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Twisted Thue equations with multiple exponents in fixed number fields

par Tobias HILGART et Volker ZIEGLER

Résumé. Soit K un corps de nombres de degré $d \geq 3$. On fixe $s \leq d-2$ éléments multiplicativement indépendants et remplissant certaines conditions techniques, qui se réduisent à une condition d'indépendance $\mathbb{Q}$-linéaire si on admet la conjecture de Schanuel. Nous considérons l'équation de Thue tordue

$$|N_{K/\mathbb{Q}}(X - \gamma_1^{t_1} \cdots \gamma_s^{t_s} Y)| = 1,$$

et prouvons qu'il n'existe qu'un nombre fini de solutions $(x, y; t_1, \ldots, t_s)$ dans $\mathbb{Z}^2 \times \mathbb{N}^s$ avec $xy \neq 0$ et $\mathbb{Q}(\gamma_1^{t_1} \cdots \gamma_s^{t_s}) = K$. Ces solutions sont effectivement calculables.

Abstract. Let K be a number field of degree $d \geq 3$ and fix $s \leq d-2$ multiplicatively independent $\gamma_1, \ldots, \gamma_s \in K^*$ that fulfil some technical requirements, which can be vastly simplified to $\mathbb{Q}$-linearly independence, given Schanuel's conjecture. We then consider the twisted Thue equation

$$|N_{K/\mathbb{Q}}(X - \gamma_1^{t_1} \cdots \gamma_s^{t_s} Y)| = 1,$$

and prove that it has only finitely many solutions $(x, y; t_1, \ldots, t_s)$ in $\mathbb{Z}^2 \times \mathbb{N}^s$ with $xy \neq 0$ and $\mathbb{Q}(\gamma_1^{t_1} \cdots \gamma_s^{t_s}) = K$, all of which are effectively computable.

1. Introduction

One of the first non-binary parametrised Thue equations to ever be solved was

$$f(X, Y; a) := X^3 - (a-1)X^2Y - (a+2)XY^2 - Y^3 = \pm 1,$$

done so by Thomas [9]. If we denote by $\alpha_1, \alpha_2, \alpha_3$ the roots of the polynomial $f(X, 1; a)$, then we can write equivalently as the norm-form equation

$$N_{K/\mathbb{Q}}(X - \alpha_1 Y) = \pm 1,$$

where for $K = \mathbb{Q}(\alpha_1)$, we denote by $N_{K/\mathbb{Q}}$ the norm relative to the field extension $K/\mathbb{Q}$.

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Levesque and Waldschmidt extended this result [7] by twisting the equation by an exponential parameter t and showed that $|N_{K/\mathbb{Q}}(X - \alpha_1^t Y)| = 1$ still only has finitely many integer solutions $(x, y;\, a, t)$ where $|y| \geq 2$. They conjectured that this still holds if the Thue equation is twisted by two exponents, i.e. $|N_{K/\mathbb{Q}}(X - \alpha_1^t \alpha_2^s Y)| = 1$, and it is reasonable to expect that if it works for two, it might work for finitely many – at least given sufficiently "nice" conditions.

In a similar but different vein, Levesque and Waldschmidt [6] also showed the following: Let $K = \mathbb{Q}(\alpha)$, with embeddings $\Phi = \{\tilde{\sigma}_1, \dots, \tilde{\sigma}_d\}$ into $\mathbb{C}$ and $0 < \nu < 1$. Furthermore, let $\varepsilon \in \mathbb{Z}_K$ be an integral unit that fulfils $\mathbb{Q}(\alpha\varepsilon) = K$ and

$$(1.1) \qquad |\sigma_1(\alpha\varepsilon)| = \max_i(|\sigma_i(\alpha\varepsilon)|),\ |\sigma_2(\alpha\varepsilon)| \geq \max_i(|\sigma_i(\alpha\varepsilon)|)^\nu$$

for two distinct embeddings $\sigma_1, \sigma_2 \in \Phi$. Then for any solution (x, y, ε) of the inequality

$$|N_{K/\mathbb{Q}}(x - \alpha\varepsilon\, y)| \leq m$$

the logarithms $\log|x|, \log|y|$, as well as the absolute logarithmic height $h(\alpha\varepsilon)$ can be bounded by m^c for some effectively computable constant c. They conjectured that the result would hold even without Condition (1.1). Proof of the finiteness of the number of solutions is already established in [5] but rests on Schmidt's subspace theorem, which does not allow for an effective upper bound for their heights.

For our result, we also fix the number field. The "base" elements of the norm-form equation are fixed, too, but can be chosen somewhat more freely. Our main result is as follows:

Theorem 1.1. *Let K be a number field of degree $d \geq 3$ and $s \leq d - 2$. Furthermore, let $\gamma_1, \dots, \gamma_s \in K^*$ be multiplicatively independent algebraic integers, such that the following condition holds:*

For each choice of $d - 1$ embeddings $\tilde{\sigma}_1, \dots, \tilde{\sigma}_{d-1} \in \mathrm{Hom}_{\mathbb{Q}}(K, \mathbb{C})$,

$$(*) \qquad \operatorname{rank} \begin{pmatrix} \log\left|\frac{\tilde{\sigma}_1(\gamma_1)}{\tilde{\sigma}_{d-1}(\gamma_1)}\right| & \cdots & \log\left|\frac{\tilde{\sigma}_1(\gamma_s)}{\tilde{\sigma}_{d-1}(\gamma_s)}\right| \\ \vdots & \ddots & \vdots \\ \log\left|\frac{\tilde{\sigma}_{d-2}(\gamma_1)}{\tilde{\sigma}_{d-1}(\gamma_1)}\right| & \cdots & \log\left|\frac{\tilde{\sigma}_{d-2}(\gamma_s)}{\tilde{\sigma}_{d-1}(\gamma_s)}\right| \end{pmatrix} = s,$$

i.e. the matrix has full column-rank s.

Then the Thue equation

$$(1.2) \qquad |N_{K/\mathbb{Q}}(X - \gamma_1^{t_1} \cdots \gamma_s^{t_s} Y)| = 1$$

has only finitely many solutions $(x, y;\, t_1, \dots, t_s) \in \mathbb{Z}^2 \times \mathbb{N}^s$, where $xy \neq 0$ and $\mathbb{Q}(\gamma_1^{t_1} \cdots \gamma_s^{t_s}) = K$, all of which can be effectively computed.

Remark 1.2. Given Schanuel's conjecture [4, p. 30], that for any $\mathbb{Q}$-linearly independent complex numbers $z_1, \dots, z_n$, the transcendence degree of the field $\mathbb{Q}(z_1, \dots, z_n, e^{z_1}, \dots, e^{z_n})$ over $\mathbb{Q}$ is at least n, our condition on the γ_i can be relaxed in the following way:

Instead of requiring Condition $(*)$, it suffices that $\gamma_1, \dots, \gamma_s$ are $\mathbb{Q}$-linearly independent. See Section 4 for more details.

The outline of the proof of Theorem 1.1 is that we follow the usual Baker method of constructing a linear form in logarithms of the hypothetical solutions to Equation (1.2) and looking at lower bounds to derive an effectively computable upper bound for $|t_1|, \dots, |t_s|$. Using Bugeaud's and Győry's explicit upper bound for solutions of Thue equations [3] allows us to bound $\log|x|, \log|y|$ in terms of $|t_1|, \dots, |t_s|$ and thus absolutely as well.

For this to work, however, we need that the embeddings of $\gamma_1^{t_1} \cdots \gamma_s^{t_s}$ are not all "close", asymptotically in terms of $\max_i |t_i|$, to a distinguished embedding, which is given by the type j of the solution. If they are, we need a different argument, for which we require the Condition $(*)$ or Schanuel's conjecture.

2. Preliminaries

We start by listing the results of Baker and Wüstholz [1], or Bugeaud and Győry [3] respectively, as well as two smaller lemmata, one of which goes back to Tijdeman [10] and is stated in a slightly different setting.

For the sake of completeness, we briefly define the absolute (or Weil) height and Mahler's measure, see, for instance, [2] or [8]. If K is a number field of degree $d = [K : \mathbb{Q}]$, and for every place ν, we write $d_\nu = [K_\nu : \mathbb{Q}_\nu]$ for the completions $K_\nu, \mathbb{Q}_\nu$ with respect to ν, then we normalise the absolute value $|\cdot|_\nu$ so that

(1) if $\nu \mid p$ for a prime number p, then $|p|_\nu = p^{-d_\nu/d}$,
(2) if $\nu \mid \infty$ and ν is real, then $|x|_\nu = |x|^{1/d}$,
(3) if $\nu \mid \infty$ and ν is complex, then $|x|_\nu = |x|^{2/d}$,

and $|x|$ denotes the Euclidian absolute value in $\mathbb{R}$ or $\mathbb{C}$. Given this normalisation, the product formula

$$\prod_\nu |\alpha|_\nu = 1$$

holds for every $\alpha \in K^*$. The absolute height of $\alpha \in K$ is then defined as

$$H(\alpha) = \prod_\nu \max(1, |\alpha|_\nu),$$

and the absolute logarithmic height as $h(\alpha) = \log H(\alpha)$. The absolute logarithmic height is then equal to the logarithm of the Mahler measure

$M(m_\alpha)$ of its minimal polynomial m_α, i.e. if the minimal polynomial of $\alpha \in K$ is $m_\alpha(X) = a_n \prod_{i=1}^n (X - \alpha_i) \in \mathbb{Z}[X]$, then

$$h(\alpha) = \frac{1}{n} \log M(m_\alpha) = \frac{1}{n}\left(\log|a_n| + \sum_{i=1}^n \log\max(1, |\alpha_i|)\right).$$

Proposition 2.1 (Baker, Wüstholz; [1]). *Let $\gamma_1, \dots, \gamma_t$ be algebraic numbers not 0 or 1 in $K = \mathbb{Q}(\gamma_1, \dots, \gamma_t)$, which is of degree D. Let $b_1, \dots, b_t \in \mathbb{Z}$ and*

$$\Lambda = b_1 \log\gamma_1 + \cdots + b_t \log\gamma_t \neq 0.$$

Then

$$\log|\Lambda| \geq -C \cdot h_1 \cdots h_t \cdot \log B,$$

where $C = 18(t+1)!t^{t+1}(32D)^{t+2}\log(2tD)$, $B \geq \max(3, |b_1|, \dots, |b_t|)$ and

$$h_i \geq \max(h(\gamma_i), \log|\gamma_i|\, D^{-1}, 0.16\, D^{-1})$$

for $i \in \{1, \dots, t\}$.

Proposition 2.2 (Bugeaud, Győry; [3]). *Let $B \geq \max(|m|, e)$, f be an irreducible polynomial with root α and $K = \mathbb{Q}(\alpha)$. Let R be the regulator of K and r be the unit rank. Let H be an upper bound to the absolute values of the coefficients of f and $n = \deg f \geq 3$. Let $F(X,Y) = Y^n f(\frac{X}{Y})$, then all solutions $(x,y) \in \mathbb{Z}^2$ of the Thue equation $F(X,Y) = m$ satisfy*

$$\log\max(|x|, |y|) \leq c \cdot R \cdot \max(\log R, 1)(R + \log(HB)),$$

where $c = 3^{r+27}(r+1)^{7r+19}n^{2n+6r+14}$.

The following lemma goes back to a result of Tijdeman [10] on the difference of consecutive numbers comprised of primes up to some given bound. We can adapt the statement to our setting–with only little alterations–and prove it almost analogously.

Lemma 2.3. *Let K be a number field of degree $d \geq s$ and $\gamma_1, \dots, \gamma_s \in K^*$ multiplicatively independent. Let $\gamma = \gamma(t_1, \dots, t_s) = \gamma_1^{t_1} \cdots \gamma_s^{t_s}$ for non-zero integers $t_1, \dots, t_s$.*

Then for any two conjugates $\gamma^{(1)}, \gamma^{(2)}$ of γ with $M = |\gamma^{(1)}| > |\gamma^{(2)}| = m$ there exists an effectively computable constant c independent of $t_1, \dots, t_s$ such that

$$M - m > \frac{M}{h(M)^c}.$$

Proof. We start with moving the conjugation of γ down to the conjugations of the individual γ_i, i.e. if $\gamma^{(1)}$ is the conjugation of γ under the embedding $\tilde{\sigma}$, let $\gamma_i^{(1)} = \tilde{\sigma}(\gamma_i)$, then we write

$$M - m = |\gamma^{(1)}| - |\gamma^{(2)}| = |(\gamma_1^{(1)})^{t_1} \cdots (\gamma_s^{(1)})^{t_s}| - |(\gamma_1^{(2)})^{t_1} \cdots (\gamma_s^{(2)})^{t_s}|,$$

and thus

$$1 - \frac{m}{M} = 1 - \underbrace{\left|(\gamma_1^{(1)})^{t_1} \cdots (\gamma_s^{(1)})^{t_s} (\gamma_1^{(2)})^{-t_1} \cdots (\gamma_s^{(2)})^{-t_s}\right|}_{=:e^{\Lambda}}.$$

We want to use Proposition 2.1 to bound Λ. We have that $h(\gamma_i) \ll 1$. If we also had $|t_i| \ll h(M)$, then we would have, by Proposition 2.1, that $|\Lambda| > h(M)^{-c}$ for some effectively computable c independent of the t_i, which is precisely the constant of the proposition multiplied with the heights $h(\gamma_i)$. And if $|\Lambda| > h(M)^{-c}$, then $1 - e^{\Lambda} \gg h(M)^{-c}$ which then proves the assumption after multiplying the inequality with M (we can make the inequality $1-e^{\Lambda} \gg h(M)^{-c}$ a strict "$>$" by allowing for a larger constant c).

So what remains is to prove that $|t_i| \ll h(M)$. For that let S be the set of all places ν of K for which $|\gamma^{(1)}|_\nu \neq 1$ but in any case includes all non-archimedean places $\infty_1, \dots, \infty_d$, that is $S = \{\nu_1, \dots, \nu_n, \infty_1, \dots, \infty_d\}$.

In the spirit of an S-adic version of Minkowski spaces, we identify γ with the $|S|$-dimensional vector of logarithms of the individual valuations, i.e.

$$\gamma \mapsto \underbrace{\begin{pmatrix} \log|\gamma|_{\infty_1} \\ \vdots \\ \log|\gamma|_{\infty_d} \\ \log|\gamma|_{\nu_1} \\ \vdots \\ \log|\gamma|_{\nu_n} \end{pmatrix}}_{=:v} = t_1 \begin{pmatrix} \log|\gamma_1|_{\infty_1} \\ \vdots \\ \log|\gamma_1|_{\infty_d} \\ \log|\gamma_1|_{\nu_1} \\ \vdots \\ \log|\gamma_1|_{\nu_n} \end{pmatrix} + \cdots + t_s \begin{pmatrix} \log|\gamma_s|_{\infty_1} \\ \vdots \\ \log|\gamma_s|_{\infty_d} \\ \log|\gamma_s|_{\nu_1} \\ \vdots \\ \log|\gamma_s|_{\nu_n} \end{pmatrix}.$$

We further see that the right-hand side is of the form $v = \Gamma(t_1, \dots, t_s)^\top$ for the $(d+n) \times s$ dimensional matrix Γ of the $\log|\gamma_i|_\nu$. Since $\gamma_1, \dots, \gamma_s$ are multiplicatively independent, the matrix Γ has full column-rank. We can thus multiply the equation with $(\Gamma^\top\Gamma)^{-1}\Gamma^\top$ and apply the l_1 norm. Using the consistency $\|Ax\| \leq \|A\| \cdot \|x\|$ and hiding the matrix norms inside a constant c that does not depend on the t_i, this gives

$$c \cdot |v|_{l_1} \geq \left| \begin{pmatrix} t_1 \\ \vdots \\ t_s \end{pmatrix} \right|_{l_1},$$

which of course implies $|t_i| \ll |v|_{l_1}$. For the final remaining estimate, note that to compute the height $h(\gamma) = \sum_\nu \max(\log|\gamma|_\nu, 0)$, we can ignore all the valuations where $|\gamma|_\nu = 1$ and thus sum precisely over all entries of the vector v and take only the positive values. But by the product formula, we have $\sum_{i=1}^{d+n} v_i = 0$, i.e. the positive entries, which contribute to the height cancel with the negative that do not. This, in turn, means that if we sum the absolute values, which is precisely what we do when calculating the l_1-norm $|v|_{l_1}$, it must be two times the sum of all positive (or negative, resp.) values, thus $|v|_{l_1} = 2h(\gamma)$. This gives us $|t_i| \ll h(\gamma)$ and thus proves the assertion. □

The final preliminary lemma gives a simple way to control the product of pairwise maxima by the global maximum in a finite sequence of positive numbers.

Lemma 2.4. *Let $a_1, \dots, a_d \in \mathbb{R}$ with $0 < a_1 \leq \dots \leq a_d$, where $a_d > 1$ and $a_1 \cdots a_d = 1$. Then we get*

$$\prod_{\substack{i \in \{1,\dots,d\} \\ i \neq j}} \max(a_i, a_j) \geq a_d^{\frac{1}{d-1}}$$

for all fixed $j \in 1, \dots, d$.

Proof. We express everything in terms of the largest number, a_d. The extreme case, where a_1 is as large as possible, is

$$\frac{1}{a_d^{\frac{1}{d-1}}} \cdots \frac{1}{a_d^{\frac{1}{d-1}}} \cdot a_d = 1,$$

where all terms except for a_d are equal and thus have to be $a_d^{-\frac{1}{d-1}}$ for the product to still be 1. In this situation, if $j = d$, then the maximum $\max(a_i, a_j)$ is a_d every time and the product is a_d^{d-1}, which is greater than our purported bound of $a_d^{\frac{1}{d-1}}$.

If $j < d$ instead, then the maximum is $a_d^{\frac{1}{d-1}}$ in $d-2$ cases and a_d once, thus

$$\prod_{\substack{i \in \{1,\dots,d\} \\ i \neq j}} \max(a_j, a_i) = \left(\frac{1}{a_d^{\frac{1}{d-1}}} \right)^{d-2} a_d = a_d^{1-\frac{d-2}{d-1}} = a_d^{\frac{1}{d-1}}.$$

Now we move from the extreme case to the general. Let $a_i = a_d^{\frac{1}{d-1}} c_i$, where $c_1 \leq \dots \leq c_n \leq 1$, and $1 \leq c_{n+1} \leq \dots \leq c_{d-1}$. For the product $a_1 \cdots a_d = 1$ to still be 1, we need $c_1 \cdots c_{d-1} = 1$ to cancel.

If $n < j < d$, then it is only the constants ≥ 1 that show up in the product $\prod_{i \neq j} \max(a_j, a_i)$, and we can ignore them to get the purported lower bound. If instead $j \leq n$, we get the constant c_j for the first j indices and c_i afterwards. But $\prod_{i=1}^{j} c_j \prod_{i=j+1}^{d} c_i \geq 1$, since we substituted the possibly smaller c_i for c_j and the product was 1 beforehand. Thus we can again ignore the constants and get the purported bound. □

3. Proof of the main theorem

Let $(x, y;\, t_1, \ldots, t_s) \in \mathbb{Z}^2 \times \mathbb{N}^s$ be a solution to Thue Equation (1.2) and assume that all t_i are non-zero, as we would otherwise carry out the proof with $s' \leq s$ many γ_i that need to be written in the equation. We also assume $xy \neq 0$, as for $y = 0$, we get a solution with $x = \pm 1$ and any choice for $(t_1, \ldots, t_s)$. Similarly, for $x = 0$, it depends on the γ_i whether there are solutions, e.g. if they are units, then $y = \pm 1$ and any $(t_1, \ldots, t_s)$ would be eligible.

Let $t := \max_{i \in \{1,\ldots,s\}} |t_i|$ and $\beta_i := \tilde{\sigma}_i(x - \gamma_1^{t_1} \cdots \gamma_s^{t_s} y)$. We also write $\sigma_i = \tilde{\sigma}_i(\gamma_1^{t_1} \cdots \gamma_s^{t_s})$ and $\gamma_r^{(i)}$ for the individual embedments $\tilde{\sigma}_i(\gamma_r)$ to make the expressions more readable.

After reshuffling the indices, we can assume that $|\sigma_1| \geq \cdots \geq |\sigma_d|$. The polynomial $f(X) = N_{K/\mathbb{Q}}(X - \gamma_1^{t_1} \cdots \gamma_s^{t_s})$ is irreducible, since we have $\mathbb{Q}(\gamma_1^{t_1} \cdots \gamma_s^{t_s}) = K$ per our requirement on $(x, y;\, t_1, \ldots, t_s)$.

We define the type of the solution to be the index j, for which the equation $|\beta_j| = \min_{i \in \{1,\ldots,d\}} |\beta_i|$ holds. We distinguish between the following two cases:

Case 1. There exist at least two distinct indices $i \in \{1, \ldots, d\} \setminus \{j\}$ such that $|\log|\frac{\sigma_i}{\sigma_j}|| \geq \kappa \log t$, where κ is a (fixed but) sufficiently large constant independent of the solution $(x, y;\, t_1, \ldots, t_s)$.

Case 2. For all but one index $i \in \{1, \ldots, d\} \setminus \{j\}$, we have that $\log|\frac{\sigma_i}{\sigma_j}| \leq \kappa \log t$.

We want to prove that $t \ll 1$ in each case, i.e. that t can be bounded by some effectively computable constant. This would imply $|x|, |y| \ll 1$ and thus prove the finiteness of the number of solutions to Thue Equation (1.2).

3.1. Case 1. Let k, l be two distinct indices that fulfil the condition of Case 1, i.e.

$$(**) \qquad \left|\log\left|\frac{\sigma_k}{\sigma_j}\right|\right| \geq \kappa \log t, \quad \left|\log\left|\frac{\sigma_l}{\sigma_j}\right|\right| \geq \kappa \log t$$

holds. We state Siegel's identity,

$$\beta_j(\sigma_k - \sigma_l) + \beta_l(\sigma_j - \sigma_k) + \beta_k(\sigma_l - \sigma_j) = 0, \tag{3.1}$$

and by dividing by the third term on the left-hand side and rearranging things slightly, this is equivalent to

$$\underbrace{\frac{\beta_j}{\beta_k} \cdot \frac{\sigma_k - \sigma_l}{\sigma_j - \sigma_l}}_{=:L} + \underbrace{\frac{\beta_l}{\beta_k} \cdot \frac{\sigma_j - \sigma_k}{\sigma_j - \sigma_l}}_{=:L'} = 1. \tag{3.2}$$

We now show that it is possible to choose k, l so that L is very small: But first, regardless of the choice of k and l, since $(x, y;\, t_1, \ldots, t_s)$ is a solution to Thue Equation (1.2), we have

$$1 = |N_{K/\mathbb{Q}}(x - \gamma_1^{t_1} \cdots \gamma_s^{t_s} y)| = |\beta_1 \cdots \beta_d|.$$

If we use the minimality of β_j, we have $2|\beta_i| \geq |\beta_i - \beta_j| = |y(\sigma_j - \sigma_i)|$ and get that

$$|\beta_j| = \prod_{\substack{i \in \{1,\dots,d\} \\ i \neq j}} \frac{1}{|\beta_i|} \ll \frac{1}{|y|^{d-1} \prod_{\substack{i \in \{1,\dots,d\} \\ i \neq j}} |\sigma_j - \sigma_i|}.$$

We apply Lemma 2.3 on each of the $|\sigma_j - \sigma_i|$ and use the equality to the Mahler measure for the height to derive $h(\sigma_i) \ll \log \max_i |\sigma_i| = \log|\sigma_1|$. We thus have

$$|\beta_j| \ll \frac{\log|\sigma_1|^{(d-1)c}}{|y|^{d-1} \prod_{\substack{i \in \{1,\dots,d\} \\ i \neq j}} \max(|\sigma_j|, |\sigma_i|)}.$$

Next, we apply Lemma 2.4 on the product of the $\max(|\sigma_j|, |\sigma_i|)$, which gives

$$\prod_{\substack{i \in \{1,\dots,d\} \\ i \neq j}} \max(|\sigma_j|, |\sigma_i|) \geq |\sigma_1|^{\frac{1}{d-1}}.$$

Combining this with the previous bound gives

$$|\beta_j| \ll \frac{\log|\sigma_1|^{(d-1)c}}{|y|^{d-1} |\sigma_1|^{\frac{1}{d-1}}}.$$

Plugging this into our expression L and once more applying the inequality $|\beta_k| \geq \frac{y}{2}|\sigma_j - \sigma_k|$ finally gives

$$|L| = |\frac{\beta_j}{\beta_k} \cdot \frac{\sigma_k - \sigma_l}{\sigma_j - \sigma_l}| \ll \frac{\log|\sigma_1|^{(d-1)c}}{|y|^d |\sigma_1|^{\frac{1}{d-1}}} \cdot \frac{|\sigma_k - \sigma_l|}{|\sigma_j - \sigma_k||\sigma_j - \sigma_l|}. \tag{3.3}$$

The first factor already looks like it could be of order e^{-ct}, but we first check that the second term cannot ruin everything, at least for specific choices for k, l.

Since $x \neq 0$, we have $|x| \geq 1$. Furthermore, $|\beta_j| = |x - \sigma_j y| \leq q < 1$ for some q and thus $1 - q \leq |x| - q \leq |\sigma_j y|$. Equivalent to this is

$$|y|^{-1} \ll |\sigma_j|. \tag{3.4}$$

In addition, we have $|\sigma_j - \sigma_k| = |\sigma_{\max}(1 - \frac{\sigma_{\min}}{\sigma_{\max}})|$ if we denote, by abuse of notation, $\sigma_{\max}$ as the larger, and $\sigma_{\min}$ as the smaller of the two. But since k fulfils Condition (**), we have $|\frac{\sigma_{\min}}{\sigma_{\max}}| \leq t^{-\kappa}$, which is less than, say, $\frac{1}{2}$, and thus

$$|\sigma_j - \sigma_k| = \left|\sigma_{\max}\left(1 - \frac{\sigma_{\min}}{\sigma_{\max}}\right)\right| \gg |\sigma_{\max}|, \tag{3.5}$$

which also holds for l.

We now differentiate between different cases for j and show that for each one, we can choose k, l that fulfil Condition $(**)$ and do not blow up the expression L.

(1) Case $j \in \{1, d\}$:

(a) Case $j = 1$: We can choose any k, l that fulfil Condition $(**)$. We have $|\sigma_k - \sigma_l| \leq 2\max(|\sigma_k|, |\sigma_l|) \ll |\sigma_1|$, and since k fulfils Condition $(**)$, we have $|\sigma_1 - \sigma_k| = |\sigma_1||1 - \frac{\sigma_k}{\sigma_1}| \gg |\sigma_1|$ by Equation (3.5), same for l.
Thus, we would even further improve Equation (3.3), i.e.

$$L \ll \frac{\log|\sigma_1|^{(d-1)c}}{|y|^d|\sigma_1|^{\frac{1}{d-1}}} \cdot \frac{|\sigma_k - \sigma_l|}{|\sigma_j - \sigma_k||\sigma_j - \sigma_l|} \ll \frac{\log|\sigma_1|^{(d-1)c}}{|y|^d|\sigma_1|^{\frac{1}{d-1}}} \cdot \frac{|\sigma_1|}{|\sigma_1||\sigma_1|} = \frac{\log|\sigma_1|^{(d-1)c}}{|y|^d|\sigma_1|^{1+\frac{1}{d-1}}}.$$

(b) Case $j = d$: We choose $k = 1$, which fulfils Condition $(**)$, and any other l that does too. Then, $|\sigma_1 - \sigma_l| \ll |\sigma_1|$ on the one hand and $|\sigma_d - \sigma_1| \gg |\sigma_1|$, $|\sigma_d - \sigma_l| \gg |\sigma_d|$ by Equation (3.5) on the other, while $|\sigma_d| \gg |y|^{-1}$ by Equation (3.4).
Thus we worsen Equation (3.3) by a factor $|y|$,

$$L \ll \frac{\log|\sigma_1|^{(d-1)c}}{|y|^d|\sigma_1|^{\frac{1}{d-1}}} \cdot \frac{|\sigma_1|}{|\sigma_1||y|^{-1}} = \frac{\log|\sigma_1|^{(d-1)c}}{|y|^{d-1}|\sigma_1|^{\frac{1}{d-1}}}.$$

(2) Case $j \notin \{1, d\}$:

(a) Case $|\sigma_j| \gg |\sigma_1|^{\frac{1}{2}}$: We choose $k = d$, which fulfils Condition $(**)$, and any other l that does too. Then, $|\sigma_d - \sigma_l| \ll |\sigma_1|$, and, as a combination of Equation (3.5) and the (sub-) case condition, $|\sigma_j - \sigma_d||\sigma_j - \sigma_l| \gg |\sigma_j|^2 \gg |\sigma_1|$. Plugging everything into Equation (3.3) gives

$$L \ll \frac{\log|\sigma_1|^{(d-1)c}}{|y|^d|\sigma_1|^{\frac{1}{d-1}}} \cdot \frac{|\sigma_1|}{|\sigma_1|} = \frac{\log|\sigma_1|^{(d-1)c}}{|y|^d|\sigma_1|^{\frac{1}{d-1}}}.$$

(b) Case $|\sigma_j| \ll |\sigma_1|^{\frac{1}{2}}$: We choose $k = 1$, which fulfils Condition $(**)$, and any other l that does too. Then, $|\sigma_1 - \sigma_l| \ll |\sigma_1|$, $|\sigma_j - \sigma_1| \gg |\sigma_1|$, while $|\sigma_j - \sigma_l| \gg |\sigma_j| \gg |y|^{-1}$, as a combination of Equation (3.5) and Equation (3.4).
Thus,

$$L \ll \frac{\log|\sigma_1|^{(d-1)c}}{|y|^d|\sigma_1|^{\frac{1}{d-1}}} \cdot \frac{|\sigma_1|}{|\sigma_1||y|^{-1}} = \frac{\log|\sigma_1|^{(d-1)c}}{|y|^{d-1}|\sigma_1|^{\frac{1}{d-1}}}.$$

In all four cases, we have at least that

$$L \ll \frac{\log|\sigma_1|^{(d-1)c}}{|y|^{d-1}|\sigma_1|^{\frac{1}{d-1}}}, \tag{3.6}$$

and we now show that this bound is exponentially small (in t). To that end, we write

$$\log|\sigma_i| = \log|\tilde{\sigma}_i(\gamma_1^{t_1}\cdots\gamma_s^{t_s})| = t_1\log|\gamma_1^{(i)}| + \cdots + t_s\log|\gamma_s^{(i)}|.$$

This gives the system of linear equations

$$\underbrace{\begin{pmatrix} \log|\gamma_1^{(1)}| & \cdots & \log|\gamma_s^{(1)}| \\ \vdots & \ddots & \vdots \\ \log|\gamma_1^{(d)}| & \cdots & \log|\gamma_s^{(d)}| \end{pmatrix}}_{=:\Gamma} \begin{pmatrix} t_1 \\ \vdots \\ t_s \end{pmatrix} = \begin{pmatrix} \log|\sigma_1| \\ \vdots \\ \log|\sigma_d| \end{pmatrix}$$

if done so for all $i \in \{1,\ldots,d\}$. We take the maximum norm and use the consistency $\|Ax\| \leq \|A\|\cdot\|x\|$ to get

$$\log|\sigma_1| \leq c_1 t, \tag{3.7}$$

where $c_1 = \|\Gamma\|_{\max}$ and does not depend on $(x, y; t_1, \ldots, t_s)$.

Similarly, since $\gamma_1, \ldots, \gamma_s$ are multiplicatively independent, the matrix Γ has full column rank. Thus, $\Gamma^\top\Gamma$ is invertible, we multiply with $\Gamma^\top$ and $(\Gamma^\top\Gamma)^{-1}$ and take the maximum norm. This gives $t \leq c_2\log|\sigma_1|$ or $|\sigma_1| \geq e^{\frac{1}{c_2}t}$.

We plug in the upper and lower bounds for $|\sigma_1|$ into Equation (3.6) and get, by basically ignoring the contribution of $|y|$ with $|y| \geq 1$,

$$L \ll \frac{\log|\sigma_1|^{(d-1)c}}{|y|^{d-1}|\sigma_1|^{\frac{1}{d-1}}} \ll \frac{(c_1t)^{(d-1)c}}{|y|^{d-1}e^{\frac{1}{c_2}t\frac{1}{d-1}}} \ll e^{-c_3t} \tag{3.8}$$

for some effectively computable constant c_3.

We now return to Siegel's Identity (3.2), apply the bound from Equation (3.8) and get $\log L' = \log|1-L| \ll e^{-c_3t}$. Also note that $L' \neq 1$, since $L = \frac{\beta_j}{\beta_k}\cdot\frac{\sigma_k-\sigma_l}{\sigma_j-\sigma_l} = 0$ would imply that $\sigma_k = \sigma_l$ but this is impossible per our requirement that $\mathbb{Q}(\gamma_1^{t_1}\cdots\gamma_s^{t_s}) = K$. Thus,

$$0 \neq |\log L'| = \left|\log\left|\frac{\beta_l}{\beta_k}\right| + \log\left|\frac{\sigma_j-\sigma_k}{\sigma_j-\sigma_l}\right|\right| \ll e^{-c_3t}.$$

Let us now call $\sigma_A = \max(|\sigma_j|, |\sigma_k|)$, $\sigma_a = \min(|\sigma_j|, |\sigma_k|)$ and $\sigma_B = \max(|\sigma_j|, |\sigma_l|)$, $\sigma_b = \min(|\sigma_j|, |\sigma_l|)$. Then

$$\log\left|\frac{\sigma_j-\sigma_k}{\sigma_j-\sigma_l}\right| = \log\frac{\sigma_A}{\sigma_B} + \log\left|\frac{1-\frac{\sigma_a}{\sigma_A}}{1-\frac{\sigma_b}{\sigma_B}}\right|,$$

and since k, l fulfil Condition $(**)$, both $\frac{\sigma_a}{\sigma_A}, \frac{\sigma_b}{\sigma_B} \leq t^{-\kappa}$. Thus, $\log\left|\frac{1-\frac{\sigma_a}{\sigma_A}}{1-\frac{\sigma_b}{\sigma_B}}\right| = O(t^{-\kappa})$, which gives

$$\Lambda = \left|\log\left|\frac{\beta_l}{\beta_k}\right| + \log\frac{\sigma_A}{\sigma_B}\right| \ll t^{-\kappa}.$$

Assume for now that $\Lambda \neq 0$. Let r be the unit rank of our number field. Then we have $\beta_k = (\eta_1^{(k)})^{b_1} \cdots (\eta_r^{(k)})^{b_r}$ in terms of the fundamental units $\eta_1, \ldots, \eta_r$ of $\mathbb{Z}_K^\times$, same for β_l, while on the other hand we can write $\sigma_A = (\gamma_1^{(A)})^{t_1} \cdots (\gamma_s^{(A)})^{t_s}$, same for σ_B. We can thus write

$$\Lambda = \left|\sum_{i=1}^{r} b_i\Big(\log\left|\eta_i^{(l)}\right| - \log\left|\eta_i^{(k)}\right|\Big) + \sum_{i=1}^{s} t_i\Big(\log\left|\gamma_i^{(A)}\right| - \log\left|\gamma_i^{(B)}\right|\Big)\right| \ll t^{-\kappa}. \tag{3.9}$$

We now argue that we can bound the b_i and thus all coefficients of Λ by t, then apply Proposition 2.1.

To that end, note that for any $i \neq j$, we have that

$$\begin{aligned}\log|\beta_i| &= \log|x - \sigma_j y + y(\sigma_j - \sigma_i)| \\ &= \log|y| + \log|\sigma_i - \sigma_j| + \log\left|1 + \frac{\beta_j}{y|\sigma_i - \sigma_j|}\right| \\ &\ll \log|y| + \log|\sigma_i - \sigma_j|.\end{aligned}$$

On the one hand, we have $|\sigma_i - \sigma_j| \leq 2\max(|\sigma_i|, |\sigma_j|) \ll |\sigma_1|$. On the other hand, we have by Lemma 2.3 and Equation (3.4) that

$$|\sigma_i - \sigma_j| \geq \max(|\sigma_i|, |\sigma_j|) - \min(|\sigma_i|, |\sigma_j|) > \frac{\max(|\sigma_i|, |\sigma_j|)}{h(\sigma_i)^c} \gg \frac{1}{|y|h(\sigma_i)^c}.$$

We thus have

$$|\log|\sigma_i - \sigma_j|| \ll \max(\log|\sigma_1|, \log|y| + \log h(\sigma_i)),$$

where we have, as a reminder, $h(\sigma_i) \ll \log|\sigma_1|$ by looking at the Mahler measure to compute the height, and $\log|\sigma_1| \ll t$ by Equation (3.7). This gives $\log|\sigma_i - \sigma_j| \ll \log y + t$ and thus

$$\log|\beta_i| \ll \log|y| + t$$

for all $i \neq j$. We also get the same bound for the positive quantity $-\log|\beta_j|$, since by the above inequality,

$$-\log|\beta_j| = \sum_{i=1, i\neq j}^{d} \log|\beta_i| \ll \log|y| + t.$$

Next, we look at the coefficients of the polynomial $(X - \sigma_1) \cdots (X - \sigma_d)$. Their absolute values can obviously be bounded by $|\sigma_1|^d$ and $|\sigma_1|^d \leq e^{d c_1 t}$

by Equation (3.7). If we then apply Proposition 2.2 with $H = e^{d c_1 t}$, since $R \ll 1$ as it does not depend on the t_i, we get that

$$\log|x|, \log|y| \ll t \tag{3.10}$$

and thus

$$|\log|\beta_i|| \ll \log|y| + t \ll t \tag{3.11}$$

for all $i \in \{1, \dots, d\}$.

We return to our decomposition into powers of fundamental units,

$$\beta_i = (\eta_1^{(i)})^{b_1} \cdots (\eta_r^{(i)})^{b_r}.$$

Doing this for all $i = 1, \dots, r$, and we only care that the unit rank $r < d$, reveals that $(b_1, \dots, b_r)$ is a solution to the system of linear equations

$$\underbrace{\begin{pmatrix} \log|\eta_1^{(1)}| & \cdots & \log|\eta_r^{(1)}| \\ \vdots & \ddots & \vdots \\ \log|\eta_1^{(r)}| & \cdots & \log|\eta_r^{(r)}| \end{pmatrix}}_{=H} \begin{pmatrix} b_1 \\ \vdots \\ b_r \end{pmatrix} = \begin{pmatrix} \log|\beta_1| \\ \vdots \\ \log|\beta_r| \end{pmatrix}.$$

Since the η_i are multiplicatively independent, the matrix H is invertible. We multiply with the inverse H^{-1} and apply the maximum norm, which gives, in combination with Equation (3.10),

$$\max_i |b_i| \leq c_4 \max_i \log|\beta_i| \ll t, \tag{3.12}$$

where $c_4 = \|H^{-1}\|_{\max}$.

If we return to Equation (3.9), we have bounded the absolute value of every coefficient by t. The heights of the $\eta_i^{(\cdot)}, \gamma_i^{(\cdot)}$ do not depend on x, y or the t_i and are thus bounded by an effectively computable constant. In the case that $\Lambda \neq 0$, we plug everything into Proposition 2.1, with $h_i \ll 1, B \ll t$ and get $\log|\Lambda| \geq c_5 \log t$ for some effectively computable constant c_5. If we compare this with the logarithm of the upper bound from Equation (3.9), we get that $\kappa \log t \leq c_5 \log t$.

Now, if κ is sufficiently large, i.e. larger than the constant c_5 which itself is independent from κ, we get that $t \ll 1$.

We now have to check what happens if the linear form vanishes instead, i.e. if $\Lambda = 0$. In this case, we have that $\frac{\beta_l}{\beta_k} = \frac{\sigma_B}{\sigma_A}$ and have to differentiate between four cases:

(1) Case $A = B = j$: This implies $\beta_l = \beta_k$ and thus $\sigma_l = \sigma_k$, which we again ruled out by requiring that $\mathbb{Q}(\gamma_1^{t_1} \cdots \gamma_s^{t_s}) = K$.
(2) Case $A = k, B = l$: This implies $\beta_l \sigma_k = \beta_k \sigma_l$, if we plug this into Siegel's Identity (3.1), it then gives $\beta_j(\sigma_k - \sigma_l) = \beta_l \sigma_j + \beta_k \sigma_j$ and

thus

$$\frac{\beta_j(\sigma_k - \sigma_l)}{\beta_l \sigma_j} - 1 = \frac{\beta_k}{\beta_l}.$$

The fraction on the left-hand side is $\ll e^{-c_6 t}$, which follows completely analogously to how we showed Equation (3.8), that $L \ll e^{-c_3 t}$. Thus,

$$\log\left|\frac{\beta_k}{\beta_l}\right| = \log(1 - O(e^{-c_6 t})) \ll e^{-c_6 t},$$

and the first equality also implies that $\log|\beta_k/\beta_l| \neq 0$. We now do the same thing, i.e. write $\beta_i = (\eta_1^{(i)})^{b_1} \cdots (\eta_r^{(i)})^{b_r}$ and apply Proposition 2.1 to the linear form $0 \neq \log|\beta_k/\beta_l| \ll e^{-c_6 t}$, since $|b_i| \ll t$ by Equation (3.12), and deduce $t \ll 1$.

(3) Case $A = k, B = j$: First, this means that $|\sigma_k| > |\sigma_j| > |\sigma_l|$. Second, this implies $\beta_l \sigma_k = \beta_k \sigma_j$ and thus $\beta_j(\sigma_k - \sigma_l) + \beta_l \sigma_j + \beta_k \sigma_l - 2\beta_k \sigma_j = 0$, if we plug it into Siegel's Identity (3.1). This gives

$$\frac{\beta_j(\sigma_k - \sigma_l)}{2\beta_k \sigma_j} + \frac{\sigma_l}{2\sigma_j} - 1 = -\frac{\beta_l}{2\beta_k}.$$

The first fraction on the left-hand side is $\ll e^{-c_7 t}$, analogously to how we showed $L \ll e^{-c_3 t}$. The second fraction $|\sigma_l/\sigma_j| \ll t^{-\kappa}$, since l fulfils Condition $(**)$ and $|\sigma_j| \geq |\sigma_l|$. This gives

$$0 \neq \left|\log\left|\frac{\beta_l}{\beta_k}\right| - \log 2\right| \ll t^{-\kappa},$$

and thus $t \ll 1$ analogously to the case $\Lambda \neq 0$.

(4) Case $A = j, B = l$: This is analogous to the previous case. We now have $|\sigma_l| \geq |\sigma_j| \geq |\sigma_k|$ and $\beta_l \sigma_j = \beta_k \sigma_l$ and get, by plugging this into Siegel's Identity (3.1),

$$-\frac{\beta_j(\sigma_k - \sigma_l)}{2\beta_l \sigma_j} + \frac{\sigma_k}{2\sigma_j} - 1 = -\frac{\beta_k}{2\beta_l}.$$

The first fraction on the left-hand side is again $\ll e^{-c_8 t}$, analogously to $L \ll e^{-c_3 t}$, while the second fraction is $\ll t^{-\kappa}$, since k fulfils Condition $(**)$ and $|\sigma_j| \geq |\sigma_k|$. This gives

$$0 \neq \left|\log\left|\frac{\beta_k}{\beta_l}\right| - \log 2\right| \ll t^{-\kappa}$$

and thus $t \ll 1$ analogously to the case $\Lambda \neq 0$.

We have proven $t \ll 1$ in all four subcases and thus throughout Case 1. If we plug this into Equation (3.10), we have $|x|, |y| \ll 1$ and thus an effectively computable upper bound to the size of the solutions, which means that there are only finitely many.

3.2. Case 2. If Condition (∗∗) does not hold, then $\log|\sigma_i/\sigma_j| \ll \log t$ holds instead for all but one index $i \in \{1,\dots,d\}\backslash\{j\}$. We rename the indices so it holds for $i = 1,\dots,d-2$, while $j = d$. We lose the ordering of $\sigma_1,\dots,\sigma_d$ but do not need this in the following arguments.

If we rewrite $\log|\sigma_i/\sigma_d| \ll \log t$ for $i \in \{1,\dots,d-2\}$, then this means that

$$\underbrace{\begin{pmatrix} \log|\frac{\gamma_1^{(1)}}{\gamma_1^{(d)}}| & \cdots & \log|\frac{\gamma_s^{(1)}}{\gamma_s^{(d)}}| \\ \vdots & \ddots & \vdots \\ \log|\frac{\gamma_1^{(d-2)}}{\gamma_1^{(d)}}| & \cdots & \log|\frac{\gamma_s^{(d-2)}}{\gamma_s^{(d)}}| \end{pmatrix}}_{=\Gamma} \begin{pmatrix} t_1 \\ \vdots \\ t_s \end{pmatrix} \ll \begin{pmatrix} \log t \\ \vdots \\ \log t \end{pmatrix} \tag{3.13}$$

holds. The matrix Γ has full column rank per Condition (∗) in our theorem. Thus $\Gamma^\top\Gamma$ is invertible. We multiply both sides first with $\Gamma^\top$ and then $(\Gamma^\top\Gamma)^{-1}$ and apply the maximum norm.

Since the matrix does not depend on $(x, y; t_1,\dots,t_s)$, neither does their norm, and after using $\|Ax\| \le \|A\| \cdot \|x\|$, we get $t \ll \log t$ and thus $t \ll 1$.

If we plug this into Equation (3.10), we can use Proposition 2.2 in Case 2 as well to derive said inequality, we get that $|x|, |y| \ll 1$, which proves Theorem 1.1.

If we assume Schanuel's conjecture, then it follows from the $\mathbb{Q}$-linear independency of $\gamma_1,\dots,\gamma_s$ that the matrix Γ from Equation (3.13) has full rank. For the sake of completeness, we state this as a second Theorem:

Theorem 3.1. *Let K be a number field of degree $d \ge 3$ and $s \le d-2$. Furthermore, let $\gamma_1,\dots\gamma_s \in K^*$ be multiplicatively independent and $\mathbb{Q}$-linearly independent algebraic integers.*

If Schanuel's conjecture holds, then the Thue equation

$$|N_{K/\mathbb{Q}}(X - \gamma_1^{t_1}\cdots\gamma_s^{t_s}Y)| = 1$$

has only finitely many solutions $(x, y; t_1,\dots,t_s) \in \mathbb{Z}^2 \times \mathbb{N}^s$, where $xy \neq 0$ and $\mathbb{Q}(\gamma_1^{t_1}\cdots\gamma_s^{t_s}) = K$, all of which can be effectively computed.

Proof. The proof for Case 1 of Theorem 1.1 can be carried over par for par. We adapt it for Case 2, i.e. let Equation (3.13) hold.

Assume that the matrix Γ does not have full rank, that there exist $x_1,\dots,x_s$, not all zero such that

$$x_1 \begin{pmatrix} \log|\frac{\gamma_1^{(1)}}{\gamma_1^{(d)}}| \\ \vdots \\ \log|\frac{\gamma_1^{(d-2)}}{\gamma_1^{(d)}}| \end{pmatrix} + \cdots + x_s \begin{pmatrix} \log|\frac{\gamma_s^{(1)}}{\gamma_s^{(d)}}| \\ \vdots \\ \log|\frac{\gamma_s^{(d-2)}}{\gamma_s^{(d)}}| \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

If we rewrite this slightly, then this means that

$$\begin{pmatrix} \log|\gamma_1^{(1)}| & \cdots & \log|\gamma_s^{(1)}| \\ \vdots & \ddots & \vdots \\ \log|\gamma_1^{(d-2)}| & \cdots & \log|\gamma_s^{(d-2)}| \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_s \end{pmatrix} = \begin{pmatrix} \lambda \\ \vdots \\ \lambda \end{pmatrix},$$

where $\lambda = x_1 \log|\gamma_1^{(d)}| + \cdots + x_s \log|\gamma_s^{(d)}| \neq 0$, since the γ_i are multiplicatively independent. Thus λ is a non-zero eigenvalue of the matrix on the left-hand side, which in turn means that λ and $\log|\gamma_1^{(i)}|, \dots, \log|\gamma_s^{(i)}|$ are algebraically dependent. But Schanuel's conjecture asserts that if $\gamma_1, \dots, \gamma_s$ are linearly independent over $\mathbb{Q}$ then $\log|\gamma_1|, \dots, \log|\gamma_s|$ are algebraically independent over $\overline{\mathbb{Q}}$. This gives the contradiction and thus the full rank for the original matrix Γ.

We can then proceed analogously to the proof of Case 1 and derive $t \ll 1$ and thus $|x|, |y| \ll 1$. □

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