\alpha^{[2]}(t)$ which deals with rational approximations to α by non-convergents. Replacing the function $\psi_\alpha(t)$ in the definition of $\mathbb{L}$ by $\psi_\alpha^{[2]}(t)$, we get a set $\mathbb{L}_2$ which is called the second Lagrange spectrum. In the present paper, we give the complete structure of the initial discrete part of $\mathbb{L}_2$.

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1. Introduction

For an irrational number $\alpha \in \mathbb{R}$ consider its irrationality measure function

$$\psi_\alpha(t) := \min_{1\le q\le t, q\in\mathbb{Z}} \|q\alpha\|, \quad \text{where} \quad \|x\| = \min_{n\in\mathbb{Z}} |x-n|.$$

The Lagrange constant of α is defined as

$$\lambda(\alpha) := \limsup_{t\to\infty} (t\cdot\psi_\alpha(t))^{-1}. \tag{1.1}$$

If the right-hand side of (1.1) is finite, the number α is called *badly approximable*. The set of values $\lambda(\alpha)$ as α runs through the set of all badly approximable numbers forms the Lagrange spectrum[1] $\mathbb{L}$.

$$\mathbb{L} := \{\lambda \mid \exists\, \alpha \in \mathbb{R}\setminus\mathbb{Q}\colon \lambda = \lambda(\alpha)\}. \tag{1.2}$$

This spectrum (along with the closely related Markoff spectrum $\mathbb{M}$) was first studied by A. Markoff [4, 5] at the end of the XIX century. In particular, he showed that $\mathbb{L}$ below 3 forms a discrete set with the accumulation point 3. Moreover, $\mathbb{L}\cap(-\infty,3)$ consists of the numbers $\sqrt{9-\frac{4}{m^2}}$, where m is a positive integer such that

$$m^2 + m_1^2 + m_2^2 = 3mm_1m_2, \quad m_1 \le m; \quad m_2 \le m \tag{1.3}$$

holds for some positive integers m_1, m_2. The equation (1.3) is known as the Markoff equation. The set of integer solutions of (1.3) was studied by many authors (see for example [1, Chapter 2]). In [4] Markoff also gave a full description of the set $\{\alpha \mid \lambda(\alpha) < 3\}$. It turned out that all such α are quadratic irrationalities, however, the structure of the set is very complicated. One can find a nice and detailed survey of the results about the Markoff and Lagrange spectra up to the 1990s in a book [1].

Recently a new interest appeared in this classical area and a dynamical approach was widely used to research the $\mathbb{M}$ and $\mathbb{L}$ spectra. See for example papers [6, 3], and a survey [7].

The Lagrange constant $\lambda(\alpha)$ can be easily expressed in terms of the continued fraction expansion of α. Suppose that

$$\alpha = [a_0; a_1, a_2, \ldots] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ldots}}, \tag{1.4}$$

$$a_0 \in \mathbb{Z}, \quad a_i \in \mathbb{Z}_+, \quad i = 1, 2, \ldots.$$

[1] Some authors define the Lagrange constant $\lambda(\alpha)$ equal to $\liminf_{t\to\infty} t\cdot\psi_\alpha(t)$. Therefore, the Lagrange spectrum, defined by these authors, contains the reciprocals of the elements of $\mathbb{L}$ from our definition.

Denote by $\frac{p_n}{q_n}$ the n-th convergent fraction to α, i.e. $\frac{p_n}{q_n} := [a_0; a_1, \ldots, a_n]$. Let us also introduce the following common notation

$$\alpha_n := [a_n; a_{n+1}, \ldots], \quad \alpha_n^* := [0; a_n, a_{n-1}, \ldots, a_1].$$

We say that two irrational numbers α and β are *equivalent* if $\alpha_n = \beta_m$ for some $n, m \in \mathbb{N}$. It is a well-known fact, usually called Perron's formula, that for any $n \in \mathbb{N}$ one has

$$\|q_n\alpha\| = \frac{1}{q_n(\alpha_{n+1} + \alpha_n^*)}. \tag{1.5}$$

As the function $\psi_\alpha(t)$ has its jumps at the points q_i, where $i = 1, 2, \ldots$ (this fact is known as Lagrange's Theorem), from (1.1) and (1.5) one can deduce that

$$\lambda(\alpha) = \limsup_{t\to\infty}(\alpha_t + \alpha_{t-1}^*). \tag{1.6}$$

Another important Diophantine constant of an irrational number α is the Dirichlet constant $\omega(\alpha)$ defined as

$$\omega(\alpha) := \limsup_{t\to\infty} t \cdot \psi_\alpha(t). \tag{1.7}$$

The corresponding Diophantine spectrum, known as Dirichlet spectrum $\mathbb{D}$, is defined as the set of values $d(\alpha) = 1/\omega(\alpha) - 1$ when α runs through the set of all badly approximable numbers. The value $d(\alpha)$ can be also expressed in terms of continued fractions:

$$d(\alpha) = \limsup_{t\to\infty}\Big(\frac{\alpha_{t+1}}{\alpha_t^*}\Big). \tag{1.8}$$

This spectrum is significantly less studied than $\mathbb{L}$, however, the discrete part of $\mathbb{D}$ is known and allows a simple and elegant description. Lesca [2] and Morimoto [8] proved that $2 + \sqrt{5}$ is the smallest accumulation point of $\mathbb{D}$. However, compared with the discrete part of the Lagrange spectrum, the description of the irrational numbers α such that $d(\alpha) < 2 + \sqrt{5}$ is very easy. Namely, if $d(\alpha) < 2 + \sqrt{5}$, then α is equivalent either to $[1; \overline{1}]$ or to $[1; \overline{1_{2k-1}, 2}]$ for some $k \geq 1$. We write $(a_1, \ldots, a_n)_m$ if the pattern $a_1, \ldots, a_n$ is repeated m times and $\overline{a_1, \ldots, a_n}$ if this pattern forms an infinite periodic sequence.

In the present paper, we study the Diophantine spectrum associated with approximation to α by non-convergents. Such approximation was considered by several authors, including Rockett and Szüsz ([10, Chapter 8]) and Moshchevitin [9]. Denote $\mathcal{Q}_\alpha := \{q_1 < q_2 < \ldots\}$ to be the set of denominators of the convergents to some irrational number α. The irrationality measure function that corresponds to approximation by non-convergents is

$$\psi_\alpha^{[2]}(t) := \min_{1\leq q\leq t, q\in\mathbb{Z}, q\notin\mathcal{Q}_\alpha} \|q\alpha\|. \tag{1.9}$$

Obviously, like $\psi_\alpha(t)$, $\psi_\alpha^{[2]}(t)$ is a non-increasing peacewise constant function. An analog of the Lagrange constant $\lambda(\alpha)$ for approximation by non-convergents is the quantity

$$\lambda^{[2]}(\alpha) := \limsup_{t\to\infty}(t\cdot\psi_\alpha^{[2]}(t))^{-1}. \tag{1.10}$$

The corresponding Diophantine spectrum is called the second Lagrange spectrum. It is defined as

$$\mathbb{L}_2 := \{\lambda \mid \exists\, \alpha\in\mathbb{R}\setminus\mathbb{Q}\colon \lambda=\lambda^{[2]}(\alpha)\}. \tag{1.11}$$

The present paper contains a complete description of the discrete part of $\mathbb{L}_2$. It turns out, that the discrete part of $\mathbb{L}_2$ is similar to that of the Dirichlet spectrum $\mathbb{D}$. In particular, we prove that if a real number α belongs to the discrete part of $\mathbb{L}_2$, then α is equivalent to one of the following numbers: $[1;\overline{1}]$, $[1;\overline{1,1,3}]$ or $[1;\overline{1,1,1,1,3,(1,1,3)_{2k-1}}]$, where $k\geq 1$. The structure of our proof is close to the argument of Lesca [2]. However, the approximation by non-convergents has its own specificity, in particular, the expression of $\lambda^{[2]}(\alpha)$ in terms of continued fractions is more complicated.

Note that in the definition (1.9) we also consider the approximations of the form $\frac{rp_k}{rq_k}$, where $r\geq 2$. Indeed,

$$\|q\alpha\| = \min_{p\in\mathbb{Z}}|q\alpha-p| = q\big(\min_{p\in\mathbb{Z}}|\alpha-p/q|\big).$$

One can consider a similar irrationality measure function that excludes such cases:

$$\psi_\alpha^{[2]*}(t) := \min_{1\leq q\leq t,(p,q)\in\mathbb{Z}^2,\ p/q\neq p_k/q_k}|q\alpha-p|. \tag{1.12}$$

As the minimum in (1.12) is considered on a smaller set than in (1.9), one can see that $\psi_\alpha^{[2]}(t)\leq\psi_\alpha^{[2]*}(t)$ for any α and t. Similarly to (1.10), one can define a Lagrange-type constant

$$\lambda^{[2]*}(\alpha) := \limsup_{t\to\infty}(t\cdot\psi_\alpha^{[2]*}(t))^{-1} \tag{1.13}$$

and the corresponding Lagrange-type spectrum $\mathbb{L}_2^*$. This spectrum was studied in [9]. Particularly, it was shown (Theorem 4) that $\mathbb{L}_2^*\subset\big[\frac{1}{\sqrt{5}},2\big]$.

2. Previous results on $\mathbb{L}_2$

We start with a lemma from [9] which is the main tool to calculate $\lambda^{[2]}(\alpha)$ for a given irrational number α. We follow the notation of this paper but, as we already mentioned, we consider the "reverse" definition of $\mathbb{L}$ and $\mathbb{L}_2$. That is why, $\varkappa_n^i(\alpha)$ in our notation equals $1/\varkappa_n^i(\alpha)$ in the notation of the paper [9].

Lemma 1. *Suppose that an irrational number α is not equivalent to $\frac{1+\sqrt{5}}{2} = [1;\overline{1}]$. Consider three quantities:*

$$\varkappa_n^1(\alpha) = \frac{\alpha_n + \alpha_{n-1}^*}{(1+\alpha_{n-1}^*)(\alpha_n - 1)}, \quad \varkappa_n^2(\alpha) = \frac{\alpha_{n+1} + \alpha_n^*}{(1-\alpha_n^*)(\alpha_{n+1}+1)}, \quad \varkappa_n^4(\alpha) = \frac{\alpha_n + \alpha_{n-1}^*}{4}. \tag{2.1}$$

Then

$$\lambda^{[2]}(\alpha) = \limsup_{n\to\infty:\, a_n \geq 2} \max(\varkappa_n^1(\alpha), \varkappa_n^2(\alpha), \varkappa_n^4(\alpha)). \tag{2.2}$$

In paper [9] by means of Lemma 1 the two smallest elements of $\mathbb{L}_2$ were calculated.

Theorem A.

(1) *The smallest element of $\mathbb{L}_2$ is $\lambda_1 := \frac{\sqrt{5}}{4} \approx 0.559016$. Moreover, if $\lambda^{[2]}(\alpha) = \lambda_1$, then $\alpha \sim \frac{1+\sqrt{5}}{2} = [1;\overline{1}]$.*

(2) *The second smallest element of $\mathbb{L}_2$ is $\lambda_2 := \frac{\sqrt{17}}{4} \approx 1.030776$. Moreover, if $\lambda^{[2]}(\alpha) = \lambda_2$, then $\alpha \sim \frac{1+\sqrt{17}}{2} = [2;\overline{1,1,3}]$.*

A little bit later in [11] P. Semenyuk calculated the third smallest element of $\mathbb{L}_2$.

Theorem B. *The third smallest element of $\mathbb{L}_2$ is $\lambda_3 := \frac{13\sqrt{173}}{164} \approx 1.042611$. Moreover, if $\lambda^{[2]}(\alpha) = \lambda_3$, then $\alpha \sim \frac{39+13\sqrt{173}}{82} = [2;\overline{1,1,1,1,3,1,1,3}]$.*

3. Main result

For each $n \geq 3$ define $\xi_n = [0;\overline{1,1,1,1,3,(1,1,3)_{2n-5}}]$ and

$$\lambda_n := \frac{[3;\overline{(1,1,3)_{2n-5},1,1,1,1,3}] + [0;\overline{1,1,1,1,(3,1,1)_{2n-5},3}]}{4}. \tag{3.1}$$

One can easily see that λ_3 defined in (3.1) is the same constant as in Theorem B. Let us also denote by λ_∞ the limit

$$\lambda_\infty := \lim_{n\to\infty} \lambda_n = \frac{[3;\overline{1,1,3}] + [0;1,1,1,1,\overline{3,1,1}]}{4} = \frac{3\sqrt{17}+21}{32} \approx 1.042791.$$

Now we are ready to formulate our main result.

Theorem 1.

(1) *The spectrum $\mathbb{L}_2$ below λ_∞ forms a discrete set*

$$(-\infty, \lambda_\infty) \cap \mathbb{L}_2 = \{\lambda_1 < \lambda_2 < \ldots < \lambda_n < \ldots\},$$

where λ_1 and λ_2 are defined in Theorem A, $\lambda_n, n \geq 3$ is defined in (3.1). Moreover, $\lambda_\infty \in \mathbb{L}_2$.

(2) *For all* $n \geq 3$ *if an irrational* α *is such that* $\lambda^{[2]}(\alpha) = \lambda_n$, *then* $\alpha \sim \xi_n$.

The proof of Theorem 1 is given in two next sections. In Section 4 we study prohibited patterns i.e. sequences $(b_1, \ldots, b_m)$ that occur in the sequence $(a_1, \ldots, a_n, \ldots)$ with $\lambda^{[2]}(\alpha) < \lambda_\infty$ only finitely many times. In the same section we deduce that if $\lambda_2 < \lambda^{[2]}(\alpha) < \lambda_\infty$, then α is equivalent to ξ_k for some $k \geq 3$. In Section 5 we show that if $\alpha \sim \xi_k$, $k \geq 3$, then

$$\limsup_{n\to\infty:\, a_n\geq 2} \max(\varkappa_n^1(\alpha), \varkappa_n^2(\alpha), \varkappa_n^4(\alpha)) = \limsup_{n\to\infty:\, a_n\geq 2} \varkappa_n^4(\alpha)$$

and complete the proof of Theorem 1.

4. Prohibited patterns

The following lemma was proved in [9] and [11]. For the sake of completeness, we give a sketch of proof of it.

Lemma 2. *Assume that the continued fraction expansion* $[a_0; a_1, \ldots, a_n, \ldots]$ *of an irrational number* α *contains infinitely many elements greater than* 3 *or infinitely many patterns of the form* 2, 33, 313, 31113, 311111, 1113111. *Then* $\lambda^{[2]}(\alpha) > \lambda_\infty$.

Proof. See Table 4.1 below. The bold script indicates the element a_n. To give an example, let us prove the estimate on the third line. Suppose that $a_n = 2$ for infinitely many n. It is clear that

$$\lambda^{[2]}(\alpha) = \limsup_{n\to\infty:\, a_n\geq 2} \max(\varkappa_n^1(\alpha), \varkappa_n^2(\alpha), \varkappa_n^4(\alpha)) \geq \limsup_{n\to\infty:\, a_n\geq 2} \varkappa_n^2(\alpha).$$

Note that the function $\varkappa_n^1(\alpha_n, \alpha_{n-1}^*)$ is decreasing on both arguments while the functions $\varkappa_n^2(\alpha_{n+1}, \alpha_n^*)$ and $\varkappa_n^4(\alpha_n, \alpha_{n-1}^*)$ are increasing on both arguments. Therefore, in order to obtain the lower estimate of $\varkappa_n^2$, we need to substitute the lower estimates of α_{n+1} and α_n^*. As we see from the lines 1 and 2 of the table below, without loss of generality one can say that $a_i \leq 3$ for all $i \geq 1$. Thus, $\alpha_{n+1} \geq [1; \overline{3, 1}]$ and $\alpha_n^* \geq [0; 2, \overline{1, 3}]$. Substituting these estimates to (2.1) yields $\varkappa_n^2(\alpha) \geq 1.116515 > \lambda_\infty$.

The lemma is proven. □

Lemma 2 states that all the patterns from the first column of the table above are prohibited. The following corollary is an immediate consequence of Lemma 2.

Corollary 1. *Suppose that* $\lambda_2 < \lambda^{[2]}(\alpha) \leq \lambda_\infty$ *for some irrational* $\alpha = [a_0; a_1, \ldots, a_n, \ldots]$. *Then there exists* N *such that the sequence* $(a_N, a_{N+1}, \ldots)$ *has the form* $(31111(311)_{n_1}31111(311)_{n_2}31111\ldots)$, *where all* $n_i \geq 1$.

Table 4.1.

Pattern	$\varkappa_n^i$ used	Lower estimate	Approx value
$a_n \geq 5$	$\varkappa_n^4$	$([5]+[0])/4$	1.25
4	$\varkappa_n^4$	$([4;\overline{4,1}]+[0;\overline{4,1}])/4$	1.103553
2	$\varkappa_n^2$	$\alpha_{n+1} \geq [1;\overline{3,1}]$; $\alpha_n^* \geq [0;2,\overline{1,3}]$	1.116515
33	$\varkappa_n^1$	$\alpha_n \leq [3;\overline{1,3}]$; $\alpha_{n-1}^* \leq [0;3,\overline{3,1}]$	1.123722
313	$\varkappa_n^4$	$([3;1,3,\overline{3,1}]+[0;1,\overline{1,3}])/4$	1.080930
31113	$\varkappa_n^4$	$([3;1,1,1,3,1,1,\overline{3,1,1,1}]+[0;1,1,\overline{3,1,1,1}])/4$	1.050188
311111	$\varkappa_n^4$	$([3;1,1,1,1,1,1,\overline{3,1,1,1}]+[0;1,1,\overline{3,1,1,1}])/4$	1.044287
111**3**111	$\varkappa_n^4$	$([3;1,1,1,1,\overline{3,1,1,1}]+[0;1,1,1,1,\overline{3,1,1,1}])/4$	1.054716

Without loss of generality one can say that number N from the previous corollary equals 1. We will frequently refer to the following classical lemma concerning difference of two continued fractions.

Lemma 3. *Let* $\alpha = [a_0; a_1, \ldots, a_n, \alpha_{n+1}]$ *and* $\beta = [a_0; a_1, \ldots, a_n, \beta_{n+1}]$ *be two continued fractions. Then*

$$\beta - \alpha = (-1)^{n+1} \frac{\beta_{n+1} - \alpha_{n+1}}{q_n^2(\alpha_{n+1} + \alpha_n^*)(\beta_{n+1} + \alpha_n^*)}, \tag{4.1}$$

where q_n *is the denominator of the convergent* $\frac{p_n}{q_n} = [a_0; a_1, \ldots, a_n]$.

Proof. Note that $\alpha_n^* = \beta_n^*$. Using Perron's formula one can easily see that

$$\begin{aligned} \beta - \alpha &= \left(\beta - \frac{p_n}{q_n}\right) - \left(\alpha - \frac{p_n}{q_n}\right) \\ &= \frac{(-1)^{n+1}}{q_n^2}\left(\frac{1}{\alpha_{n+1} + \alpha_n^*} - \frac{1}{\beta_{n+1} + \alpha_n^*}\right) \\ &= \frac{(-1)^{n+1}}{q_n^2} \frac{\beta_{n+1} - \alpha_{n+1}}{(\alpha_{n+1} + \alpha_n^*)(\beta_{n+1} + \alpha_n^*)}. \end{aligned} \tag{4.2}$$

□

One can consider the value q_t as the *continuant* of the sequence $(a_1, \ldots, a_t)$. Let us give some definition. Suppose that A is an arbitrary (possibly empty) finite sequence of positive integers. By $\langle A \rangle$ we denote its continuant. It is defined as follows: continuant of an empty sequence $\langle \cdot \rangle$ equals 1, $\langle a_1 \rangle = a_1$, if $t \geq 2$ then one has

$$\langle a_1, a_2, \ldots, a_t \rangle = a_t \langle a_1, a_2, \ldots, a_{t-1} \rangle + \langle a_1, a_2, \ldots, a_{t-2} \rangle. \tag{4.3}$$

One can see that

$$[a_0; a_1, \dots, a_t] = \frac{\langle a_0, a_1, a_2, \dots, a_t\rangle}{\langle a_1, a_2, \dots, a_t\rangle} = \frac{p_t}{q_t}. \tag{4.4}$$

Formula (4.3) leads to the following more general equality

$$\begin{aligned} \langle a_1, a_2, \dots, a_t, a_{t+1}, \dots, a_s\rangle &= \langle a_1, a_2, \dots, a_t\rangle\langle a_{t+1}, \dots, a_s\rangle + \langle a_1, a_2, \dots, a_{t-1}\rangle\langle a_{t+2}, \dots, a_s\rangle \\ &= \langle a_1, a_2, \dots, a_t\rangle\langle a_{t+1}, a_{t+2}, \dots, a_s\rangle \\ &\quad \times (1 + [0; a_t, a_{t-1}, \dots, a_1][0; a_{t+1}, a_{t+2}, \dots, a_s]). \end{aligned} \tag{4.5}$$

Denote $\alpha_\infty := [3; \overline{1, 1, 3}]$ and $\alpha_\infty^* := [0; 1, 1, 1, 1, \overline{3, 1, 1}]$. Then $\lambda_\infty = \frac{\alpha_\infty + \alpha_\infty^*}{4}$.

Lemma 4. *The pattern*

$$111(311)_{2m}3111 \tag{4.6}$$

is prohibited for all $m \geq 0$.

Proof. We will prove this statement by induction on m. The base case $m = 0$ is already considered in Lemma 2. Suppose that for all $m \leq k$ the pattern (4.6) is prohibited. Let us also assume that the pattern (4.6) for $m = k + 1$ occurs in the sequence $(a_1, \dots, a_n, \dots)$ infinitely many times but $\lambda^{[2]}(\alpha) \leq \lambda_\infty$. Let n be an index of the first "3" in the group $(311)_{2k+2}$. Due to Lemma 1, to get a contradiction it is enough to show that $\varkappa_n^4(\alpha) > \lambda_\infty$. This is equivalent to the inequality

$$\alpha_n - \alpha_\infty > \alpha_\infty^* - \alpha_{n-1}^*. \tag{4.7}$$

Let us obtain the lower estimate of $\alpha_n - \alpha_\infty$ using Lemma 3. We know that

$$\begin{aligned} &\alpha_n - \alpha_\infty \\ &= [\underbrace{(3; 1, 1)_{2k+2}, 3, 1, 1}_{\text{coinciding part}}, 1, 1, \dots] - [\underbrace{(3; 1, 1)_{2k+2}, 3, 1, 1}_{\text{coinciding part}}, 3, 1, 1, \dots]. \end{aligned} \tag{4.8}$$

The first different partial quotient has index $6k + 9$, hence by Lemma 3 one has

$$\begin{aligned} &\alpha_n - \alpha_\infty \\ &= \frac{[3; 1, 1, \dots] - [1; 1, 1, \dots]}{\langle 1, 1, (3, 1, 1)_{2k+2}\rangle^2([3; 1, 1, \dots] + [0; 1, 1, 3, \dots])([1; 1, 1, \dots] + [0; 1, 1, 3, \dots])} \\ &> \frac{[3; 1, 1] - [1; 1]}{\langle 1, 1, (3, 1, 1)_{2k+2}\rangle^2([3; 1] + [0; 1])([1; 1] + [0; 1])} \\ &= \frac{1}{10\langle 1, 1, (3, 1, 1)_{2k+2}\rangle^2}. \end{aligned} \tag{4.9}$$

Now we obtain the lower estimate of $\alpha^*_{n-1} = [0;1,1,1,1,3,\dots]$. As the patterns 33, 313, 311111, and 1113111 are prohibited, the continued fraction expansion of α^*_{n-1} extends as

$$\alpha^*_{n-1} = [0;1,1,1,1,(3,1,1)_2,\dots].$$

We need the lower estimate for α^*_{n-1}, so we write

$$\alpha^*_{n-1} \ge [0;1,1,1,1,(3,1,1)_2,3,\dots].$$

Taking into account the next partial quotients we see that

$$\alpha^*_{n-1} \ge [0;1,1,1,1,(3,1,1)_2,3,1,1,\dots]. \tag{4.10}$$

We will show that for $0 \le j \le k$ one has

$$\alpha^*_{n-1} \ge [0;1,1,1,1,(3,1,1)_{2j+2},\dots].$$

For $j = 0$ this estimate is provided by (4.10). Since by the induction hypothesis, the pattern $111(311)_{2m}3111$ is prohibited for $0 \le m \le k$ (as well as the patterns 33, 313, and 2), it follows that if $j < k$ one has

$$\begin{aligned}\alpha^*_{n-1} \ge {} & [0;1,1,1,1,(3,1,1)_{2j+2},3,1,1,3,1,1,\dots] \\ & = [0;1,1,1,1,(3,1,1)_{2j+4},\dots].\end{aligned} \tag{4.11}$$

Applying (4.11) for $j = k$ we obtain the lower bound

$$\alpha^*_{n-1} \ge [0;1,1,1,1,(3,1,1)_{2k+2},3,1,1,1,\dots].$$

Let us now estimate the difference $\alpha^*_\infty - \alpha^*_{n-1}$ from above. We see that

$$\begin{aligned}\alpha^*_\infty - \alpha^*_{n-1} \le {} & [\underbrace{0;1,1,1,1,(3,1,1)_{2k+2},3,1,1}_{\text{coinciding part}},3,1,1,\dots] \\ & - [\underbrace{0;1,1,1,1,(3,1,1)_{2k+2},3,1,1}_{\text{coinciding part}},1,1,\dots].\end{aligned} \tag{4.12}$$

These expansions differ for the first time at the partial quotient number $6k + 14$, hence by Lemma 3 one has

$$\begin{aligned}&\alpha^*_\infty - \alpha^*_{n-1} \\ &\le \frac{[3;1,1,\dots] - [1;1,1,\dots]}{\langle 1,1,1,1,(3,1,1)_{2k+3}\rangle^2([3;1,1,\dots]+[0;1,1,3,\dots])([1;1,1,\dots]+[0;1,1,3,\dots])} \\ &< \frac{[3;1] - [1;1,1]}{\langle 1,1,1,1,(3,1,1)_{2k+3}\rangle^2([3;1,1]+[0;1,1])([1;1,1]+[0;1,1])} \\ &= \frac{5}{16\langle 1,1,1,1,(3,1,1)_{2k+3}\rangle^2}.\end{aligned} \tag{4.13}$$

Comparing (4.9) and (4.13), one can see that in order to obtain (4.7) it suffices to show that

$$\frac{\langle 1,1,1,1,(3,1,1)_{2k+3}\rangle}{\langle 1,1,(3,1,1)_{2k+2}\rangle} > \sqrt{\frac{25}{8}}.$$

The last inequality easily follows from the trivial estimates

$$\langle 1,1,1,1,(3,1,1)_{2k+3}\rangle > \langle 1,1,1,1,3,1,1\rangle\langle (3,1,1)_{2k+2}\rangle = 41\langle (3,1,1)_{2k+2}\rangle$$

and

$$\langle 1,1,(3,1,1)_{2k+2}\rangle < 3\langle (3,1,1)_{2k+2}\rangle.$$

Lemma 4 is proven. □

Lemma 5. *The pattern*

$$31111(311)_{2m+1}31111(311)_{2k+1}31111, \tag{4.14}$$

where $k > m \geq 0$ is prohibited.

Proof. The proof is similar to that of Lemma 4. Our goal is to show that the inequality

$$\alpha_\infty - \alpha_n < \alpha^*_{n-1} - \alpha^*_\infty \tag{4.15}$$

holds for infinitely many n. Let n be an index of the first "3" in the group $(311)_{2k+1}$. Once again, we obtain the upper estimate of $\alpha_\infty - \alpha_n$ using Lemma 3. We see that

$$\alpha_\infty - \alpha_n \geq [\underbrace{(3;1,1)_{2k+1},3,1,1}_{\text{coinciding part}},3,1,1,3,1,1,\ldots] - [\underbrace{(3;1,1)_{2k+1},3,1,1}_{\text{coinciding part}},1,1,3,1,1,\ldots]. \tag{4.16}$$

These expansions differ for the first time at the partial quotient number $6k+6$. Using the estimates from (4.13), one can see that

$$\alpha_\infty - \alpha_n < \frac{5}{16\langle 1,1,(3,1,1)_{2k+1}\rangle^2}. \tag{4.17}$$

Now we need a lower estimate for the difference $\alpha^*_{n-1} - \alpha^*_\infty$. We have

$$\alpha^*_{n-1} - \alpha^*_\infty \geq [\underbrace{0;1,1,1,1,(3,1,1)_{2m+1},3,1,1}_{\text{coinciding part}},1,1,\ldots] - [\underbrace{0;1,1,1,1,(3,1,1)_{2m+1},3,1,1}_{\text{coinciding part}},3,1,1,\ldots]. \tag{4.18}$$

These expansions differ for the first time at the partial quotient number $6m+11$. Using the estimates from (4.9), one can see that

$$\alpha^*_{n-1} - \alpha^*_\infty > \frac{1}{10\langle 1,1,1,1,(3,1,1)_{2m+2}\rangle^2}. \tag{4.19}$$

Comparing (4.17) and (4.19), we deduce that in order to obtain (4.15) it suffices to show that

$$\frac{\langle 1,1,(3,1,1)_{2k+1}\rangle}{\langle 1,1,1,1,(3,1,1)_{2m+2}\rangle} > \sqrt{\frac{25}{8}}.$$

As $k \geq m+1$, it is enough to verify the inequality

$$\frac{\langle 1,1,(3,1,1)_{2m+3}\rangle}{\langle 1,1,1,1,(3,1,1)_{2m+2}\rangle} > \sqrt{\frac{25}{8}} \tag{4.20}$$

for all $m \geq 0$. As

$$\langle 1,1,(3,1,1)_{2m+3}\rangle \geq \langle 1,1,(3,1,1)_{2m+2}\rangle\langle 3,1,1\rangle = 7\langle 1,1,(3,1,1)_{2m+2}\rangle$$

and

$$\langle 1,1,1,1,(3,1,1)_{2m+2}\rangle \leq 3\langle 1,1,(3,1,1)_{2m+2}\rangle,$$

the inequality (4.20) holds and the lemma is proven. □

As an immediate consequence of Lemmas 4 and 5, we deduce the following corollary.

Corollary 2. *Suppose that* $\lambda_2 < \lambda^{[2]}(\alpha) < \lambda_\infty$ *for some irrational* $\alpha = [a_0; a_1, \ldots, a_n, \ldots]$. *Then* $\alpha \sim \xi_n$ *for some* $n \geq 3$.

5. Upper estimates

In this section we show that $\lambda^{[2]}(\xi_i) = \lambda_i$ and thus complete the proof of Theorem 1. First, we prove two technical lemmas.

Lemma 6. *Let* $\alpha = [a_0; a_1, \ldots]$ *be an arbitrary irrational number. If* a_n *is the middle "3" in the pattern* 311**3**113, *then*

$$\max(\varkappa_n^1(\alpha), \varkappa_n^2(\alpha), \varkappa_n^4(\alpha)) < 1.04.$$

Proof. As the function $\varkappa_n^1(\alpha_n, \alpha_{n-1}^*)$ is decreasing on both arguments and

$$\alpha_n \geq [3;1,1,\overline{3,1,1,1}], \qquad \alpha_{n-1}^* \geq [0;1,1,\overline{3,1,1,1}],$$

we see that

$$\varkappa_n^1(\alpha) \leq \frac{[3;1,1,\overline{3,1,1,1}] + [0;1,1,\overline{3,1,1,1}]}{(1+[0;1,1,\overline{3,1,1,1}])([3;1,1,\overline{3,1,1,1}]-1)} \approx 1.031440. \tag{5.1}$$

The quantities $\varkappa_n^2(\alpha)$ and $\varkappa_n^4(\alpha)$ are estimated in a similar way. As

$$\alpha_{n+1} \leq [1;1,\overline{3,1,1,1}], \quad \alpha_n^* \leq [0;3,1,1,\overline{3,1,1,1}],$$

we get the following estimate

$$\varkappa_n^2(\alpha) \leq \frac{[1;1,\overline{3,1,1,1}] + [0;3,1,1,\overline{3,1,1,1}]}{(1+[1;1,\overline{3,1,1,1}])(1-[0;3,1,1,\overline{3,1,1,1}])} \approx 1.031440. \tag{5.2}$$

Finally, as

$$\alpha_n \leq [3;1,1,3,1,1,\overline{3,1,1,1}], \qquad \alpha_{n-1}^* \leq [0;1,1,3,1,1,\overline{3,1,1,1}]$$

we see that

(5.3)
$$\varkappa_n^4(\alpha) \le \frac{[3;1,1,3,1,1,\overline{3,1,1,1}]+[0;1,1,3,1,1,\overline{3,1,1,1}]}{4} \approx 1.030785.$$

Combining the estimates (5.1), (5.2), and (5.3) yields the statement of the lemma. □

Lemma 7. *Let $\alpha = [a_0; a_1, \ldots]$ be an arbitrary irrational number. If a_n is the middle "3" in the pattern 31111**3**113 or 311**3**11113, then*

$$\varkappa_n^4(\alpha) > \max(\varkappa_n^1(\alpha), \varkappa_n^2(\alpha)).$$

Proof. We will give a proof for the pattern 31111**3**113 only, as the proof for the second pattern is exactly same. One can easily see that $\varkappa_n^4(\alpha) > \varkappa_n^1(\alpha)$ if and only if the inequality

$$(1+\alpha_{n-1}^*)(\alpha_n - 1) > 4 \tag{5.4}$$

holds. As

$$\alpha_n \ge [3;1,1,\overline{3,1,1,1}], \quad \alpha_{n-1}^* \ge [0;1,1,1,1,\overline{3,1,1,1}],$$

we have the following estimate

$$(1+\alpha_{n-1}^*)(\alpha_n - 1) \ge ([1;1,1,1,1,\overline{3,1,1,1}])([2;1,1,\overline{3,1,1,1}]) \approx 4.120747$$

and the inequality (5.4) holds.

Using the obvious properties $\alpha_n = a_n + 1/\alpha_{n+1}$ and $\alpha_n^* = 1/(a_n + \alpha_{n-1}^*)$ and the fact that $a_n = 3$, one can transform $\varkappa_n^2$ and $\varkappa_n^4$ as follows:

$$\varkappa_n^2(\alpha) = \frac{3\alpha_{n+1} + \alpha_{n+1}\alpha_{n-1}^* + 1}{(2+\alpha_{n-1}^*)(\alpha_{n+1}+1)}, \quad \varkappa_n^4(\alpha) = \frac{3\alpha_{n+1} + \alpha_{n+1}\alpha_{n-1}^* + 1}{4\alpha_{n+1}}.$$

Hence $\varkappa_n^4(\alpha) > \varkappa_n^2(\alpha)$ if and only if

$$\alpha_{n+1}\alpha_{n-1}^* + \alpha_{n-1}^* - 2\alpha_{n+1} + 2 > 0. \tag{5.5}$$

We know that

$$\alpha_{n+1} \le [1;1,\overline{3,1,1,1}], \quad \alpha_{n-1}^* \ge [0;1,1,1,1,\overline{3,1,1,1}].$$

Substituting these estimates to (5.5), we can see that the inequality is satisfied and therefore $\varkappa_n^4(\alpha) > \varkappa_n^2(\alpha)$. Lemma is proven. □

Corollary 3. $\lambda^{[2]}(\xi_i) = \lambda_i$.

Proof. It follows directly from Lemmas 6 and 7. □

Corollary 4. $\lambda_\infty \in \mathbb{L}_2$.

Proof. Consider

$$\alpha = [0;(3,1,1)_{2n_1+1},3,1,1,1,1,(3,1,1)_{2n_2+1},3,1,1,1,1,(3,1,1)_{2n_3+1},\ldots], \tag{5.6}$$

where n_i is an arbitrary sequence of natural numbers that tends to infinity. By Lemmas 6 and 7, $\lambda^{[2]}(\alpha) = \lambda_\infty$. □

Thus we completed the proof of Theorem 1. Also, from (5.6) one can see that the set of real numbers satisfying $\lambda^{[2]}(\alpha) = \lambda_\infty$ has continuum many elements. Note that the set of numbers satisfying $\lambda^{[2]}(\alpha) = \lambda_n$ for each n is countable as all such numbers are equivalent to ξ_n.

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