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Uniqueness of L-functions and meromorphic functions under sharing of sets

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Résumé. Dans [4], les auteurs ont démontré que l'ensemble des zéros d'un polynôme d'unicité, satisfaisant certaines conditions supplémentaires, est un ensemble d'unicité pour les fonctions L. Ils ont également déterminé les conditions sous lesquelles un polynôme est un polynôme d'unicité forte pour les fonctions L. Ces résultats sont une version améliorée d'un résultat de [3]. Dans cet article, nous obtenons un certain nombre de théorèmes d'unicité pour les fonctions L appartenant à la classe de Selberg étendue, qui étendent, de manière significative, les résultats de [3] et [4] dans une nouvelle direction et les améliorons dans certains cas. En utilisant ces résultats, nous pouvons exhiber certaines classes d'ensembles d'unicité pour les fonctions L, qui ne peuvent pas être trouvées avec les résultats de [3] et [4].

Abstract. In [4], the authors proved that the zero set of a uniqueness polynomial, satisfying some additional conditions, becomes a unique range set for L-functions. They also determined the conditions under which a polynomial becomes a strong uniqueness polynomial for L-functions. These results are the improved version of one result of [3]. In this paper we obtain a number of uniqueness theorems for L-functions in the extended Selberg class, which significantly extend the results of [3] and [4] in a new direction and improve them in some cases. From our results we can show some classes of unique range sets for L-functions which cannot be found by the results of [3] and [4].

1. Introduction and Main results

In 1989, Selberg defined a rather general class $\mathcal{S}$ of Dirichlet series having an Euler product, analytic continuation and a functional equation of Riemann type, to formulate some fundamental conjectures concerning them. His main purpose was to study the value-distribution of linear combinations of L-functions. In the meantime, this so-called Selberg class became an important object of research, but still it is not understood very well. The Selberg class $\mathcal{S}$ of L-functions, with the Riemann zeta function $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$

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as the prototype, is the class of functions $\mathcal{L}(s) = \sum_{n=1}^{\infty} a(n)n^{-s}$ of a complex variable, satisfying the following hypotheses (see [11]):

(i) *Ramanujan hypothesis:* $a(n) \ll n^{\epsilon}$ for every $\epsilon > 0$.

(ii) *Analytic continuation:* There is a non-negative integer k such that $(s-1)^k\mathcal{L}(s)$ is an entire function of finite order.

(iii) *Functional equation:* $\mathcal{L}$ satisfies a functional equation of type

$$\Lambda_{\mathcal{L}}(s) = \omega\overline{\Lambda_{\mathcal{L}}(1-\bar{s})},$$

where

$$\Lambda_{\mathcal{L}}(s) = \mathcal{L}(s)Q^s\prod_{j=1}^{K}\Gamma(\lambda_j s + \nu_j)$$

with positive real numbers Q, λ_j and complex numbers ν_j, ω with $\operatorname{Re}\nu_j \geq 0$ and $|\omega| = 1$.

(iv) *Euler product hypothesis:* $\mathcal{L}$ can be written for $\operatorname{Re}(s) > 1$

$$\mathcal{L}(s) = \prod_p \exp\left(\sum_{k=1}^{\infty} b(p^k)/p^{ks}\right)$$

with suitable coefficients $b(p^k)$ satisfying $b(p^k) \ll p^{k\theta}$ for some $\theta < 1/2$ where the product is taken over all prime numbers p.

The degree $d_{\mathcal{L}}$ of an L-function $\mathcal{L}$ is defined to be

$$d_{\mathcal{L}} = 2\sum_{j=1}^{K}\lambda_j,$$

where λ_j and K are respectively the positive real number and the positive integer as in axiom (iii) above.

In 1999, Kaczorowski–Perelli introduced the extended Selberg class $\mathcal{S}^{\#}$, defined as the collection of not identically vanishing L-functions $\mathcal{L}$ which satisfy the axioms (i)–(iii) above. Naturally, a function is called L-function if it satisfies the Euler product hypothesis, but in $\mathcal{S}^{\#}$ we can have some functions with degree zero which do not have an Euler product, and so it is worthwhile to study the extended Selberg class.

By the analytic continuation axiom, an L-function can be analytically continued as a meromorphic function in the complex plane $\mathbb{C}$. In the last few years value distribution of L-functions has become an interesting area of research. Readers can have a look on [7, 8, 16].

In this paper we are going to discuss some results in value distribution of L-functions in $\mathcal{S}^{\#}$. Before entering into the detail literature, let us assume $\mathcal{M}(\mathbb{C})$ as the field of meromorphic functions over $\mathbb{C}$. To prove the main results we use Nevanlinna theory. So it is assumed that the readers are familiar with standard notations like the characteristic function

$T(r, f)$, the proximity function $m(r, f)$, counting (reduced counting) function $N(r, f)$ ($\overline{N}(r, f)$) that are also explained in [14]. By $S(r, f)$ we mean any quantity that satisfies $S(r, f) = O(\log(rT(r, f)))$ when$r \to \infty$, except possibly on a set of finite Lebesgue measure. When f has finite order, then $S(r, f) = O(\log r)$ for all r.

Let us take $f \in \mathcal{M}(\mathbb{C})$, then the order of f is defined as

$$\rho(f) := \limsup_{r\to\infty} \frac{\log T(r, f)}{\log r}.$$

Throughout this paper by an L-function we mean an L-function in $\mathcal{S}^{\#}$ of non-zero degree with the normalized condition $a(1) = 1$.

Before proceeding further, we require the following definitions.

Definition 1.1. Let f and g be two non-constant meromorphic functions in $\mathcal{M}(\mathbb{C})$ and let S be a subset of $\mathbb{C}$. For some $a \in \mathbb{C} \cup \{\infty\}$, we define $E_f(S) = \cup_{a\in S}\{z : f(z) - a = 0\}$, where each point is counted according to its multiplicity. If we do not count the multiplicity, then the set $\cup_{a\in S}\{z : f(z) - a = 0\}$ is denoted by $\overline{E}_f(S)$. If $E_f(S) = E_g(S)$ then we say f and g share the set S CM. On the other hand, if $\overline{E}_f(S) = \overline{E}_g(S)$ then we say f and g share the set S IM.

Definition 1.2. Let f be a non constant meromorpic function. We denote by $S(f)$ the set of all small functions of f, i.e., $S(f) = \{a \in \mathcal{M}(\mathbb{C}) : T(r, a) = S(r, f)\ as\ r \to \infty\}$.

Definition 1.3 ([17])**.** Let k be a positive integer, $a \in S(f)$ and $E_k(0; f-a)$ be the set of all zeros of $f - a$, where a zero of multiplicity p is counted p times if $p \leq k$, and $k + 1$ times if $p > k$. If $E_k(0; f - a) = E_k(0; g - a)$, we say that $f - a$, $g - a$ share the 0 with weight k and we write it as $f - a$ and $g - a$ share $(0, k)$.

Clearly if $f - a$, $g - a$ share $(0, k)$ then $f - a$, $g - a$ share $(0, p)$ for any $0 \leq p < k$. Also we note that $f - a$, $g - a$ share a value 0 IM or CM if and only if $f - a$, $g - a$ share $(0, 0)$ or $(0, \infty)$ respectively.

For the sake of convenience, when a is chosen from $\mathbb{C} \cup \{\infty\}$, $f - a$ and $g - a$ share $(0, k)$, we say that f, g share (a, k).

Definition 1.4 ([6])**.** Let $a \in S(f)$, by $N(r, 0; f - a \,|\geq s)$ ($N(r, 0; f - a \,|\leq s)$) we denote the counting function of those zeros of $f - a$ of multiplicity $\geq s$ ($\leq s$). Also $\overline{N}(r, 0; f - a \,|\geq s)$ ($\overline{N}(r, 0; f - a \,|\leq s)$) are defined analogously

Definition 1.5. A polynomial P is called a uniqueness polynomial for meromorphic functions, if for any two non-constant meromorphic functions $f, g \in \mathcal{M}(\mathbb{C})$, the condition $P(f) = P(g)$ implies $f = g$. On the other hand, if for a non-zero constant c, the condition $P(f) = cP(g)$ implies $f = g$, then P is called a strong uniqueness polynomial.

Definition 1.6 ([9])**.** Let f be a non-constant meromorphic function. $a \in \mathbb{C}$ is said to be a generalized Picard exceptional value of f, if $f - a$ has at most finitely many zeros in $\mathbb{C}$.

In 2007, Steuding [12, p. 152] obtained a uniqueness result between two L-functions under sharing of values. It showed that unlike the case of uniqueness of meromorphic functions, in the case of L-functions, the number of shared values can be reduced significantly. Below we invoke the result.

Theorem 1.7 ([12])**.** *If two L-functions $\mathcal{L}_1$ and $\mathcal{L}_2$ share a complex value $c\ (\neq \infty)$ CM, then $\mathcal{L}_1 = \mathcal{L}_2$.*

Remark 1.8. Hu–Li [2] pointed out that in Theorem 1.7 one should add the condition $c \neq 1$.

In 2015, in case of sharing a finite subset of $\mathbb{C}\backslash\{1\}$ CM, Hu–Wu [3] obtained some uniqueness theorems for L-functions.

Theorem 1.9 ([3])**.** *Let n be a positive integer and consider a subset $S = \{c_1, \ldots, c_n\} \subset \mathbb{C}\backslash\{1\}$ of distinct complex numbers, satisfying*

$$n + (n-1)\sigma_1(c_1, \ldots, c_n) + \cdots + 2\sigma_{n-2}(c_1, \ldots, c_n) + \sigma_{n-1}(c_1, \ldots, c_n) \neq 0, \tag{1.1}$$

where σ_j are the elementary symmetric polynomials, defined by

$$\sigma_j(c_1, \ldots, c_n) = (-1)^j \sum_{1 \le i_1 < i_2 < \cdots < i_j \le n} c_{i_1} c_{i_2} \ldots c_{i_j} \quad j = 1, 2, \ldots, n-1.$$

If two L-functions share S CM, then they are equal.

Theorem 1.10 ([3])**.** *Let c_1, c_2 be two distinct complex numbers and take two positive integers k_1, k_2 with $k_1 k_2 > 1$. If two L-functions $\mathcal{L}_1$ and $\mathcal{L}_2$ share (c_1, k_1), (c_2, k_2), then $\mathcal{L}_1 = \mathcal{L}_2$.*

Theorem 1.11 ([3])**.** *Let us consider $c \in \mathbb{C}$ and $S = \{c_1, \ldots, c_n\} \subset \mathbb{C}\backslash\{1, c\}$ of distinct complex numbers, satisfying* (1.1)*.If two L-functions share (S, k_1), (c, k_2) and $k_1 k_2 > 1$, then they are equal.*

Notices that there exists a large class of unique range sets for L-functions, which do not satisfy the condition (1.1) (see [4]). Then in [4] the authors focused to determine certain conditions, such that the zero set of an arbitrary polynomial becomes unique range set for L-functions and a strong uniqueness polynomial for L-functions. Their results are given below.

Theorem 1.12 ([4])**.** *Let P be a uniqueness polynomial for L-functions. Suppose that P has no multiple zeros, and $P(1) \neq 0$. Then the zero set of P is a unique range set for L-functions, counting multiplicities.*

Theorem 1.13 ([4]). *Let P be a polynomial, satisfying $P(1)P'(1) \neq 0$. Then P is a strong uniqueness polynomial for L-functions.*

However, from Theorem 1.13, we cannot assure whether a polynomial P, satisfying $P(1) = 0$, is a strong uniqueness polynomial for L-functions or not. For example, let $n \geq 13$, consider the polynomial

$$P(z) = z^n - 2z^{n-2} + 1.$$

In view of Theorem 1.7, (see [15]), zero set S of P is a unique range set for meromorphic functions and also a uniqueness polynomial but P does not satisfy the hypothesis of Theorem 1.12. Analogously, the set $S = \{z \mid z^n - nz + 1 = 0\}$, $(n \geq 13)$, is a unique range set for meromorphic functions. But the polynomial $P(z) = z^n - nz + 1$ does not satisfy the hypothesis of Theorem 1.13, although it is a strong uniqueness polynomial for meromorphic functions (see [1]) and of course, for L-functions. So we see that regarding the statement of Theorems 1.12, 1.13 further investigations are needed.

In this paper we will show that it is possible to remove the condition over $P(1)$ provided that the L-functions satisfy the same functional equation.

To proceed further, let us consider the polynomials

$$(1.2) \quad P(w) = a_0 + a_1 w + a_2 w^2 + \cdots + a_n w^n \quad \text{and} \quad Q(w) = P(w) - a_0,$$

where P has all simple zeros and Q is a polynomial with m $(\leq n)$ distinct zeros. Also let us denote

$$(1.3) \quad P_1(w) = a_n w^{n-1} + (a_n + a_{n-1}) w^{n-2} + (a_n + a_{n-1} + a_{n-2}) w^{n-3} + \cdots + (a_n + \cdots + a_1),$$

for some $a_i \in \mathbb{C}$ $(i = 1, 2, \ldots, n)$ and $a_n \neq 0$. Clearly here $P'(1) = P_1(1)$.

Through out the following discussion by P, Q and P_1 we will denote the polynomials as defined in (1.2), (1.3).

Theorem 1.14. *Let P be a polynomial with simple zeros and $P'(0) \neq 0$ or $P'(1) \neq 0$. Then P is a strong uniqueness polynomial for those L-functions which satisfy the same functional equation.*

Again from the proof of Theorems 1.12, 1.13 we can easily conclude that if two non-constant L-functions share the zero set of a polynomial P CM, with $P(1) \cdot P'(1) \neq 0$, then they are identical. We will show that these results are also obtainable for L-functions satisfying the same functional type equation, where the restrictions on the polynomial, namely $P(1) \cdot P'(1) \neq 0$, are no longer required. So it will be interesting to consider the cases: (i) $P(1) = 0$ but $P'(1) \neq 0$; (ii) $P'(1) = 0$ but $P(1) \neq 0$. As both the cases fall under $P(1) \cdot P'(1) = 0$, it will be interesting to investigate the result when $\mathcal{L}_1$ and $\mathcal{L}_2$ share the zero set of a polynomial P CM. Here with the help of Theorem 1.14 we can obtain our next result. In fact, in the following result, for L-functions having the same functional type equation, the range of URS (Unique Range Set) is extended.

Theorem 1.15. *Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two L-functions satisfying the same functional equation. Also let P be a polynomial with simple zeros and $P'(0) \neq 0$ or $P'(1) \neq 0$. If $\mathcal{L}_1$ and $\mathcal{L}_2$ share $S = \{w : P(w) = 0\}$ CM, then $\mathcal{L}_1 = \mathcal{L}_2$.*

From Theorems 1.14, 1.15, the following corollaries can be obtained immediately.

Corollary 1.16. *Let n be a positive integer and consider a subset $S = \{c_1, \ldots, c_n\} \subset \mathbb{C}$ of distinct complex numbers, satisfying (1.1) or $\sigma_{n-1} \neq 0$. If two L-functions, satisfying the same functional equation, share S CM, then they are equal.*

Corollary 1.17. *Let us consider $c \in \mathbb{C}$ and $S = \{c_1, \ldots, c_n\} \subset \mathbb{C}\backslash\{c\}$ of distinct complex numbers, satisfying (1.1) or $\sigma_{n-1} \neq 0$. If two L-functions, satisfying the same functional equation, share (S, k_1), (c, k_2) and $k_1 k_2 > 1$, then they are equal.*

In Theorems 1.9–1.11 the authors actually show that if two L-functions share a set CM, then under certain suppositions, they become identical. As we know, L-functions possess meromorphic continuations, so naturally a question arises whether it is possible to find a uniqueness relation between a meromorphic function f and an L-function $\mathcal{L}$, while they share some sets. In this paper, inspired by the previous results, imposing some restrictions on a meromorphic function f, we obtain some relations between f and $\mathcal{L}$ while they are sharing some sets. In particular, if we choose $f = \mathcal{L}_1$ and $\mathcal{L} = \mathcal{L}_2$, where both $\mathcal{L}_1$, $\mathcal{L}_2$ are L-functions, then we get all the earlier results.

Theorem 1.18. *Let f be a meromorphic function of finite order having a pole at $s = 1$ at most, and let $\lim_{\mathrm{Re}(s)\to\infty} f(s) = A$ $(A \neq \infty)$. Also let $\mathcal{L}$ be a non-constant L-function and P be a polynomial such that $P(1) \cdot P(A) \neq 0$ and $P_1(A) \neq 0$, and f, $\mathcal{L}$ share the set $S = \{w : P(w) = 0\}$ CM. Then $f = \mathcal{L}$ if one of the following conditions is satisfied:*

(i) *There exists a $s_0 \in \mathbb{C}$ where f, $\mathcal{L}$ assume the same value c with $P(c) \neq 0$,*

(ii) *$n > 2m$, where n, m are defined in (1.2).*

Corollary 1.19. *Let f be a meromorphic function of finite order having pole at $s = 1$ at most, and $\lim_{\mathrm{Re}(s)\to\infty} f(s) = A$ $(A \neq \infty)$. Also let $\mathcal{L}$ be a non-constant L-function and P be a polynomial such that $P(1) \cdot P(A) \neq 0$ and $P_1(A) \neq 0$.*

If f and $\mathcal{L}$ share $S = \{w : P(w) = 0\}$ CM, then either $f = \mathcal{L}$ or $P(f) = CP(\mathcal{L})$ for the constant $C = \frac{P(A)}{P(1)}$.

Theorem 1.20. *Let f be a meromorphic function of finite order with finitely many poles and $\lim_{\mathrm{Re}(s)\to\infty} f(s) = A$ $(A \neq \infty)$. Also let $\mathcal{L}$ be a non-constant L-function and P be a polynomial such that $P(1)\cdot P(A) \neq 0$ and $P_1(A) \neq 0$. If f and $\mathcal{L}$ share $(S = \{w : P(w) = 0\}, k_1)$ and (b, k_2) $(b \in \mathbb{C}; P(b) \neq 0)$, where $k_1k_2 > 1$, then $f = \frac{\alpha_1 - b}{\alpha_2 - b}(\mathcal{L} - b) + b$ for some α_1, α_2; with $P(\alpha_1) = 0 = P(\alpha_2)$.*

Remark 1.21. Clearly Theorem 1.20 improves Theorem 1.11.

Theorem 1.22. *Let f be a meromorphic function having finitely many poles and $\mathcal{L}$ be a non-constant L-function. Also let us consider a_i $(i = 1, 2)$ be two distinct rational functions. If $f - a_i$ and $\mathcal{L} - a_i$ share $(0, k_i)$ for $i = 1, 2$ where $k_1k_2 > 1$ then $f = \mathcal{L}$.*

Remark 1.23. Theorem 1.22 improves Theorem 1.10.

In the following example we will show that the condition $P(1)\cdot P(A) \neq 0$ cannot be dropped.

Example 1.24. Let $\mathcal{L} = \zeta$ and $f = \omega\zeta$, where $\omega^n = 1$. Clearly f and $\mathcal{L}$ share $\{a : a^n - 1 = 0\}$ and both have simple pole only at $s = 1$, $\lim_{\sigma\to\infty} f = \omega$, but $f \neq \mathcal{L}$.

2. Lemmas

Lemma 2.1 ([13])**.** *Let f be a non-constant meromorphic function on $\mathbb{C}$ and also let a_j $(j = 1, 2, \ldots, q)$ be q distinct elements in $S(f) \cup \{\infty\}$. Then we have*

$$(q - 2)T(r, f) \leq \sum_{j=1}^{q} \overline{N}(r, f = a_j).$$

Lemma 2.2 (see [14, Theorem 1.14]))**.** *Let f, $g \in \mathcal{M}(\mathbb{C})$ and let $\rho(f)$, $\rho(g)$ be the order of f and g, respectively. Then*

$$\rho(f\cdot g) \leq \max\{\rho(f), \rho(g)\}.$$

Lemma 2.3. *Let $\mathcal{L}$ be a non-constant L-function and α be a rational function. Then $\mathcal{L} - \alpha$ has no generalized Picard exceptional values in $\mathbb{C}$.*

Proof. On the contrary, assume that $\mathcal{L} - \alpha$ has finitely many zeros. Let $a_1, a_2, \ldots, a_t$ be the zeros of $\mathcal{L} - \alpha$ with multiplicity $k_1, k_2, \ldots, k_t$, respectively. Again, since $\mathcal{L}$ has only one pole and $N(r, \infty; \alpha) = O(\log r)$, then suppose $b_1, b_2, \ldots, b_p$ are the poles of $\mathcal{L} - \alpha$ with multiplicity $l_1, l_2, \ldots, l_p$. Then we have,

$$\frac{(s - b_1)^{l_1}(s - b_2)^{l_2}\ldots(s - b_p)^{l_p}(\mathcal{L}(s) - \alpha(s))}{(s - a_1)^{k_1}(s - a_2)^{k_2}\ldots(s - a_t)^{k_t}} = e^{p(s)}. \tag{2.1}$$

From (2.1)

$$\mathcal{L}(s)=\alpha(s)+\frac{(s-a_1)^{k_1}(s-a_2)^{k_2}\dots(s-a_t)^{k_t}.e^{p(s)}}{(s-b_1)^{l_1}(s-b_2)^{l_2}\dots(s-b_p)^{l_p}}. \tag{2.2}$$

Therefore from (2.2) we have

$$T(r,\mathcal{L})=r^{\deg(p)}+O(\log r). \tag{2.3}$$

Also by Steuding [12, p. 150] we have

$$T(r,\mathcal{L})=\frac{d_{\mathcal{L}}}{\pi}r\log r+O(r), \tag{2.4}$$

where $d_{\mathcal{L}}$ is the degree of $\mathcal{L}$.

Then clearly $\rho(\mathcal{L})=1$, and since the order of rational function is zero, from (2.1) and Lemma 2.2 we see that p is a polynomial of degree ≤ 1.

So (2.3) implies $T(r,\mathcal{L})\leq O(r)$. Hence from (2.4) we arrive at a contradiction. Therefore $\mathcal{L}-\alpha$ has infinitely many zeros. □

Lemma 2.4 ([10]). *Let f be a non-constant meromorphic function and let*

$$R(f)=\frac{\sum_{k=0}^{n}a_kf^k}{\sum_{j=0}^{m}b_jf^j}$$

be an irreducible rational function in f with constant coefficients $\{a_k\}$ and $\{b_j\}$, where $a_n\neq 0$ and $b_m\neq 0$. Then

$$T(r,R(f))=dT(r,f)+S(r,f),$$

where $d=\max\{n,m\}$.

3. Proofs of the theorems

Proof of Theorem 1.18. Because f and $\mathcal{L}$ share the zero set of P CM, $P(f)$ and $P(\mathcal{L})$ share 0 CM. Let us consider the following function

$$F=\frac{P(f)}{P(\mathcal{L})}.$$

Clearly F has finitely many zeros and poles, which come from the poles of $\mathcal{L}$ and f, respectively. Also it is given that f is an entire function or a meromorphic function having pole only at $s=1$. Then we can write F as

$$F=\frac{P(f)}{P(\mathcal{L})}=(s-1)^ke^{p(s)}, \tag{3.1}$$

for some integer k. Since f is of finite order, from (2.4) we see that $\mathcal{L}$ is of order 1, then from Lemma 2.2, it is obvious that $\rho(e^p)$ is finite, and hence $p(s)$ is a polynomial of finite degree.

Now we can write,

$$(s-1)^ke^{p(s)}=(\sigma+it-1)^ke^{p(\sigma+it)},\quad \sigma,t\in\mathbb{R}. \tag{3.2}$$

Suppose

$$\operatorname{Re}(p(s)) = a_0(t) + a_1(t)\sigma + \cdots + a_p\sigma^p, \tag{3.3}$$

where $a_i(t)$ $(i = 0, 1, \ldots, p)$ are polynomials in t.

Here we have

$$\lim_{\operatorname{Re}(s)\to\infty} \frac{P(f)}{P(\mathcal{L})} = \frac{P(A)}{P(1)}, \tag{3.4}$$

which together with (3.1) and (3.2) implies for every fixed t

$$\lim_{\operatorname{Re}(s)\to\infty} |(s-1)^k e^{p(s)}| = \lim_{\sigma\to\infty} |(\sigma - 1 + it)|^k e^{\operatorname{Re}(p(\sigma+it))} = \left|\frac{P(A)}{P(1)}\right|. \tag{3.5}$$

Now we prove that $a_i(t) = 0$, for $i = 1, 2, \ldots, p$. If not, let us take $a_p(t) \not\equiv 0$ for some $p = 1, 2, \ldots$. Without loss of generality let us take some $t = t_0$ where $a_p(t_0) \neq 0$. Then we can have from (3.5),

$$\lim_{\sigma\to\infty} |(\sigma - 1 + it_0)|^k e^{\operatorname{Re}(p(\sigma+it_0))} = 0 \text{ or } \infty,$$

according as $a_p(t_0) < 0$ or $a_p(t_0) > 0$, a contradiction. Hence $a_p(t_0) = 0$, and since t_0 is arbitrary, our claim is proved. Hence $\operatorname{Re}(p) = a_0(t)$. Then from above we must have $k = 0$, otherwise, $\lim_{\sigma\to\infty} |(\sigma - 1 + it)|^k e^{a_0(t)} = 0$ or ∞ according as $k < 0$ or $k > 0$.

Therefore we must have $|(s-1)^k e^{p(s)}| = e^{a_0(t)}$ and from (3.1) and (3.5) we get $\lim_{\operatorname{Re}(s)\to\infty} e^{a_0(t)} = \left|\frac{P(A)}{P(1)}\right|$ $\forall\, t \in \mathbb{R}$. Hence $e^{a_0(t)} =$ constant. Here p is a polynomial with constant real part, therefore, by the Cauchy–Riemann equations, p is a constant polynomial.

Therefore, from (3.1) we have $\frac{P(f)}{P(\mathcal{L})} =$ constant. Again $\lim_{\operatorname{Re}(s)\to\infty} \frac{P(f)}{P(\mathcal{L})} = \frac{P(A)}{P(1)}$, and hence finally we have $\frac{P(f)}{P(\mathcal{L})} = \frac{P(A)}{P(1)}$, i.e., $P(f) = \frac{P(A)}{P(1)} P(\mathcal{L})$.

Let us assume that there is some $s_0 \in \mathbb{C}$ such that $f(s_0) = \mathcal{L}(s_0) = c$. Then we have $P(f) = P(\mathcal{L})$, and

$$(f - \mathcal{L})(a_n(f^{n-1} + f\mathcal{L}^{n-2} + \cdots + \mathcal{L}^{n-1}) + \cdots + a_2(\mathcal{L} + f) + a_1) = 0. \tag{3.6}$$

From (3.6) we cannot have

$$a_n(f^{n-1} + f^{n-2}\mathcal{L} + \cdots + \mathcal{L}^{n-1}) + \cdots + a_2(\mathcal{L} + f) + a_1 = 0,$$

otherwise taking $\lim_{\operatorname{Re}(s)\to\infty}$ we get $P_1(A) = 0$. Therefore we must have $f = \mathcal{L}$.

Let us consider the case when $n > 2m$, where m is the number of distinct zeros of Q as defined in (1.2). Then for $C = \frac{P(A)}{P(1)}$, we can write

$$P(f) = CP(\mathcal{L})$$
$$Q(f) + a_0 = C(Q(\mathcal{L}) + a_0),$$

i.e., $Q(f) - CQ(\mathcal{L}) = a_0(C - 1)$.

Using Lemma 2.4, we obtain

$$T(r,f)=T(r,\mathcal{L})+O(\log r). \tag{3.7}$$

Using the Second Fundamental Theorem and (3.7), if $C\neq 1$ we have,

$$\begin{aligned}nT(r,f)&=T(r,Q(f))\\&\leq \overline{N}(r,\infty;Q(f))+\overline{N}(r,0;Q(f))+\overline{N}(r,a_0(C-1);Q(f))+S(r,f)\\&\leq \overline{N}(r,0;Q(f))+\overline{N}(r,0;Q(\mathcal{L}))+O(\log r)\\&\leq 2mT(r,f)+O(\log r),\end{aligned}$$

a contradiction. Therefore $C=1$, and hence $P(f)=P(\mathcal{L})$. Then using the fact that $P_1(A)\neq 0$ and doing the same as in (3.6) we get $f=\mathcal{L}$. □

Proof of Theorem 1.20. It is given that f and $\mathcal{L}$ share (S,k_1), (b,k_2). It follows that $P(f)$ and $P(\mathcal{L})$ share $(0,k_1)$. Using the Second Fundamental Theorem we have,

$$\begin{aligned}nT(r,\mathcal{L})&\leq \overline{N}(r,0;P(\mathcal{L}))+\overline{N}(r,b;\mathcal{L})+\overline{N}(r,\infty;\mathcal{L})+S(r,\mathcal{L})\\&\leq \overline{N}(r,0;P(f))+\overline{N}(r,b;f)+O(\log r)+S(r,\mathcal{L})\\&\leq (n+1)T(r,f)+S(r,\mathcal{L}).\end{aligned} \tag{3.8}$$

(3.8) implies $\rho(\mathcal{L})\leq\rho(f)$.

Similarly, we get

$$nT(r,f)\leq (n+1)T(r,\mathcal{L})+O(\log r)+S(r,f), \tag{3.9}$$

and thus $\rho(\mathcal{L})\geq\rho(f)$. Finally we have $\rho(\mathcal{L})=\rho(f)$. From (2.4), $\rho(\mathcal{L})=1$, and hence $S(r,f)=S(r,\mathcal{L})=O(\log r)$.

Now let us consider the auxiliary functions:

$$\phi=\frac{f'}{f-b}-\frac{\mathcal{L}'}{\mathcal{L}-b},\quad \psi=\frac{(P(f))'}{P(f)}-\frac{(P(\mathcal{L}))'}{P(\mathcal{L})}=\frac{P'(f)f'}{P(f)}-\frac{P'(\mathcal{L})\mathcal{L}'}{P(\mathcal{L})},$$

here it is given that $P(b)\neq 0$. We consider the following two cases:

Case-I. $\phi\cdot\psi\neq 0$. Let us assume that z_0 is a zero of $P(f)=(f-\alpha_1)\times(f-\alpha_2)\ldots(f-\alpha_n)$ (say), of order $\geq k_1+1$. Then it becomes a zero of $P(\mathcal{L})$ of order $\geq k_1+1$. Suppose that z_0 is a zero of $f-\alpha_i$ and $\mathcal{L}-\alpha_j$ for some $1\leq i,j\leq n$. Then z_0 is a zero of f' and $\mathcal{L}'$ of order $\geq k_1$, and hence it is a zero of ϕ of order $\geq k_1$. Therefore, $k_1\overline{N}(r,0;P(f)\mid\geq k_1+1)\leq N(r,0;\phi)$.

Again, let z_1 be a zero of $f-b$ of order $\geq k_2+1$. Since f, $\mathcal{L}$ share (b,k_2), then z_1 becomes a zero of $\mathcal{L}-b$ of order $\geq k_2+1$, and is a zero of f' and $\mathcal{L}'$ of order $\geq k_2$. It is assumed that $P(b)\neq 0$, then it can be verified that z_1 is a zero of ψ of order at least k_2, and hence, $k_2\overline{N}(r,b;f\mid\geq k_2+1)\leq N(r,0;\psi)$.

Hence we have

$$\begin{aligned}(3.10)\quad k_1\overline{N}(r,0;P(f)\mid\geq k_1+1) &\leq N(r,0;\phi)\\ &\leq T(r,\phi)+S(r,f)\\ &\leq \overline{N}(r,\infty;\phi)+S(r,f)\\ &\leq \overline{N}(r,b;f\mid\geq k_2+1)+O(\log r)\\ &\leq \frac{1}{k_2}N(r,0;\psi)+O(\log r)\\ &\leq \frac{1}{k_2}T(r,\psi)+O(\log r)\\ &\leq \frac{1}{k_2}\overline{N}(r,0;P(f)\mid\geq k_1+1)+O(\log r).\end{aligned}$$

Since $k_1k_2>1$, from (3.10) we get,

$$\overline{N}(r,0;P(f)\mid\geq k_1+1)=O(\log r).$$

Similarly, it can be shown that $\overline{N}(r,b;f\mid\geq k_2+1)=O(\log r)$.

So $T(r,\phi)=O(\log r)=T(r,\psi)$, i.e., ϕ and ψ are rationals. Then it can be written as

$$\phi=\frac{f'}{f-b}-\frac{\mathcal{L}'}{\mathcal{L}-b}=P_1+\sum_{i=1}^{p}\frac{m_i}{s-\alpha_i} \tag{3.11}$$

and

$$\psi=\frac{(P(f))'}{P(f)}-\frac{(P(\mathcal{L}))'}{P(\mathcal{L})}=P_2+\sum_{i=1}^{q}\frac{n_i}{s-\beta_i}, \tag{3.12}$$

where P_1, P_2 are two polynomials, m_i $(i=1,2,\dots,p)$, n_i $(i=1,2,\dots,q)$ are some constants, α_i $(i=1,2,\dots,p)$, β_i $(i=1,2,\dots,q)$ are the poles of ϕ, ψ, respectively. Note that some α_i's may coincide with some β_i's.

Integrating (3.11) and (3.12) we obtain

$$\phi_o=\frac{f-b}{\mathcal{L}-b}=c_1\prod_{i=1}^{p}(s-\alpha_i)^{m_i}e^{\int P_1}=Q_1e^{\hat{P}_1}, \tag{3.13}$$

$$\psi_o=\frac{P(f)}{P(\mathcal{L})}=c_2\prod_{i=1}^{q}(s-\beta_i)^{n_i}e^{\int P_2}=Q_2e^{\hat{P}_2}, \tag{3.14}$$

where c_i's $(i=1,2)$ are non-zero constants. From (3.13), (3.14) and Lemma 2.2 we must have $\deg\widehat{P}_i\leq 1$. Let us consider $\widehat{P}_i(s)=a_is+b_i$, for some complex constants a_i,b_i $(i=1,2)$.

Also $\lim_{\mathrm{Re}(s)\to\infty}\psi_o=\frac{P(A)}{P(1)}$, it implies $\lim_{\mathrm{Re}(s)\to\infty}Q_2e^{\hat{P}_2}=\frac{P(A)}{P(1)}$, $\forall\ s=\sigma+it\in\mathbb{C}$.

Therefore we must have $a_2=0$, and then $\lim_{\sigma\to\infty}Q_2$ is finite.

Again in that case for the existence of the limit as $\sigma \to \infty$, from above we get $Q_2 = c_2 \frac{\prod_{i=1}^{j}(\sigma-\beta_i)^{m_i}}{\prod_{k=j+1}^{q}(\sigma-\beta_k)^{-m_k}}$, where $\sum_{i=1}^{j} m_i = -\sum_{k=j+1}^{q} m_k$ and $\lim_{\mathrm{Re}(s)\to\infty} Q_2 = c_2$. Therefore, $c_2 e^{b_2} = \frac{P(A)}{P(1)}$, and finally we have

$$\psi_o = \frac{P(f)}{P(\mathcal{L})} = \frac{1}{c_2}\frac{P(A)}{P(1)}Q_2. \tag{3.15}$$

Suppose $\frac{Q_2}{c_2} - \frac{P(1)}{P(A)} \neq 0$. Now, let z_2 be a zero of $f-b$ and $\mathcal{L}-b$, then $P(f(z_2)) = P(b) = P(\mathcal{L}(z_2))$. Therefore from (3.15), z_2 becomes a zero of $\frac{P(b)}{P(b)} - \frac{1}{c_2}\frac{P(A)}{P(1)}Q_2$, i.e., z_2 is a zero of the rational function $\frac{P(1)}{P(A)} - \frac{Q_2}{c_2}$. Hence, $\overline{N}(r,b;f) \leq \overline{N}(r,0;\frac{Q_2}{c_2} - \frac{P(1)}{P(A)}) = O(\log r)$. From Lemma 2.3, this implies a contradiction as $d_{\mathcal{L}} > 0$. Hence, we must have $\frac{Q_2}{c_2} - \frac{P(1)}{P(A)} = 0$ and then $P(f) = P(\mathcal{L})$. Since $P_1(A) \neq 0$, from (3.6) we get $f = \mathcal{L}$, which is a special case of the conclusion of the theorem, under $\alpha_1 = \alpha_2$.

Case-II. $\phi \cdot \psi = 0$. Suppose $\psi = 0$ i.e., $\frac{P'(f)}{P(f)} - \frac{P'(\mathcal{L})}{P(\mathcal{L})} = 0$. Then we have $\frac{P(f)}{P(\mathcal{L})} =$ constant. There exist $s_1 \in \mathbb{C}$ such that $f(s_1) = \mathcal{L}(s_1) = b$, and using this we get $P(f) = P(\mathcal{L})$, and hence $f = \mathcal{L}$.

If $\phi = 0$, then we have $\frac{f-b}{\mathcal{L}-b} = c$, a constant. Also as f and $\mathcal{L}$ share the set of zeros of P, there exists some s_0 such that $f(s_0) = \alpha_1$ and $\mathcal{L}(s_0) = \alpha_2$, where $P(\alpha_1) = 0 = P(\alpha_2)$. Then we get $c = \frac{\alpha_1-b}{\alpha_2-b}$, and thus we have the result.

It is easy to verify that if $b \neq 1, A$, then we have $c = \frac{\alpha_1-b}{\alpha_2-b} = \frac{A-b}{1-b}$. □

Proof of Theorem 1.22. It is given that $f-a_1$, $\mathcal{L}-a_1$ share $(0,k_1)$ and $f-a_2$, $\mathcal{L}-a_2$ share $(0,k_2)$. Using Lemma 2.1 and doing the same as in (3.8)–(3.9), we have $\rho(f) = \rho(\mathcal{L})$ $(=1)$. Hence $S(r,f) = S(r,\mathcal{L}) = O(\log r)$. Let us consider the following functions

$$\widehat{F} = \frac{f'-a_1'}{f-a_1} - \frac{\mathcal{L}'-a_1'}{\mathcal{L}-a_1}, \quad \widehat{G} = \frac{f'-a_2'}{f-a_2} - \frac{\mathcal{L}'-a_2'}{\mathcal{L}-a_2}.$$

First assume that $\widehat{F} \neq 0$ and $\widehat{G} \neq 0$. Proceeding similarly as done in (3.10) we get $\overline{N}(r,0;f-a_1 \mid \geq k_1+1) = O(\log r)$ and $\overline{N}(r,0;f-a_2 \mid \geq k_2+1) = O(\log r)$.

Let us consider the following functions

$$H = Q\frac{f-a_1}{\mathcal{L}-a_1},$$

where Q is a rational function such that H has neither a pole nor a zero in $\mathbb{C}$. Such a Q does exist, since $\overline{N}(r,\infty;f) = \overline{N}(r,\infty;\mathcal{L}) = \overline{N}(r,0;$

$f-a_1\mid\geq k_1+1)=\overline{N}(r,0;\mathcal{L}-a_1\mid\geq k_1+1)=O(\log r)$. Thus, H is an entire function without any zeros. Hence, it can be expressed as

$$H=Q\frac{f-a_1}{\mathcal{L}-a_1}=e^{\eta},$$

for some entire function η. Since $\rho(f)=\rho(\mathcal{L})=1$, from Lemma 2.2 we can write $\eta=\alpha s+\beta$ for some $\alpha,\beta\in\mathbb{C}$. Let,

$$I=\Bigg(\frac{(f-a_2)(f'-a_1')-(f-a_1)(f'-a_2')}{(f-a_1)(f-a_2)} \\ -\frac{(\mathcal{L}-a_2)(\mathcal{L}'-a_1')-(\mathcal{L}-a_1)(\mathcal{L}'-a_2')}{(\mathcal{L}-a_1)(\mathcal{L}-a_2)}\Bigg)(f-\mathcal{L}).$$

Assume that $I\neq 0$. It is easy to verify that

$$T(r,I)=m(r,I)+N(r,I)\leq O(T(r,H))+O(\log r)\leq O(r).$$

Using the Second Fundamental Theorem for small functions we have

$$\begin{aligned}(3.16)\quad T(r,\mathcal{L})&\leq\overline{N}(r,0;f-a_2)+\overline{N}(r,0;f-a_1)+\overline{N}(r,\infty;\mathcal{L})+S(r,\mathcal{L})\\ &\leq\overline{N}\left(r,1;\frac{H}{Q}\right)+\overline{N}(r,0;I)+O(\log r)\\ &\leq O\left(T(r,e^{\alpha s+\beta})\right)+O(\log r)\leq O(r).\end{aligned}$$

Hence (3.16) contradicts (2.4).

Therefore either $\frac{H}{Q}=1$ or $I=0$. In case $\frac{H}{Q}=1$ we have immediately that $f=\mathcal{L}$. Suppose that $I=0$ and that by absurd $f\neq\mathcal{L}$. Then we have,

$$\frac{f'-a_1'}{f-a_1}-\frac{f'-a_2'}{f-a_2}-\frac{\mathcal{L}'-a_1'}{\mathcal{L}-a_1}+\frac{\mathcal{L}'-a_2'}{\mathcal{L}-a_2}=0$$

Integrating we have

$$\frac{f-a_1}{f-a_2}=c\frac{\mathcal{L}-a_1}{\mathcal{L}-a_2},$$

for some constant c. Clearly from above $T(r,f)=T(r,\mathcal{L})+O(\log r)$.

If $c=1$ then we get $f=\mathcal{L}$. Again if $c\neq 1$ then from above we get $\frac{2f-(a_1+a_2)}{a_2-a_1}=\frac{(c+1)\mathcal{L}-(ca_1+a_2)}{(c-1)\mathcal{L}-ca_1+a_2}$, and since $N(r,\infty;f)=O(\log r)$, we have $\overline{N}(r,0;\mathcal{L}-\frac{ca_1-a_2}{c-1})=O(\log r)$, a contradiction to Lemma 2.3, as $d_{\mathcal{L}}>0$.

Now assume at least one of $\widehat{F}$, $\widehat{G}$ is equivalent to zero. Let us assume, $\widehat{F}=0$, i.e., $\frac{f'-a_1'}{f-a_1}-\frac{\mathcal{L}'-a_1'}{\mathcal{L}-a_1}=0$. Integrating we have $f-a_1=c_1(\mathcal{L}-a_1)$ for some non-zero constant. We have some s_0, s.t $f(s_0)=\mathcal{L}(s_0)=a_2(s_0)$, and it implies $c_1=1$. Hence we get, $f=\mathcal{L}$.

Similarly, if $\widehat{G}=0$ we get $f=\mathcal{L}$. □

Proof of Theorem 1.14. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two L-functions, satisfying equation $P(\mathcal{L}_1) = cP(\mathcal{L}_2)$, where $P(s) = a_0 + a_1 s + \cdots + a_n s^n$ and c $(\neq 0)$ is some constant. Then we have,

$$\frac{P(\mathcal{L}_1)}{P(\mathcal{L}_2)} = c. \tag{3.17}$$

Since $\mathcal{L}_1$ and $\mathcal{L}_2$ are of non-zero degree and satisfy the same functional equation, $\mathcal{L}_1 - \mathcal{L}_2$ also satisfies the same functional equation. From the proof of the (a)-part of the Lemma in [5, p. 2487], we know that for some sufficiently large constant $\kappa(> 0)$ and large $\kappa_0(> 0)$, $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}_1 - \mathcal{L}_2$ and the Gamma function $\prod_{j=1}^{K}(\Gamma(\lambda_j s + \nu_j))^{-1}$ in the functional equation of axiom (iii) have the same zeros in the region $\operatorname{Re} s < -\kappa_0$ and $|\operatorname{Im} s| < \kappa$, and $\mathcal{L}_1$, $\mathcal{L}_2$, while $\mathcal{L}_1 - \mathcal{L}_2$ have no zeros in $\operatorname{Re} s \geq \kappa_0$.

Next we let us consider the following two cases:

Case-I. $a_0 \neq 0$. Let s_0 be a common zero of $\mathcal{L}_1$ and $\mathcal{L}_2$ in the region $\operatorname{Re} s < -\kappa_0$ and $|\operatorname{Im} s| < \kappa$. Putting $s = s_0$ in (3.17) we have $c = 1$. Hence,

$$P(\mathcal{L}_1) = P(\mathcal{L}_2) \tag{3.18}$$

This implies,

$$a_n(\mathcal{L}_1^n - \mathcal{L}_2^n) + \cdots + a_2(\mathcal{L}_1^2 - \mathcal{L}_2^2) + a_1(\mathcal{L}_1 - \mathcal{L}_2) = 0$$

i.e.,

$$\begin{aligned}(\mathcal{L}_1 - \mathcal{L}_2)(a_n(\mathcal{L}_1^{n-1} + \mathcal{L}_1^{n-2}\mathcal{L}_2 + \cdots + \mathcal{L}_2^{n-1})\\ + a_{n-1}(\mathcal{L}_1^{n-2} + \mathcal{L}_1^{n-3}\mathcal{L}_2 + \cdots + \mathcal{L}_2^{n-2}) + \cdots + a_1) = 0,\end{aligned}$$

If $\mathcal{L}_1 - \mathcal{L}_2 \neq 0$ then from above we must have

$$\sum_{k=1}^{n} a_k(\mathcal{L}_1^{k-1} + \mathcal{L}_1^{k-2}\mathcal{L}_2 + \cdots + \mathcal{L}_1\mathcal{L}_2^{k-2} + \mathcal{L}_2^{k-1}) = 0. \tag{3.19}$$

Considering limit as $\operatorname{Re}(s) \to \infty$ from (3.19) we have

$$\lim_{\operatorname{Re}(s)\to\infty} \sum_{k=1}^{n} a_k(\mathcal{L}_1^{k-1} + \mathcal{L}_1^{k-2}\mathcal{L}_2 + \ldots \mathcal{L}_1 L_2^{k-2} + \mathcal{L}_2^{k-1}) = 0,$$

i.e., $na_n + \cdots + 2a_2 + a_1 = P^{'}(1) = 0$. Also putting $s = s_0$ in (3.19) we get $a_1 = P'(0) = 0$. Hence we see that both $P'(1)$ and $P'(0)$ are simultaneously equal to zero, a contradiction. Therefore we must have $\mathcal{L}_1 = \mathcal{L}_2$.

Case-II. $a_0 = 0$. Since P has no multiple zero, $a_1 \neq 0$. Clearly then $P(\mathcal{L}_i) = \mathcal{L}_i(a_1 + a_2\mathcal{L}_i + \cdots + a_n\mathcal{L}_i^{n-1})$ for $i = 1, 2$.

The multiplicative inverse of Gamma function has zeros at $s = -1, -2, \ldots, -n, \ldots$, therefore $\prod_{j=1}^{K}(\Gamma(\lambda_j s + \nu_j))^{-1}$ has zeros at $s = \frac{-n-\nu_j}{\lambda_j}$ for $n = 1, 2, \ldots$ and $j = 1, 2, \ldots, K$. Since it is assumed that $d_{\mathcal{L}} > 0$, we must have $K > 0$. Here for some j or fixing some j it is possible to get a infinite sequence $\left\{s_n = \frac{-N-\nu_j}{\lambda_j}\right\}_{n=1}^{\infty}$ of common zeros of $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}_1 - \mathcal{L}_2$ and $\prod_{j=1}^{K}(\Gamma(\lambda_j s + \nu_j))^{-1}$ in $\mathrm{Re}(s) < -\kappa_0$ and $|\mathrm{Im}(s)| < \kappa$, where $N = n+$ constant, while no-one of $\mathcal{L}_1, \mathcal{L}_2$, $\mathcal{L}_1 - \mathcal{L}_2$ vanishes for $\mathrm{Re}(s) \geq \kappa_0$. Here $\mathrm{Re}(s_n) = -N - \mathrm{Re}(\nu_j) < -\kappa_0 < 0$, we have $s_n \to -\infty$ as n diverges.

Since $\mathcal{L}_1$, $\mathcal{L}_2$ and $\mathcal{L}_1 - \mathcal{L}_2$ satisfy the same functional equation, it follows that they have the same degree $d_{\mathcal{L}} > 0$. From the functional equation we have also that

$$\mathcal{L}_i(s) = \chi(s)\overline{\mathcal{L}_i(1-\bar{s})}, \quad \text{for } i = 1, 2$$

and

$$\begin{aligned}\mathcal{L}_1(s) - \mathcal{L}_2(s) &= \chi(s)\left(\overline{\mathcal{L}_1(1-\bar{s})} - \overline{\mathcal{L}_2(1-\bar{s})}\right)\\ &= \chi(s)\left(\overline{\mathcal{L}_1(1-\bar{s}) - \mathcal{L}_2(1-\bar{s})}\right),\end{aligned}$$

where $\chi(s) = \frac{\omega Q^{1-2s}\prod_{j=1}^{K}\Gamma(\lambda_j(1-s)+\bar{\nu}_j)}{\prod_{j=1}^{K}\Gamma(\lambda_j s+\nu_j)}$.

Clearly, here $\mathcal{L}_i(1-\overline{s_n}) \neq 0$ for $i = 1, 2$; for all n, since $\mathcal{L}_i$ is zero free in the right half plane. Also from the construction of s_n, $\mathcal{L}_1(1-\overline{s_n}) - \mathcal{L}_2(1-\overline{s_n}) \neq 0$ for all n.

Here we have,

$$\frac{\mathcal{L}_1(s) - \mathcal{L}_2(s)}{\mathcal{L}_2(s)} = \frac{\overline{\mathcal{L}_1(1-\bar{s})} - \overline{\mathcal{L}_2(1-\bar{s})}}{\overline{\mathcal{L}_2(1-\bar{s})}}. \tag{3.20}$$

Again from (3.17) we have,

$$\frac{P(\mathcal{L}_1)}{P(\mathcal{L}_2)} - 1 = c - 1, \tag{3.21}$$

i.e., $\left|\frac{\mathcal{L}_1-\mathcal{L}_2}{\mathcal{L}_2}\right| \left|\frac{a_n(\mathcal{L}_1^{n-1}+\mathcal{L}_1^{n-2}\mathcal{L}_2+\cdots+\mathcal{L}_2^{n-1})+a_{n-1}(\mathcal{L}_1^{n-2}+\mathcal{L}_1^{n-3}\mathcal{L}_2+\cdots+\mathcal{L}_2^{n-2})+\cdots+a_1}{(a_1+a_2\mathcal{L}_2+\cdots+a_n\mathcal{L}_2^{n-1})}\right| = |c-1|$.

Using (3.20) in above, with $s = s_n$ we get,

$$\left|\frac{\overline{\mathcal{L}_1(1-\overline{s_n})} - \overline{\mathcal{L}_2(1-\overline{s_n})}}{\overline{\mathcal{L}_2(1-\overline{s_n})}}\right| = |c-1|.$$

Again,

$$\lim_{n\to\infty}\left|\frac{\mathcal{L}_1(1-\overline{s_n}) - \mathcal{L}_2(1-\overline{s_n})}{\mathcal{L}_2(1-\overline{s_n})}\right| = 0.$$

Hence $c = 1$, therefore $P(\mathcal{L}_1) = P(\mathcal{L}_2)$. Then proceeding the same as we have done in Case-I after (3.18), we get that $\mathcal{L}_1 = \mathcal{L}_2$. □

Proof of Theorem 1.15. It is given that $\mathcal{L}_1$, $\mathcal{L}_2$ share $S = \{w : P(w) = 0\}$ CM. Clearly then we can set the auxiliary function

$$\frac{P(\mathcal{L}_1)}{P(\mathcal{L}_2)} = (s-1)^k e^{as+b}, \tag{3.22}$$

for some $k \in \mathbb{Z}$ and $a, b \in \mathbb{C}$.

Case-I. $P(1) \neq 0$. Taking the limit $\sigma \to \infty$ in (3.22) we deduce that $k = a = 0$, so that $P(\mathcal{L}_1) = e^b P(\mathcal{L}_2)$, then the claim follows from Theorem 1.14.

Case-II. $P(1) = 0$ *and* $P(0) \neq 0$. Let $P(\mathcal{L}_i) = a_0 + a_1\mathcal{L}_i + \cdots + a_n\mathcal{L}_i^n = (\mathcal{L}_i - 1)(\mathcal{L}_i - \alpha_1)\ldots(\mathcal{L}_i - \alpha_{n-1})$ for $i = 1, 2$.

Each $\mathcal{L}_i$ for $i = 1, 2$ tends to 1 as σ diverges. Now $\mathcal{L}_i$ can be represented by Dirichlet series, i.e., $\mathcal{L}_i(s) = \sum_{n=1}^{\infty} \frac{a_i(n)}{n^s}$, $i = 1, 2$, absolutely convergent for $\sigma > 1$ where $a_i(1) = 1$. Also let n_1, n_2 be two integers such that $n_i = \min\{n\ (\geq 2) : a_i(n) \neq 0\}$ for $i = 1, 2$.

Hence we have that

$$\frac{\mathcal{L}_1(s) - 1}{\mathcal{L}_2(s) - 1} = \frac{\frac{1}{n_1^s}\left(a_1(n_1) + \sum_{n>n_1}^{\infty} a_1(n)(\frac{n_1}{n})^s\right)}{\frac{1}{n_2^s}\left(a_2(n_2) + \sum_{n>n_2}^{\infty} a_2(n)(\frac{n_2}{n})^s\right)} = \left(\frac{n_2}{n_1}\right)^s G(s),$$

where

$$G(s) = \frac{a_1(n_1) + \sum_{n>n_1}^{\infty} a_1(n)(\frac{n_1}{n})^s}{a_2(n_2) + \sum_{n>n_2}^{\infty} a_2(n)(\frac{n_2}{n})^s} = \left(\frac{n_1}{n_2}\right)^s \frac{\mathcal{L}_1(s) - 1}{\mathcal{L}_2(s) - 1}. \tag{3.23}$$

From (3.23) we have, $\lim_{\sigma \to +\infty} G(s) = \frac{a_1(n_1)}{a_2(n_2)} =$ non-zero constant.

$$\lim_{\mathrm{Re}(s)\to\infty} \frac{\mathcal{L}_1(s) - 1}{\mathcal{L}_2(s) - 1} \cdot \left(\frac{n_1}{n_2}\right)^s = A_1, \tag{3.24}$$

where $A_1 = \frac{a_1(n_1)}{a_2(n_2)}$.

We have that

$$\frac{P(\mathcal{L}_1)}{P(\mathcal{L}_2)} = \frac{(\mathcal{L}_1 - 1)(\mathcal{L}_1 - \alpha_1)\ldots(\mathcal{L}_1 - \alpha_{n-1})}{(\mathcal{L}_2 - 1)(\mathcal{L}_2 - \alpha_1)\ldots(\mathcal{L}_1 - \alpha_{n-1})} = G\left(\frac{n_2}{n_1}\right)^s \frac{(\mathcal{L}_1 - \alpha_1)\ldots(\mathcal{L}_1 - \alpha_{n-1})}{(\mathcal{L}_2 - \alpha_1)\ldots(\mathcal{L}_2 - \alpha_{n-1})}, \tag{3.25}$$

$\forall\ s = \sigma + it \in \mathbb{C}\ (\sigma, t \in \mathbb{R})$.

Again $\lim_{\sigma\to\infty} \frac{(\mathcal{L}_1-\alpha_1)\ldots(\mathcal{L}_1-\alpha_{n-1})}{(\mathcal{L}_2-\alpha_1)\ldots(\mathcal{L}_1-\alpha_{n-1})} = 1$.

Let us consider the function

$$\mathcal{G} = G\frac{(\mathcal{L}_1 - \alpha_1)\ldots(\mathcal{L}_1 - \alpha_{n-1})}{(\mathcal{L}_2 - \alpha_1)\ldots(\mathcal{L}_1 - \alpha_{n-1})} = \left(\frac{n_1}{n_2}\right)^s \frac{(\mathcal{L}_1 - 1)(\mathcal{L}_1 - \alpha_1)\ldots(\mathcal{L}_1 - \alpha_{n-1})}{(\mathcal{L}_2 - 1)(\mathcal{L}_2 - \alpha_1)\ldots(\mathcal{L}_1 - \alpha_{n-1})} = \left(\frac{n_1}{n_2}\right)^s \frac{P(\mathcal{L}_1)}{P(\mathcal{L}_2)}. \tag{3.26}$$

Using (3.23) we get from the last equality of (3.26) that

$$\begin{aligned}\mathcal{G} &= \left(\frac{n_1}{n_2}\right)^s e^{as+b}(s-1)^k \\ &= q^s e^{as+b}(s-1)^k \\ &= e^{b_1 s+b}(s-1)^k,\end{aligned} \tag{3.27}$$

where $q = \frac{n_1}{n_2} \in \mathbb{Q}^+$ and $b_1 = a + \log q$. Next we will show that $b_1 = 0 = k$.

Clearly from the first equality of (3.26) that $\lim_{\mathrm{Re}(s)\to\infty} \mathcal{G} = A_1$. Using this, from the last equality of (3.27) we have,

$$\lim_{\mathrm{Re}(s)\to\infty} |(s-1)^k e^{b_1 s+b}| = |A_1|.$$

Since $A_1 \in \mathbb{C}\backslash\{0\}$, we must have $k = 0 = b_1$.

Hence we have $a = -\log q$, i.e., $e^{-a} = q$ and $\mathcal{G}(s) = e^b$.

Therefore from the last equality of (3.26) we have,

$$\frac{P(\mathcal{L}_1)}{P(\mathcal{L}_2} = \frac{(\mathcal{L}_1 - 1)(\mathcal{L}_1 - \alpha_1)\ldots(\mathcal{L}_1 - \alpha_{n-1})}{(\mathcal{L}_2 - 1)(\mathcal{L}_2 - \alpha_1)\ldots(\mathcal{L}_2 - \alpha_{n-1})} = e^b q^{-s}, \tag{3.28}$$

for some $q \in \mathbb{Q}^+$.

$\mathcal{L}_1$ and $\mathcal{L}_2$ are two non-trivial L-functions in $\mathcal{S}^{\#}$, satisfying the same functional type equation, and hence they have the same set of trivial zeros in the negative half plane which appear due to the poles of Gamma function. Now just as in Case-II of the proof of Theorem 1.14 we can construct a infinite sequence $\{s_n\}$ of common trivial zeros of $\mathcal{L}_1$, $\mathcal{L}_2$ in some region $\mathrm{Re}\, s < -\kappa_0$ and $|\mathrm{Im}\, s| < \kappa$, such that $s_n \to -\infty$, as $n \to \infty$, and no one of $\mathcal{L}_1$, $\mathcal{L}_2$ vanishes for $\mathrm{Re}(s) > \kappa_0$.

From (3.28), putting $s = s_n$, we have $e^b q^{-s_n} = 1$ then we have $e^{\mathrm{Re}(b)} q^{\mathrm{Re}(-s_n)} = 1$ for $n = 1, 2, \ldots$. Taking limit as $n \to \infty$, it implies $q^{\mathrm{Re}(-s_n)} \to \infty$ or 0 according as $q > 1$ or < 1, but $\{e^{\mathrm{Re}(b)} q^{\mathrm{Re}(-s_n)}\}_{n=1}^{\infty}$ is a constant sequence. Hence we must have $q = 1$ and then $e^b = 1$.

Finally from (3.28) we have $P(\mathcal{L}_1) = P(\mathcal{L}_2)$. From Theorem 1.14 the result follows immediately.

Case-III. $P(1) = 0 = P(0)$. Without loss of generality let us assume $\alpha_1 = 0$. From (3.28) we have,

$$\frac{P(\mathcal{L}_1)}{P(\mathcal{L}_2)} = \frac{\mathcal{L}_1(\mathcal{L}_1 - 1)(\mathcal{L}_1 - \alpha_2)\ldots(\mathcal{L}_1 - \alpha_{n-1})}{\mathcal{L}_2(\mathcal{L}_2 - 1)(\mathcal{L}_2 - \alpha_2)\ldots(\mathcal{L}_2 - \alpha_{n-1})} = e^b q^{-s}.$$

Then we get,

$$\left|\frac{\mathcal{L}_1(s)(\mathcal{L}_1(s) - 1)(\mathcal{L}_1(s) - \alpha_2)\ldots(\mathcal{L}_1(s) - \alpha_{n-1})}{\mathcal{L}_2(s)(\mathcal{L}_2(s) - 1)(\mathcal{L}_2(s) - \alpha_2)\ldots(\mathcal{L}_2(s) - \alpha_{n-1})}\right| = e^{\mathrm{Re}(b)} q^{-\mathrm{Re}(s)}. \tag{3.29}$$

Therefore, $\frac{\mathcal{L}_1(s)}{\mathcal{L}_2(s)} = \overline{\frac{\mathcal{L}_1(1-\bar{s})}{\mathcal{L}_2(1-\bar{s})}}$ and using this, from (3.29) we get,

$$\left|\frac{\mathcal{L}_1(1-\bar{s})(\mathcal{L}_1(s)-1)(\mathcal{L}_1(s)-\alpha_2)\dots(\mathcal{L}_1(s)-\alpha_{n-1})}{\mathcal{L}_2(1-\bar{s})(\mathcal{L}_2(s)-1)(\mathcal{L}_2(s)-\alpha_2)\dots(\mathcal{L}_2(s)-\alpha_{n-1})}\right| = e^{\mathrm{Re}(b)}q^{-\mathrm{Re}(s)}.$$

Considering the sequence $\{s_n\}_{n=1}^{\infty}$ of trivial zeros of $\mathcal{L}_1$ and $\mathcal{L}_2$ as discussed in Case-II of Theorem 1.14, and putting $s = s_n$ in previous formula, we have that

$$\left|\frac{\mathcal{L}_1(1-\overline{s_n})}{\mathcal{L}_2(1-\overline{s_n})}\right| = e^{\mathrm{Re}(b)}q^{\mathrm{Re}(-s_n)}. \tag{3.30}$$

Taking limit as $n \to \infty$ from (3.30) we have $\lim_{n\to\infty} e^{\mathrm{Re}(b)}q^{\mathrm{Re}(-s_n)} = 1$, but $q^{\mathrm{Re}(-s_n)} \to \infty$ or 0 according as $q > 1$ or < 1, a contradiction. Hence q must be 1, and hence $e^{\mathrm{Re}(b)} = 1$.

Therefore we finally get $P(\mathcal{L}_1) = e^b P(\mathcal{L}_2)$, and from Theorem 1.14 we get $\mathcal{L}_1 = \mathcal{L}_2$. □

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