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Résumé. Pour un corps de nombres quelconque, nous prouvons qu'il existe un champs algébrique de Deligne–Mumford propre, lisse, géométriquement connexe de dimension 1 et de genre 1/2, défini sur l'anneau des entiers du corps et violant le principe local-global pour les points entiers.

Abstract. For any number field, we prove that there exists a stacky curve of genus 1/2 defined over the ring of its integers violating the local-global principle for integral points.

1. Introduction

Given a number field K, let $\mathcal{O}_K$ be the ring of its integers, and let Ω_K be the set of all its nontrivial places. Let K_v be the completion of K at $v \in \Omega_K$. For a finite place v, let $\mathcal{O}_v$ be the valuation ring of K_v. For an archimedean place v, let $\mathcal{O}_v = K_v$. Let X be an algebraic stack of finite type over $\mathcal{O}_K$. If the set $X(\mathcal{O}_K) \neq \emptyset$, then the set $X(\mathcal{O}_v) \neq \emptyset$ for all $v \in \Omega_K$. The converse does not always hold. We say that X violates the *local-global principle for integral points* if $X(\mathcal{O}_v) \neq \emptyset$ for all $v \in \Omega_K$, whereas $X(\mathcal{O}_K) = \emptyset$. For $K = \mathbb{Q}$, Darmon and Granville [3] implicitly gave an example of a stacky curve violating the local-global principle for integral points. In the paper [1], Bhargava and Poonen proved that any stacky curve over $\mathcal{O}_K$ of genus less than 1/2 satisfies the local-global principle for integral points. For $K = \mathbb{Q}$, they gave an example of a genus-1/2 stacky curve violating the local-global principle for integral points in loc. cit.

Our goal is to generalize their counterexample to any number field. We will prove the following theorem.

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Mots-clefs. stacky curves, local points, integral points, local-global principle for integral points.

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Theorem 1.1 (Theorem 5.1). *For any number field K, there exists a stacky curve of genus-$1/2$ over $\mathcal{O}_K$ violating the local-global principle for integral points.*

The way to prove this theorem is to give an explicit construction of a genus-1/2 stacky curve violating the local-global principle for integral points. The paper is organised as follows. In Section 2, we set up the background by recalling some facts on stacky curves. Then we introduce a class of genus-1/2 stacky curves in Section 3. In Section 4, we prove that the stacky curves given in Section 3 have local integral points. Finally, in Section 5, we put some restrictions on the stacky curves given in Section 3 so that they do not have integral points. Then Theorem 5.1 holds.

2. Notation and preliminaries

2.1. Notation. Given a number field K, let $\mathcal{O}_K$ be the ring of its integers, and let Ω_K be the set of all its nontrivial places. Let $\infty_K^r \subset \Omega_K$ be the subset of all real places, and let $\Omega_K^f \subset \Omega_K$ be the subset of all finite places. Let K_v be the completion of K at $v \in \Omega_K$. For a finite place v, let $\mathcal{O}_v$ be the valuation ring of K_v, and let $\mathbb{F}_v$ be the residue field. For an archimedean place v, let $\mathcal{O}_v = K_v$. We say that an element is a *prime element*, if the ideal generated by this element is a prime ideal. If an element $p \in \mathcal{O}_K$ is a prime element, we denote its associated valuation by v_p, and its associated valuation ring (field) by $\mathcal{O}_p$ (respectively K_p). Let $\overline{K}$ be an algebraic closure of K.

2.2. Stacky curves. In this subsection, we briefly recall some facts on stacky curves. We refer to [5], [8] and [1] for more details.

We say that X is a *stacky curve* over K, if X is a smooth, proper and geometrically connected 1-dimensional Deligne–Mumford stack over K that contains a nonempty open substack isomorphic to a scheme, cf. [8, Definiton 5.2.1]. Given a stacky curve X over a number field K, by [4, Theorem 1.1], let X_{coarse} be its coarse moduli space, which is a smooth, projective and geometrically connected curve over K. Let $\pi\colon X \to X_{\text{coarse}}$ be the coarse space morphism. For any finite extension L/K and any closed point $P \in X_{\text{coarse}}(L)$, let G_P be the stabilizer of X above P, which is a finite group scheme over K. Let $\mathcal{P} \subset X_{\text{coarse}}$ be the reduced finite subsheme above which the stabilizer is nontrivial. Hence, π is an isomorphism over the open subscheme $X_{\text{coarse}} \backslash \mathcal{P}$. Motivated by the Riemann–Hurwitz formula, the genus of X is defined by

$$g(X)\colon = g(X_{\text{coarse}}) + \frac{1}{2}\sum_{P\in\mathcal{P}}\left(1 - \frac{1}{\deg G_P}\right)\deg P. \tag{2.1}$$

This formula is stable under base field change. It can be defined using the geometrically closed points of $\mathcal{P}$ by

$$g(X):=g(X_{\text{coarse}})+\frac{1}{2}\sum_{\overline{P}\in\mathcal{P}(\overline{K})}\left(1-\frac{1}{\deg G_{\overline{P}}}\right). \tag{2.2}$$

In particular, the genus is a nonnegative rational number. From this formula, one deduces the following lemma.

Lemma 2.1 ([6, Lemma 6 and Proposition 8]). *Let X be a stacky curve over a number field K, then $g(X)\geq 0$. If $g(X)<1$, then $g(X_{\text{coarse}})=0$ and X is geometrically isomorphic to $\mathbb{P}^1$.*

It follows from the Hasse–Minkowski theorem that for a stacky curve of genus less than one over a number field, the local-global principle for rational points always holds. Bhargava and Poonen [6, Theorem 5] proved that the local-global principle for integral points always holds for a stacky curve of genus less than 1/2 over a number field. Furthermore, Christensen [2, Theorem 13.0.6] proved that it satisfies strong approximation. Because of these, we consider the local-global principle for integral points of genus-1/2 stacky curves. We say that $\mathcal{X}$ is a *stacky curve* over $\mathcal{O}_K$, if $\mathcal{X}$ is a proper algebraic stack over $\mathcal{O}_K$ whose generic fiber is a stacky curve over K. For any $\mathcal{O}_K$-algebra R, let $\mathcal{X}(R)$ be the set of isomorphism classes of $\mathcal{O}_K$-morphisms $\operatorname{Spec} R\to\mathcal{X}$.

3. A class of genus-1/2 stacky curves

Let K be a number field. Let

$$\mu_2:=\operatorname{Spec}\mathcal{O}_K[\lambda]/(\lambda^2-1)\subset\mathbb{G}_m:=\operatorname{Spec}\mathcal{O}_K[\lambda,1/\lambda]$$

be the closed subgroup scheme. Let

$$\mathbb{Z}/2\mathbb{Z}:=\operatorname{Spec}\mathcal{O}_K[\lambda]/(\lambda-1)\bigsqcup\operatorname{Spec}\mathcal{O}_K[\lambda]/(\lambda+1).$$

The following lemma states that these two finite group schemes are isomorphic over $\mathcal{O}_K[1/2]$.

Lemma 3.1. *Given a number field field K, the natural morphism $\mathbb{Z}/2\mathbb{Z}\to\mu_2$ given by*

$$\mathcal{O}_K[\lambda]/(\lambda^2-1)\longrightarrow\mathcal{O}_K[\lambda]/(\lambda-1)\times\mathcal{O}_K[\lambda]/(\lambda+1):\ \bar{\lambda}\longmapsto(\bar{\lambda},\bar{\lambda})$$

is a group homomorphism. And it is an isomorphism over $\mathcal{O}_K[1/2]$.

Proof. By a direct check of group operators of these two group schemes, this is a group homomorphism. After base change to $\mathcal{O}_K[1/2]$, the ring homomorphism is an isomorphism. $\square$

Let $p,\ q$ be two coprime integers in K. Let $z^2-px^2-qy^2$ be a homogeneous polynomial in $\mathcal{O}_K[x,y,z]$ with homogeneous coordinates $(x:y:z)$. Let $\mathcal{Y}_{(p,q)} := \operatorname{Proj}\mathcal{O}_K[x,y,z]/(z^2-px^2-qy^2)$, and let $Y_{(p,q)}$ be its base change to K. We define a μ_2-action on $\mathcal{Y}_{(p,q)}$ by letting $\lambda\in\mu_2$ act as $(x:y:z)\mapsto(x:y:\lambda z)$. Let $[\mathcal{Y}_{(p,q)}/\mu_2]$ and $[Y_{(p,q)}/\mu_2]$ be the quotient stacks over $\mathcal{O}_K$ and K respectively.

Proposition 3.2. *The quotient stack $[\mathcal{Y}_{(p,q)}/\mu_2]$ is a Deligne–Mumford stack over $\mathcal{O}_K[1/2]$. The quotient stack $[Y_{(p,q)}/\mu_2]$ is a genus-$1/2$ stacky curve.*

Proof. Since $[\mathcal{Y}_{(p,q)}/(\mathbb{Z}/2\mathbb{Z})]$ is a Deligne–Mumford stack over $\mathcal{O}_K$, the first argument follows from Lemma 3.1. In particular, the quotient stack $[Y_{(p,q)}/\mu_2]$ is a Deligne–Mumford stack. For a Deligne–Mumford stack, the properties of being smooth, proper and geometrically connected of dimension one follow from these properties of $Y_{(p,q)}$. Let $\mathcal{P}_{z=0}\subset Y_{(p,q)}$ be the finite K-subscheme defined by $z=0$. The group μ_2 acts freely on $\operatorname{Proj}K[x,y,z]/(z^2-px^2-qy^2)\backslash\mathcal{P}_{z=0}$, so the stack $(\operatorname{Proj}K[x,y,z]/(z^2-px^2-qy^2)\backslash\mathcal{P}_{z=0})/\mu_2$ is representable by a scheme, which is an open substack of $[Y_{(p,q)}/\mu_2]$. Since $\operatorname{Proj}K[x,y,z]/(z^2-px^2-qy^2)\backslash\mathcal{P}_{z=0}$ is geometrically isomorphic to $\mathbb{G}_m$, geometrically this action over it can be viewed as the action from the Kummer sequence $1\to\mu_2\to\mathbb{G}_m\to\mathbb{G}_m\to 1$. Hence the stack $(\operatorname{Proj}K[x,y,z]/(z^2-px^2-qy^2)\backslash\mathcal{P}_{z=0})/\mu_2$ is geometrically isomorphic to $\mathbb{G}_m$. So $[Y_{(p,q)}/\mu_2]$ is a stacky curve and $g([Y_{(p,q)}/\mu_2]_{\text{coarse}})=0$. As μ_2 acts trivially on $\mathcal{P}_{z=0}$ containing two geometric points, by the genus formula (2.2), we have $g([Y_{(p,q)}/\mu_2])=1/2$. □

The stacky curves that we consider in this paper, are the quotient stacks of the form $[Y_{(p,q)}/\mu_2]$. We denote $[\mathcal{Y}_{(p,q)}/\mu_2]$ by $\mathcal{X}_{(p,q)}$.

4. Existence of local points

In this section, we prove that the stacky curve $\mathcal{X}_{(p,q)}$ has local integral points, i.e. the set $\mathcal{X}_{(p,q)}(\mathcal{O}_v)\neq\emptyset$ for all $v\in\Omega_K$.

Lemma 4.1. *Given a number field K, let $p,\ q$ be two coprime integers in K. Let $S=\infty_K^r\cup\{v\in\Omega_K^f|v(2pq)\neq 0\}$ be a finite set. Then the set $\mathcal{Y}_{(p,q)}(\mathcal{O}_v)\neq\emptyset$ for all $v\in\Omega_K\backslash S$.*

Proof. For any finite place $v\in\Omega_K$, by Chevalley–Warning theorem (cf. [7, Chapter I §2, Corollary 2]), the set $\mathcal{Y}_{(p,q)}(\mathbb{F}_v)\neq\emptyset$. For any $v\in\Omega_K\backslash S$, the scheme $\mathcal{Y}_{(p,q)}$ is smooth over $\mathcal{O}_v$. By the smooth lifting theorem, the set $\mathcal{Y}_{(p,q)}(\mathcal{O}_v)\neq\emptyset$ for all $v\in\Omega_K\backslash S$. □

Remark 4.2. Consider the quotient morphism: $\mathcal{Y}_{(p,q)}\to\mathcal{X}_{(p,q)}$. Then this lemma implies that the set $\mathcal{X}_{(p,q)}(\mathcal{O}_v)\neq\emptyset$ for all $v\in\Omega_K\backslash S$.

In order to prove that the stacky curves $\mathcal{X}_{(p,q)}$ has local integral points. We need to check that the set $\mathcal{X}_{(p,q)}(\mathcal{O}_v) \neq \emptyset$ for all $v \in S$.

Let R be a principal ideal domain over $\mathcal{O}_K$. We analyze the set $\mathcal{X}_{(p,q)}(R)$ first.

By definition of the quotient stack, a morphism $\operatorname{Spec} R \to \mathcal{X}_{(p,q)}$ is given by a μ_2-torsor T over R equipped with a μ_2-equivariant morphism $T \to \mathcal{Y}_{(p,q)}$. The torsors are classified by $\mathrm{H}^1_{\mathrm{fppf}}(R, \mu_2)$, which is isomorphic to $R^\times/R^{\times 2}$, since $\mathrm{H}^1_{\mathrm{fppf}}(R, \mathbb{G}_m) = \operatorname{Pic} R = 0$. Explicitly, if $t \in R^\times$, the corresponding μ_2-torsor is $T_t := \operatorname{Spec} R[u]/(u^2 - t)$ and the μ_2-action on T_t is given by letting $\lambda \in \mu_2$ act as $u \mapsto \lambda u$. Let $\mathcal{Y}_{(p,q)t} := \operatorname{Proj} R[x,y,z']/(tz'^2 - px^2 - qy^2)$ be the twist of $\mathcal{Y}_{(p,q)}$ by t. Consider the μ_2-torsor $\mathcal{Y}_{(p,q)t} \times_{\operatorname{Spec} R} T_t$ over $\mathcal{Y}_{(p,q)t}$. Define a morphism $\mathcal{Y}_{(p,q)t} \times_{\operatorname{Spec} R} T_t \to \mathcal{Y}_{(p,q)}$ given by

$$\mathcal{O}_K[x,y,z]/(z^2 - px^2 - qy^2) \longrightarrow R[x,y,z',u]/(tz'^2 - px^2 - qy^2, u^2 - t)$$
$$(x,y,z) \longmapsto (x,y,uz').$$

It is a μ_2-equivariant morphism. This gives a morphism $\pi_t\colon \mathcal{Y}_{(p,q)t} \to \mathcal{X}_{(p,q)}$. To give a μ_2-equivariant morphism $T_t \to \mathcal{Y}_{(p,q)}$ is the same as giving a triple $(a_1, a_2, a_3) \in R^3$, and the μ_2-equivariant morphism is given by

$$\mathcal{O}_K[x,y,z]/(z^2 - px^2 - qy^2) \longrightarrow R[u]/(u^2 - t)$$
$$(x,y,z) \longmapsto (a_1, a_2, a_3 u).$$

And the triple (a_1, a_2, a_3) gives a morphism $\operatorname{Spec} R \to \mathcal{Y}_{(p,q)t}$ defined by

$$R[x,y,z']/(tz'^2 - px^2 - qy^2) \longrightarrow R$$
$$(x,y,z') \longmapsto (a_1, a_2, a_3).$$

Hence, to give a μ_2-equivariant morphism $T_t \to \mathcal{Y}_{(p,q)}$ is the same as giving a morphism $\operatorname{Spec} R \to \mathcal{Y}_{(p,q)t}$. Thus we obtain

$$\mathcal{X}_{(p,q)}(R) = \coprod_{t \in R^\times/R^{\times 2}} \pi_t(\mathcal{Y}_{(p,q)t}(R)). \tag{4.1}$$

With this preparation, we have the following proposition.

Proposition 4.3. *Given a number field K, let p, q be two coprime integers in K. Then the set $\mathcal{X}_{(p,q)}(\mathcal{O}_v) \neq \emptyset$ for all $v \in \Omega_K$.*

Proof. By Lemma 4.1, we need to check that the set $\mathcal{X}_{(p,q)}(\mathcal{O}_v) \neq \emptyset$ for all $v \in S$.

Suppose that $v \in \infty_K^r$ or $v \nmid q$. Then $q \in \mathcal{O}_v^\times$. Since $qz^2 - px^2 - qy^2 = 0$ has a nontrivial solution $(x:y:z) = (0:1:1)$, we have $\mathcal{Y}_{(p,q)q}(\mathcal{O}_v) \neq \emptyset$. Hence, the set $\mathcal{X}_{(p,q)}(\mathcal{O}_v) \neq \emptyset$.

Similarly, suppose that $v \nmid p$. Then sets $\mathcal{Y}_{(p,q)p}(\mathcal{O}_v) \neq \emptyset$ and $\mathcal{X}_{(p,q)}(\mathcal{O}_v) \neq \emptyset$.

Since p and q are two coprime integers, the set $\mathcal{X}_{(p,q)}(\mathcal{O}_v) \neq \emptyset$ for all $v \in \Omega_K$. ☐

5. Genus 1/2-stacky curves violating the local-global principle for integral points

Given a number field K, we put some restrictions on the choice of integers p, q so that the stacky curve $\mathcal{X}_{(p,q)}$ has no integral points, i.e. the set $\mathcal{X}_{(p,q)}(\mathcal{O}_K) = \emptyset$. We choose p, q in the following way.

5.1. Choosing prime elements. Given a number field K, since the ideal class group of K is finite, we take a positive integer N such that $\mathcal{O}_K[1/N]$ is a principal ideal domain. By Dirichlet's unit theorem, the group $\mathcal{O}_K[1/N]^\times$ is a finitely generated abelian group. We assume that it is generated by $\{a_i\}$ for $i = 1, \ldots, n$. By Čebotarev's density theorem and global class field theory applied to a ray class field, we can find a pair of two different odd prime elements (p, q) such that

(1) $a_i \in K_p^{\times 2}$ for all $i = 1, \ldots, n$,
(2) $q \notin K_p^{\times 2}$.

We refer to [9, Section 4.2.2] and [10, Proposition 5.3] for more details. Then we have the following theorem.

Theorem 5.1. *Let K be a number field. Let a positive integer N and a pair of two different odd prime elements (p, q) be chosen as in Subsection 5.1. Let $\mathcal{X}_{(p,q)}$ be the stacky curve defined in Section 3. Then $\mathcal{X}_{(p,q)}$ is a stacky curve of genus-$1/2$ over $\mathcal{O}_K$ violating the local-global principle for integral points.*

Proof. By Proposition 3.2, the genus of $\mathcal{X}_{(p,q)}$ is $1/2$. By Proposition 4.3, the set $\mathcal{X}_{(p,q)}(\mathcal{O}_v) \neq \emptyset$ for any $v \in \Omega_K$.

Next, we prove that the set $\mathcal{X}_{(p,q)}(\mathcal{O}_K[1/N]) = \emptyset$. Since the ring $\mathcal{O}_K[1/N]$ is a principal ideal domain, in order to prove that $\mathcal{X}_{(p,q)}(\mathcal{O}_K[1/N]) = \emptyset$, by the equality of sets (4.1), it will be sufficient to prove that for any $t \in \mathcal{O}_K[1/N]^\times$, the set $\mathcal{Y}_{(p,q)t}(\mathcal{O}_K[1/N]) = \emptyset$. Since $\mathcal{O}_K[1/N]^\times$ is generated by $\{a_i\}$ for $i = 1, \ldots, n$, and by the chosen condition of Subsection 5.1 that $a_i \in K_p^{\times 2}$, we have $\mathcal{Y}_{(p,q)t}$ is isomorphic to $\mathcal{Y}_{(p,q)}$ over K_p for all $t \in \mathcal{O}_K[1/N]^\times$. By the choice of elements q, the set $\mathcal{Y}_{(p,q)}(K_p) = \emptyset$. So $\mathcal{X}_{(p,q)}(\mathcal{O}_K[1/N]) = \emptyset$, which implies that $\mathcal{X}_{(p,q)}(\mathcal{O}_K) = \emptyset$.

So the stacky curve $\mathcal{X}_{(p,q)}$ is of genus-$1/2$, and violates the local-global principle for integral points. □

Remark 5.2. This theorem implies that the chosen stacky curve $\mathcal{X}_{(p,q)}$ violates strong approximation in the sense of [2]. In addition, this stacky curve $\mathcal{X}_{(p,q)}$ is smooth over $\mathcal{O}_K[\frac{1}{2pq}]$.

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