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The Hybrid Euler–Hadamard Product Formula for Dirichlet L-functions in $\mathbb{F}_q[T]$

par Michael YIASEMIDES

Résumé. Nous donnons une formule de produit d'Euler–Hadamard hybride pour les fonctions L de Dirichlet du corps $\mathbb{F}_q[T]$. Nous déterminons explicitement le terme principal du moment d'ordre $2k$ du produit eulérien et, en utilisant la théorie des matrices aléatoires, proposons une formule conjecturale pour le moment d'ordre $2k$ du produit d'Hadamard. Avec une conjecture de scindage, ça nous ramène à une conjecture sur le moment d'ordre $2k$ des fonctions L de Dirichlet. En faveur de la conjecture de scindage, nous démontrons qu'elle est vraie pour $k = 1, 2$. Ce travail est l'analogue pour les corps de fonctions du travail de Bui et Keating. La différence la plus importante est que dans notre cas la formule de produit d'Euler–Hadamard est exacte (sans terme d'erreur).

Abstract. For Dirichlet L-functions in $\mathbb{F}_q[T]$ we obtain a hybrid Euler–Hadamard product formula. We explicitly obtain the main term of the $2k$-th moment of the Euler product, and we conjecture via random matrix theory the main term of the $2k$-th moment of the Hadamard product. Then making a splitting conjecture, this leads to a conjecture for the $2k$-th moment of Dirichlet L-functions. Finally, we lend support for the splitting conjecture by proving the cases $k = 1, 2$. This work is the function field analogue of the work of Bui and Keating, with the most notable difference being that the Euler–Hadamard product formula is exact in this setting (no error term).

1. Introduction and Statement of Results

Moments of L-functions are natural statistics to study if one wishes to understand the L-functions; and they have several important applications such as non-vanishing results, zero-density estimates, and the proportion of zeros on the critical line [13]. Asymptotic results have been obtained

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Mots-clefs. hybrid Euler–Hadamard product, moments, Dirichlet L-functions, function fields, random matrix theory.

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up to the fourth moment, while for higher moments only bounds have been rigorously obtained. Conjectures have been made for the asymptotic behaviour of higher moments. To illustrate the progress, let us consider the Riemann zeta-function and its behaviour on the critical line. Hardy and Littlewood [15] showed that

$$\frac{1}{T}\int_{t=0}^{T}\left|\zeta\left(\frac{1}{2}+it\right)\right|^{2}\mathrm{d}t \sim \log T,$$

as $T \to \infty$, and it was shown by Ingham [19] that

$$\frac{1}{T}\int_{t=0}^{T}\left|\zeta\left(\frac{1}{2}+it\right)\right|^{4}\mathrm{d}t \sim \frac{1}{12}\frac{6}{\pi^2}(\log T)^4$$

as $T \to \infty$. For higher moments we have what is often referred to as a folklore conjecture: For integers $k \geq 0$,

$$\lim_{T\to\infty}\frac{1}{(\log T)^{k^2}}\frac{1}{T}\int_{t=0}^{T}\left|\zeta\left(\frac{1}{2}+it\right)\right|^{2k}\mathrm{d}t = f(k)a(k), \tag{1.1}$$

where $f(k)$ is a real-valued function and

$$a(k) := \prod_{p}\left(\left(1-\frac{1}{p}\right)^{k^2}\sum_{m=0}^{\infty}\frac{d_k(p^m)^2}{p^m}\right).$$

We have $a(0) = 1, a(1) = 1, a(2) = \frac{1}{\zeta(2)} = \frac{6}{\pi^2}$, and we have an understanding of $a(k)$ for higher values of k. The factor $f(k)$ is more elusive. Clearly, from the results described above, we have $f(0) = 1$, $f(1) = 1$, $f(2) = \frac{1}{12}$. It has been conjectured via number-theoretic means that $f(3) = \frac{42}{9!}$ [10] and $f(4) = \frac{24024}{16!}$ [11]. For conjectures on higher powers one can use the random matrix theory approach of Keating and Snaith [22] or the recipe developed by Conrey, Farmer, Keating, Rubinstein, and Snaith [9]. All conjectures are in agreement with each other and the established rigorous results.

In this paper we focus on the random matrix theory aspect. Montgomery and Dyson observed that the pair correlation of the non-trivial zeros of $\zeta(s)$ behaves similarly to the pair correlation of eigenvalues of a random Hermitian matrix [23]. Given that the eigenvalues of a matrix are the zeros of its characteristic polynomial, one can consider modelling $\zeta(s)$ on the critical line with the characteristic polynomials of unitary matrices. Indeed, by calculating the moments of these characteristic polynomials, Keating and Snaith [22] conjectured that $f(k) := \prod_{j=0}^{k-1}\frac{j!}{(j+k)!}$. Note this agrees with the results and conjectures that were established previously for $k = 2, 4, 6, 8$.

However, this approach did not introduce the factor $a(k)$ in (1.1) in a natural way. In effect, the random matrices did not model $\zeta(s)$ in its entirety. This was addressed by Gonek, Hughes, and Keating [14] who expressed $\zeta(s)$ as a hybrid Euler–Hadamard product: $\zeta(s) \approx P_X(s)Z_X(s)$, where $P_X(s)$ is a roughly a partial Euler product and $Z_X(s)$ is roughly a partial Hadamard product (a product over the zeros of $\zeta(s)$). The variable X determines the contribution of each factor. They conjectured that, asymptotically, the $2k$-th moment of $\zeta(s)$ on the critical line can be factored into the $2k$-th moment of $P_X(s)$ multiplied by the $2k$-th moment of $Z_X(s)$ (known as the splitting conjecture); and they showed that the former contributes the factor $a(k)$ in (1.1) and conjectured via random matrix theory that the latter contributes the factor $f(k)$. That is, they obtained a conjecture for the $2k$-th moment of $\zeta(s)$ in a way that the factor $a(k)$ appears naturally. They also lent support for the splitting conjecture by demonstrating that it holds for the cases $k = 1, 2$.

This approach, using an Euler–Hadamard hybrid formula, was later applied to discrete moments of the derivative of the Riemann zeta-function by Bui, Gonek, and Milinovich [6].

Furthermore, the relationship between random matrix theory and the Riemann zeta-function extends to families of L-functions [20]. Indeed, the proportion of L-functions in a certain family that have j-th zero in some interval $[a, b]$ appears to be the same as the proportion of matrices in a certain matrix ensemble that have j-th eigenvalue in $[a, b]$. The ensemble depends on the family; the size of the matrices depends on the conductor q of the family; and the observation is made as $q \to \infty$.

We can consider, for example, the family of Dirichlet L-functions. The associated ensemble of matrices is the unitary matrices [8, p. 887]. By making use of this relationship, and using the Euler–Hadamard product approach described above, Bui and Keating [7] conjectured that

$$\frac{1}{\phi^*(q)} \sideset{}{^*}\sum_{\chi \bmod q} \left| L\left(\frac{1}{2}, \chi\right) \right|^{2k} \sim a(k) \frac{G^2(k+1)}{G(2k+1)} \prod_{p \mid q} \left(\sum_{m=0}^{\infty} \frac{d_k(p^m)^2}{p^m} \right)^{-1} (\log q)^{k^2} \tag{1.2}$$

as $q \to \infty$. $L(s, \chi)$ is the Dirichlet L-function associated with the Dirichlet character χ; $\phi^*(q)$ is the number of primitive Dirichlet characters of modulus q; the star in the sum indicates the sum is over primitive characters only; and $G(z)$ is the Barnes G-function. This had been conjectured previously (see [21]), but this approach allows for all the factors to appear naturally.

One can consider the above problems in the function field setting. In fact, it is this setting that gives some insight into the relationship between random matrix theory and number theory [20, Section 3]. In function fields, Bui and Florea [5] developed the hybrid Euler–Hadamard product model for the family of quadratic Dirichlet L-functions. In this paper we do the same for Dirichlet L-functions of any primitive character, which is the function field analogue of the work of Bui and Keating described above.

In what follows, we define $\mathcal{A} := \mathbb{F}_q[T]$, where q is a prime power (not to be confused with q previously which represents the conductor); and $L(s,\chi)$ is the Dirichlet L-function associated to the Dirichlet character χ on $\mathcal{A}$. The set of monics are represented by $\mathcal{M}$, and the set of monic primes by $\mathcal{P}$. For a general $\mathcal{S} \subset \mathcal{A}$, the restriction to elements of degree n is denoted by $\mathcal{S}_n$. For $A \in \mathcal{A}\backslash\{0\}$ we define $|A| := q^{\deg A}$, and we take $|0| := 0$. The aim of this paper is to provide support for the following conjecture, which is the analogue of (1.2), in such a way that all factors appear naturally.

Conjecture 1.1. *For all non-negative integers k, it is conjectured that*

$$\frac{1}{\phi^*(R)} \sideset{}{^*}\sum_{\chi \bmod R} \left| L\left(\frac{1}{2}, \chi\right)\right|^{2k} \sim f(k)a(k) \prod_{P \mid R} \left(\sum_{m=0}^{\infty} \frac{d_k(P^m)^2}{|P|^m} \right)^{-1} (\deg R)^{k^2},$$

as $\deg R \to \infty$, where

$$a(k) := \prod_{P \in \mathcal{P}} \left(\left(1 - \frac{1}{|P|}\right)^{k^2} \sum_{m=0}^{\infty} \frac{d_k(P^m)^2}{|P|^m} \right).$$

and

$$f(k) := \frac{G^2(k+1)}{G(2k+1)} = \prod_{i=0}^{k-1} \frac{i!}{(i+k)!},$$

where G is the Barnes G-function. (Again, the star indicates the sum is over primitive characters, and $\phi^*(R)$ is the number of primitive Dirichlet characters of modulus R).

This conjecture has been verified for the cases $k = 1, 2$ by Andrade and Yiasemides [2]: As $\deg R \to \infty$,

$$\frac{1}{\phi^*(R)} \sideset{}{^*}\sum_{\chi \bmod R} \left| L\left(\frac{1}{2}, \chi\right)\right|^{2} \sim \frac{\phi(R)}{|R|} \deg R \tag{1.3}$$

and

$$\frac{1}{\phi^*(R)} \sideset{}{^*}\sum_{\chi \bmod R} \left| L\left(\frac{1}{2}, \chi\right)\right|^{4} \sim \frac{1 - q^{-1}}{12} \prod_{P \mid R} \left(\frac{\left(1 - |P|^{-1}\right)^3}{1 + |P|^{-1}} \right) (\deg R)^4. \tag{1.4}$$

It can be shown that $a(2) = 1 - q^{-1}$ and $f(2) = \frac{1}{12}$, and so we have agreement with Conjecture 1.1.

First we will require an Euler–Hadamard hybrid formula, which we prove in Section 2.

Theorem 1.2. *Let* $X \geq 1$ *be an integer and let* $u(x)$ *be a positive* C^∞-*function with support in* $[e, e^{1+q^{-X}}]$. *Let*

$$v(x) = \int_{t=x}^{\infty} u(t)\mathrm{d}t$$

and take u *to be normalised so that* $v(0) = 1$. *Furthermore, for* $y \in \mathbb{C}\backslash\{0\}$ *with* $\arg(y) \neq \pi$, *we define* $E_1(y) := \int_{w=y}^{y+\infty} \frac{e^{-w}}{w}\mathrm{d}w$; *and for* $z \in \mathbb{C}\backslash\{0\}$ *with* $\arg(z) \neq \pi$, *we define*

$$U(z) := \int_{x=0}^{\infty} u(x)E_1(z\log x)\mathrm{d}x.$$

Let χ *be a primitive Dirichlet character of modulus* $R \in \mathcal{M}\backslash\{1\}$, *and let* $\rho_n = \frac{1}{2} + i\gamma_n$ *be the* n-*th zero of* $L(s,\chi)$. *Then, for all* $s \in \mathbb{C}$ *we have*

$$L(s,\chi) = P_X(s,\chi)Z_X(s,\chi), \tag{1.5}$$

where

$$P_X(s,\chi) = \exp\left(\sum_{\substack{A\in\mathcal{M}\\ \deg A\leq X}} \frac{\chi(A)\Lambda(A)}{|A|^s \log|A|}\right)$$

and

$$Z_X(s,\chi) = \exp\left(-\sum_{\rho_n} U\Big((s-\rho_n)(\log q)X\Big)\right).$$

Strictly speaking, if $s = \rho$ *or* $\arg(s-\rho) = \pi$ *for some zero* ρ *of* $L(s,\chi)$, *then* $Z_X(s,\chi)$ *is not well defined. In this case, we take*

$$Z_X(s,\chi) = \lim_{s_0\to s} Z_X(s_0,\chi)$$

and we show that this is well defined.

Remark 1.3. We note that our hybrid Euler–Hadamard product formula, (1.5), does not involve an error term, unlike the analogous Theorem 1 in [14] and Theorem 1 in [7]. This is due to the fact that we are working in the function field setting. Indeed, this can also be seen in [1, 5], where they consider the Euler–Hadamard product formula over function fields for quadratic Dirichlet L-functions.

We also note that $Z_X(s,\chi)$ is expressed in terms of $u(x)$. Whereas, $P_X(s,\chi)$ and $L(s,\chi)$ are independent of $u(x)$. Thus, given the equality (1.5), we can see that, as long as $u(x)$ satisfies the conditions in the theorem, the value of $Z_X(s,\chi)$ is independent of any further restrictions made on $u(x)$. Ultimately, this is due to the fact that we are working in the function field setting and due to our choice of support for $u(x)$. Indeed, this is why our support for $u(x)$ is not quite the exact analogy to the support of $u(x)$ in Theorem 1 of [7]. We note that in Theorem 1 in [7], $P_X(s,\chi)$ and $L(s,\chi)$

also do not depend on $u(x)$, but this is because the dependency exists in the error term.

Next, we make a splitting conjecture.

Conjecture 1.4 (Splitting Conjecture). *For integers $k \geq 0$, we have*

$$\frac{1}{\phi^*(R)} \sum_{\chi \bmod R}^{*} \left| L\left(\frac{1}{2}, \chi\right) \right|^{2k} \sim \left(\frac{1}{\phi^*(R)} \sum_{\chi \bmod R}^{*} \left| P_X\left(\frac{1}{2}, \chi\right) \right|^{2k} \right) \cdot \left(\frac{1}{\phi^*(R)} \sum_{\chi \bmod R}^{*} \left| Z_X\left(\frac{1}{2}, \chi\right) \right|^{2k} \right)$$

as $X, \deg R \to \infty$ with $X \leq \log_q \deg R$.

We then obtain the $2k$-th moment of the partial Euler product in Section 3, and we use a random matrix theory model to conjecture the $2k$-th moment of the Hadamard product in Section 4:

Theorem 1.5. *For positive integers k, we have*

$$\frac{1}{\phi^*(R)} \sum_{\chi \bmod R}^{*} \left| P_X\left(\frac{1}{2}, \chi\right) \right|^{2k} \sim a(k) \left[\prod_{\substack{\deg P \leq X \\ P \mid R}} \left(\sum_{m=0}^{\infty} \frac{d_k(P^m)^2}{|P|^m} \right)^{-1} \right] \left(e^{\gamma} X \right)^{k^2}$$

as $X, \deg R \to \infty$ with $X \leq (2-\delta) \log_q \deg R$, where $\delta > 0$ can be taken to be arbitrarily small. Here, γ is the Euler–Mascheroni constant, and $a(k)$ is an in Conjecture 1.1.

Conjecture 1.6. *For integers $k \geq 0$, we have*

$$\frac{1}{\phi^*(R)} \sum_{\chi \bmod R}^{*} \left| Z_X\left(\frac{1}{2}, \chi\right) \right|^{2k} \sim \frac{G^2(k+1)}{G(2k+1)} \left(\frac{\deg R}{e^{\gamma} X} \right)^{k^2},$$

as $\deg R \to \infty$, where γ is the Euler–Mascheroni constant and G is the Barnes G-function.[1]

Similarly as in [7], Conjecture 1.4, Theorem 1.5, and Conjecture 1.6 together reproduce Conjecture 1.1 as desired, but only for certain cases, such as when the largest prime divisor of R has degree less than X, or when R is prime.

In Sections 5 and 7 we rigorously obtain the second and fourth moments of the Hadamard product, respectively.

[1] Recall that for integers $k \geq 0$ we have $\frac{G^2(k+1)}{G(2k+1)} = \prod_{i=0}^{k-1} \frac{i!}{(i+k)!}$.

Theorem 1.7. *We have that*

$$\frac{1}{\phi^*(R)}\sum_{\chi \bmod R}{}^{*}\left|Z_X\Big(\frac{1}{2},\chi\Big)\right|^2 = \frac{1}{\phi^*(R)}\sum_{\chi \bmod R}{}^{*}\left|L\Big(\frac{1}{2},\chi\Big)P_X\Big(\frac{1}{2},\chi\Big)^{-1}\right|^2 \sim \frac{\deg R}{e^\gamma X}\prod_{\substack{\deg P > X\\ P\mid R}}\left(1-\frac{1}{|P|}\right)$$

as $X, \deg R \to \infty$ with $X \leq (2-\delta)\log_q \deg R$, where $\delta > 0$ can be taken to be arbitrarily small.

Theorem 1.8. *We have*

$$\frac{1}{\phi^*(R)}\sum_{\chi \bmod R}{}^{*}\left|Z_X\Big(\frac{1}{2},\chi\Big)\right|^4 = \frac{1}{\phi^*(R)}\sum_{\chi \bmod R}{}^{*}\left|L\Big(\frac{1}{2},\chi\Big)P_X\Big(\frac{1}{2},\chi\Big)^{-1}\right|^4 \sim \frac{1}{12}\left(\frac{\deg R}{e^\gamma X}\right)^4\prod_{\substack{\deg P > X\\ P\mid R}}\frac{(1-|P|^{-1})^3}{1+|P|^{-1}}$$

as $X, \deg R \to \infty$ with $X \leq \log_q \log \deg R$.

In Theorems 1.7 and 1.8, a product over P appears on the right side that does not appear in Conjecture 1.6 (which is based on random matrix theory). Given certain restrictions on R, the product is asymptotic to 1, and thus we have agreement with Conjecture 1.6, but this does not hold generally. Regardless of this discrepancy between our results and the conjecture, we can see that Theorems 1.7 and 1.8, along with Theorem 1.5, (1.3), and (1.4) verify the Splitting Conjecture for the cases $k = 1, 2$.

Note that in Theorem 1.8 we required the condition $X \leq \log_q \log \deg R$ which is more restrictive than the condition $X \leq (2-\delta)\log_q \deg R$ in the Splitting Conjecture. However, given recent progress [16] and the results that have been establish in the area of twisted moments (see, for example, [3, 12, 18, 24] for $\zeta(s)$ and [17, 27] for Dirichlet L-functions), we expect that one can improve upon this restriction for Theorem 1.8.

Before proceeding, let us make a brief notational remark required in later sections. Let $a \in \mathbb{C}$ and $b \in \mathbb{C}\backslash\{0\}$, and let f be an integrable complex function. The integral $\int_{t=a}^{a+b\infty} f(t)\mathrm{d}t$ is defined to be over the straight line starting at a and in the direction of b. That is, $\int_{t=a}^{a+b\infty} f(t)\mathrm{d}t = \int_{s=0}^{\infty} f\big(a + \frac{b}{|b|}s\big)\mathrm{d}s$. If $a = 0$ then we will simply write $\int_{t=0}^{b\infty} f(t)\mathrm{d}t$, and if $b = \pm 1$ then we will write $\int_{t=a}^{a\pm\infty} f(t)\mathrm{d}t$.

2. The Hybrid Euler–Hadamard Product Formula

The proof of Theorem 1.2 follows by some alterations to results given in [4, 7, 14]. However, to demonstrate that the product formula is exact in the function field setting, and in the interest of clarity, we provide the full proof. First, let us recall that for Dirichlet characters χ and $\mathrm{Re}(s) > 1$, taking the logarithmic derivative of $L(s, \chi)$ gives

$$-\frac{L'}{L}(s, \chi) = \sum_{A \in \mathcal{M}} \frac{\chi(A)\Lambda(A)}{|A|^s}.$$

It is well-known [26] that for non-trivial characters of modulus $R \in \mathcal{M}$, $L(s, \chi)$ is a finite polynomial in q^{-s}:

$$L(s, \chi) = \sum_{A \in \mathcal{M}} \frac{\chi(A)}{|A|^s} = \sum_{\substack{A \in \mathcal{M} \\ \deg A < \deg R}} \frac{\chi(A)}{|A|^s}.$$

From this, we can deduce that as $\mathrm{Re}(s) \to \infty$,

$$\frac{L'}{L}(-s, \chi) = O_\chi(1). \tag{2.1}$$

Lemma 2.1. *Let X be a positive integer, and let $u(x)$ be a positive C^∞-function with support in $[e, e^{1+q^{-X}}]$. Let $\widetilde{u}(s)$ be its Mellin transform. That is,*

$$\widetilde{u}(s) = \int_{x=0}^{\infty} x^{s-1} u(x) \mathrm{d}x$$

and

$$u(x) = \frac{1}{2\pi i} \int_{\mathrm{Re}(s)=c} x^{-s} \widetilde{u}(s) \mathrm{d}s,$$

where c can take any value in $\mathbb{R}$ (due to our restrictions on the support of u, we can see that $\widetilde{u}(s)$ is well-defined for all $s \in \mathbb{C}$, and so, by the Mellin inversion theorem, c can take any value in $\mathbb{R}$). Then,

$$\widetilde{u}(s) \ll \begin{cases} \frac{1}{|s|+1} \max_x\{|u'(x)|\} e^{2\,\mathrm{Re}(s)} & \textit{if } \mathrm{Re}(s) > 0 \\ \frac{1}{|s|+1} \max_x\{|u'(x)|\} e^{\mathrm{Re}(s)} & \textit{if } \mathrm{Re}(s) \leq 0. \end{cases}$$

Proof. We have, by integration by parts, that

$$\widetilde{u}(s) = \int_{x=e}^{e^{1+q^{-X}}} x^{s-1} u(x) \mathrm{d}x = -\frac{1}{s} \int_{x=e}^{e^{1+q^{-X}}} x^{s} u'(x) \mathrm{d}x.$$

If $|s| > 1$, then it is not difficult to deduce that the above is

$$\ll \begin{cases} \frac{1}{|s|+1} \max_x\{|u'(x)|\} e^{2\,\mathrm{Re}(s)} & \text{if } \mathrm{Re}(s) > 0 \\ \frac{1}{|s|+1} \max_x\{|u'(x)|\} e^{\mathrm{Re}(s)} & \text{if } \mathrm{Re}(s) \leq 0. \end{cases}$$

If $|s| \leq 1$, then, by using the fact that $\int_{x=e}^{e^{1+q^{-X}}} u'(x)\mathrm{d}x = 0$, we obtain

$$\widetilde{u}(s) = \int_{x=e}^{e^{1+q^{-X}}} \frac{1-x^s}{s} u'(x)\mathrm{d}x = -\int_{x=e}^{e^{1+q^{-X}}} \Big(\int_{y=1}^{x} y^{s-1}\mathrm{d}y\Big) u'(x)\mathrm{d}x$$
$$\ll \int_{x=e}^{e^{1+q^{-X}}} |u'(x)|\mathrm{d}x \ll \max_x\{|u'(x)|\},$$

from which the result follows. □

Lemma 2.2. *Let X be a positive integer, and let $u(x)$ be a positive C^∞-function with support in $[e, e^{1+q^{-X}}]$, and let $\widetilde{u}(s)$ be its Mellin transform. Let*

$$v(x) = \int_{t=x}^{\infty} u(t)\mathrm{d}t$$

and take u to be normalised so that $v(0) = 1$. Note that its Mellin transform is

$$\widetilde{v}(s) = \frac{\widetilde{u}(s+1)}{s}.$$

Let χ be a primitive Dirichlet character of modulus $R \in \mathcal{M}\backslash\{1\}$. Then, for $s \in \mathbb{C}$ not being a zero of $L(s,\chi)$, we have

$$-\frac{L'}{L}(s,\chi) = \sum_{\substack{A\in\mathcal{M}\\ \deg A \leq X}} \frac{\chi(A)\Lambda(A)}{|A|^s} + \sum_{\rho_n} \frac{\widetilde{u}(1+(\rho_n - s)(\log q)X)}{\rho_n - s}, \tag{2.2}$$

where $\rho_n = \frac{1}{2} + i\gamma_n$ is the n-th zero of $L(s,\chi)$. Note that, by Lemma 2.1, we can see that the sum over the zeros is absolutely convergent.

Proof. Let $c > \max\{0, (1-\mathrm{Re}(s))(\log q)X\}$. By the Mellin inversion theorem, we have

$$\sum_{A\in\mathcal{M}} \frac{\chi(A)\Lambda(A)}{|A|^s} v\Big(e^{\frac{\deg A}{X}}\Big)$$
$$= \frac{1}{2\pi i} \sum_{A\in\mathcal{M}} \frac{\chi(A)\Lambda(A)}{|A|^s} \int_{\mathrm{Re}(w)=c} \frac{\widetilde{u}(w+1)}{w} |A|^{-\frac{w}{(\log q)X}} \mathrm{d}w$$
$$= \frac{1}{2\pi i} \int_{\mathrm{Re}(w)=c} \frac{\widetilde{u}(w+1)}{w} \sum_{A\in\mathcal{M}} \frac{\chi(A)\Lambda(A)}{|A|^{s+\frac{w}{(\log q)X}}} \mathrm{d}w$$
$$= -\frac{1}{2\pi i} \int_{\mathrm{Re}(w)=c} \frac{\widetilde{u}(w+1)}{w} \frac{L'}{L}\Big(s + \frac{w}{(\log q)X}, \chi\Big) \mathrm{d}w.$$

The interchange of integral and summation is justified by absolute convergence, which holds because $c > (1-\mathrm{Re}(s))(\log q)X$ and by Lemma 2.1.

We now shift the line of integration to $\mathrm{Re}(w) = -M$, for some $M > \max\{0, \mathrm{Re}(s)(\log q)X\}$, giving

$$\sum_{A \in \mathcal{M}} \frac{\chi(A)\Lambda(A)}{|A|^s} v\Big(e^{\frac{\deg A}{X}}\Big) = -\frac{L'}{L}(s,\chi) - \sum_{\rho_n} \frac{\widetilde{u}(1 + (\rho_n - s)(\log q)X)}{\rho_n - s} - \frac{1}{2\pi i}\int_{\mathrm{Re}(w)=-M} \frac{\widetilde{u}(w+1)}{w} \frac{L'}{L}\Big(s + \frac{w}{(\log q)X}, \chi\Big)\mathrm{d}w,$$

where the sum over the zeros counts multiplicities. This requires some justification. We make use of the contour that is the rectangle with vertices at

$$c \pm i\Big(\big(d - \mathrm{Im}(s)\big)(\log q)X + 2\pi n X\Big)$$

and

$$-M \pm i\Big(\big(d - \mathrm{Im}(s)\big)(\log q)X + 2\pi n X\Big).$$

Here, $d > 0$ is such that $\frac{1}{2} + id$ is not a pole of $\frac{L'}{L}(s,\chi)$ (that is, not a zero of $L(s,\chi)$). It is clear that as $n \to \infty$ we capture all the poles and the left edge tends to the integral over $\mathrm{Re}(w) = -M$. Due to the vertical periodicity of $\frac{L'}{L}$, and our choice of d, we can see that the top and bottom integrals are equal to $O_{c,M}(n^{-1})$, which vanishes as $n \to \infty$. By (2.1) and Lemma 2.1, if we let $M \to \infty$ then we see that the integral over $\mathrm{Re}(w) = -M$ vanishes. Finally, we note that

$$v\Big(e^{\frac{\deg A}{X}}\Big) = \begin{cases} 1 & \text{if } \deg A \leq X \\ 0 & \text{if } \deg A \geq X(1 + q^{-X}). \end{cases}$$

Also, since X is a positive integer, there are no integers in the interval $\big(X, X(1+q^{-X})\big) \subseteq \big(X, X + \frac{1}{2}\big)$, and so there are no $A \in \mathcal{A}$ that have degree in this interval. It follows that

$$\sum_{A \in \mathcal{M}} \frac{\chi(A)\Lambda(A)}{|A|^s} v\Big(e^{\frac{\log|A|}{(\log q)X}}\Big) = \sum_{\substack{A \in \mathcal{M} \\ \deg A \leq X}} \frac{\chi(A)\Lambda(A)}{|A|^s}. \qquad \square$$

Lemma 2.3. *Suppose $u(x)$ has support in $[e, e^{1+q^{-X}}]$. For all $z \in \mathbb{C}\backslash\{0\}$ with $\arg(z) \neq \pi$ we define*

$$U(z) := \int_{x=0}^{\infty} u(x) E_1(z \log x)\mathrm{d}x.$$

(Recall, for $y \in \mathbb{C}\backslash\{0\}$ with $\arg(y) \neq \pi$, we define $E_1(y) := \int_{w=y}^{y+\infty} \frac{e^{-w}}{w}\mathrm{d}w$). Let χ be a primitive Dirichlet character of modulus $R \in \mathcal{M}\backslash\{1\}$, and suppose ρ is a zero of $L(s,\chi)$ and $s \in \mathbb{C}\backslash\{\rho\}$ with $\arg(s-\rho) \neq \pi$. Then,

$$\int_{s_0=s}^{s+\infty} \frac{\widetilde{u}\big(1+(\rho-s_0)(\log q)X\big)}{\rho-s_0}\mathrm{d}s_0 = -U\Big((s-\rho)(\log q)X\Big).$$

Proof. We have

$$\begin{aligned}
&\int_{s_0=s}^{s+\infty} \frac{\widetilde{u}\big(1+(\rho-s_0)(\log q)X\big)}{\rho-s_0}\mathrm{d}s_0 \\
&\qquad = \int_{s_0=s}^{s+\infty} \frac{1}{\rho-s_0} \int_{x=0}^{\infty} x^{(\rho-s_0)(\log q)X} u(x)\mathrm{d}x\mathrm{d}s_0 \\
&\qquad = \int_{x=0}^{\infty} u(x) \int_{s_0=s}^{s+\infty} \frac{e^{(\rho-s_0)(\log q)X\log x}}{\rho-s_0}\mathrm{d}s_0\mathrm{d}x \\
&\qquad = -\int_{x=0}^{\infty} u(x) \int_{w=(s-\rho)(\log q)X\log x}^{(s-\rho)(\log q)X\log x+\infty} \frac{e^{-w}}{w}\mathrm{d}w\mathrm{d}x \\
&\qquad = -\int_{x=0}^{\infty} u(x)E_1\Big((s-\rho)(\log q)X\log x\Big)\mathrm{d}x \\
&\qquad = -U\Big((s-\rho)(\log q)X\Big).
\end{aligned}$$

The interchange of integration is justified by absolute convergence, which holds for $X > 1$. □

We can now proceed with the proof of Theorem 1.2.

Proof of Theorem 1.2. Suppose $s \in \mathbb{C}$ is not a zero of $L(s,\chi)$ and $\arg(s-\rho) \neq \pi$ for all zeros ρ of $L(s,\chi)$. We recall that (2.2) gives us

$$-\frac{L'}{L}(s_0,\chi) = \sum_{\substack{A\in\mathcal{M}\\ \deg A\leq X}} \frac{\chi(A)\Lambda(A)}{|A|^{s_0}} + \sum_{\rho_n} \frac{\widetilde{u}\big(1+(\rho_n-s_0)(\log q)X\big)}{\rho_n-s_0},$$

to which we apply the integral $\int_{s_0=s}^{s+\infty}\mathrm{d}s_0$ to both sides to obtain

$$\log L(s,\chi) = \sum_{\substack{A\in\mathcal{M}\\ \deg A\leq X}} \frac{\chi(A)\Lambda(A)}{|A|^{s}\log|A|} - \sum_{\rho} U\Big((s-\rho)(\log q)X\Big). \tag{2.3}$$

For the integral over the sum over zeros, we applied Lemma 2.3, after an interchange of summation and integration that is justified by Lemma 2.1.

We now take exponentials of both sides of (2.3) to obtain

$$\begin{aligned} L(s,\chi) &= \exp\Bigg(\sum_{\substack{A \in \mathcal{M} \\ \deg A \leq X}} \frac{\chi(A)\Lambda(A)}{|A|^s \log|A|} \Bigg) \exp\Big(- \sum_{\rho} U\Big((s-\rho)(\log q) X \Big) \Big) \\ &= P_X(s,\chi) Z_X(s,\chi). \end{aligned}$$

Now suppose we have $s \in \mathbb{C}$, not being a zero of $L(s,\chi)$, but with $\arg(s-\rho) = \pi$ for some zero ρ of $L(s,\chi)$. We can see that $\lim_{s_0 \to s} L(s_0,\chi) = L(s,\chi)$ and $\lim_{s_0 \to s} P_X(s_0,\chi) = P_X(s,\chi) \neq 0$. The latter is non-zero as $P_X(s,\chi)$ is the exponential of a polynomial. From this, we can deduce that $\lim_{s_0 \to s} Z_X(s_0,\chi) = L(s,\chi)\big(P_X(s,\chi)\big)^{-1} \in \mathbb{C}$. Similarly, if s is a zero of $L(s,\chi)$, then we can see that $\lim_{s_0 \to s} Z_X(s_0,\chi) = L(s,\chi)\big(P_X(s,\chi)\big)^{-1} = 0$. This completes the proof. □

3. Moments of the Partial Euler Product

Recall the prime polynomial theorem (see [26]):

$$|\mathcal{P}_n| = \frac{1}{n} \sum_{d \mid n} \mu(d) q^{\frac{n}{d}} = \frac{q^n}{n} + O\bigg(\frac{q^{\frac{n}{2}}}{n} \bigg). \tag{3.1}$$

Let us define

$$\begin{aligned} \mathcal{S}(X) &:= \{ A \in \mathcal{A} : P \mid A \longrightarrow \deg P \leq X \}, \\ \mathcal{S}_{\mathcal{M}}(X) &:= \{ A \in \mathcal{M} : P \mid A \longrightarrow \deg P \leq X \}. \end{aligned}$$

Furthermore, for all $\operatorname{Re}(s) > 0$ and primitive characters χ we define

$$P_X^*(s,\chi) := \prod_{\deg P \leq X} \bigg(1 - \frac{\chi(P)}{|P|^s} \bigg)^{-1} \prod_{\frac{X}{2} < \deg P \leq X} \bigg(1 + \frac{\chi(P)^2}{2|P|^{2s}} \bigg)^{-1}, \tag{3.2}$$

and for positive integers k and $A \in \mathcal{S}_{\mathcal{M}}(X)$ we define $\alpha_k(A)$ by

$$P_X^*(s,\chi)^k = \sum_{A \in \mathcal{S}_{\mathcal{M}}(X)} \frac{\alpha_k(A)\chi(A)}{|A|^s}.$$

To prove Theorem 1.5, we require the following Lemma.

Lemma 3.1. *For positive integers k, we have*

$$\begin{aligned} P_X\Big(\frac{1}{2},\chi\Big)^k &= \Big(1 + O_k(X^{-1})\Big) P_X^*\Big(\frac{1}{2},\chi\Big)^k \\ &= \Big(1 + O_k(X^{-1})\Big) \sum_{A \in \mathcal{S}_{\mathcal{M}}(X)} \frac{\alpha_k(A)\chi(A)}{|A|^{\frac{1}{2}}}. \end{aligned} \tag{3.3}$$

We also have that

$$\begin{aligned} \alpha_k(A) &= d_k(A) \quad \text{if } A \in \mathcal{S}_{\mathcal{M}}\left(\frac{X}{2}\right) \text{ or } A \text{ is prime} \\ 0 \le \alpha_k(A) &\le d_k(A) \quad \text{if } A \notin \mathcal{S}_{\mathcal{M}}\left(\frac{X}{2}\right) \text{ and } A \text{ is not prime.} \end{aligned} \tag{3.4}$$

Proof. First we note that

$$P_X\left(\frac{1}{2},\chi\right) = \exp\left(\sum_{\substack{A\in\mathcal{M}\\ \deg A\le X}} \frac{\chi(A)\Lambda(A)}{|A|^{\frac{1}{2}}\log|A|}\right) = \exp\left(\sum_{\deg P\le X}\sum_{j=1}^{N_P}\frac{\chi(P)^j}{j|P|^{\frac{j}{2}}}\right),$$

where $N_P := \left\lfloor \frac{X}{\deg P}\right\rfloor$. Also, by using the Taylor series for log, we have

$$P_X^*\left(\frac{1}{2},\chi\right) = \exp\left(\sum_{\deg P\le X}\sum_{j=1}^{\infty}\frac{\chi(P)^j}{j|P|^{\frac{j}{2}}} + \sum_{\frac{X}{2}<\deg P\le X}\sum_{j=1}^{\infty}\frac{(-1)^j\chi(P)^{2j}}{j2^j|P|^j}\right).$$

Hence,

$$\begin{aligned} &P_X\left(\frac{1}{2},\chi\right)P_X^*\left(\frac{1}{2},\chi\right)^{-1} \\ &\qquad = \exp\left(-\sum_{\deg P\le X}\sum_{j=N_p+1}^{\infty}\frac{\chi(P)^j}{j|P|^{\frac{j}{2}}} - \sum_{\frac{X}{2}<\deg P\le X}\sum_{j=1}^{\infty}\frac{(-1)^j\chi(P)^{2j}}{j2^j|P|^j}\right). \end{aligned}$$

We now show that the terms inside the exponential are equal to $O(X^{-1})$, from which we easily deduce

$$P_X\left(\frac{1}{2},\chi\right)^k = \Big(1+O_k(X^{-1})\Big)P_X^*\left(\frac{1}{2},\chi\right)^k.$$

To this end, using the prime polynomial theorem for the last line below, we have

$$
\begin{aligned}
(3.5)\quad \sum_{\deg P \leq X} \sum_{j=N_P+1}^{\infty} \frac{\chi(P)^j}{j|P|^{\frac{j}{2}}} &+ \sum_{\frac{X}{2} < \deg P \leq X} \sum_{j=1}^{\infty} \frac{(-1)^j \chi(P)^{2j}}{j 2^j |P|^j} \\
&= \sum_{\deg P \leq \frac{X}{2}} \sum_{j=N_P+1}^{\infty} \frac{\chi(P)^j}{j|P|^{\frac{j}{2}}} + \sum_{\frac{X}{2} < \deg P \leq X} \sum_{j=2}^{\infty} \frac{\chi(P)^j}{j|P|^{\frac{j}{2}}} \\
&\quad + \sum_{\frac{X}{2} < \deg P \leq X} \sum_{j=1}^{\infty} \frac{(-1)^j \chi(P)^{2j}}{j 2^j |P|^j} \\
&= \sum_{\deg P \leq \frac{X}{2}} \sum_{j=N_P+1}^{\infty} \frac{\chi(P)^j}{j|P|^{\frac{j}{2}}} + \sum_{\frac{X}{2} < \deg P \leq X} \sum_{j=3}^{\infty} \frac{\chi(P)^j}{j|P|^{\frac{j}{2}}} \\
&\quad + \sum_{\frac{X}{2} < \deg P \leq X} \sum_{j=2}^{\infty} \frac{(-1)^j \chi(P)^{2j}}{j 2^j |P|^j} \\
&\ll \sum_{\deg P \leq \frac{X}{2}} |P|^{-\frac{N_P+1}{2}} + \sum_{\frac{X}{2} < \deg P \leq X} |P|^{-\frac{3}{2}} \\
&\ll q^{-\frac{X}{2}} \sum_{\deg P \leq \frac{X}{2}} 1 + \sum_{\frac{X}{2} < n \leq X} \frac{q^{-\frac{n}{2}}}{n} \\
&\ll \frac{1}{X}.
\end{aligned}
$$

We now proceed to prove (3.4). The first case is clear, so assume that $A \notin \mathcal{S}_{\mathcal{M}}(\frac{X}{2})$ and A is not prime. We note that

$$
\begin{aligned}
&\left(1 - \frac{\chi(P)}{|P|^{\frac{1}{2}}}\right)^{-1} \left(1 + \frac{\chi(P)^2}{2|P|}\right)^{-1} \\
&\qquad = \left(1 + \frac{\chi(P)}{|P|^{\frac{1}{2}}} + \frac{\chi(P)^2}{|P|} + \dots\right) \left(1 - \frac{\chi(P)^2}{2|P|} + \frac{\chi(P)^4}{2^2|P|^2} - \dots\right) \\
&\qquad = \sum_{r=0}^{\infty} \left(\sum_{\substack{r_1, r_2 \geq 0 \\ r_1 + 2r_2 = r}} \left(-\frac{1}{2}\right)^{r_2} \right) \frac{\chi(P)^r}{|P|^{\frac{r}{2}}} \\
&\qquad = \sum_{r=0}^{\infty} \frac{2}{3} \left(1 - \left(-\frac{1}{2}\right)^{\lfloor \frac{r}{2} \rfloor + 1}\right) \frac{\chi(P)^r}{|P|^{\frac{r}{2}}}.
\end{aligned}
$$

Since

$$0 \le \frac{2}{3}\Big(1-\Big(-\frac{1}{2}\Big)^{\lfloor \frac{r}{2}\rfloor+1}\Big) \le 1$$

for all $r \ge 0$, the result follows. ☐

We can now prove Theorem 1.5, but before doing so let us recall Mertens's Third Theorem in $\mathbb{F}_q[T]$, the proof of which is very similar to that of Theorem 3 in [25]: As $n \to \infty$,

$$\prod_{\deg P \le n}\Big(1-\frac{1}{|P|}\Big)^{-1} \sim e^{\gamma} n. \tag{3.6}$$

Proof of Theorem 1.5. Throughout this proof, any asymptotic relations are to be taken as $X, \deg R \to \infty$ with $X \le (2-\delta)\log_q \deg R$. By Lemma 3.1 it suffices to prove that

$$\frac{1}{\phi^*(R)} \sideset{}{^*}\sum_{\chi \bmod R} \Bigg| \sum_{A \in \mathcal{S}_{\mathcal{M}}(X)} \frac{\alpha_k(A)\chi(A)}{|A|^{\frac{1}{2}}}\Bigg|^2$$
$$\sim a(k) \prod_{\substack{\deg P \le X \\ P \mid R}} \Bigg(\sum_{m=0}^{\infty} \frac{d_k(P^m)^2}{|P|^m}\Bigg)^{-1} \Big(e^{\gamma} X\Big)^{k^2}.$$

We will truncate our Dirichlet series. This will allow us to bound the lower order terms later. We have

$$\sum_{A \in \mathcal{S}_{\mathcal{M}}(X)} \frac{\alpha_k(A)\chi(A)}{|A|^{\frac{1}{2}}} = \sum_{\substack{A \in \mathcal{S}_{\mathcal{M}}(X) \\ \deg A \le \frac{1}{4}\deg R}} \frac{\alpha_k(A)\chi(A)}{|A|^{\frac{1}{2}}} + O_{\delta}\Big(|R|^{-\frac{\delta}{6}}\Big). \tag{3.7}$$

This makes use of the following, where $\epsilon = \frac{\delta}{5}$:

$$\begin{aligned}
\sum_{\substack{A \in \mathcal{S}_{\mathcal{M}}(X) \\ \deg A > \frac{1}{4}\deg R}} \frac{\alpha_k(A)\chi(A)}{|A|^{\frac{1}{2}}} &\le |R|^{-\epsilon} \sum_{A \in \mathcal{S}_{\mathcal{M}}(X)} \frac{d_k(A)}{|A|^{\frac{1}{2}-\epsilon}} \\
&= |R|^{-\epsilon} \prod_{\deg P \le X}\Big(1-\frac{1}{|P|^{\frac{1}{2}-\epsilon}}\Big)^{-k} \\
&= |R|^{-\epsilon} \exp\Bigg(\sum_{\deg P \le X} -k\log\Big(1-\frac{1}{|P|^{\frac{1}{2}-\epsilon}}\Big)\Bigg) \\
&= |R|^{-\epsilon} \exp\Bigg(kO\Bigg(\sum_{\deg P \le X} \frac{1}{|P|^{\frac{1}{2}-\epsilon}}\Bigg)\Bigg)
\end{aligned} \tag{3.8}$$

$$= |R|^{-\epsilon} \exp\bigg(kO\Big(\frac{q^{(\frac{1}{2}+\epsilon)X+\sqrt{X}}}{\sqrt{X}} \Big) \bigg)$$
$$= O_{\delta}\Big(|R|^{-\frac{\delta}{6}} \Big),$$

where, using the prime polynomial theorem, the fifth relation is justified by the following:

$$\begin{aligned}
\sum_{\deg P \leq X} \frac{1}{|P|^{\frac{1}{2}-\epsilon}} &\leq \sum_{k=0}^{\sqrt{X}} \sum_{kX \leq \deg P \leq (k+1)X} \frac{1}{|P|^{\frac{1}{2}-\epsilon}} \\
&\leq \sum_{k=0}^{\sqrt{X}} \frac{1}{q^{(\frac{1}{2}-\epsilon)k\sqrt{X}}} \frac{q^{(k+1)\sqrt{X}}}{(k+1)\sqrt{X}} \\
&= \frac{q^{\sqrt{X}}}{\sqrt{X}} \sum_{k=0}^{\sqrt{X}} \frac{q^{k(\frac{1}{2}+\epsilon)\sqrt{X}}}{k+1} \leq \frac{q^{(\frac{1}{2}+\epsilon)X+\sqrt{X}}}{\sqrt{X}}.
\end{aligned}$$

By the Cauchy–Schwarz inequality, it suffices to prove that

$$\frac{1}{\phi^*(R)} \sideset{}{^*}\sum_{\chi \bmod R} \bigg| \sum_{\substack{A \in \mathcal{S}_{\mathcal{M}}(X) \\ \deg A \leq \frac{1}{4} \deg R}} \frac{\alpha_k(A)\chi(A)}{|A|^{\frac{1}{2}}} \bigg|^2 \sim a(k) \prod_{\substack{\deg P \leq X \\ P \mid R}} \bigg(\sum_{m=0}^{\infty} \frac{d_k(P^m)^2}{|P|^m} \bigg)^{-1} \Big(e^{\gamma} X \Big)^{k^2}.$$

Now, we have that

$$\begin{aligned}
(3.9)\quad & \frac{1}{\phi^*(R)} \sideset{}{^*}\sum_{\chi \bmod R} \bigg| \sum_{\substack{A \in \mathcal{S}_{\mathcal{M}}(X) \\ \deg A \leq \frac{1}{4} \deg R}} \frac{\alpha_k(A)\chi(A)}{|A|^{\frac{1}{2}}} \bigg|^2 \\
&= \frac{1}{\phi^*(R)} \sum_{\substack{A,B \in \mathcal{S}_{\mathcal{M}}(X) \\ \deg A, \deg B \leq \frac{1}{4} \deg R \\ (AB,R)=1}} \frac{\alpha_k(A)\alpha_k(B)}{|AB|^{\frac{1}{2}}} \sum_{\substack{EF=R \\ F \mid (A-B)}} \mu(E)\phi(F) \\
&= \frac{1}{\phi^*(R)} \sum_{EF=R} \mu(E)\phi(F) \sum_{\substack{A,B \in \mathcal{S}_{\mathcal{M}}(X) \\ \deg A, \deg B \leq \frac{1}{4} \deg R \\ (AB,R)=1 \\ A \equiv B (\bmod F)}} \frac{\alpha_k(A)\alpha_k(B)}{|AB|^{\frac{1}{2}}}
\end{aligned}$$

$$= \sum_{\substack{A\in\mathcal{S}_{\mathcal{M}}(X)\\ \deg A\leq\frac{1}{4}\deg R\\ (A,R)=1}} \frac{\alpha_k(A)^2}{|A|} + \frac{1}{\phi^*(R)}\sum_{EF=R}\mu(E)\phi(F) \sum_{\substack{A,B\in\mathcal{S}_{\mathcal{M}}(X)\\ \deg A,\deg B\leq\frac{1}{4}\deg R\\ (AB,R)=1\\ A\equiv B(\mathrm{mod}\, F)\\ A\neq B}} \frac{\alpha_k(A)\alpha_k(B)}{|AB|^{\frac{1}{2}}}.$$

We first consider the second term on the far right side: The off-diagonal terms. We note that the inner sum is zero if $\deg F > \frac{1}{4}\deg R$, and we also make use of (3.4), to obtain

$$\begin{aligned}
&\frac{1}{\phi^*(R)}\sum_{EF=R}\mu(E)\phi(F) \sum_{\substack{A,B\in\mathcal{S}_{\mathcal{M}}(X)\\ \deg A,\deg B\leq\frac{1}{4}\deg R\\ (AB,R)=1\\ A\equiv B(\mathrm{mod}\, F)\\ A\neq B}} \frac{\alpha_k(A)\alpha_k(B)}{|AB|^{\frac{1}{2}}}\\
&\qquad\ll \frac{1}{\phi^*(R)} \sum_{\substack{EF=R\\ \deg F\leq\frac{1}{4}\deg R}} \phi(F) \sum_{A,B\in\mathcal{S}_{\mathcal{M}}(X)} \frac{d_k(A)d_k(B)}{|AB|^{\frac{1}{2}}}\\
&\qquad\leq \frac{1}{\phi^*(R)} \prod_{\deg P\leq X}\Big(1-|P|^{-\frac{1}{2}}\Big)^{-2k} \sum_{\substack{EF=R\\ \deg F\leq\frac{1}{4}\deg R}} \phi(F)\\
&\qquad\leq \frac{1}{\phi^*(R)} \prod_{\deg P\leq X}\Big(1-|P|^{-\frac{1}{2}}\Big)^{-2k} \sum_{\substack{F\in\mathcal{M}\\ \deg F\leq\frac{1}{4}\deg R}} |R|^{\frac{1}{4}}\\
&\qquad\leq \frac{|R|^{\frac{1}{2}}}{\phi^*(R)} \prod_{\deg P\leq X}\Big(1-|P|^{-\frac{1}{2}}\Big)^{-2k}\\
&\qquad= o(1).
\end{aligned}$$

The last relation makes use of a similar result to (3.8). Now we consider the first term on the far right side of (3.9): The diagonal terms. We required a truncated sum only for the off-diagonal terms, and so we extend our sum using similar means as in (3.8):

$$\sum_{\substack{A\in\mathcal{S}_{\mathcal{M}}(X)\\ \deg A\leq\frac{1}{4}\deg R\\ (A,R)=1}} \frac{\alpha_k(A)^2}{|A|} = \sum_{\substack{A\in\mathcal{S}_{\mathcal{M}}(X)\\ (A,R)=1}} \frac{\alpha_k(A)^2}{|A|} + O\Big(|R|^{-\frac{\delta}{6}}\Big).$$

Now, using (3.4) for the first relation below (and part of the second relation), we have that

$$\sum_{\substack{A \in \mathcal{S}_{\mathcal{M}}(X) \\ (A,R)=1}} \frac{\alpha_k(A)^2}{|A|} = \prod_{\substack{\deg P \leq X \\ P \nmid R}} \left(\sum_{m=0}^{\infty} \frac{\alpha_k(P^m)^2}{|P|^m} \right) \tag{3.10}$$

$$= \prod_{\substack{\deg P \leq X \\ P \nmid R}} \left(\sum_{m=0}^{\infty} \frac{d_k(P^m)^2}{|P|^m} \right) \prod_{\substack{\frac{X}{2} < \deg P \leq X \\ P \nmid R}} \left(\frac{1 + \frac{d_k(P)^2}{|P|} + \sum_{m=2}^{\infty} \frac{\alpha_k(P^m)^2}{|P|^m}}{1 + \frac{d_k(P)^2}{|P|} + \sum_{m=2}^{\infty} \frac{d_k(P^m)^2}{|P|^m}} \right)$$

$$= \prod_{\substack{\deg P \leq X \\ P \mid R}} \left(\sum_{m=0}^{\infty} \frac{d_k(P^m)^2}{|P|^m} \right)^{-1} \prod_{\deg P \leq X} \left(\left(1 - \frac{1}{|P|}\right)^{k^2} \sum_{m=0}^{\infty} \frac{d_k(P^m)^2}{|P|^m} \right)$$

$$\cdot \prod_{\deg P \leq X} \left(1 - \frac{1}{|P|}\right)^{-k^2} \prod_{\frac{X}{2} < \deg P \leq X} \left(1 + O_k\left(\frac{1}{|P|^2} \right) \right)$$

$$= (1 + o(1)) a(k) \prod_{\substack{\deg P \leq X \\ P \mid R}} \left(\sum_{m=0}^{\infty} \frac{d_k(P^m)^2}{|P|^m} \right)^{-1} \left(e^{\gamma} X \right)^{k^2}.$$

For the last equality, we used (3.6). The proof follows. □

4. Moments of the Hadamard Product

In this section we provide support for the Conjecture 1.6. The approach is similar to [7, 14], but we provide some additional heuristic support in Remark 4.2. We require the following lemma.

Lemma 4.1. *For real $y > 0$ define*

$$\mathrm{Ci}(y) := -\int_{t=y}^{\infty} \frac{\cos(t)}{t} \mathrm{d}t,$$

and let x be real and non-zero. Then,

$$\operatorname{Re} E_1(ix) = -\mathrm{Ci}(|x|).$$

Proof. If $x > 0$, then

$$\begin{aligned} \operatorname{Re} E_1(ix) &= \operatorname{Re} \int_{w=ix}^{ix+\infty} \frac{e^{-w}}{w} \mathrm{d}w = \operatorname{Re} \int_{w=ix}^{i\infty} \frac{e^{-w}}{w} \mathrm{d}w = \operatorname{Re} \int_{t=x}^{\infty} \frac{e^{-it}}{t} \mathrm{d}t \\ &= -\mathrm{Ci}(|x|), \end{aligned}$$

where the second relation follows from a contour shift. Similarly, if $x < 0$, then

$$\begin{aligned}\operatorname{Re} E_1(ix) &= \operatorname{Re}\int_{w=ix}^{ix+\infty} \frac{e^{-w}}{w}\mathrm{d}w = \operatorname{Re}\int_{w=ix}^{-i\infty} \frac{e^{-w}}{w}\mathrm{d}w = \operatorname{Re}\int_{t=|x|}^{\infty} \frac{e^{it}}{t}\mathrm{d}t \\ &= -\operatorname{Ci}(|x|).\end{aligned}$$

□

Now, writing $\gamma_n(\chi)$ for the imaginary part of the n-th zero of $L(s,\chi)$, we can see that

$$\begin{aligned}(4.1)\quad & \frac{1}{\phi^*(R)} \sideset{}{^*}\sum_{\chi \bmod R} \left| Z_X\left(\frac{1}{2},\chi\right)\right|^{2k} \\ &= \frac{1}{\phi^*(R)} \sideset{}{^*}\sum_{\chi \bmod R} \exp\left(-2k \operatorname{Re} \sum_{\gamma_n(\chi)} U\Big(-i\gamma_n(\chi)(\log q)X\Big)\right) \\ &= \frac{1}{\phi^*(R)} \sideset{}{^*}\sum_{\chi \bmod R} \exp\left(-2k \operatorname{Re} \sum_{\gamma_n(\chi)} \int_{x=0}^{\infty} u(x) E_1(-i\gamma_n(\chi)(\log q)X\log x)\mathrm{d}x\right) \\ &= \frac{1}{\phi^*(R)} \sideset{}{^*}\sum_{\chi \bmod R} \exp\left(2k \sum_{\gamma_n(\chi)} \int_{x=0}^{\infty} u(x) \operatorname{Ci}\big(|\gamma_n(\chi)|(\log q)X\log x\big)\mathrm{d}x\right).\end{aligned}$$

We note that the terms in the exponential tend to zero as $|\gamma_n(\chi)|$ tends to infinity, and so the above is primarily concerned with the zeros close to $\frac{1}{2}$. As described in Section 1, there is a relationship between the zeros of Dirichlet L-functions near $\frac{1}{2}$ and the eigenphases of random unitary matrices near 0: The proportion of Dirichlet L-functions of modulus R that have j-th zero (that is, its imaginary part) in some interval $[a,b]$ appears to be the same as the proportion of unitary $N(R)\times N(R)$ matrices that have j-th eigenphase in $[a,b]$ (at least, this is the case in an appropriate limit). Naturally, one asks what value $N(R)$ should take in terms of R. We note that the mean spacing between zeros of Dirichlet L-functions of modulus R is $\frac{2\pi}{\log q \deg R}$, while the mean spacing between eigenphases of unitary $N\times N$ matrices is $\frac{2\pi}{N}$. Therefore, we take $N(R) = \lfloor \log q \deg R \rfloor$. So, we replace the imaginary parts of the zeros with eigenphases of $N(R)\times N(R)$ unitary matrices, and instead of averaging over primitive characters we average over unitary matrices. That is, we conjecture

$$
\begin{aligned}
(4.2)\quad & \frac{1}{\phi^*(R)} \sideset{}{^*}\sum_{\chi \bmod R} \left| Z_X\left(\frac{1}{2}, \chi\right) \right|^{2k} \\
&= \frac{1}{\phi^*(R)} \sideset{}{^*}\sum_{\chi \bmod R} \exp\left(2k \sum_{\gamma_n(\chi)} \int_{x=0}^{\infty} u(x) \operatorname{Ci}\big(|\gamma_n(\chi)|(\log q) X \log x\big) \mathrm{d}x \right) \\
&\sim \int_{A \in U(N(R))} \exp\left(2k \sum_{\theta_n(A)} \int_{x=0}^{\infty} u(x) \operatorname{Ci}\big(|\theta_n(A)|(\log q) X \log x\big) \mathrm{d}x \right) \mathrm{d}A
\end{aligned}
$$

as $\deg R \to \infty$, where the integral is with respect to the Haar measure, and $\theta_n(A)$ is the n-th eigenphase of A. The eigenphases are periodic with period 2π, and these periodicised eigenphases are included in the sum. An asymptotic evaluation of the right side can be made identically as in Section 4 of [14]; but we simply replace their $\log X$ with our $(\log q)X$, and we replace their $N = \lfloor \log T \rfloor$ with our $N(R) = \lfloor \log q \deg R \rfloor$. This leads us to the conjecture that

$$
\frac{1}{\phi^*(R)} \sideset{}{^*}\sum_{\chi \bmod R} \left| Z_X\left(\frac{1}{2}, \chi\right) \right|^{2k} \sim \frac{G^2(k+1)}{G(2k+1)} \left(\frac{\deg R}{e^{\gamma} X} \right)^{k^2},
$$

as $\deg R \to \infty$. We note that in [14], their $u(x)$ has a slightly different support than the support of our $u(x)$. However, this does not affect the result.

Remark 4.2. We will provide further justification for one of the steps above, which is not given in [14]. In the middle line of (4.2) we have a sum over all $\gamma_n(\chi)$. This includes zeros that are far away from $\frac{1}{2}$. We mentioned previously that their contribution is small, but a closer inspection reveals that we cannot dismiss them so easily, and so we must justify replacing them with the eigenphases of our unitary matrices. For the zeros close to $\frac{1}{2}$ (that is, for $\gamma_n(\chi)$ close to 0) we have already provided this justification. For the zeros further away, one can argue that the zeros of a typical Dirichlet L-function are equidistributed in some manner, and that the eigenphases of a typical unitary matrix are also equidistributed in some manner. Thus, we could replace the former with the latter. This is based on the idea that if you sum a function over a set of equidistributed points on some interval I, then the result is roughly equal to the integral over I of that function multiplied by the reciprocal of the mean spacing of the points. Recall that the mean spacing of our eigenphases is equal to that of our zeros. Naturally, one asks why we do not use the same justification for the zeros close to $\frac{1}{2}$. The answer is that the function $\operatorname{Ci}(x)$ has a discontinuity at $x = 0$, and so we require a stronger justification for the zeros near $\frac{1}{2}$ (that is, the $\gamma_n(\chi)$

close to 0). Finally, we remark that we do not provide any rigorous support for the claims on equidistribution above.

5. The Second Hadamard Moment

Before proving Theorem 1.7, we prove several lemmas. First, by (3.3) we have

$$P_X\Big(\frac{1}{2},\chi\Big) = \Big(1+O(X^{-1})\Big)P_X^*\Big(\frac{1}{2},\chi\Big).$$

Rearranging and using (3.2) gives

$$\begin{aligned}(5.1)\quad & P_X\Big(\frac{1}{2},\chi\Big)^{-1} \\ &= \Big(1+O(X^{-1})\Big)P_X^*\Big(\frac{1}{2},\chi\Big)^{-1} \\ &= \Big(1+O(X^{-1})\Big)\prod_{\deg P\leq X}\Big(1-\frac{\chi(P)}{|P|^{\frac{1}{2}}}\Big)\prod_{\frac{X}{2}<\deg P\leq X}\Big(1+\frac{\chi(P)^2}{2|P|}\Big) \\ &= \Big(1+O(X^{-1})\Big)\sum_{A\in\mathcal{S}_{\mathcal{M}}(X)}\frac{\alpha_{-1}(A)\chi(A)}{|A|^{\frac{1}{2}}},\end{aligned}$$

where α_{-1} is defined multiplicatively by

$$\begin{aligned}\alpha_{-1}(P) &:= \begin{cases}-1 & \text{if } \deg P\leq X\\ 0 & \text{if } \deg P>X;\end{cases}\\ \alpha_{-1}(P^2) &:= \begin{cases}0 & \text{if } \deg P\leq \frac{X}{2}\\ \frac{1}{2} & \text{if } \frac{X}{2}<\deg P\leq X\\ 0 & \text{if } \deg P>X;\end{cases}\\ \alpha_{-1}(P^3) &:= \begin{cases}0 & \text{if } \deg P\leq \frac{X}{2}\\ -\frac{1}{2} & \text{if } \frac{X}{2}<\deg P\leq X\\ 0 & \text{if } \deg P>X;\end{cases}\\ \alpha_{-1}(P^m) &:= 0 \text{ for } m\geq 4.\end{aligned}$$

Later, we will require the following two simple bounds: For all $R \in \mathcal{M}$, as $X \to \infty$,

$$\sum_{\substack{HST \in \mathcal{S}_{\mathcal{M}}(X) \\ (S,T)=1 \\ (HST,R)=1 \\ \deg HS, \deg HT \leq \frac{1}{10} \deg R}} \frac{|\alpha_{-1}(HS)\alpha_{-1}(HT)|}{|HST|} \ll \bigg(\sum_{H \in \mathcal{S}_{\mathcal{M}}(X)} \frac{1}{|H|} \bigg)^3 = \prod_{\deg P \leq X} \Big(1 - |P|^{-1} \Big)^{-3} \ll X^3, \tag{5.2}$$

where we have used (3.6); and

$$\sum_{\substack{HST \in \mathcal{S}_{\mathcal{M}}(X) \\ (S,T)=1 \\ (HST,R)=1 \\ \deg HS, \deg HT \leq \frac{1}{10} \deg R}} \frac{\alpha_{-1}(HS)\alpha_{-1}(HT)}{|HST|} \deg ST \ll \sum_{H \in \mathcal{S}_{\mathcal{M}}(X)} \frac{1}{|H|} \sum_{S,T \in \mathcal{S}_{\mathcal{M}}(X)} \frac{\deg ST}{|ST|} \ll X^4, \tag{5.3}$$

where the last equality is obtained by taking the derivative of $f(s) := \sum_{S,T \in \mathcal{S}_{\mathcal{M}}(X)} \frac{1}{|ST|^s} = \prod_{\deg P \leq X} \big(1 - |P|^{-s}\big)^{-2}$, evaluating at 1, and using the prime polynomial theorem to get

$$\sum_{S,T \in \mathcal{S}_{\mathcal{M}}(X)} \frac{\deg ST}{|ST|} = 2 \prod_{\deg P \leq X} \Big(1 - |P|^{-1} \Big)^{-2} \sum_{\deg P \leq X} \frac{\deg P}{|P| - 1} \ll X^3.$$

Lemma 5.1. *Let $V \in \mathcal{M}$. V may or may not depend on R. As $X, \deg R \to \infty$ with $X \leq (2-\delta) \log_q \deg R$, we have*

$$\begin{aligned} \sum_{\substack{HST \in \mathcal{S}_{\mathcal{M}}(X) \\ (S,T)=1 \\ (HST,V)=1 \\ \deg HS, \deg HT \leq \frac{1}{10} \deg R}} \frac{\alpha_{-1}(HS)\alpha_{-1}(HT)}{|HST|} &= \Big(1 + O\Big(q^{-\frac{X}{2}} \Big) \Big) \prod_{\substack{\deg P \leq X \\ P \nmid V}} \Big(1 - \frac{1}{|P|} \Big) + O\Big(\frac{1}{|R|^{\frac{1}{21}}} \Big) \\ &\sim \prod_{\substack{\deg P \leq X \\ P \nmid V}} \Big(1 - \frac{1}{|P|} \Big). \end{aligned}$$

Proof. The second relation in the Lemma follows easily from (3.6). We will prove the first. In this proof, all asymptotic relations are to be taken as $X, \deg R \to \infty$ with $X \leq (2-\delta)\log_q \deg R$. Similar to (3.7), we can remove the conditions $\deg HS, \deg HT \leq \frac{1}{10}\deg R$ from the sum and this only adds an $O\big(|R|^{-\frac{1}{40}}\big)$ term. Now, writing $C = HS$ and $D = HT$, we have

$$\begin{aligned}
\sum_{\substack{HST\in\mathcal{S}_{\mathcal{M}}(X)\\(S,T)=1\\(HST,V)=1}} \frac{\alpha_{-1}(HS)\alpha_{-1}(HT)}{|HST|} &= \sum_{\substack{CD\in\mathcal{S}_{\mathcal{M}}(X)\\(CD,V)=1}} \frac{\alpha_{-1}(C)\alpha_{-1}(D)}{|CD|}|(C,D)| \\
&= \sum_{\substack{CD\in\mathcal{S}_{\mathcal{M}}(X)\\(CD,V)=1}} \frac{\alpha_{-1}(C)\alpha_{-1}(D)}{|CD|} \sum_{G|(C,D)} \phi(G) \\
&= \sum_{\substack{G\in\mathcal{S}_{\mathcal{M}}(X)\\(G,V)=1}} \frac{\phi(G)}{|G|^2}\Bigg(\sum_{\substack{C\in\mathcal{S}_{\mathcal{M}}(X)\\(C,V)=1}} \frac{\alpha_{-1}(CG)}{|C|}\Bigg)^2 .
\end{aligned}$$

Before continuing, let us make a definition: For all $A \in \mathcal{M}$ and all $P \in \mathcal{P}$, let $e_P(A)$ be the largest integer such that $P^{e_P(A)} \mid A$. Continuing, we note that we can restrict the sums to polynomials that are fourth power free. Indeed, $\alpha_{-1}(P^m) = 0$ for all $P \in \mathcal{P}$ and all $m \geq 4$. Note that if $P \mid G$ then we must have that $0 \leq e_P(C) \leq 3 - e_P(G)$, while if $P \nmid G$ then $0 \leq e_P(C) \leq 3$. So, we have

$$\begin{aligned}
&\sum_{\substack{C\in\mathcal{S}_{\mathcal{M}}(X)\\(C,V)=1}} \frac{\alpha_{-1}(CG)}{|C|} \\
&= \prod_{P|G}\Bigg(\sum_{j=0}^{3-e_P(G)} \frac{\alpha_{-1}(P^{j+e_P(G)})}{|P|^j}\Bigg) \prod_{\substack{\deg P\leq X\\ P\nmid G\\ P\nmid V}} \Bigg(\sum_{j=0}^{3}\frac{\alpha_{-1}(P^j)}{|P|^j}\Bigg) \\
&= \prod_{\substack{\deg P\leq X\\ P\nmid V}} \Bigg(\sum_{j=0}^{3}\frac{\alpha_{-1}(P^j)}{|P|^j}\Bigg) \prod_{P|G}\Bigg(\sum_{j=0}^{3-e_P(G)} \frac{\alpha_{-1}(P^{j+e_P(G)})}{|P|^j}\Bigg) \\
&\qquad \times \prod_{P|G}\Bigg(\sum_{j=0}^{3}\frac{\alpha_{-1}(P^j)}{|P|^j}\Bigg)^{-1} .
\end{aligned}$$

So,

$$\begin{aligned}
&\sum_{\substack{G \in \mathcal{S}_{\mathcal{M}}(X) \\ (G,V)=1}} \frac{\phi(G)}{|G|^2} \Bigg(\sum_{\substack{C \in \mathcal{S}_{\mathcal{M}}(X) \\ (C,V)=1}} \frac{\alpha_{-1}(CG)}{|C|} \Bigg)^2 \\
&= \prod_{\substack{\deg P \leq X \\ P \nmid V}} \bigg(\sum_{j=0}^{3} \frac{\alpha_{-1}(P^j)}{|P|^j} \bigg)^2 \\
&\quad \times \prod_{\substack{\deg P \leq X \\ P \nmid V}} \Bigg(\sum_{i=0}^{3} \frac{\phi(P^i)}{|P|^{2i}} \bigg(\sum_{j=0}^{3-i} \frac{\alpha_{-1}(P^{j+i})}{|P|^j} \bigg)^2 \bigg(\sum_{j=0}^{3} \frac{\alpha_{-1}(P^j)}{|P|^j} \bigg)^{-2} \Bigg) \\
&= \prod_{\substack{\deg P \leq X \\ P \nmid V}} \Bigg(\sum_{i=0}^{3} \frac{\phi(P^i)}{|P|^{2i}} \bigg(\sum_{j=0}^{3-i} \frac{\alpha_{-1}(P^{j+i})}{|P|^j} \bigg)^2 \Bigg) \\
&= \prod_{\substack{\deg P \leq X \\ P \nmid V}} \Bigg(\sum_{i=0}^{3} \sum_{j=0}^{3-i} \sum_{k=0}^{3-i} \frac{\phi(P^i) \alpha_{-1}(P^{j+i}) \alpha_{-1}(P^{k+i})}{|P^{2i+j+k}|} \Bigg) \\
&= \prod_{\substack{\deg P \leq X \\ P \nmid V}} \Big(1 - \frac{1}{|P|} \Big) \prod_{\substack{\frac{X}{2} \leq \deg P \leq X \\ P \nmid V}} \Big(1 + O\Big(\frac{1}{|P^2|} \Big) \Big) \\
&= \Big(1 + O\Big(q^{-\frac{X}{2}} \Big) \Big) \prod_{\substack{\deg P \leq X \\ P \nmid V}} \Big(1 - \frac{1}{|P|} \Big).
\end{aligned}$$

The result follows. □

Lemma 5.2. *Let $R \in \mathcal{M}$. Suppose $Z_1 \leq \deg R$ and $F \mid R$. Further, suppose $C, D \in \mathcal{S}_{\mathcal{M}}(X)$ with $\deg C, \deg D \leq \frac{1}{10} \deg R$. Then, we have*

$$\sum_{\substack{A,B \in \mathcal{M} \\ \deg AB = Z_1 \\ AC \equiv BD (\operatorname{mod} F) \\ AC \neq BD \\ (AB,R)=1}} \frac{1}{|AB|^{\frac{1}{2}}} \ll \frac{q^{\frac{Z_1}{2}} (Z_1 + 1) |CD|}{|F|}.$$

Proof. Consider the case where $\deg AC > \deg BD$, and suppose that $\deg A = i$. We have that $AC = LF + BD$ for some $L \in \mathcal{M}$ with $\deg L = \deg AC - \deg F = i + \deg C - \deg F$, and $\deg B = Z_1 - \deg A = Z_1 - i$.

Hence,

$$\sum_{\substack{A,B\in\mathcal{M}\\ \deg AB=Z_1\\ AC\equiv BD(\mathrm{mod}\, F)\\ (AB,R)=1\\ \deg AC>\deg BD}} \frac{1}{|AB|^{\frac{1}{2}}} \leq q^{-\frac{Z_1}{2}} \sum_{i=0}^{Z_1} \sum_{\substack{L\in\mathcal{M}\\ \deg L=i+\deg C-\deg F}} \sum_{\substack{B\in\mathcal{M}\\ \deg B=Z_1-i}} 1$$

$$= q^{\frac{Z_1}{2}} \sum_{i=0}^{Z_1} \sum_{\substack{L\in\mathcal{M}\\ \deg L=i+\deg C-\deg F}} q^{-i} = \frac{q^{\frac{Z_1}{2}}|C|}{|F|} \sum_{i=0}^{Z_1} 1 = \frac{q^{\frac{Z_1}{2}}(Z_1+1)|C|}{|F|}.$$

Similarly, when $\deg BD > \deg AC$ we have

$$\sum_{\substack{A,B\in\mathcal{M}\\ \deg AB=Z_1\\ AC\equiv BD(\mathrm{mod}\, F)\\ (AB,R)=1\\ \deg AC>\deg BD}} \frac{1}{|AB|^{\frac{1}{2}}} \leq \frac{q^{\frac{Z_1}{2}}(Z_1+1)|D|}{|F|}.$$

Suppose now that $\deg AC = \deg BD = i$. Then, $2i = \deg ABCD = Z_1 + \deg CD$. We have $\deg B = i - \deg D = \frac{Z_1+\deg C-\deg D}{2}$, and $AC = LF + BD$ for some $L \in \mathcal{A}$ with $\deg L < i - \deg F = \frac{Z_1+\deg CD}{2} - \deg F$. Hence,

$$\sum_{\substack{A,B\in\mathcal{M}\\ \deg AB=Z_1\\ AC\equiv BD(\mathrm{mod}\, F)\\ (AB,R)=1\\ \deg AC=\deg BD}} \frac{1}{|AB|^{\frac{1}{2}}} \leq^{-\frac{Z_1}{2}} \sum_{\substack{B\in\mathcal{M}\\ \deg B=\frac{Z_1+\deg C-\deg D}{2}}} \sum_{\substack{L\in\mathcal{A}\\ \deg L<\frac{Z_1+\deg CD}{2}-\deg F}} 1$$

$$= \frac{|CD|^{\frac{1}{2}}}{|F|} \sum_{\substack{B\in\mathcal{M}\\ \deg B=\frac{Z_1+\deg C-\deg D}{2}}} 1 = \frac{q^{\frac{Z_1}{2}}|C|}{|F|}.$$

The result follows. □

Lemma 5.3. *Let χ a primitive character of modulus $R \neq 1$. Then,*

$$\left|L\left(\frac{1}{2},\chi\right)\right|^2 = 2 \sum_{\substack{A,B\in\mathcal{M}\\ \deg AB<\deg R}} \frac{\chi(A)\overline{\chi}(B)}{|AB|^{\frac{1}{2}}} + c(\chi),$$

where, if χ is odd, we define

$$c(\chi) := -\sum_{\substack{A,B\in\mathcal{M}\\ \deg AB=\deg R-1}} \frac{\chi(A)\overline{\chi}(B)}{|AB|^{\frac{1}{2}}},$$

and if χ is even we define

$$\begin{aligned} c(\chi) := &- \frac{q}{\left(q^{\frac{1}{2}}-1\right)^2} \sum_{\substack{A,B\in\mathcal{M} \\ \deg AB = \deg R - 2}} \frac{\chi(A)\overline{\chi}(B)}{|AB|^{\frac{1}{2}}} \\ &- \frac{2q^{\frac{1}{2}}}{q^{\frac{1}{2}}-1} \sum_{\substack{A,B\in\mathcal{M} \\ \deg AB = \deg R - 1}} \frac{\chi(A)\overline{\chi}(B)}{|AB|^{\frac{1}{2}}} \\ &+ \frac{1}{\left(q^{\frac{1}{2}}-1\right)^2} \sum_{\substack{A,B\in\mathcal{M} \\ \deg AB = \deg R}} \frac{\chi(A)\overline{\chi}(B)}{|AB|^{\frac{1}{2}}}. \end{aligned}$$

Proof. See Lemmas 3.10 and 3.11 in [2]. □

Lemma 5.4. *Let $R \in \mathcal{M}$ and let x be a positive integer. Then,*

$$\sum_{\substack{A\in\mathcal{M} \\ \deg A \leq x \\ (A,R)=1}} \frac{1}{|A|} = \begin{cases} \frac{\phi(R)}{|R|}x + O\Big(\frac{\phi(R)}{|R|}\log\omega(R)\Big) & \text{if } x \geq \deg R, \\ \frac{\phi(R)}{|R|}x + O\Big(\frac{\phi(R)}{|R|}\log\omega(R)\Big) + O\Big(\frac{2^{\omega(R)}x}{q^x}\Big) & \text{if } x < \deg R. \end{cases}$$

Proof. See Lemma 4.12 in [2]. This result is slightly stronger, but the proof is identical. □

The big O terms in Lemma hold for any x is the given ranges, and no limits are required. Note that, in certain cases these terms are larger than the first term. Regardless, this does not cause us any problems, and we only require the two cases in the following corollary.

Corollary 5.5. *If $a > 0$ and $x = a \deg R$, then,*

$$\sum_{\substack{A\in\mathcal{M} \\ \deg A \leq x \\ (A,R)=1}} \frac{1}{|A|} = \frac{\phi(R)}{|R|}x + O_a\Big(\frac{\phi(R)}{|R|}\log\omega(R)\Big).$$

If $b > 2$ and $x = \log_q b^{\omega(R)}$, then

$$\sum_{\substack{A\in\mathcal{M} \\ \deg A \leq x \\ (A,R)=1}} \frac{1}{|A|} = \frac{\phi(R)}{|R|}x + O_b\Big(\frac{\phi(R)}{|R|}\log\omega(R)\Big).$$

Proof. First consider the case where $x = a \deg R$. If $q > e^{\frac{4\log 2}{a}}$, then

$$\frac{2^{\omega(R)}x}{q^x} \ll \frac{2^{\omega(R)}}{q^{\frac{x}{2}}} \leq q^{\frac{\log 2}{\log q}\deg R - \frac{a}{2}\deg R} < q^{-\frac{a}{4}\deg R} \ll_a \frac{\phi(R)}{|R|}.$$

If $q \leq e^{\frac{4\log 2}{a}}$, then

$$\frac{2^{\omega(R)}x}{q^x} \ll \frac{2^{\omega(R)}}{q^{\frac{x}{2}}} = q^{O\left(\frac{\deg R}{\log \deg R}\right)-\frac{a}{2}\deg R} \leq q^{-\frac{a}{4}\deg R} \ll_a \frac{\phi(R)}{|R|},$$

where the second relation holds for $\deg R > c_a$, where c_a is some constant that is dependent on a, but independent of q. Finally, there are only a finite number of cases where $q \leq e^{\frac{4\log 2}{a}}$ and $\deg R \leq c_a$, and so

$$\frac{2^{\omega(R)}x}{q^x} \ll_a \frac{\phi(R)}{|R|}$$

for these cases too. The proof follows from Lemma 5.4.

Now consider the case where $x = \log_q b^{\omega(R)}$. We have that

$$\frac{2^{\omega(R)}x}{q^x} = \frac{2^{\omega(R)}(\log_q b)\omega(R)}{b^{\omega(R)}} \ll_b \frac{2^{\omega(R)}}{\left(\frac{b+2}{2}\right)^{\omega(R)}} = \left(\frac{4}{b+2}\right)^{\omega(R)}$$
$$= \prod_{P|R}\left(\frac{4}{b+2}\right) \ll_b \prod_{P|R}\left(1-\frac{1}{|P|}\right) \ll_b \frac{\phi(R)}{|R|}.$$

Again, the proof follows from Lemma 5.4. □

We can now prove Theorem 1.7.

Proof of Theorem 1.7. Throughout the proof, all asymptotic relations will be taken as $X, \deg R \to \infty$ with $X \leq (2-\delta)\log_q \deg R$. Now, by (5.1), we have

$$\frac{1}{\phi^*(R)} \sideset{}{^*}\sum_{\chi \bmod R} \left| L\left(\frac{1}{2},\chi\right) P_X\left(\frac{1}{2},\chi\right)^{-1}\right|^2 \sim \frac{1}{\phi^*(R)} \sideset{}{^*}\sum_{\chi \bmod R} \left| L\left(\frac{1}{2},\chi\right) P_X^*\left(\frac{1}{2},\chi\right)^{-1}\right|^2. \tag{5.4}$$

Similar to (3.7), we truncate our sum:

$$P_X^*\left(\frac{1}{2},\chi\right)^{-1} = \sum_{\substack{C\in\mathcal{S}_{\mathcal{M}}(X)\\ \deg C \leq \frac{1}{10}\deg R}} \frac{\alpha_{-1}(C)\chi(C)}{|C|^{\frac{1}{2}}} + O_\delta\left(|R|^{-\frac{\delta}{6}}\right).$$

Using this, the Cauchy–Schwarz inequality, and (1.3), it suffices to prove that

$$\frac{1}{\phi^*(R)} \sideset{}{^*}\sum_{\chi \bmod R} \left| L\left(\frac{1}{2}, \chi\right) \right|^2 \sum_{\substack{C,D \in \mathcal{S}_{\mathcal{M}}(X) \\ \deg C, \deg D \leq \frac{1}{10} \deg R}} \frac{\alpha_{-1}(C)\alpha_{-1}(D)\chi(C)\overline{\chi}(D)}{|CD|^{\frac{1}{2}}} \sim \frac{\deg R}{e^{\gamma} X} \prod_{\substack{\deg P > X \\ P \mid R}} \left(1 - \frac{1}{|P|}\right). \tag{5.5}$$

Now, by Lemma 5.3, we have

$$\begin{aligned} &\frac{1}{\phi^*(R)} \sideset{}{^*}\sum_{\chi \bmod R} \left| L\left(\frac{1}{2}, \chi\right) \right|^2 \sum_{\substack{C,D \in \mathcal{S}_{\mathcal{M}}(X) \\ \deg C, \deg D \leq \frac{1}{10} \deg R}} \frac{\alpha_{-1}(C)\alpha_{-1}(D)\chi(C)\overline{\chi}(D)}{|CD|^{\frac{1}{2}}} \\ &= \frac{1}{\phi^*(R)} \sideset{}{^*}\sum_{\chi \bmod R} \big(a(\chi) + c(\chi)\big) \sum_{\substack{C,D \in \mathcal{S}_{\mathcal{M}}(X) \\ \deg C, \deg D \leq \frac{1}{10} \deg R}} \frac{\alpha_{-1}(C)\alpha_{-1}(D)\chi(C)\overline{\chi}(D)}{|CD|^{\frac{1}{2}}}, \end{aligned}$$

where

$$a(\chi) := 2 \sum_{\substack{A,B \in \mathcal{M} \\ \deg AB < \deg R}} \frac{\chi(A)\overline{\chi}(B)}{|AB|^{\frac{1}{2}}}$$

and $c(\chi)$ is defined in Lemma 5.3. We first consider the case with $a(\chi)$. We have

$$\begin{aligned} &\frac{1}{\phi^*(R)} \sideset{}{^*}\sum_{\chi \bmod R} a(\chi) \sum_{\substack{C,D \in \mathcal{S}_{\mathcal{M}}(X) \\ \deg C, \deg D \leq \frac{1}{10} \deg R}} \frac{\alpha_{-1}(C)\alpha_{-1}(D)\chi(C)\overline{\chi}(D)}{|CD|^{\frac{1}{2}}} \\ &= \frac{2}{\phi^*(R)} \sideset{}{^*}\sum_{\chi \bmod R} \sum_{\substack{A,B \in \mathcal{M} \\ C,D \in \mathcal{S}_{\mathcal{M}}(X) \\ \deg AB < \deg R \\ \deg C, \deg D \leq \frac{1}{10} \deg R}} \frac{\alpha_{-1}(C)\alpha_{-1}(D)\chi(AC)\overline{\chi}(BD)}{|ABCD|^{\frac{1}{2}}} \\ &= \frac{2}{\phi^*(R)} \sum_{EF = R} \mu(E)\phi(F) \sum_{\substack{A,B \in \mathcal{M} \\ C,D \in \mathcal{S}_{\mathcal{M}}(X) \\ \deg AB < \deg R \\ \deg C, \deg D \leq \frac{1}{10} \deg R \\ (ABCD, R) = 1 \\ AC \equiv BD (\bmod F)}} \frac{\alpha_{-1}(C)\alpha_{-1}(D)}{|ABCD|^{\frac{1}{2}}} \end{aligned} \tag{5.6}$$

$$= 2 \sum_{\substack{A,B\in\mathcal{M}\\ C,D\in\mathcal{S}_{\mathcal{M}}(X)\\ \deg AB<\deg R\\ \deg C,\deg D\leq\frac{1}{10}\deg R\\ (ABCD,R)=1\\ AC=BD}} \frac{\alpha_{-1}(C)\alpha_{-1}(D)}{|ABCD|^{\frac{1}{2}}}$$

$$+ \frac{2}{\phi^*(R)} \sum_{EF=R} \mu(E)\phi(F) \sum_{\substack{A,B\in\mathcal{M}\\ C,D\in\mathcal{S}_{\mathcal{M}}(X)\\ \deg AB<\deg R\\ \deg C,\deg D\leq\frac{1}{10}\deg R\\ (ABCD,R)=1\\ AC\equiv BD(\mathrm{mod}\, F)\\ AC\neq BD}} \frac{\alpha_{-1}(C)\alpha_{-1}(D)}{|ABCD|^{\frac{1}{2}}}.$$

For the first term on the right side, the diagonal terms, we write $A = GS$, $B = GT$, $C = HT$, $D = HS$ where $G, H, S, T \in \mathcal{M}$ and $(S, T) = 1$, giving

$$(5.7)\quad 2 \sum_{\substack{A,B,C,D\in\mathcal{M}\\ C,D\in\mathcal{S}_{\mathcal{M}}(X)\\ \deg AB<\deg R\\ \deg C,\deg D\leq\frac{1}{10}\deg R\\ (ABCD,R)=1\\ AC=BD}} \frac{\alpha_{-1}(C)\alpha_{-1}(D)}{|ABCD|^{\frac{1}{2}}}$$

$$= 2 \sum_{\substack{G\in\mathcal{M}\\ H,S,T\in\mathcal{S}_{\mathcal{M}}(X)\\ \deg G^2ST<\deg R\\ \deg HS,\deg HT\leq\frac{1}{10}\deg R\\ (GHST,R)=1\\ (S,T)=1}} \frac{\alpha_{-1}(HT)\alpha_{-1}(HS)}{|GHST|}$$

$$= 2 \sum_{\substack{H,S,T\in\mathcal{S}_{\mathcal{M}}(X)\\ \deg HS,\deg HT\leq\frac{1}{10}\deg R\\ (HST,R)=1\\ (S,T)=1}} \frac{\alpha_{-1}(HS)\alpha_{-1}(HT)}{|HST|} \sum_{\substack{G\in\mathcal{M}\\ \deg G\leq\frac{\deg R-\deg ST}{2}\\ (G,R)=1}} \frac{1}{|G|}.$$

By Corollary 5.5, (5.2), (5.3), and Lemma 5.1 we obtain the asymptotic relation below. The final relation uses (3.6).

$$2 \sum_{\substack{A,B\in\mathcal{M} \\ C,D\in\mathcal{S}_{\mathcal{M}}(X) \\ \deg AB<\deg R \\ \deg C,\deg D\leq\frac{1}{10}\deg R \\ (ABCD,R)=1 \\ AC=BD}} \frac{\alpha_{-1}(C)\alpha_{-1}(D)}{|ABCD|^{\frac{1}{2}}} \sim \frac{\phi(R)}{|R|}\deg R \prod_{\substack{\deg P\leq X \\ P\nmid R}} \left(1-\frac{1}{|P|}\right) \sim \frac{\deg R}{e^{\gamma}X} \prod_{\substack{\deg P>X \\ P\mid R}} \left(1-\frac{1}{|P|}\right). \tag{5.8}$$

For the second term on the far right side of (5.6), the off-diagonal terms, we use Lemma 5.2 to obtain

$$\begin{aligned} &\frac{2}{\phi^*(R)} \sum_{EF=R} \mu(E)\phi(F) \sum_{\substack{A,B\in\mathcal{M} \\ C,D\in\mathcal{S}_{\mathcal{M}}(X) \\ \deg AB<\deg R \\ \deg C,\deg D\leq\frac{1}{10}\deg R \\ (ABCD,R)=1 \\ AC\equiv BD(\mathrm{mod}\, F) \\ AC\neq BD}} \frac{\alpha_{-1}(C)\alpha_{-1}(D)}{|ABCD|^{\frac{1}{2}}} \\ &= \frac{2}{\phi^*(R)} \sum_{\substack{C,D\in\mathcal{S}_{\mathcal{M}}(X) \\ \deg C,\deg D\leq\frac{1}{10}\deg R \\ (CD,R)=1}} \frac{\alpha_{-1}(C)\alpha_{-1}(D)}{|CD|^{\frac{1}{2}}} \\ &\quad\times \sum_{EF=R} \mu(E)\phi(F) \sum_{\substack{A,B\in\mathcal{M} \\ \deg AB<\deg R \\ (AB,R)=1 \\ AC\equiv BD(\mathrm{mod}\, F) \\ AC\neq BD}} \frac{1}{|AB|^{\frac{1}{2}}} \\ &\ll \frac{|R|^{\frac{1}{2}}\deg R}{\phi^*(R)} \sum_{\substack{C,D\in\mathcal{S}_{\mathcal{M}}(X) \\ \deg C,\deg D\leq\frac{1}{10}\deg R}} |CD|^{\frac{1}{2}} \sum_{EF=R} |\mu(E)| \frac{\phi(F)}{|F|} \\ &\ll \frac{|R|^{\frac{4}{5}} 2^{\omega(R)} \deg R}{\phi^*(R)} = o(1). \end{aligned} \tag{5.9}$$

Finally, for the case with $c(\chi)$, we can proceed similarly as above to show that it's contribution is $\ll X^3$. $\square$

6. Preliminary Results for the Fourth Hadamard Moment

In this section we develop the preliminary results that are required for the proof of Theorem 1.8. We begin with two results that will simplify the problem.

Lemma 6.1. *For $X \geq 12$, we have that*

$$P_X\Big(\frac{1}{2},\chi\Big)^{-2} = (1+O(X^{-1}))P_X{}^{**}\Big(\frac{1}{2},\chi\Big),$$

where

$$P_X^{**}\Big(\frac{1}{2},\chi\Big) := \sum_{A\in\mathcal{S}_{\mathcal{M}}(X)} \frac{\beta(A)\chi(A)}{|A|^{\frac{1}{2}}}$$

and β is defined multiplicatively by

$$\beta(P) := \begin{cases} -2 & \textit{if} \deg P \leq X \\ 0 & \textit{if} \deg P > X \end{cases}$$

$$\beta(P^2) := \begin{cases} 1 & \textit{if} \deg P \leq \frac{X}{2} \\ 2 & \textit{if } \frac{X}{2} < \deg P \leq X \\ 0 & \textit{if} \deg P > X \end{cases} \tag{6.1}$$

$$\beta(P^k) := 0 \textit{ for } k \geq 3.$$

Proof. By Lemma 3.1 we have

$$P_X\Big(\frac{1}{2},\chi\Big)^{-2} = (1+O(X^{-1}))\prod_{\deg P\leq X}\Big(1-\frac{\chi(P)}{|P|^{\frac{1}{2}}}\Big)^2 \prod_{\frac{X}{2}<\deg P\leq X}\Big(1+\frac{\chi(P)^2}{2|P|}\Big)^2.$$

By writing $P_X^{**}\big(\frac{1}{2},\chi\big)$ as an Euler product, we see that

$$\begin{aligned}&\prod_{\deg P\leq X}\Big(1-\frac{\chi(P)}{|P|^{\frac{1}{2}}}\Big)^2 \prod_{\frac{X}{2}<\deg P\leq X}\Big(1+\frac{\chi(P)^2}{2|P|}\Big)^2\\ &= P_X^{**}\Big(\frac{1}{2},\chi\Big)\prod_{\frac{X}{2}<\deg P\leq X}\left(1+\frac{-\frac{2\chi(P)^3}{|P|^{\frac{3}{2}}}+\frac{5\chi(P)^4}{4|P|^2}-\frac{\chi(P)^5}{2|P|^{\frac{5}{2}}}+\frac{\chi(P)^6}{4|P|^6}}{1-\frac{2\chi(P)}{|P|^{\frac{1}{2}}}+\frac{2\chi(P)^2}{|P|}}\right)\\ &= P_X^{**}\Big(\frac{1}{2},\chi\Big)\prod_{\frac{X}{2}<\deg P\leq X}\Big(1+O\Big(|P|^{-\frac{3}{2}}\Big)\Big)\end{aligned}$$

$$= P_X^{**}\Big(\frac{1}{2}, \chi\Big) \exp\bigg(O\bigg(\sum_{\frac{X}{2} < \deg P \leq X} |P|^{-\frac{3}{2}} \bigg)\bigg)$$
$$= \Big(1 + O\big(X^{-1} q^{-\frac{X}{4}}\big)\Big) P_X^{**}\Big(\frac{1}{2}, \chi\Big).$$

The result follows. The requirement that $X \geq 12$ is so that the factor $\Big(1 - \frac{2\chi(P)}{|P|^{\frac{1}{2}}} + \frac{2\chi(P)^2}{|P|}\Big)^{-1}$ in the second line is guaranteed to be non-zero. □

Lemma 6.2. *We define*

$$\widehat{P_x^{**}}\Big(\frac{1}{2}, \chi\Big) := \sum_{\substack{A \in \mathcal{S}_{\mathcal{M}}(X) \\ \deg A \leq \frac{1}{8} \log_q \deg R}} \frac{\beta(A)\chi(A)}{|A|^{\frac{1}{2}}}.$$

Then, as $X, \deg R \to \infty$ with $X \leq \log_q \log \deg R$,

$$P_X^{**}\Big(\frac{1}{2}, \chi\Big) = \widehat{P_x^{**}}\Big(\frac{1}{2}, \chi\Big) + O\Big((\deg R)^{-\frac{1}{33}}\Big).$$

Proof. We have, as $X, \deg R \to \infty$ with $X \leq \log_q \log \deg R$,

$$\sum_{\substack{A \in \mathcal{S}_{\mathcal{M}}(X) \\ \deg A > \frac{1}{8} \log_q \deg R}} \frac{\beta(A)\chi(A)}{|A|^{\frac{1}{2}}} \ll \frac{1}{(\deg R)^{\frac{1}{32}}} \sum_{A \in \mathcal{S}_{\mathcal{M}}(X)} \frac{|\beta(A)|}{|A|^{\frac{1}{4}}}$$
$$= (\deg R)^{-\frac{1}{32}} \prod_{\deg P \leq X} \Big(1 + 2|P|^{-\frac{1}{4}} + 2|P|^{-\frac{1}{2}}\Big)$$
$$= (\deg R)^{-\frac{1}{32}} \exp\bigg(O\bigg(\sum_{\deg P \leq X} |P|^{-\frac{1}{4}}\bigg)\bigg)$$
$$= (\deg R)^{-\frac{1}{32}} \exp\bigg(O\bigg(\frac{q^{\frac{7}{8}X}}{X}\bigg)\bigg)$$
$$\leq (\deg R)^{-\frac{1}{33}}.$$ □

We now prove several results that will be used to obtain the main asymptotic term in Theorem 1.8. The following two results are generally known in the field, but as far as the author is aware they are not given explicitly anywhere in the literature. As they are non-trivial, we provide them here for completeness.

Lemma 6.3. *Suppose $A_1, A_2, A_3, B_1, B_2, B_3 \in \mathcal{M}$ satisfy $A_1A_2A_3 = B_1B_2B_3$. Then, there are $G_1, G_2, G_3, V_{1,2}, V_{1,3}, V_{2,1}, V_{2,3}, V_{3,1}, V_{3,2} \in \mathcal{M}$, satisfying $(V_{i,j}, V_{k,l}) = 1$ when both $i \neq k$ and $j \neq l$ hold, such that*

$$\begin{aligned} A_1 &= G_1V_{1,2}V_{1,3} \qquad & B_1 &= G_1V_{2,1}V_{3,1} \\ A_2 &= G_2V_{2,1}V_{2,3} \qquad & B_2 &= G_2V_{1,2}V_{3,2} \\ A_3 &= G_3V_{3,1}V_{3,2} \qquad & B_3 &= G_3V_{1,3}V_{2,3}. \end{aligned}$$

Furthermore, this is a bijective correspondence. To clarify, G_i is the highest common divisor of A_i and B_i; and in $V_{i,j}$ the subscript i indicates that $V_{i,j}$ divides A_i and the subscript j indicates that $V_{i,j}$ divides B_j.

Proof. Let us write $A_i = G_iS_i$ and $B_i = G_iT_i$, where

$$\begin{aligned} G_i &= (A_i, B_i) \\ (S_i, T_i) &= 1. \end{aligned} \tag{6.2}$$

Since $A_1A_2A_3 = B_1B_2B_3$, we must have that

$$S_1S_2S_3 = T_1T_2T_3. \tag{6.3}$$

First we note that, due to (6.3) and the coprimality relations in (6.2), we have that $S_i \mid T_jT_k$ and $T_i \mid S_jS_k$ for i, j, k distinct.

Second, again due to (6.3) and (6.2), we must have that (S_1, S_2, S_3), $(T_1, T_2, T_3) = 1$.

Third, for $i \neq j$, we define $S_{i,j} := (S_i, S_j)$ and $T_{i,j} := (T_i, T_j)$. Again due to (6.3) and (6.2), we have $(S_{i,j})^2 \mid T_k$ and $(T_{i,j})^2 \mid S_k$ for i, j, k distinct. Furthermore, $(S_{i_1,j_1}, S_{i_2,j_2}) = 1$ and $(T_{i_1,j_1}, T_{i_2,j_2}) = 1$ for all $\{i_1, j_1\} \neq \{i_2, j_2\}$, and $(S_{i_1,j_1}, T_{i_2,j_2}) = 1$ for all i_1, j_1, i_2, j_2.

From these three points we can deduce that

$$\begin{aligned} S_1 &= S_{1,2}S_{1,3}(T_{2,3})^2S_1{}' \qquad & T_1 &= T_{1,2}T_{1,3}(S_{2,3})^2T_1{}' \\ S_2 &= S_{1,2}S_{2,3}(T_{1,3})^2S_2{}' \qquad & T_2 &= T_{1,2}T_{2,3}(S_{1,3})^2T_2{}' \\ S_3 &= S_{1,3}S_{2,3}(T_{1,2})^2S_3{}' \qquad & T_3 &= T_{1,3}T_{2,3}(S_{1,2})^2T_3{}' \end{aligned}$$

for some $S_i{}'$ and $T_i{}'$ satisfying $(S_i{}', T_i{}') = 1$ for all i and $(S_i{}', S_j{}'), (T_i{}', T_j{}') = 1$ for $i \neq j$. By (6.3) we have that $S_1{}'S_2{}'S_3{}' = T_1{}'T_2{}'T_3{}'$. From these points we can deduce that

$$\begin{aligned} S_1{}' &= U_{1,2}U_{1,3} \qquad & T_1{}' &= U_{2,1}U_{3,1} \\ S_2{}' &= U_{2,1}U_{2,3} \qquad & T_2{}' &= U_{1,2}U_{3,2} \\ S_3{}' &= U_{3,1}U_{3,2} \qquad & T_3{}' &= U_{1,3}U_{2,3} \end{aligned}$$

where the $U_{i,j}$ are pairwise coprime. Also, for i, j, k distinct, because $U_{i,j} \mid T_j$ and $(S_j, T_j) = 1$, we have that $(U_{i,j}, S_j) = 1$, and hence $(U_{i,j}, S_{j,k})$, $(U_{i,j}, S_{j,i}) = 1$. Similarly, for i, j, k distinct, we have $(U_{i,j}, T_{i,k}), (U_{i,j}, T_{i,j}) = 1$.

So, by defining

$$\begin{aligned} V_{1,2} &= S_{1,3}T_{2,3}U_{1,2} & V_{2,1} &= S_{2,3}T_{1,3}U_{2,1} & V_{3,1} &= S_{2,3}T_{1,2}U_{3,1} \\ V_{1,3} &= S_{1,2}T_{2,3}U_{1,3} & V_{2,3} &= S_{1,2}T_{1,3}U_{2,3} & V_{3,2} &= S_{1,3}T_{1,2}U_{3,2} \end{aligned}$$

we complete the proof for the existence claim.

Uniqueness follows from the following observation: If we have G_i and $V_{i,j}$ satisfying the conditions in the Lemma, then we can deduce

$$G_i = (A_i, B_i) \quad \text{for all } i, \text{ and}$$

$$V_{i,j} = \left(V_{i,j}V_{k,j}, \frac{V_{i,j}V_{k,j}V_{j,i}V_{k,i}}{V_{k,i}V_{k,j}} \right) = \left(\widehat{B}_j, \frac{\widehat{B}_i \widehat{B}_j}{\widehat{A}_k} \right) \quad \text{for } i,j,k \text{ distinct},$$

where we define $\widehat{B}_i, \widehat{A}_i$ by $B_i = G_i \widehat{B}_i = (A_i, B_i)\widehat{B}_i$ and $A_i = G_i \widehat{A}_i = (A_i, B_i)\widehat{A}_i$ for all i. Since the far right side of each line above is expressed entirely in terms of $A_1, A_2, A_3, B_1, B_2, B_3$, we must have uniqueness. □

Lemma 6.4. *Suppose* $V_{1,3}, V_{2,3}, V_{3,1}, V_{3,2} \in \mathcal{M}$*, and* $(V_{1,3}, V_{3,1}V_{3,2}) = 1$ *and* $(V_{2,3}, V_{3,1}V_{3,2}) = 1$*. Then,*

$$\Big\{ (V_{1,2}, V_{2,1}) \in \mathcal{M}^2 : (V_{1,2}, V_{2,3}V_{3,1}) = 1, (V_{2,1}, V_{1,3}V_{3,2}) = 1, (V_{1,2}, V_{2,1}) = 1 \Big\}$$
$$= \bigcup_{\substack{V \in \mathcal{M} \\ \big(V, (V_{1,3}V_{3,1}, V_{2,3}V_{3,2})\big) = 1}} \left\{ (V_{1,2}, V_{2,1}) \in \mathcal{M}^2 : \begin{array}{l} V_{1,2}V_{2,1} = V, (V_{1,2}, V_{2,3}V_{3,1}) = 1, \\ (V_{2,1}, V_{1,3}V_{3,2}) = 1, (V_{1,2}, V_{2,1}) = 1 \end{array} \right\},$$

and for each such V *we have*

$$\# \left\{ (V_{1,2}, V_{2,1}) \in \mathcal{M}^2 : \begin{array}{l} V_{1,2}V_{2,1} = V, (V_{1,2}, V_{2,3}V_{3,1}) = 1, \\ (V_{2,1}, V_{1,3}V_{3,2}) = 1, (V_{1,2}, V_{2,1}) = 1 \end{array} \right\} = 2^{\omega(V) - \omega\Big(\big(V, V_{1,3}V_{2,3}V_{3,1}V_{3,2} \big) \Big)}.$$

Proof. For the first claim we note that $(V_{1,2}, V_{2,3}V_{3,1}) = 1$ and $(V_{2,1}, V_{1,3}V_{3,2}) = 1$ imply that

$$\Big(V, (V_{1,3}, V_{2,3}) \cdot (V_{3,1}, V_{3,2}) \Big) = 1,$$

and, due to the given coprimality relations of $V_{1,3}$,$V_{2,3}$,$V_{3,1}$, and $V_{3,2}$ given in Lemma 6.3, we have

$$(V_{1,3}, V_{2,3}) \cdot (V_{3,1}, V_{3,2}) = (V_{1,3}V_{3,1}, V_{2,3}V_{3,2}).$$

The first claim follows.

We now look at the second claim. For $A, B \in \mathcal{M}$, we define A_B to be the maximal divisor of A that is coprime to B, and we define A^B by $A = A_B A^B$. We then have that

$$V = V_{V_{1,3}V_{2,3}V_{3,1}V_{3,2}} V^{V_{1,3}V_{2,3}V_{3,1}V_{3,2}} = V_{V_{1,3}V_{2,3}V_{3,1}V_{3,2}} V^{V_{1,3}} V^{V_{2,3}} V^{V_{3,1}} V^{V_{3,2}},$$

where the last equality follows from $\big(V, (V_{1,3}V_{3,1}, V_{2,3}V_{3,2})\big) = 1$ and the fact that $(V_{1,3}, V_{3,1}) = 1$ and $(V_{2,3}, V_{3,2}) = 1$. Now, $V = V_{1,2}V_{2,1}$ and by the coprimality relations we must have that $V^{V_{1,3}}V^{V_{3,2}} \mid V_{1,2}$ and $V^{V_{2,3}}V^{V_{3,1}} \mid V_{2,1}$. So, we see that

$$
\begin{aligned}
&\#\left\{(V_{1,2}, V_{2,1}) \in \mathcal{M}^2 : \begin{array}{l} V_{1,2}V_{2,1} = V, (V_{1,2}, V_{2,3}V_{3,1}) = 1, \\ (V_{2,1}, V_{1,3}V_{3,2}) = 1, (V_{1,2}, V_{2,1}) = 1 \end{array}\right\} \\
&= \#\left\{(V_{1,2}, V_{2,1}) \in \mathcal{M}^2 : \begin{array}{l} V_{1,2}V_{2,1} = V_{V_{1,3}V_{2,3}V_{3,1}V_{3,2}} V^{V_{1,3}}V^{V_{2,3}}V^{V_{3,1}}V^{V_{3,2}}, \\ V^{V_{1,3}}V^{V_{3,2}} | V_{1,2}\,,\ V^{V_{2,3}}V^{V_{3,1}} | V_{2,1}\,,\ (V_{1,2}, V_{2,1}) = 1 \end{array}\right\} \\
&= 2^{\omega\left(V_{V_{1,3}V_{2,3}V_{3,1}V_{3,2}}\right)} = 2^{\omega(V) - \omega\left(\left(V, V_{1,3}V_{2,3}V_{3,1}V_{3,2}\right)\right)}. \qquad \square
\end{aligned}
$$

Lemma 6.5. *Let* $R, M \in \mathcal{M}$ *with* $\deg M \leq \deg R$*,* k *be a non-negative integer, and* z *be an integer-valued function of* R *such that* $z \sim \deg R$ *as* $\deg R \to \infty$*. Then, as* $\deg R \to \infty$*, we have*

$$
\begin{aligned}
&\sum_{\substack{N \in \mathcal{M} \\ \deg N \leq z \\ (N,R)=1}} \frac{2^{\omega(N) - \omega\left((N,M)\right)}}{|N|} (z - \deg N)^k \\
&\qquad \frac{(1 - q^{-1})}{(k+2)(k+1)} \prod_{P \mid MR} \left(\frac{1 - |P|^{-1}}{1 + |P|^{-1}}\right) \prod_{\substack{P \mid M \\ P \nmid R}} \left(\frac{1}{1 - |P|^{-1}}\right) \\
&\qquad\qquad \times \Big(z^{k+2} + O_k\big(z^{k+1} \log \deg R\big)\Big)
\end{aligned}
$$

and

$$
\begin{aligned}
&\sum_{\substack{N \in \mathcal{M} \\ \deg N \leq z \\ (N,R)=1}} \frac{2^{\omega(N) - \omega\left((N,M)\right)}}{|N|} (\deg N)^k \\
&\qquad = \frac{(1 - q^{-1})}{(k+2)} \prod_{P \mid MR} \left(\frac{1 - |P|^{-1}}{1 + |P|^{-1}}\right) \prod_{\substack{P \mid M \\ P \nmid R}} \left(\frac{1}{1 - |P|^{-1}}\right) \\
&\qquad\qquad \times \Big(z^{k+2} + O_k\big(z^{k+1} \log \deg R\big)\Big).
\end{aligned}
$$

Proof. The second result follows easily from the first by using the binomial theorem. The first result is proved using the standard approach (see [7], for example). We take

$$F(s) = \sum_{\substack{N \in \mathcal{M} \\ (N,R)=1}} \frac{2^{\omega(N) - \omega\big((N,M)\big)}}{|N|^s}$$
$$= \frac{\zeta_{\mathcal{A}}(s)^2}{\zeta_{\mathcal{A}}(2s)} \prod_{P \mid MR} \Big(\frac{1 - |P|^{-s}}{1 + |P|^{-s}} \Big) \prod_{\substack{P \mid M \\ P \nmid R}} \Big(\frac{1}{1 - |P|^{-s}} \Big).$$

Then, for a positive c we have

$$\frac{1}{2\pi i} \int_{c - i\infty}^{c + i\infty} F(1+s) \frac{y^s}{s^{k+1}} \mathrm{d}s$$
$$= \frac{(\log q)^k}{k!} \sum_{\substack{N \in \mathcal{M} \\ \deg N \leq z \\ (N,R)=1}} \frac{2^{\omega(N) - \omega\big((N,M)\big)}}{|N|} \big(z - \deg N \big)^k,$$

where we have used Perron's formula and the summation representation of $F(1+s)$. This gives the left-side of the first result in the lemma. The right side is obtained by using the product representation of $F(1+s)$ and shifting the line of integration to $\mathrm{Re}(s) = \frac{1}{4}$. □

Lemma 6.6. *Suppose ν is a multiplicative function on $\mathcal{A}$ and that there exists a non-negative integer r such that $\nu(P^k) = O(k^r)$ for all primes P (the implied constant is independent of P). Furthermore, suppose there is an $\eta > 0$ such that $\nu(A) \ll_{\eta} |A|^{\eta}$ as $\deg A \to \infty$.*

Let $R \in \mathcal{M}$ be a variable, $a, b > 0$ be constants, and $X = X(R), y = y(R)$ be non-negative, increasing, integer-valued functions such that $X \leq a \log_q \log \deg R$ and $y \geq b \log_q \deg R$ for large enough $\deg R$.

Let c and ϵ be such that $c > \epsilon > \max\big\{0, 1 - \frac{1}{a}\big\}$ and $c > \eta$, and let $\theta > 0$ be small. Finally, let $S \in \mathcal{M}$; S may depend on R. We then have that

$$\sum_{\substack{A \in \mathcal{S}_{\mathcal{M}}(X) \\ \deg A \leq y \\ (A,S)=1}} \frac{\nu(A)}{|A|^c}$$
$$= \prod_{\substack{\deg P \leq X \\ (P,S)=1}} \Bigg(1 + \frac{\nu(P)}{|P|^c} + \frac{\nu(P^2)}{|P|^{2c}} + \dots \Bigg) + O_{q,a,b,c,r,\epsilon,\theta}\Big((\deg R)^{-b(c-\epsilon)(1-\theta)} \Big)$$

as $\deg R \to \infty$.

Proof. The proof is similar to Lemma 6.5. We begin with

$$F(s) := \sum_{\substack{A \in \mathcal{S}_{\mathcal{M}}(X) \\ (A,S)=1}} \frac{\nu(A)}{|A|^{s+c}} = \prod_{\substack{\deg P \leq X \\ (P,S)=1}} \left(1 + \frac{\nu(P)}{|P|^{s+c}} + \frac{\nu(P^2)}{|P|^{2(s+c)}} + \dots\right).$$

By letting $d \geq 2$ and using Perron's formula on

$$\frac{1}{2\pi i} \int_{d-i\infty}^{d+i\infty} \sum_{\substack{A \in \mathcal{S}_{\mathcal{M}}(X) \\ (A,S)=1}} \frac{\nu(A)}{|A|^{s+c}} \frac{q^{(y+\frac{1}{2})s}}{s} \mathrm{d}s,$$

we get the left side of the lemma. Using the Euler product representation of $F(s)$ and a contour shift to $\mathrm{Re}(s) = -c + \epsilon$ gives the right side. The main term comes from the singularity at 0. □

We now prove a result that is required to bound the lower order terms in the proof of Theorem 1.8, but first we require three results from [2]. (See Theorem 6.1, Lemma 7.7, and Lemma 7.8 in [2])

Theorem 6.7. *Suppose α, β are fixed and satisfy $0 < \alpha < \frac{1}{2}$ and $0 < \beta < \frac{1}{2}$. Let $X \in \mathcal{M}$ and y be a positive integer satisfying $\beta \deg X < y \leq \deg X$. Also, let $A \in \mathcal{A}$ and $G \in \mathcal{M}$ satisfy $(A, G) = 1$ and $\deg G < (1-\alpha)y$. Then, we have that*

$$\sum_{\substack{N \in \mathcal{M} \\ \deg(N-X)<y \\ N \equiv A (\mathrm{mod}\, G)}} d(N) \ll_{\alpha,\beta} \frac{q^y \deg X}{\phi(G)}.$$

Lemma 6.8. *Let $F, K \in \mathcal{M}$, $x \geq 0$, and $a \in \mathbb{F}_q^*$. Suppose also that $\frac{1}{2}x < \deg KF \leq \frac{3}{4}x$. Then,*

$$\sum_{\substack{N \in \mathcal{M} \\ \deg N = x - \deg KF \\ (N,F)=1}} d(N) d(KF + aN) \ll q^x x^2 \frac{1}{|KF|} \sum_{\substack{H | K \\ \deg H \leq \frac{x - \deg KF}{2}}} \frac{d(H)}{|H|}.$$

Lemma 6.9. *Let $F \in \mathcal{M}$, $K \in \mathcal{A} \backslash \{0\}$, and $x \geq 0$ satisfy $\deg KF < x$. Then,*

$$\sum_{\substack{N \in \mathcal{M} \\ \deg N = x \\ (N,F)=1}} d(N) d(KF + N) \ll q^x x^2 \sum_{\substack{H | K \\ \deg H \leq \frac{x}{2}}} \frac{d(H)}{|H|}.$$

Lemma 6.10. *Let $F \in \mathcal{M}$, $A_3, B_3 \in \mathcal{S}_{\mathcal{M}}(X)$ with $(A_3 B_3, F) = 1$, and z_1, z_2 be non-negative integers. Also, we define*

$$\widehat{\deg}(A) := \begin{cases} 1 & \text{if } \deg A = 0 \\ \deg A & \text{if } \deg A \geq 1. \end{cases}$$

Then, for all $\epsilon > 0$ we have the following:

$$\sum_{\substack{A_1,A_2,B_1,B_2 \in \mathcal{M} \\ \deg A_1 B_1 = z_1 \\ \deg A_2 B_2 = z_2 \\ (A_1 A_2 B_1 B_2, F)=1 \\ A_1 A_2 A_3 \equiv B_1 B_2 B_3 (\operatorname{mod} F) \\ A_1 A_2 A_3 \neq B_1 B_2 B_3}} 1 \ll_\epsilon \left(q^{z_1} q^{z_2} \right)^{1+\epsilon} |A_3 B_3| \frac{\widehat{\deg}(A_3 B_3)}{|F|}$$

if $z_1 + z_2 + \deg A_3 B_3 \leq \frac{19}{10} \deg F$*; and*

$$\sum_{\substack{A_1,A_2,B_1,B_2 \in \mathcal{M} \\ \deg A_1 B_1 = z_1 \\ \deg A_2 B_2 = z_2 \\ (A_1 A_2 B_1 B_2, F)=1 \\ A_1 A_2 A_3 \equiv B_1 B_2 B_3 (\operatorname{mod} F) \\ A_1 A_2 A_3 \neq B_1 B_2 B_3}} 1 \ll q^{z_1+z_2} |A_3 B_3| (z_1 + z_2 + \deg A_3 B_3)^3 \frac{1}{\phi(F)}$$

if $z_1 + z_2 + \deg A_3 B_3 > \frac{19}{10} \deg F$.

Proof. We can split the sum into the cases $\deg A_1 A_2 A_3 > \deg B_1 B_2 B_3$, $\deg A_1 A_2 A_3 < \deg B_1 B_2 B_3$, and $\deg A_1 A_2 A_3 = \deg B_1 B_2 B_3$ with $A_1 A_2 A_3 \neq B_1 B_2 B_3$.

When $\deg A_1 A_2 A_3 > \deg B_1 B_2 B_3$, we have that $A_1 A_2 A_3 = KF + B_1 B_2 B_3$ where $K \in \mathcal{M}$ and $\deg KF > \deg B_1 B_2 B_3$. Furthermore,

$$\begin{aligned} 2 \deg KF = 2 \deg A_1 A_2 A_3 &> \deg A_1 A_2 A_3 + \deg B_1 B_2 B_3 \\ &= \deg A_1 B_1 + \deg A_2 B_2 + \deg A_3 B_3 = z_1 + z_2 + \deg A_3 B_3, \end{aligned}$$

from which we deduce that

$$a_0 := \frac{z_1 + z_2 + \deg A_3 B_3}{2} < \deg KF \leq z_1 + z_2 + \deg A_3 =: a_1.$$

Also,

$$\deg KF + \deg B_1 B_2 = \deg A_1 A_2 A_3 + \deg B_1 B_2 = z_1 + z_2 + \deg A_3,$$

from which we deduce that

$$\deg B_1 B_2 = z_1 + z_2 + \deg A_3 - \deg KF.$$

Similarly, if $\deg A_1 A_2 A_3 < \deg B_1 B_2 B_3$, we can show that

$$b_0 := \frac{z_1 + z_2 + \deg A_3 B_3}{2} < \deg KF \leq z_1 + z_2 + \deg B_3 =: b_1$$

and

$$\deg A_1 A_2 = z_1 + z_2 + \deg B_3 - \deg KF.$$

When $\deg A_1A_2A_3 = \deg B_1B_2B_3$, we must have that

$$\deg A_1A_2 = \frac{z_1+z_2+\deg B_3-\deg A_3}{2},$$
$$\deg B_1B_2 = \frac{z_1+z_2+\deg A_3-\deg B_3}{2}.$$

Also, we can write $A_1A_2A_3 = KF + B_1B_2B_3$, where $\deg KF < \deg B_1B_2B_3 = \frac{z_1+z_2+\deg A_3B_3}{2}$ and $K \neq 0$ need not be monic.

So, writing $N = B_1B_2$ when $\deg A_1A_2A_3 \geq \deg B_1B_2B_3$, and $N = A_1A_2$ when $\deg A_1A_2A_3 < \deg B_1B_2B_3$, we have that

$$(6.4)\quad \sum_{\substack{A_1,A_2,B_1,B_2\in\mathcal{M}\\ \deg A_1B_1=z_1\\ \deg A_2B_2=z_2\\ (A_1A_2B_1B_2,F)=1\\ A_1A_2A_3\equiv B_1B_2B_3 (\mathrm{mod}\, F)\\ A_1A_2A_3\neq B_1B_2B_3}} 1$$
$$\leq \sum_{\substack{K\in\mathcal{M}\\ a_0<\deg KF\leq a_1}} \sum_{\substack{N\in\mathcal{M}\\ \deg N=z_1+z_2+\deg A_3-\deg KF\\ (N,F)=1}} d(N)d\Big((KF+NB_3){A_3}^{-1}\Big)$$
$$+ \sum_{\substack{K\in\mathcal{M}\\ b_0<\deg KF\leq b_1}} \sum_{\substack{N\in\mathcal{M}\\ \deg N=z_1+z_2+\deg B_3-\deg KF\\ (N,F)=1}} d(N)d\Big((KF+NA_3){B_3}^{-1}\Big)$$
$$+ \sum_{\substack{K\in\mathcal{A}\backslash\{0\}\\ \deg KF<a_0}} \sum_{\substack{N\in\mathcal{M}\\ \deg N=\frac{z_1+z_2+\deg A_3-\deg B_3}{2}\\ (N,F)=1}} d(N)d\Big((KF+NB_3){A_3}^{-1}\Big).$$

We must remark that if $A_3 \mid (KF+NB_3)$ then we define $(KF+NB_3){A_3}^{-1}$ by $(KF+NB_3){A_3}^{-1}\cdot A_3 = (KF+NB_3)$. If $A_3 \nmid (KF+NB_3)$, then we ignore the term with $(KF+NB_3){A_3}^{-1}$ in the sum; that is, we take the definition $d\big((KF+NB_3){A_3}^{-1}\big) := 0$. We do the same for $(KF+NA_3){B_3}^{-1}$.

Step 1. Let us consider the case when $z_1 + z_2 + \deg A_3 B_3 \leq \frac{19}{10} \deg F$. By using well known bounds on the divisor function, we have that

$$\sum_{\substack{K \in \mathcal{M} \\ a_0 < \deg KF \leq a_1}} \sum_{\substack{N \in \mathcal{M} \\ \deg N = z_1 + z_2 + \deg A_3 - \deg KF \\ (N,F)=1}} d(N) d\Big((KF + NB_3) {A_3}^{-1} \Big)$$

$$\ll_{\epsilon} \Big(q^{z_1} q^{z_2} \Big)^{\frac{\epsilon}{2}} \sum_{\substack{K \in \mathcal{M} \\ a_0 < \deg KF \leq a_1}} \sum_{\substack{N \in \mathcal{M} \\ \deg N = z_1 + z_2 + \deg A_3 - \deg KF \\ (N,F)=1}} 1$$

$$\leq \Big(q^{z_1} q^{z_2} \Big)^{1+\frac{\epsilon}{2}} |A_3| \sum_{\substack{K \in \mathcal{M} \\ a_0 < \deg KF \leq a_1}} \frac{1}{|KF|}$$

$$\ll \Big(q^{z_1} q^{z_2} \Big)^{1+\frac{\epsilon}{2}} |A_3| \frac{z_1 + z_2 + \deg A_3}{|F|} \ll_{\epsilon} \Big(q^{z_1} q^{z_2} \Big)^{1+\epsilon} |A_3| \frac{\widehat{\deg} A_3}{|F|}.$$

Similarly,

$$\sum_{\substack{K \in \mathcal{M} \\ b_0 < \deg KF \leq b_1}} \sum_{\substack{N \in \mathcal{M} \\ \deg N = z_1 + z_2 + \deg B_3 - \deg KF \\ (N,F)=1}} d(N) d\Big((KF + NA_3) {B_3}^{-1} \Big)$$
$$\ll_{\epsilon} \Big(q^{z_1} q^{z_2} \Big)^{1+\epsilon} |B_3| \frac{\widehat{\deg} B_3}{|F|}.$$

As for the sum

$$\sum_{\substack{K \in \mathcal{A} \backslash \{0\} \\ \deg KF < a_0}} \sum_{\substack{N \in \mathcal{M} \\ \deg N = \frac{z_1 + z_2 + \deg A_3 - \deg B_3}{2} \\ (N,F)=1}} d(N) d\Big((KF + NB_3) {A_3}^{-1} \Big),$$

we note that it does not apply to this case where $z_1 + z_2 + \deg A_3 B_3 \leq \frac{19}{10} \deg F$ because this would imply $\deg KF \geq \deg F \geq \frac{20}{19} a_0$, which does not overlap with range $\deg KF < a_0$ in the sum. Hence,

$$\sum_{\substack{A_1, A_2, B_1, B_2 \in \mathcal{M} \\ \deg A_1 B_1 = z_1 \\ \deg A_2 B_2 = z_2 \\ (A_1 A_2 B_1 B_2, F) = 1 \\ A_1 A_2 A_3 \equiv B_1 B_2 B_3 (\operatorname{mod} F) \\ A_1 A_2 A_3 \neq B_1 B_2 B_3}} 1 \ll_{\epsilon} \Big(q^{z_1} q^{z_2} \Big)^{1+\epsilon} |A_3 B_3| \frac{\widehat{\deg} (A_3 B_3)}{|F|}$$

for $z_1 + z_2 + \deg A_3 B_3 \leq \frac{19}{10} \deg F$.

Step 2. We now consider the case when $z_1 + z_2 + \deg A_3B_3 > \frac{19}{10}\deg F$.

Step 2.1. We consider the subcase where $a_0 < \deg KF \leq \frac{3}{2}a_0$. This allows us to apply Lemma 6.8 for the second relation below.

$$\begin{aligned}
&\sum_{\substack{K\in\mathcal{M}\\ a_0<\deg KF\leq\frac{3}{2}a_0}} \sum_{\substack{N\in\mathcal{M}\\ \deg N=z_1+z_2+\deg A_3-\deg KF\\ (N,F)=1}} d(N)d\Big((KF+NB_3){A_3}^{-1}\Big)\\
&\leq \sum_{\substack{K\in\mathcal{M}\\ a_0<\deg KF\leq\frac{3}{2}a_0}} \sum_{\substack{N\in\mathcal{M}\\ \deg N=2a_0-\deg KF\\ (N,F)=1}} d(N)d\Big(KF+N\Big)\\
&\ll q^{z_1}q^{z_2}|A_3B_3|(z_1+z_2+\deg A_3B_3)^2\\
&\qquad\times\frac{1}{|F|}\sum_{\substack{K\in\mathcal{M}\\ a_0<\deg KF\leq\frac{3}{2}a_0}}\frac{1}{|K|}\sum_{\substack{H|K\\ \deg H\leq\frac{2a_0-\deg KF}{2}}}\frac{d(H)}{|H|}\\
&\leq q^{z_1}q^{z_2}|A_3B_3|(z_1+z_2+\deg A_3B_3)^2\frac{1}{|F|}\sum_{\substack{K\in\mathcal{M}\\ \deg KF\leq 2a_0}}\frac{1}{|K|}\sum_{H|K}\frac{d(H)}{|H|}\\
&\leq q^{z_1}q^{z_2}|A_3B_3|(z_1+z_2+\deg A_3B_3)^2\frac{1}{|F|}\sum_{\substack{H\in\mathcal{M}\\ \deg H\leq 2a_0}}\frac{d(H)}{|H|}\sum_{\substack{K\in\mathcal{M}\\ \deg K\leq 2a_0\\ H|K}}\frac{1}{|K|}\\
&\leq q^{z_1}q^{z_2}|A_3B_3|(z_1+z_2+\deg A_3B_3)^3\frac{1}{|F|}\sum_{\substack{H\in\mathcal{M}\\ \deg H\leq 2a_0}}\frac{d(H)}{|H|^2}\\
&\ll q^{z_1}q^{z_2}|A_3B_3|(z_1+z_2+\deg A_3B_3)^3\frac{1}{|F|}.
\end{aligned}$$

Similarly,

$$\sum_{\substack{K\in\mathcal{M}\\ b_0<\deg KF\leq\frac{3}{2}b_0}} \sum_{\substack{N\in\mathcal{M}\\ \deg N=z_1+z_2+\deg B_3-\deg KF\\ (N,F)=1}} d(N)d\Big((KF+NA_3){B_3}^{-1}\Big) \ll \frac{q^{z_1}q^{z_2}|A_3B_3|(z_1+z_2+\deg A_3B_3)^3}{|F|}.$$

Step 2.2. Now we consider the subcase where $\frac{3}{2}a_0 < \deg KF \leq a_1$. We have that

$$\begin{aligned}
&\sum_{\substack{K \in \mathcal{M} \\ \frac{3}{2}a_0 < \deg KF \leq a_1}} \sum_{\substack{N \in \mathcal{M} \\ \deg N = z_1 + z_2 + \deg A_3 - \deg KF \\ (N,F)=1}} d(N) d\Big((KF + NB_3){A_3}^{-1} \Big) \\
&\qquad \leq \sum_{\substack{K \in \mathcal{M} \\ \frac{3}{2}a_0 < \deg KF \leq a_1}} \sum_{\substack{N \in \mathcal{M} \\ \deg N = 2a_0 - \deg KF \\ (N,F)=1}} d(N) d(KF + N) \\
&\qquad \leq \sum_{\substack{N \in \mathcal{M} \\ \deg N < \frac{a_0}{2} \\ (N,F)=1}} \sum_{\substack{K \in \mathcal{M} \\ \deg KF = 2a_0 - \deg N}} d(N) d(KF + N) \\
&\qquad \leq \sum_{\substack{N \in \mathcal{M} \\ \deg N < \frac{a_0}{2} \\ (N,F)=1}} d(N) \sum_{\substack{M \in \mathcal{M} \\ \deg(M - X_{(N)}) < 2a_0 - \deg N \\ M \equiv N (\mathrm{mod}\, F)}} d(M)
\end{aligned}$$

where we define $X_{(N)} := T^{2a_0 - \deg N}$ (The monic polynomial of degree $2a_0 - \deg N$ with all non-leading coefficients equal to 0). We can now apply Theorem 6.7. One may wish to note that

$$y := 2a_0 - \deg N \geq \frac{3}{4}(z_1 + z_2 + \deg A_3 B_3) \geq \frac{3}{4}\frac{19}{10} \deg F$$

and so $\deg F \leq \frac{40}{57} y = (1 - \alpha) y$, where $0 < \alpha < \frac{1}{2}$, as required. Hence, we have that

$$\begin{aligned}
&\sum_{\substack{K \in \mathcal{M} \\ \frac{3}{2}a_0 < \deg KF \leq a_1}} \sum_{\substack{N \in \mathcal{M} \\ \deg N = z_1 + z_2 + \deg A_3 - \deg KF \\ (N,F)=1}} d(N) d\Big((KF + NB_3){A_3}^{-1} \Big) \\
&\qquad \leq q^{z_1} q^{z_2} |A_3 B_3| (z_1 + z_2 + \deg A_3 B_3) \frac{1}{\phi(F)} \sum_{\substack{N \in \mathcal{M} \\ \deg N < \frac{a_0}{2} \\ (N,F)=1}} \frac{d(N)}{|N|} \\
&\qquad \leqq^{z_1} q^{z_2} |A_3 B_3| (z_1 + z_2 + \deg A_3 B_3)^3 \frac{1}{\phi(F)}.
\end{aligned}$$

Similarly, if $\frac{3}{2}b_0 < \deg KF \leq b_1$ then

$$\sum_{\substack{K\in\mathcal{M} \\ \frac{3}{2}b_0<\deg KF\leq b_1}} \sum_{\substack{N\in\mathcal{M} \\ \deg N = z_1+z_2+\deg B_3-\deg KF \\ (N,F)=1}} d(N) d\Big((KF+NA_3){B_3}^{-1}\Big)$$
$$\leq q^{z_1} q^{z_2} |A_3B_3| (z_1+z_2+\deg A_3B_3)^3 \frac{1}{\phi(F)}.$$

Step 2.3. We now look at the sum

$$\sum_{\substack{K\in\mathcal{A}\backslash\{0\} \\ \deg KF<a_0}} \sum_{\substack{N\in\mathcal{M} \\ \deg N = \frac{z_1+z_2+\deg A_3-\deg B_3}{2} \\ (N,F)=1}} d(N) d\Big((KF+NB_3){A_3}^{-1}\Big).$$

By Lemma 6.9 we have that

$$\sum_{\substack{K\in\mathcal{A}\backslash\{0\} \\ \deg KF<a_0}} \sum_{\substack{N\in\mathcal{M} \\ \deg N = \frac{z_1+z_2+\deg A_3-\deg B_3}{2} \\ (N,F)=1}} d(N) d\Big((KF+NB_3){A_3}^{-1}\Big)$$
$$\leq \sum_{\substack{K\in\mathcal{A}\backslash\{0\} \\ \deg KF<a_0}} \sum_{\substack{N\in\mathcal{M} \\ \deg N = a_0 \\ (N,F)=1}} d(N) d(KF+N)$$
$$\ll q^{\frac{z_1+z_2}{2}} |A_3B_3|^{\frac{1}{2}} (z_1+z_2+\deg A_3B_3)^2 \sum_{\substack{K\in\mathcal{A}\backslash\{0\} \\ \deg KF<a_0}} \sum_{H|K} \frac{d(H)}{|H|}$$
$$\leq q^{z_1+z_2-1} |A_3B_3| (z_1+z_2+\deg A_3B_3)^2 \frac{1}{|F|} \sum_{\substack{K\in\mathcal{A}\backslash\{0\} \\ \deg KF<a_0}} \frac{1}{|K|} \sum_{H|K} \frac{d(H)}{|H|}$$
$$\leq q^{z_1+z_2} |A_3B_3| (z_1+z_2+\deg A_3B_3)^3 \frac{1}{|F|},$$

where the second-to-last relation uses the fact that a_0 is an integer (since $\deg A_1A_2A_3 = \deg B_1B_2B_3$) and so $\deg KF < a_0$ implies $\deg KF \leq a_0 - 1$, and the last relation uses a similar calculation as that in Step 2.1.

Step 2.4. We apply Steps 2.1, 2.2, and 2.3 to (6.4) and we see that, for $z_1+z_2+\deg A_3B_3 > \frac{19}{10}\deg F$,

$$\sum_{\substack{A_1,A_2,B_1,B_2\in\mathcal{M} \\ \deg A_1B_1=z_1 \\ \deg A_2B_2=z_2 \\ (A_1A_2B_1B_2,F)=1 \\ A_1A_2A_3\equiv B_1B_2B_3 (\mathrm{mod}\, F) \\ A_1A_2A_3\neq B_1B_2B_3}} 1 \ll q^{z_1+z_2} |A_3B_3| (z_1+z_2+\deg A_3B_3)^3 \frac{1}{\phi(F)}. \quad \square$$

7. The Fourth Hadamard Moment

We can now prove Theorem 1.8.

Proof of Theorem 1.8. In this proof, we assume all asymptotic relations are as $X, \deg R \to \infty$ with $X \leq \log_q \log \deg R$. Using Lemmas 6.1 and 6.2, we have

$$\begin{aligned}
&\frac{1}{\phi^*(R)} \sideset{}{^*}\sum_{\chi \bmod R} \left| L\Big(\frac{1}{2}, \chi\Big) P_X\Big(\frac{1}{2}, \chi\Big)^{-1} \right|^4 \\
&\qquad \sim \frac{1}{\phi^*(R)} \sideset{}{^*}\sum_{\chi \bmod R} \left| L\Big(\frac{1}{2}, \chi\Big) \right|^4 \left| P_X^{**}\Big(\frac{1}{2}, \chi\Big) \right|^2 \\
&\qquad = \frac{1}{\phi^*(R)} \sideset{}{^*}\sum_{\chi \bmod R} \left| L\Big(\frac{1}{2}, \chi\Big) \right|^4 \left| \widehat{P_X^{**}}\Big(\frac{1}{2}, \chi\Big) + O\Big((\deg R)^{-\frac{1}{33}}\Big) \right|^2 .
\end{aligned}$$

By the Cauchy–Schwarz inequality, (1.3), and (3.6), it suffices to prove

$$\begin{aligned}
&\frac{1}{\phi^*(R)} \sideset{}{^*}\sum_{\chi \bmod R} \left| L\Big(\frac{1}{2}, \chi\Big) \right|^4 \left| \widehat{P_X^{**}}\Big(\frac{1}{2}, \chi\Big) \right|^2 \\
&\qquad \sim \frac{1}{12} (\deg R)^4 \prod_{\substack{\deg P > X \\ P \mid R}} \left(\frac{(1-|P|^{-1})^3}{1+|P|^{-1}} \right) \prod_{\deg P \leq X} (1-|P|^{-1})^4 .
\end{aligned}$$

By Lemma 5.3, we have

$$\begin{aligned}
&\frac{1}{\phi^*(R)} \sideset{}{^*}\sum_{\chi \bmod R} \left| L\Big(\frac{1}{2}, \chi\Big) \right|^4 \left| \widehat{P_X^{**}}\Big(\frac{1}{2}, \chi\Big) \right|^2 \\
&\qquad = \frac{1}{\phi^*(R)} \sideset{}{^*}\sum_{\chi \bmod R} \Big(2a(\chi) + 2b(\chi) + c(\chi) \Big)^2 \left| \widehat{P_X^{**}}\Big(\frac{1}{2}, \chi\Big) \right|^2 ,
\end{aligned}$$

where $c(\chi)$ is as in Lemma 5.3 and

$$\begin{aligned}
z_R &:= \deg R - \log_q 2^{\omega(R)} ; \\
a(\chi) &:= \sum_{\substack{A,B \in \mathcal{M} \\ \deg AB \leq z_R}} \frac{\chi(A)\overline{\chi}(B)}{|AB|^{\frac{1}{2}}} ; \\
b(\chi) &:= \sum_{\substack{A,B \in \mathcal{M} \\ z_R < \deg AB < \deg R}} \frac{\chi(A)\overline{\chi}(B)}{|AB|^{\frac{1}{2}}} .
\end{aligned}$$

By symmetry in A, B, the terms $a(\chi)$, $b(\chi)$, and $c(\chi)$ are equal to their conjugates and so they are real. Hence, by the Cauchy–Schwarz inequality, it suffices to obtain the asymptotic main term of

$$\frac{4}{\phi^*(R)} \sideset{}{^*}\sum_{\chi \bmod R} a(\chi)^2 \left| \widehat{P_X^{**}}\left(\frac{1}{2}, \chi\right) \right|^2 \tag{7.1}$$

and show that

$$\frac{1}{\phi^*(R)} \sideset{}{^*}\sum_{\chi \bmod R} b(\chi)^2 \left| \widehat{P_X^{**}}\left(\frac{1}{2}, \chi\right) \right|^2 \quad \text{and} \quad \frac{1}{\phi^*(R)} \sideset{}{^*}\sum_{\chi \bmod R} c(\chi)^2 \left| \widehat{P_X^{**}}\left(\frac{1}{2}, \chi\right) \right|^2$$

are of lower order. The reason we express the sum in terms of $a(\chi)$ and $b(\chi)$ is because the fact that $a(\chi)$ is truncated allows us to bound the lower order terms that it contributes. We cannot do this with $b(\chi)$ but, because $b(\chi)$ is a relatively short sum, we can apply others methods to bound it.

Step 1: the asymptotic main term of $\frac{4}{\phi^*(R)} \sum^*_{\chi \bmod R} a(\chi)^2 \left| \widehat{P_X^{**}}\left(\frac{1}{2}, \chi\right) \right|^2$. Recall the following two orthogonality relations: Let $R \in \mathcal{M}$ and $A, B \in \mathcal{A}$. Then,

$$\sideset{}{^*}\sum_{\chi \bmod R} \chi(A)\widetilde{\chi}(B) = \begin{cases} \sum_{\substack{EF=R \\ F \mid (A-B)}} \mu(E)\phi(F) & \text{if } (AB, R) = 1, \\ 0 & \text{otherwise;} \end{cases}$$

$$\sideset{}{^*}\sum_{\substack{\chi \bmod R \\ \chi \text{ even}}} \chi(A)\overline{\chi}(B) = \begin{cases} \frac{1}{q-1} \sum_{a \in \mathbb{F}_q{}^*} \sum_{\substack{EF=R \\ F \mid (A-aB)}} \mu(E)\phi(F) & \text{if } (AB, R) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

By taking the trivial character, the first orthogonality relation gives $\phi^*(R) = \sum_{EF=R} \mu(E)\phi(F)$. Using these, we have

$$\begin{aligned} &\frac{1}{\phi^*(R)} \sideset{}{^*}\sum_{\chi \bmod R} a(\chi)^2 \left| \widehat{P_X^{**}}\left(\frac{1}{2}, \chi\right) \right|^2 \\ &= \frac{1}{\phi^*(R)} \sideset{}{^*}\sum_{\chi \bmod R} \sum_{\substack{A_1, A_2, B_1, B_2 \in \mathcal{M} \\ A_3, B_3 \in \mathcal{S}_{\mathcal{M}}(X) \\ \deg A_1B_1, \deg A_2B_2 \leq z_R \\ \deg A_3, \deg B_3 \leq \frac{1}{8} \log_q \deg R}} \frac{\beta(A_3)\beta(B_3)\chi(A_1A_2A_3)\overline{\chi}(B_1B_2B_3)}{|A_1A_2A_3B_1B_2B_3|^{\frac{1}{2}}} \end{aligned} \tag{7.2}$$

$$= \sum_{\substack{A_1,A_2,B_1,B_2 \in \mathcal{M} \\ A_3,B_3 \in \mathcal{S}_{\mathcal{M}}(X) \\ \deg A_1 B_1, \deg A_2 B_2 \leq z_R \\ \deg A_3, \deg B_3 \leq \frac{1}{8} \log_q \deg R \\ (A_1 A_2 A_3 B_1 B_2 B_3, R)=1 \\ A_1 A_2 A_3 = B_1 B_2 B_3}} \frac{\beta(A_3)\beta(B_3)}{|A_1 A_2 A_3 B_1 B_2 B_3|^{\frac{1}{2}}}$$

$$+ \frac{1}{\phi^*(R)} \sum_{EF=R} \mu(E)\phi(F) \sum_{\substack{A_1,A_2,B_1,B_2 \in \mathcal{M} \\ A_3,B_3 \in \mathcal{S}_{\mathcal{M}}(X) \\ \deg A_1 B_1, \deg A_2 B_2 \leq z_R \\ \deg A_3, \deg B_3 \leq \frac{1}{8} \log_q \deg R \\ (A_1 A_2 A_3 B_1 B_2 B_3, R)=1 \\ A_1 A_2 A_3 \equiv B_1 B_2 B_3 (\operatorname{mod} F) \\ A_1 A_2 A_3 \neq B_1 B_2 B_3}} \frac{\beta(A_3)\beta(B_3)}{|A_1 A_2 A_3 B_1 B_2 B_3|^{\frac{1}{2}}}.$$

Step 1.1. Consider the first term on the far right side of (7.2): the diagonal terms. Lemma 6.3 gives

$$\sum_{\substack{A_1,A_2,B_1,B_2 \in \mathcal{M} \\ A_3,B_3 \in \mathcal{S}_{\mathcal{M}}(X) \\ \deg A_1 B_1, \deg A_2 B_2 \leq z_R \\ \deg A_3, \deg B_3 \leq \frac{1}{8} \log_q \deg R \\ (A_1 A_2 A_3 B_1 B_2 B_3, R)=1 \\ A_1 A_2 A_3 = B_1 B_2 B_3}} \frac{\beta(A_3)\beta(B_3)}{|A_1 A_2 A_3 B_1 B_2 B_3|^{\frac{1}{2}}}$$

$$= \sum_{\substack{G_1,G_2,V_{1,2},V_{2,1} \in \mathcal{M} \\ G_3,V_{1,3},V_{2,3},V_{3,1},V_{3,2} \in \mathcal{S}_{\mathcal{M}}(X) \\ \deg (G_1)^2 V_{1,2} V_{1,3} V_{2,1} V_{3,1} \leq z_R \\ \deg (G_2)^2 V_{2,1} V_{2,3} V_{1,2} V_{3,2} \leq z_R \\ \deg G_3 V_{3,1} V_{3,2} \leq \frac{1}{8} \log_q \deg R \\ \deg G_3 V_{1,3} V_{2,3} \leq \frac{1}{8} \log_q \deg R \\ (G_i, R), (V_{j,k}, R)=1 \ \forall i,j,k \\ (V_{i,j}, V_{k,l})=1 \text{ for } (i \neq k \wedge j \neq l)}} \frac{\beta(G_3 V_{3,1} V_{3,2})\beta(G_3 V_{1,3} V_{2,3})}{|G_1 G_2 G_3 V_{1,2} V_{1,3} V_{2,1} V_{2,3} V_{3,1} V_{3,2}|}$$

$$= \sum_{\substack{G_3,V_{1,3},V_{2,3},V_{3,1},V_{3,2} \in \mathcal{S}_{\mathcal{M}}(X) \\ \deg G_3 V_{3,1} V_{3,2} \leq \frac{1}{8} \log_q \deg R \\ \deg G_3 V_{1,3} V_{2,3} \leq \frac{1}{8} \log_q \deg R \\ (G_3 V_{1,3} V_{2,3} V_{3,1} V_{3,2}, R)=1 \\ (V_{1,3} V_{2,3}, V_{3,1} V_{3,2})=1}} \frac{\beta(G_3 V_{3,1} V_{3,2})\beta(G_3 V_{1,3} V_{2,3})}{|G_3 V_{1,3} V_{2,3} V_{3,1} V_{3,2}|}$$

$$\times \sum_{\substack{V_{1,2},V_{2,1} \in \mathcal{M} \\ \deg V_{1,2} V_{2,1} \leq z_R - \deg V_{1,3} V_{3,1} \\ \deg V_{1,2} V_{2,1} \leq z_R - \deg V_{2,3} V_{3,2} \\ (V_{1,2} V_{2,1}, R)=1 \\ (V_{1,2}, V_{2,3} V_{3,1})=1 \\ (V_{2,1}, V_{3,2} V_{1,3})=1 \\ (V_{1,2}, V_{2,1})=1}} \frac{1}{|V_{1,2} V_{2,1}|} \sum_{\substack{G_1,G_2 \in \mathcal{M} \\ \deg G_1 \leq \frac{z_R - \deg V_{1,2} V_{2,1} V_{1,3} V_{3,1}}{2} \\ \deg G_2 \leq \frac{z_R - \deg V_{1,2} V_{2,1} V_{2,3} V_{3,2}}{2} \\ (G_1 G_2, R)=1}} \frac{1}{|G_1 G_2|}.$$

By Lemma 6.4 we have

$$\sum_{\substack{V_{1,2},V_{2,1}\in\mathcal{M}\\ \deg V_{1,2}V_{2,1}\le z_R-\deg V_{1,3}V_{3,1}\\ \deg V_{1,2}V_{2,1}\le z_R-\deg V_{2,3}V_{3,2}\\ (V_{1,2}V_{2,1},R)=1\\ (V_{1,2},V_{2,3}V_{3,1})=1\\ (V_{2,1},V_{3,2}V_{1,3})=1\\ (V_{1,2},V_{2,1})=1}} \frac{1}{|V_{1,2}V_{2,1}|} \sum_{\substack{G_1,G_2\in\mathcal{M}\\ \deg G_1\le\frac{z_R-\deg V_{1,2}V_{2,1}V_{1,3}V_{3,1}}{2}\\ \deg G_2\le\frac{z_R-\deg V_{1,2}V_{2,1}V_{2,3}V_{3,2}}{2}\\ (G_1G_2,R)=1}} \frac{1}{|G_1G_2|}$$

$$= \sum_{\substack{V\in\mathcal{M}\\ \deg V\le z_R-\deg V_{1,3}V_{3,1}\\ \deg V\le z_R-\deg V_{2,3}V_{3,2}\\ \left(V,R(V_{1,3}V_{3,1},V_{2,3}V_{3,2})\right)=1}} \frac{1}{|V|} \sum_{\substack{V_{1,2},V_{2,1}\in\mathcal{M}\\ V_{1,2}V_{2,1}=V\\ (V_{1,2},V_{2,1})=1\\ (V_{1,2},V_{2,3}V_{3,1})=1\\ (V_{2,1},V_{3,2}V_{1,3})=1}} \sum_{\substack{G_1,G_2\in\mathcal{M}\\ \deg G_1\le\frac{z_R-\deg VV_{1,3}V_{3,1}}{2}\\ \deg G_2\le\frac{z_R-\deg VV_{2,3}V_{3,2}}{2}\\ (G_1G_2,R)=1}} \frac{1}{|G_1G_2|}$$

$$= \sum_{\substack{V\in\mathcal{M}\\ \deg V\le z_R-\deg V_{1,3}V_{3,1}\\ \deg V\le z_R-\deg V_{2,3}V_{3,2}\\ \left(V,R(V_{1,3}V_{3,1},V_{2,3}V_{3,2})\right)=1}} \frac{2^{\omega(V)-\omega\left(\left(V,V_{1,3}V_{2,3}V_{3,1}V_{3,2}\right)\right)}}{|V|} \sum_{\substack{G_1,G_2\in\mathcal{M}\\ \deg G_1\le\frac{z_R-\deg VV_{1,3}V_{3,1}}{2}\\ \deg G_2\le\frac{z_R-\deg VV_{2,3}V_{3,2}}{2}\\ (G_1G_2,R)=1}} \frac{1}{|G_1G_2|}.$$

So, we have

$$(7.3)\qquad \sum_{\substack{A_1,A_2,B_1,B_2\in\mathcal{M}\\ A_3,B_3\in\mathcal{S}_{\mathcal{M}}(X)\\ \deg A_1B_1,\deg A_2B_2\le z_R\\ \deg A_3,\deg B_3\le\frac{1}{8}\log_q\deg R\\ (A_1A_2A_3B_1B_2B_3,R)=1\\ A_1A_2A_3=B_1B_2B_3}} \frac{\beta(A_3)\beta(B_3)}{|A_1A_2A_3B_1B_2B_3|^{\frac{1}{2}}}$$

$$= \sum_{\substack{G_3,V_{1,3},V_{2,3},V_{3,1},V_{3,2}\in\mathcal{S}_{\mathcal{M}}(X)\\ \deg G_3V_{3,1}V_{3,2}\le\frac{1}{8}\log_q\deg R\\ \deg G_3V_{1,3}V_{2,3}\le\frac{1}{8}\log_q\deg R\\ (G_3V_{1,3}V_{2,3}V_{3,1}V_{3,2},R)=1\\ (V_{1,3}V_{2,3},V_{3,1}V_{3,2})=1}} \frac{\beta(G_3V_{3,1}V_{3,2})\beta(G_3V_{1,3}V_{2,3})}{|G_3V_{1,3}V_{2,3}V_{3,1}V_{3,2}|}$$

$$\times \sum_{\substack{V\in\mathcal{M}\\ \deg V\le z_R-\deg V_{1,3}V_{3,1}\\ \deg V\le z_R-\deg V_{2,3}V_{3,2}\\ \left(V,R(V_{1,3}V_{3,1},V_{2,3}V_{3,2})\right)=1}} \frac{2^{\omega(V)-\omega\left(\left(V,V_{1,3}V_{2,3}V_{3,1}V_{3,2}\right)\right)}}{|V|} \sum_{\substack{G_1,G_2\in\mathcal{M}\\ \deg G_1\le\frac{z_R-\deg VV_{1,3}V_{3,1}}{2}\\ \deg G_2\le\frac{z_R-\deg VV_{2,3}V_{3,2}}{2}\\ (G_1G_2,R)=1}} \frac{1}{|G_1G_2|}.$$

Now, by Corollary 5.5, if

$$\frac{z_R - \deg VV_{1,3}V_{3,1}}{2} \geq \log_q 3^{\omega(R)}$$

that is,

$$\deg V \leq \deg R - \log_q 18^{\omega(R)} - \deg V_{1,3}V_{3,1}$$

then

$$\sum_{\substack{G_1 \in \mathcal{M} \\ \deg G_1 \leq \frac{z_R - \deg VV_{1,3}V_{3,1}}{2} \\ (G_1,R)=1}} \frac{1}{|G_1|} = \frac{\phi(R)}{2|R|}(z_R - \deg VV_{1,3}V_{3,1}) + O\left(\frac{\phi(R)}{|R|}\log \omega(R)\right) = \frac{\phi(R)}{2|R|}\Big(\deg R - \deg V + O\big(\log \deg R + \omega(R)\big)\Big). \tag{7.4}$$

If

$$\deg V > \deg R - \log_q 18^{\omega(R)} - \deg V_{1,3}V_{3,1},$$

then

$$\sum_{\substack{G_1 \in \mathcal{M} \\ \deg G_1 \leq \frac{z_R - \deg VV_{1,3}V_{3,1}}{2} \\ (G_1,R)=1}} \frac{1}{|G_1|} \leq \sum_{\substack{G_1 \in \mathcal{M} \\ \deg G_1 \leq \log_q 3^{\omega(R)} \\ (G_1,R)=1}} \frac{1}{|G_1|} \ll \frac{\phi(R)}{|R|}\omega(R). \tag{7.5}$$

Similar results hold for the sum over G_2. So, let us define

$$m_0 := \min\left\{\begin{matrix} \deg R - \log_q 18^{\omega(R)} - \deg V_{1,3}V_{3,1} \ , \\ \deg R - \log_q 18^{\omega(R)} - \deg V_{2,3}V_{3,2} \end{matrix}\right\},$$

$$m_1 := \max\left\{\begin{matrix} \deg R - \log_q 18^{\omega(R)} - \deg V_{1,3}V_{3,1} \ , \\ \deg R - \log_q 18^{\omega(R)} - \deg V_{2,3}V_{3,2} \end{matrix}\right\}.$$

Then, by (7.4) and (7.5), we have

$$\sum_{\substack{V \in \mathcal{M} \\ \deg V \leq z_R - \deg V_{1,3}V_{3,1} \\ \deg V \leq z_R - \deg V_{2,3}V_{3,2} \\ \left(V, R(V_{1,3}V_{3,1}, V_{2,3}V_{3,2})\right)=1}} \frac{2^{\omega(V) - \omega\left(\left(V, V_{1,3}V_{2,3}V_{3,1}V_{3,2}\right)\right)}}{|V|} \sum_{\substack{G_1, G_2 \in \mathcal{M} \\ \deg G_1 \leq \frac{z_R - \deg VV_{1,3}V_{3,1}}{2} \\ \deg G_2 \leq \frac{z_R - \deg VV_{2,3}V_{3,2}}{2} \\ (G_1G_2,R)=1}} \frac{1}{|G_1G_2|} \tag{7.6}$$

$$= \frac{\phi(R)^2}{4|R|^2} \sum_{\substack{V\in\mathcal{M}\\ \deg V\leq m_0\\ \left(V,R(V_{1,3}V_{3,1},V_{2,3}V_{3,2})\right)=1}} \frac{2^{\omega(V)-\omega\left(\left(V,V_{1,3}V_{2,3}V_{3,1}V_{3,2}\right)\right)}}{|V|} \times \Big(\deg R - \deg V + O\big(\log\deg R + \omega(R)\big)\Big)^2 + l_1(R,V_{1,3},V_{3,1},V_{2,3},V_{3,2}),$$

where

$$l_1(R,V_{1,3},V_{3,1},V_{2,3},V_{3,2}) \ll \frac{\phi(R)^2\omega(R)\deg R}{2|R|^2} \sum_{\substack{V\in\mathcal{M}\\ m_0<\deg V\leq m_1\\ \left(V,R(V_{1,3}V_{3,1},V_{2,3}V_{3,2})\right)=1}} \frac{2^{\omega(V)-\omega\left(\left(V,V_{1,3}V_{2,3}V_{3,1}V_{3,2}\right)\right)}}{|V|} + \frac{\phi(R)^2\omega(R)^2}{|R|^2} \sum_{\substack{V\in\mathcal{M}\\ m_1<\deg V\leq \deg R\\ \left(V,R(V_{1,3}V_{3,1},V_{2,3}V_{3,2})\right)=1}} \frac{2^{\omega(V)-\omega\left(\left(V,V_{1,3}V_{2,3}V_{3,1}V_{3,2}\right)\right)}}{|V|}. \tag{7.7}$$

We now apply Corollary 6.5 to both terms on the right side of (7.6). For the second term, which is (7.7), it is just two direct applications. For the first term, we must expand $\big(\deg R - \deg V + O(\log\deg R + \omega(R))\big)^2$ and use Corollary 6.5 on each of the resulting terms. We obtain

$$\sum_{\substack{V\in\mathcal{M}\\ \deg V\leq z_R-\deg V_{1,3}V_{3,1}\\ \deg V\leq z_R-\deg V_{2,3}V_{3,2}\\ \left(V,R(V_{1,3}V_{3,1},V_{2,3}V_{3,2})\right)=1}} \frac{2^{\omega(V)-\omega\left(\left(V,V_{1,3}V_{2,3}V_{3,1}V_{3,2}\right)\right)}}{|V|} \sum_{\substack{G_1,G_2\in\mathcal{M}\\ \deg G_1\leq\frac{z_R-\deg VV_{1,3}V_{3,1}}{2}\\ \deg G_2\leq\frac{z_R-\deg VV_{2,3}V_{3,2}}{2}\\ (G_1G_2,R)=1}} \frac{1}{|G_1G_2|} \tag{7.8}$$

$$= \frac{1-q^{-1}}{48}(\deg R)^4\bigg(1+O\Big(\frac{\omega(R)+\log\deg R}{\deg R}\Big)\bigg)\prod_{P|R}\bigg(\frac{(1-|P|^{-1})^3}{1+|P|^{-1}}\bigg) \times \prod_{P|V_{1,3}V_{2,3}V_{3,1}V_{3,2}}\bigg(\frac{1-|P|^{-1}}{1+|P|^{-1}}\bigg) \prod_{\substack{P|V_{1,3}V_{2,3}V_{3,1}V_{3,2}\\ P\nmid(V_{1,3},V_{2,3}),(V_{3,1},V_{3,2})}}\bigg(\frac{1}{1-|P|^{-1}}\bigg) =: l_2(R,V_{1,3},V_{2,3},V_{3,1},V_{3,2}).$$

Before proceeding let us make the following definitions: For $A\in\mathcal{A}\backslash\{0\}$ and $P\in\mathcal{P}$ we define $e_P(A)$ to be the largest non-negative integer such that

$P^{e_P(A)} \mid A$, and

$$\gamma(A) := \prod_{P \mid A} \left(1 + e_P(A) \frac{1 - |P|^{-1}}{1 + |P|^{-1}} \right). \tag{7.9}$$

Then, we can see that

$$\begin{aligned}
&\sum_{\substack{V_{1,3}, V_{2,3} \in \mathcal{S}_{\mathcal{M}}(X) \\ V_{1,3} V_{2,3} = B_3{}'}} \prod_{P \mid V_{1,3} V_{2,3}} \left(\frac{1 - |P|^{-1}}{1 + |P|^{-1}} \right) \prod_{\substack{P \mid V_{1,3} V_{2,3} \\ P \nmid (V_{1,3}, V_{2,3})}} \left(\frac{1}{1 - |P|^{-1}} \right) \qquad (7.10)\\
&= \prod_{P \mid B_3{}'} \left(\frac{1 - |P|^{-1}}{1 + |P|^{-1}} \right) \sum_{\substack{W_1 W_2 = B_3{}' \\ (W_1, W_2) = 1}} \sum_{\substack{V_{1,3}, V_{2,3} \in \mathcal{S}_{\mathcal{M}}(X) \\ V_{1,3} V_{2,3} = B_3{}' \\ \mathrm{rad}(V_{1,3}, V_{2,3}) = \mathrm{rad}\, W_1}} \prod_{P \mid W_2} \left(\frac{1}{1 - |P|^{-1}} \right) \\
&= \prod_{P \mid B_3{}'} \left(\frac{1 - |P|^{-1}}{1 + |P|^{-1}} \right) \sum_{\substack{W_1 W_2 = B_3{}' \\ (W_1, W_2) = 1}} \prod_{P \mid W_2} \left(\frac{1}{1 - |P|^{-1}} \right) 2^{\omega(W_2)} \prod_{P \mid W_1} \Big(e_P(B_3{}') - 1 \Big) \\
&= \prod_{P \mid B_3{}'} \left(\frac{1 - |P|^{-1}}{1 + |P|^{-1}} \right) \prod_{P \mid B_3{}'} \left(\frac{2}{1 - |P|^{-1}} + \Big(e_P(B_3{}') - 1 \Big) \right) \\
&= \prod_{P \mid B_3{}'} \left(1 + e_P(B_3{}') \frac{1 - |P|^{-1}}{1 + |P|^{-1}} \right) \\
&= \gamma(B_3{}').
\end{aligned}$$

Similarly,

$$\sum_{\substack{V_{3,1}, V_{3,2} \in \mathcal{S}_{\mathcal{M}}(X) \\ V_{3,1} V_{3,2} = A_3{}'}} \prod_{P \mid V_{3,1} V_{3,2}} \left(\frac{1 - |P|^{-1}}{1 + |P|^{-1}} \right) \prod_{\substack{P \mid V_{3,1} V_{3,2} \\ P \nmid (V_{3,1}, V_{3,2})}} \left(\frac{1}{1 - |P|^{-1}} \right) = \gamma(A_3{}'). \tag{7.11}$$

We now substitute (7.8) to (7.3) and apply (7.10) and (7.11) to obtain

$$\sum_{\substack{A_1, A_2, B_1, B_2 \in \mathcal{M} \\ A_3, B_3 \in \mathcal{S}_{\mathcal{M}}(X) \\ \deg A_1 B_1, \deg A_2 B_2 \leq z_R \\ \deg A_3, \deg B_3 \leq \frac{1}{8} \log_q \deg R \\ (A_1 A_2 A_3 B_1 B_2 B_3, R) = 1 \\ A_1 A_2 A_3 = B_1 B_2 B_3}} \frac{\beta(A_3) \beta(B_3)}{|A_1 A_2 A_3 B_1 B_2 B_3|^{\frac{1}{2}}} \tag{7.12}$$

$$= \sum_{\substack{G_3,V_{1,3},V_{2,3},V_{3,1},V_{3,2}\in\mathcal{S}_{\mathcal{M}}(X)\\ \deg G_3V_{3,1}V_{3,2}\le\frac{1}{8}\log_q\deg R\\ \deg G_3V_{1,3}V_{2,3}\le\frac{1}{8}\log_q\deg R\\ (G_3V_{1,3}V_{2,3}V_{3,1}V_{3,2},R)=1\\ (V_{1,3}V_{2,3},V_{3,1}V_{3,2})=1}} \frac{\beta(G_3V_{3,1}V_{3,2})\beta(G_3V_{1,3}V_{2,3})}{|G_3V_{1,3}V_{2,3}V_{3,1}V_{3,2}|} \times l_2(R,V_{1,3},V_{2,3},V_{3,1},V_{3,2})$$

$$= \sum_{\substack{G_3,A_3',B_3'\in\mathcal{S}_{\mathcal{M}}(X)\\ \deg G_3A_3'\le\frac{1}{8}\log_q\deg R\\ \deg G_3B_3'\le\frac{1}{8}\log_q\deg R\\ (G_3A_3'B_3',R)=1\\ (A_3',B_3')=1}} \frac{\beta(G_3A_3')\beta(G_3B_3')}{|G_3A_3'B_3'|} \times \sum_{\substack{V_{3,1},V_{3,2}\in\mathcal{S}_{\mathcal{M}}(X)\\ V_{3,1}V_{3,2}=A_3'}} \sum_{\substack{V_{1,3},V_{2,3}\in\mathcal{S}_{\mathcal{M}}(X)\\ V_{1,3}V_{2,3}=B_3'}} l_2(R,V_{1,3},V_{2,3},V_{3,1},V_{3,2})$$

$$= \frac{1-q^{-1}}{48}\prod_{P|R}\left(\frac{\left(1-|P|^{-1}\right)^3}{1+|P|^{-1}}\right)(\deg R)^4 \times \sum_{\substack{G_3,A_3',B_3'\in\mathcal{S}_{\mathcal{M}}(X)\\ \deg G_3A_3'\le\frac{1}{8}\log_q\deg R\\ \deg G_3B_3'\le\frac{1}{8}\log_q\deg R\\ (G_3A_3'B_3',R)=1\\ (A_3',B_3')=1}} \frac{\beta(G_3A_3')\beta(G_3B_3')}{|G_3A_3'B_3'|}\gamma(A_3')\gamma(B_3') + l_3(R),$$

where

$$l_3(R) \ll \prod_{P|R}\left(\frac{\left(1-|P|^{-1}\right)^3}{1+|P|^{-1}}\right)(\deg R)^3\left(\omega(R)+\log\deg R\right) \times \sum_{\substack{G_3,A_3',B_3'\in\mathcal{S}_{\mathcal{M}}(X)\\ \deg G_3A_3'\le\frac{1}{8}\log_q\deg R\\ \deg G_3B_3'\le\frac{1}{8}\log_q\deg R\\ (G_3A_3'B_3',R)=1\\ (A_3',B_3')=1}} \frac{|\beta(G_3A_3')\beta(G_3B_3')|}{|G_3A_3'B_3'|}\gamma(A_3')\gamma(B_3'). \tag{7.13}$$

Consider the first term on the far right side of (7.12). We recall that $\beta(A)=0$ if A is divisible by P^3 for any prime P. Hence, defining $\Pi_{\mathcal{P},X} := \prod_{\deg P\le X} P$, we may assume that $G_3 = IJ^2$ where $I, J \mid \Pi_{\mathcal{P},X}$, $(IJ,R)=1$, and $(I,J)=1$. By similar reasoning, we may assume that $A_3' = KA_3''$ where $K \mid I$, $(A_3'', RIJ)=1$; and $B_3' = LB_3''$ where $L \mid I$, $(L,K)=1$ and

$(B_3{}'', RIJA_3{}'') = 1$. Then, by the multiplicativity of β and γ, we have

$$
\begin{aligned}
(7.14)\quad & \sum_{\substack{G_3, A_3{}', B_3{}' \in \mathcal{S}_{\mathcal{M}}(X) \\ \deg G_3 A_3{}' \leq \frac{1}{8} \log_q \deg R \\ \deg G_3 B_3{}' \leq \frac{1}{8} \log_q \deg R \\ (G_3 A_3{}' B_3{}', R)=1 \\ (A_3{}', B_3{}')=1}} \frac{\beta(G_3 A_3{}') \beta(G_3 B_3{}')}{|G_3 A_3{}' B_3{}'|} \gamma(A_3{}') \gamma(B_3{}') \\
&= \sum_{\substack{I | \Pi_{\mathcal{P},X} \\ \deg I \leq \frac{1}{8} \log_q \deg R \\ (I,R)=1}} \frac{\beta(I)^2}{|I|} \sum_{\substack{J | \Pi_{\mathcal{P},X} \\ \deg J \leq \frac{1}{16} \log_q \deg R - \frac{\deg I}{2} \\ (J,RI)=1}} \frac{\beta(J^2)^2}{|J|^2} \sum_{K|I} \frac{\beta(K^2)\gamma(K)}{\beta(K)|K|} \\
&\quad \times \sum_{\substack{L|I \\ (L,K)=1}} \frac{\beta(L^2)\gamma(L)}{\beta(L)|L|} \sum_{\substack{A_3{}'' | (\Pi_{\mathcal{P},X})^2 \\ \deg A_3{}'' \leq \frac{1}{8} \log_q \deg R - \deg IJ^2 K \\ (A_3{}'', RIJ)=1}} \frac{\beta(A_3{}'')\gamma(A_3{}'')}{|A_3{}''|} \\
&\quad \times \sum_{\substack{B_3{}'' | (\Pi_{\mathcal{P},X})^2 \\ \deg B_3{}'' \leq \frac{1}{8} \log_q \deg R - \deg IJ^2 L \\ (B_3{}'', RIJA_3{}'')=1}} \frac{\beta(B_3{}'')\gamma(B_3{}'')}{|B_3{}''|}.
\end{aligned}
$$

Consider the case where $\deg I > \frac{1}{64} \log_q \deg R$ or $\deg J > \frac{1}{64} \log_q \deg R$. Without loss of generality, suppose the former. Then, all the sums above, except that over I, can be bounded by $O\big((\log_q \log \deg R)^c\big)$ for some constant $c > 0$, while the sum over I can be bounded by $O\big((\deg R)^{-\frac{1}{66}}\big)$ (this is obtained in the same way we have done several times before, such as in (3.8)). So, with these restrictions, we have that the above is $O\big((\deg R)^{-\frac{1}{67}}\big)$.

Now consider the case where $\deg I \leq \frac{1}{64} \log_q \deg R$ and $\deg J \leq \frac{1}{64} \log_q \deg R$. Then,

$$\frac{1}{8} \log_q \deg R - \deg IJ^2 K \geq \frac{1}{16} \log_q \deg R$$

and

$$\frac{1}{8} \log_q \deg R - \deg IJ^2 L \geq \frac{1}{16} \log_q \deg R.$$

In particular, we can apply Lemma 6.6 to the last two summations of (7.14):

$$\sum_{\substack{A_3{}'' | (\Pi_{\mathcal{P},X})^2 \\ \deg A_3{}'' \leq \frac{1}{8} \log_q \deg R - \deg IJ^2 K \\ (A_3{}'', RIJ)=1}} \frac{\beta(A_3{}'')\gamma(A_3{}'')}{|A_3{}''|} \sum_{\substack{B_3{}'' | (\Pi_{\mathcal{P},X})^2 \\ \deg B_3{}'' \leq \frac{1}{8} \log_q \deg R - \deg IJ^2 L \\ (B_3{}'', RIJA_3{}'')=1}} \frac{\beta(B_3{}'')\gamma(B_3{}'')}{|B_3{}''|}$$

$$= \prod_{\substack{\deg P \leq X \\ (P,R)=1}} \left(1 + \frac{\beta(P)\gamma(P)}{|P|} + \frac{\beta(P^2)\gamma(P^2)}{|P^2|}\right)$$

$$\times \prod_{P|IJ} \left(1 + \frac{\beta(P)\gamma(P)}{|P|} + \frac{\beta(P^2)\gamma(P^2)}{|P^2|}\right)^{-1} \sum_{\substack{A_3'' | (\Pi_{\mathcal{P},X})^2 \\ \deg A_3'' \leq \frac{1}{8}\log_q \deg R - \deg IJ^2K \\ (A_3'', RIJ)=1}} \frac{\beta(A_3'')\gamma(A_3'')}{|A_3''|}$$

$$\times \prod_{P|A_3''} \left(1 + \frac{\beta(P)\gamma(P)}{|P|} + \frac{\beta(P^2)\gamma(P^2)}{|P^2|}\right)^{-1}$$

$$+ O\Big((\deg R)^{-\frac{1}{17}}\Big)$$

Hence,

$$\sum_{\substack{A_3'' | (\Pi_{\mathcal{P},X})^2 \\ \deg A_3'' \leq \frac{1}{8}\log_q \deg R - \deg IJ^2K \\ (A_3'', RIJ)=1}} \frac{\beta(A_3'')\gamma(A_3'')}{|A_3''|} \sum_{\substack{B_3'' | (\Pi_{\mathcal{P},X})^2 \\ \deg B_3'' \leq \frac{1}{8}\log_q \deg R - \deg IJ^2L \\ (B_3'', RIJA_3'')=1}} \frac{\beta(B_3'')\gamma(B_3'')}{|B_3''|}$$

$$= \prod_{\substack{\deg P \leq X \\ (P,R)=1}} \left(1 + \frac{\beta(P)\gamma(P)}{|P|} + \frac{\beta(P^2)\gamma(P^2)}{|P^2|}\right)$$

$$\times \prod_{P|IJ} \left(1 + \frac{\beta(P)\gamma(P)}{|P|} + \frac{\beta(P^2)\gamma(P^2)}{|P^2|}\right)^{-1}$$

$$\times \prod_{\substack{\deg P \leq X \\ (P,R)=1}} \left(1 + \left(\frac{\beta(P)\gamma(P)}{|P|} + \frac{\beta(P^2)\gamma(P^2)}{|P^2|}\right) \times \left(1 + \frac{\beta(P)\gamma(P)}{|P|} + \frac{\beta(P^2)\gamma(P^2)}{|P^2|}\right)^{-1}\right)$$

$$\times \prod_{P|IJ} \left(1 + \left(\frac{\beta(P)\gamma(P)}{|P|} + \frac{\beta(P^2)\gamma(P^2)}{|P^2|}\right) \times \left(1 + \frac{\beta(P)\gamma(P)}{|P|} + \frac{\beta(P^2)\gamma(P^2)}{|P^2|}\right)^{-1}\right)^{-1} + O\Big((\deg R)^{-\frac{1}{17}}\Big)$$

and so

$$
(7.15) \quad \sum_{\substack{A_3{}'' | (\Pi_{\mathcal{P},X})^2 \\ \deg A_3{}'' \leq \frac{1}{8} \log_q \deg R - \deg IJ^2K \\ (A_3{}'', RIJ)=1}} \frac{\beta(A_3{}'')\gamma(A_3{}'')}{|A_3{}''|} \sum_{\substack{B_3{}'' | (\Pi_{\mathcal{P},X})^2 \\ \deg B_3{}'' \leq \frac{1}{8} \log_q \deg R - \deg IJ^2L \\ (B_3{}'', RIJA_3{}'')=1}} \frac{\beta(B_3{}'')\gamma(B_3{}'')}{|B_3{}''|}
$$

$$
= \prod_{\substack{\deg P \leq X \\ (P,R)=1}} \left(1 + \frac{2\beta(P)\gamma(P)}{|P|} + \frac{2\beta(P^2)\gamma(P^2)}{|P^2|}\right)
$$

$$
\times \prod_{P|IJ} \left(1 + \frac{2\beta(P)\gamma(P)}{|P|} + \frac{2\beta(P^2)\gamma(P^2)}{|P^2|}\right)^{-1} + O\Big((\deg R)^{-\frac{1}{17}}\Big).
$$

Consider now the two middle summations on the right side of (7.14). We have

$$
(7.16) \quad \sum_{K|I} \frac{\beta(K^2)\gamma(K)}{\beta(K)|K|} \sum_{\substack{L|I \\ (L,K)=1}} \frac{\beta(L^2)\gamma(L)}{\beta(L)|L|}
$$

$$
= \prod_{P|I} \left(1 + \frac{\beta(P^2)\gamma(P)}{\beta(P)|P|}\right) \sum_{K|I} \frac{\beta(K^2)\gamma(K)}{\beta(K)|K|} \prod_{P|K} \left(1 + \frac{\beta(P^2)\gamma(P)}{\beta(P)|P|}\right)^{-1}
$$

$$
= \prod_{P|I} \left(1 + \frac{\beta(P^2)\gamma(P)}{\beta(P)|P|}\right) \prod_{P|I} \left(1 + \frac{\beta(P^2)\gamma(P)}{\beta(P)|P|} \left(1 + \frac{\beta(P^2)\gamma(P)}{\beta(P)|P|}\right)^{-1}\right)
$$

$$
= \prod_{P|I} \left(1 + \frac{2\beta(P^2)\gamma(P)}{\beta(P)|P|}\right).
$$

Applying (7.15) and (7.16) to (7.14), we obtain

$$
\sum_{\substack{G_3, A_3{}', B_3{}' \in \mathcal{S}_{\mathcal{M}}(X) \\ \deg G_3 A_3{}' \leq \frac{1}{8} \log_q \deg R \\ \deg G_3 B_3{}' \leq \frac{1}{8} \log_q \deg R \\ (G_3 A_3{}' B_3{}', R)=1 \\ (A_3{}', B_3{}')=1}} \frac{\beta(G_3 A_3{}')\beta(G_3 B_3{}')}{|G_3 A_3{}' B_3{}'|} \gamma(A_3{}')\gamma(B_3{}')
$$

$$
= \prod_{\substack{\deg P \leq X \\ (P,R)=1}} \left(1 + \frac{2\beta(P)\gamma(P)}{|P|} + \frac{2\beta(P^2)\gamma(P^2)}{|P^2|}\right) \sum_{\substack{I | \Pi_{\mathcal{P},X} \\ \deg I \leq \frac{1}{64} \log_q \deg R \\ (I,R)=1}} \frac{\beta(I)^2}{|I|}
$$

$$
\times \sum_{\substack{J|\Pi_{\mathcal{P},X} \\ \deg J \leq \frac{1}{64}\log_q \deg R \\ (J,RI)=1}} \frac{\beta(J^2)^2}{|J|^2} \prod_{P|I}\left(1+\frac{2\beta(P^2)\gamma(P)}{\beta(P)|P|}\right)
$$

$$
\times \prod_{P|IJ}\left(1+\frac{2\beta(P)\gamma(P)}{|P|}+\frac{2\beta(P^2)\gamma(P^2)}{|P^2|}\right)^{-1} + O\Big((\deg R)^{-\frac{1}{67}}\Big)
$$

$$
= \prod_{\substack{\deg P \leq X \\ (P,R)=1}}\left(1+\frac{2\beta(P)\gamma(P)}{|P|}+\frac{2\beta(P^2)\gamma(P^2)}{|P^2|}+\frac{\beta(P^2)^2}{|P|^2}\right)
$$

$$
\times \sum_{\substack{I|\Pi_{\mathcal{P},X} \\ \deg I \leq \frac{1}{64}\log_q \deg R \\ (I,R)=1}} \frac{\beta(I)^2}{|I|} \prod_{P|I}\Bigg(\left(1+\frac{2\beta(P^2)\gamma(P)}{\beta(P)|P|}\right)
$$

$$
\times \left(1+\frac{2\beta(P)\gamma(P)}{|P|}+\frac{2\beta(P^2)\gamma(P^2)}{|P^2|}+\frac{\beta(P^2)^2}{|P|^2}\right)^{-1}\Bigg)
$$

$$
+ O\Big((\deg R)^{-\frac{1}{67}}\Big)
$$

$$
= \prod_{\substack{\deg P \leq X \\ (P,R)=1}}\Bigg(1+\frac{2\beta(P)\gamma(P)}{|P|}+\frac{2\beta(P^2)\gamma(P^2)}{|P^2|}+\frac{\beta(P^2)^2}{|P|^2}
$$

$$
+ \frac{\beta(P)^2}{|P|}\left(1+\frac{2\beta(P^2)\gamma(P)}{\beta(P)|P|}\right)\Bigg)
$$

$$
+ O\Big((\deg R)^{-\frac{1}{67}}\Big).
$$

Now, recalling the definitions of β, γ (equations (6.1) and (7.9), respectively) we see that the product above is equal to

$$
\prod_{\substack{\deg P \leq X \\ P \nmid R}}\left(\frac{\left(1-|P|^{-1}\right)^3}{1+|P|^{-1}}\right) \prod_{\substack{\frac{X}{2} < \deg P \leq X \\ P \nmid R}}\Big(1+O\big(|P|^{-2}\big)\Big)
$$

$$
\sim \prod_{P|R}\left(\frac{\left(1-|P|^{-1}\right)^3}{1+|P|^{-1}}\right)^{-1} \prod_{\substack{\deg P > X \\ P|R}}\left(\frac{\left(1-|P|^{-1}\right)^3}{1+|P|^{-1}}\right) \prod_{\deg P \leq X}\left(\frac{\left(1-|P|^{-1}\right)^3}{1+|P|^{-1}}\right)
$$

$$
\begin{aligned}
&= \prod_{P \mid R} \left(\frac{(1-|P|^{-1})^3}{1+|P|^{-1}} \right)^{-1} \prod_{\substack{\deg P > X \\ P \mid R}} \left(\frac{(1-|P|^{-1})^3}{1+|P|^{-1}} \right) \prod_{\deg P \leq X} (1-|P|^{-1})^4 \\
&\qquad \times \prod_{\deg P \leq X} (1-|P|^{-2})^{-1} \\
&\sim (1-q^{-1})^{-1} \prod_{P \mid R} \left(\frac{(1-|P|^{-1})^3}{1+|P|^{-1}} \right)^{-1} \prod_{\substack{\deg P > X \\ P \mid R}} \left(\frac{(1-|P|^{-1})^3}{1+|P|^{-1}} \right) \left(\frac{1}{e^{\gamma} X} \right)^4 ,
\end{aligned}
$$

where we have used (3.6) for the last equality. Recall that the above is to be applied to the first term on the far right side of (7.12). We now consider $l_3(R)$: the second term on the far right side of (7.12). By means similar to those described in the paragraph after (7.14), we can show that there is some constant $c > 0$ such that

$$
\begin{aligned}
&\sum_{\substack{G_3, A_3', B_3' \in \mathcal{S}_{\mathcal{M}}(X) \\ \deg G_3 A_3' \leq \frac{1}{8} \log_q \deg R \\ \deg G_3 B_3' \leq \frac{1}{8} \log_q \deg R \\ (G_3 A_3' B_3', R) = 1 \\ (A_3', B_3') = 1}} \frac{|\beta(G_3 A_3') \beta(G_3 B_3')|}{|G_3 A_3' B_3'|} \gamma(A_3') \gamma(B_3') \ll X^c \\
&\qquad\qquad \ll (\log_q \log \deg R)^c .
\end{aligned}
$$

We apply this to (7.13) to obtain a bound for $l_3(R)$.

Hence, considering all of the above, (7.12) becomes

$$
\begin{aligned}
&\sum_{\substack{A_1, A_2, B_1, B_2 \in \mathcal{M} \\ A_3, B_3 \in \mathcal{S}_{\mathcal{M}}(X) \\ \deg A_1 B_1, \deg A_2 B_2 \leq z_R \\ \deg A_3, \deg B_3 \leq \frac{1}{8} \log_q \deg R \\ (A_1 A_2 A_3 B_1 B_2 B_3, R) = 1 \\ A_1 A_2 A_3 = B_1 B_2 B_3}} \frac{\beta(A_3) \beta(B_3)}{|A_1 A_2 A_3 B_1 B_2 B_3|^{\frac{1}{2}}} \\
&\qquad\qquad \sim \frac{1}{48} \left(\frac{\deg R}{e^{\gamma} X} \right)^4 \prod_{\substack{\deg P > X \\ P \mid R}} \left(\frac{(1-|P|^{-1})^3}{1+|P|^{-1}} \right)
\end{aligned}
\tag{7.17}
$$

Step 1.2. We consider the second term on the far right side of (7.2): the off-diagonal terms. We have

$$\begin{aligned}
&\sum_{EF=R}\mu(E)\phi(F)\sum_{\substack{A_1,A_2,B_1,B_2\in\mathcal{M}\\ A_3,B_3\in\mathcal{S}_{\mathcal{M}}(X)\\ \deg A_1B_1,\deg A_2B_2\le z_R\\ \deg A_3,\deg B_3\le\frac{1}{8}\log_q\deg R\\ (A_1A_2A_3B_1B_2B_3,R)=1\\ A_1A_2A_3\equiv B_1B_2B_3(\mathrm{mod}\,F)\\ A_1A_2A_3\ne B_1B_2B_3}}\frac{\beta(A_3)\beta(B_3)}{|A_1A_2A_3B_1B_2B_3|^{\frac{1}{2}}}\\
&\quad\le\sum_{\substack{A_3,B_3\in\mathcal{S}_{\mathcal{M}}(X)\\ \deg A_3,\deg B_3\le\frac{1}{8}\log_q\deg R\\ (A_3B_3,R)=1}}\frac{|\beta(A_3)\beta(B_3)|}{|A_3B_3|^{\frac{1}{2}}}\sum_{EF=R}|\mu(E)|\phi(F)\\
&\qquad\times\sum_{z_1,z_2=0}^{z_R}q^{-\frac{z_1+z_2}{2}}\sum_{\substack{A_1,A_2,B_1,B_2\in\mathcal{M}\\ \deg A_1B_1=z_1\\ \deg A_2B_2=z_2\\ (A_1A_2B_1B_2,R)=1\\ A_1A_2A_3\equiv B_1B_2B_3(\mathrm{mod}\,F)\\ A_1A_2A_3\ne B_1B_2B_3}}1.
\end{aligned}$$

By Lemma 6.10 we have, for $\epsilon=\frac{1}{40}$,

$$\begin{aligned}
&\sum_{z_1,z_2=0}^{z_R}q^{-\frac{z_1+z_2}{2}}\sum_{\substack{A_1,A_2,B_1,B_2\in\mathcal{M}\\ \deg A_1B_1=z_1\\ \deg A_2B_2=z_2\\ (A_1A_2B_1B_2,R)=1\\ A_1A_2A_3\equiv B_1B_2B_3(\mathrm{mod}\,F)\\ A_1A_2A_3\ne B_1B_2B_3}}1\\
&\quad\ll\frac{|A_3B_3|^{1+\frac{\epsilon}{2}}}{|F|}\sum_{\substack{z_1,z_2=0\\ z_1+z_2+\deg A_3B_3\le\frac{19}{10}\deg F}}^{z_R}q^{(z_1+z_2)\left(\frac{1}{2}+\frac{\epsilon}{2}\right)}\\
&\qquad+\frac{|A_3B_3|}{\phi(F)}\sum_{\substack{z_1,z_2=0\\ z_1+z_2+\deg A_3B_3>\frac{19}{10}\deg F}}^{z_R}q^{\frac{z_1+z_2}{2}}(z_1+z_2+\deg A_3B_3)^3\\
&\quad\ll\frac{|A_3B_3|^{1+\epsilon}}{|F|^{\frac{1}{20}-\epsilon}}+\frac{|A_3B_3|}{\phi(F)}q^{z_R}(\deg R)^3.
\end{aligned}$$

We also have

$$\begin{aligned}
&\sum_{EF=R} |\mu(E)|\phi(F) \left(\frac{|A_3B_3|^{1+\epsilon}}{|F|^{\frac{1}{20}-\epsilon}} + \frac{|A_3B_3|}{\phi(F)} q^{z_R} (\deg R)^3 \right) \\
&\quad = |A_3B_3|^{1+\epsilon} \sum_{EF=R} |\mu(E)| \frac{\phi(F)}{|F|^{\frac{1}{20}-\epsilon}} + |A_3B_3| q^{z_R} (\deg R)^3 \sum_{EF=R} |\mu(E)| \\
&\quad \ll |A_3B_3|^{1+\epsilon} |R| + |A_3B_3R| (\deg R)^3,
\end{aligned}$$

where the last relation uses

$$\begin{aligned}
&\sum_{EF=R} |\mu(E)| \frac{\phi(F)}{|F|^{\frac{1}{20}-\epsilon}} \leq \sum_{EF=R} |\mu(E)|\phi(F) \\
&= \phi(R) \sum_{EF=R} |\mu(E)| \prod_{\substack{P|E \\ P^2|R}} \left(\frac{1}{|P|} \right) \prod_{\substack{P|E \\ P^2\nmid R}} \left(\frac{1}{|P|-1} \right) \\
&\leq \phi(R) \sum_{EF=R} |\mu(E)| \prod_{P|E} \left(\frac{1}{|P|-1} \right) = \phi(R) \prod_{P|R} \left(1 + \frac{1}{|P|-1} \right) = \phi(R) \frac{|R|}{\phi(R)} \\
&= |R|.
\end{aligned}$$

Finally, using the fact that

$$\begin{aligned}
&\sum_{\substack{A_3,B_3 \in \mathcal{S}_{\mathcal{M}}(X) \\ \deg A_3, \deg B_3 \leq \frac{1}{8} \log_q \deg R \\ (A_3B_3,R)=1}} |\beta(A_3)\beta(B_3)| |A_3B_3|^{\frac{1}{2}+\epsilon} \\
&\leq \Bigg(\sum_{\substack{A \in \mathcal{M} \\ \deg A \leq \frac{1}{8} \log_q \deg R}} |\beta(A)| |A|^{\frac{1}{2}+\epsilon} \Bigg)^2 \leq \Bigg(\sum_{\substack{A \in \mathcal{M} \\ \deg A \leq \frac{1}{8} \log_q \deg R}} 2^{\omega(A)} |A|^{\frac{1}{2}+\epsilon} \Bigg)^2 \\
&\leq \Bigg(\sum_{\substack{A \in \mathcal{M} \\ \deg A \leq \frac{1}{8} \log_q \deg R}} d(A) |A|^{\frac{1}{2}+\epsilon} \Bigg)^2 \leq \Bigg(\sum_{\substack{A \in \mathcal{M} \\ \deg A \leq \frac{1}{8} \log_q \deg R}} |A|^{\frac{1}{2}+\epsilon} \Bigg)^4 \\
&\leq (\deg R)^{\frac{7}{8}},
\end{aligned}$$

we see that

$$\frac{1}{\phi^*(R)}\sum_{EF=R}\mu(E)\phi(F)\sum_{\substack{A_1,A_2,B_1,B_2\in\mathcal{M}\\ A_3,B_3\in\mathcal{S}_{\mathcal{M}}(X)\\ \deg A_1B_1,\deg A_2B_2\le z_R\\ \deg A_3,\deg B_3\le\frac{1}{8}\log_q\deg R\\ (A_1A_2A_3B_1B_2B_3,R)=1\\ A_1A_2A_3\equiv B_1B_2B_3 (\mathrm{mod}\, F)\\ A_1A_2A_3\neq B_1B_2B_3}}\frac{\beta(A_3)\beta(B_3)}{|A_1A_2A_3B_1B_2B_3|^{\frac{1}{2}}} \ll \frac{|R|}{\phi^*(R)}(\deg R)^{3+\frac{7}{8}}.$$

This is indeed of lower order than (7.17); Section 4 of [2] provides the necessary results to confirm this.

Step 2: the asymptotic main term of $\frac{1}{\phi^*(R)}\sum^*_{\chi \bmod R} b(\chi)^2\big|\widehat{P^{**}_X}(\frac{1}{2},\chi)\big|^2$.
We have that

$$(7.18)\quad \frac{1}{\phi^*(R)}\sideset{}{^*}\sum_{\chi \bmod R} b(\chi)^2\left|\widehat{P^{**}_X}\left(\frac{1}{2},\chi\right)\right|^2$$
$$\le \frac{1}{\phi^*(R)}\sum_{\chi \bmod R} b(\chi)^2\left|\widehat{P^{**}_X}\left(\frac{1}{2},\chi\right)\right|^2$$
$$\le \frac{1}{\phi^*(R)}\sum_{\chi \bmod R}\sum_{\substack{A_1,A_2,B_1,B_2\in\mathcal{M}\\ A_3,B_3\in\mathcal{S}_{\mathcal{M}}(X)\\ z_R<\deg A_1B_1,\deg A_2B_2<\deg R\\ \deg A_3,\deg B_3\le\frac{1}{8}\log_q\deg R}}\frac{\beta(A_3)\beta(B_3)\chi(A_1A_2A_3)\overline{\chi}(B_1B_2B_3)}{|A_1A_2A_3B_1B_2B_3|^{\frac{1}{2}}}$$
$$= \frac{\phi(R)}{\phi^*(R)}\sum_{\substack{A_1,A_2,B_1,B_2\in\mathcal{M}\\ A_3,B_3\in\mathcal{S}_{\mathcal{M}}(X)\\ z_R<\deg A_1B_1,\deg A_2B_2<\deg R\\ \deg A_3,\deg B_3\le\frac{1}{8}\log_q\deg R\\ (A_1A_2A_3B_1B_2B_3,R)=1\\ A_1A_2A_3=B_1B_2B_3}}\frac{\beta(A_3)\beta(B_3)}{|A_1A_2A_3B_1B_2B_3|^{\frac{1}{2}}}$$
$$+\frac{\phi(R)}{\phi^*(R)}\sum_{\substack{A_1,A_2,B_1,B_2\in\mathcal{M}\\ A_3,B_3\in\mathcal{S}_{\mathcal{M}}(X)\\ z_R<\deg A_1B_1,\deg A_2B_2<\deg R\\ \deg A_3,\deg B_3\le\frac{1}{8}\log_q\deg R\\ (A_1A_2A_3B_1B_2B_3,R)=1\\ A_1A_2A_3\equiv B_1B_2B_3 (\mathrm{mod}\, R)\\ A_1A_2A_3\neq B_1B_2B_3}}\frac{\beta(A_3)\beta(B_3)}{|A_1A_2A_3B_1B_2B_3|^{\frac{1}{2}}}.$$

Step 2.1. For the diagonal term, by similar means as in (7.3), we obtain

$$\sum_{\substack{A_1,A_2,B_1,B_2\in\mathcal{M}\\ A_3,B_3\in\mathcal{S}_{\mathcal{M}}(X)\\ z_R<\deg A_1B_1,\deg A_2B_2<\deg R\\ \deg A_3,\deg B_3\leq\frac{1}{8}\log_q\deg R\\ (A_1A_2A_3B_1B_2B_3,R)=1\\ A_1A_2A_3=B_1B_2B_3}} \frac{\beta(A_3)\beta(B_3)}{|A_1A_2A_3B_1B_2B_3|^{\frac{1}{2}}} \tag{7.19}$$

$$= \sum_{\substack{G_3,V_{1,3},V_{2,3},V_{3,1},V_{3,2}\in\mathcal{S}_{\mathcal{M}}(X)\\ \deg G_3V_{3,1}V_{3,2}\leq\frac{1}{8}\log_q\deg R\\ \deg G_3V_{1,3}V_{2,3}\leq\frac{1}{8}\log_q\deg R\\ (G_3V_{1,3}V_{2,3}V_{3,1}V_{3,2},R)=1\\ (V_{1,3}V_{2,3},V_{3,1}V_{3,2})=1}} \frac{\beta(G_3V_{3,1}V_{3,2})\beta(G_3V_{1,3}V_{2,3})}{|G_3V_{1,3}V_{2,3}V_{3,1}V_{3,2}|}$$

$$\times \sum_{\substack{V\in\mathcal{M}\\ \deg V\leq\deg R-\deg V_{1,3}V_{3,1}\\ \deg V\leq\deg R-\deg V_{2,3}V_{3,2}\\ \big(V,R(V_{1,3}V_{3,1},V_{2,3}V_{3,2})\big)=1}} \frac{2^{\omega(V)-\omega\big((V,V_{1,3}V_{2,3}V_{3,1}V_{3,2})\big)}}{|V|}$$

$$\times \sum_{\substack{G_1,G_2\in\mathcal{M}\\ \max\left\{0,\frac{z_R-\deg VV_{1,3}V_{3,1}}{2}\right\}<\deg G_1<\frac{\deg R-\deg VV_{1,3}V_{3,1}}{2}\\ \max\left\{0,\frac{z_R-\deg VV_{2,3}V_{3,2}}{2}\right\}<\deg G_2<\frac{\deg R-\deg VV_{2,3}V_{3,2}}{2}\\ (G_1G_2,R)=1}} \frac{1}{|G_1G_2|}.$$

Now, if $\frac{z_R-\deg VV_{1,3}V_{3,1}}{2}\leq\log_q 3^{\omega(R)}$ then

$$\frac{\deg R-\deg VV_{1,3}V_{3,1}}{2}\leq\log_q 3^{\omega(R)}+\frac{1}{2}\log_q 2^{\omega(R)}<\log_q 6^{\omega(R)},$$

and so, by Corollary 5.5, we have

$$\sum_{\substack{G_1\in\mathcal{M}\\ \max\left\{0,\frac{z_R-\deg VV_{1,3}V_{3,1}}{2}\right\}<\deg G_1<\frac{\deg R-\deg VV_{1,3}V_{3,1}}{2}\\ (G_1,R)=1}} \frac{1}{|G_1|} \leq \sum_{\substack{G_1\in\mathcal{M}\\ \deg G_1<\log_q 6^{\omega(R)}\\ (G_1,R)=1}} \frac{1}{|G_1|}$$

$$\ll \frac{\phi(R)}{|R|}\omega(R).$$

If $\frac{z_R-\deg VV_{1,3}V_{3,1}}{2} > \log_q 3^{\omega(R)}$ then

$$\sum_{\substack{G_1\in\mathcal{M}\\ \max\left\{0,\frac{z_R-\deg VV_{1,3}V_{3,1}}{2}\right\}<\deg G_1<\frac{\deg R-\deg VV_{1,3}V_{3,1}}{2}\\ (G_1,R)=1}} \frac{1}{|G_1|}$$

$$= \sum_{\substack{G_1\in\mathcal{M}\\ \deg G_1<\frac{\deg R-\deg VV_{1,3}V_{3,1}}{2}\\ (G_1,R)=1}} \frac{1}{|G_1|} - \sum_{\substack{G_1\in\mathcal{M}\\ \deg G_1<\frac{z_R-\deg VV_{1,3}V_{3,1}}{2}\\ (G_1,R)=1}} \frac{1}{|G_1|} \ll \frac{\phi(R)}{|R|}\omega(R),$$

where we have used Corollary 5.5 twice for the last relation. Similar results hold for the sum over G_2. Hence, proceeding similarly as we did for the diagonal terms of $\frac{1}{\phi^*(R)}\sum^*_{\chi \bmod R} a(\chi)^2\left|\widehat{P^{**}_X}(\frac{1}{2},\chi)\right|^2$, we see that there is a constant c such that

$$\frac{\phi(R)}{\phi^*(R)} \sum_{\substack{A_1,A_2,B_1,B_2\in\mathcal{M}\\ A_3,B_3\in\mathcal{S}_{\mathcal{M}}(X)\\ z<\deg A_1B_1,\deg A_2B_2<\deg R\\ \deg A_3,\deg B_3\leq\frac{1}{8}\log_q\deg R\\ (A_1A_2A_3B_1B_2B_3,R)=1\\ A_1A_2A_3=B_1B_2B_3}} \frac{\beta(A_3)\beta(B_3)}{|A_1A_2A_3B_1B_2B_3|^{\frac{1}{2}}}$$

$$\ll \frac{\phi(R)^3}{|R|^2\phi^*(R)}\omega(R)^2(\deg R)^2\prod_{P|R}\left(\frac{(1-|P|^{-1})^3}{1+|P|^{-1}}\right)(\log_q\log\deg R)^c.$$

Step 2.2. We now look at the second term on the far right side of (7.18): the off-diagonal terms. Using Lemma 6.10, we have

$$\frac{\phi(R)}{\phi^*(R)} \sum_{\substack{A_1,A_2,B_1,B_2\in\mathcal{M}\\ A_3,B_3\in\mathcal{S}_{\mathcal{M}}(X)\\ z_R<\deg A_1B_1,\deg A_2B_2<\deg R\\ \deg A_3,\deg B_3\leq\frac{1}{8}\log_q\deg R\\ (A_1A_2A_3B_1B_2B_3,R)=1\\ A_1A_2A_3\equiv B_1B_2B_3 (\mathrm{mod}\, R)\\ A_1A_2A_3\neq B_1B_2B_3}} \frac{\beta(A_3)\beta(B_3)}{|A_1A_2A_3B_1B_2B_3|^{\frac{1}{2}}}$$

$$= \frac{\phi(R)}{\phi^*(R)} \sum_{\substack{A_3,B_3 \in \mathcal{S}_{\mathcal{M}}(X) \\ \deg A_3, \deg B_3 \leq \frac{1}{8} \log_q \deg R \\ (A_3 B_3, R)=1}} \frac{\beta(A_3)\beta(B_3)}{|A_3 B_3|^{\frac{1}{2}}}$$

$$\times \sum_{z_R < z_1, z_2 < \deg R} q^{-\frac{z_1+z_2}{2}} \sum_{\substack{A_1,A_2,B_1,B_2 \in \mathcal{M} \\ \deg A_1 B_1 = z_1 \\ \deg A_2 B_2 = z_2 \\ (A_1 A_2 A_3 B_1 B_2 B_3, R)=1 \\ A_1 A_2 A_3 \equiv B_1 B_2 B_3 (\mathrm{mod}\, R) \\ A_1 A_2 A_3 \neq B_1 B_2 B_3}} 1$$

$$\ll \frac{(\deg R)^3}{\phi^*(R)} \sum_{\substack{A_3,B_3 \in \mathcal{S}_{\mathcal{M}}(X) \\ \deg A_3, \deg B_3 \leq \frac{1}{8} \log_q \deg R \\ (A_3 B_3, R)=1}} |\beta(A_3)\beta(B_3)| |A_3 B_3|^{\frac{1}{2}} \sum_{z_R < z_1, z_2 < \deg R} q^{\frac{z_1+z_2}{2}}$$

$$\ll \frac{|R|(\deg R)^3}{\phi^*(R)} \sum_{\substack{A_3,B_3 \in \mathcal{S}_{\mathcal{M}}(X) \\ \deg A_3, \deg B_3 \leq \frac{1}{8} \log_q \deg R \\ (A_3 B_3, R)=1}} |\beta(A_3)\beta(B_3)| |A_3 B_3|^{\frac{1}{2}} \ll \frac{|R|(\deg R)^{3+\frac{3}{4}}}{\phi^*(R)}.$$

Step 3. By similar means as in Steps 1 and 2, we can show

$$\frac{1}{\phi^*(R)} \sideset{}{^*}\sum_{\chi \,\mathrm{mod}\, R} c(\chi)^2 \left| \widehat{P_X^{**}}\left(\frac{1}{2}, \chi\right) \right|^2 \ll \frac{|R|(\deg R)^{3+\frac{3}{4}}}{\phi^*(R)}.$$

Thus, considering this, and the other bounds we have established in Steps 1 and 2, we can see the main term comes from (7.17), and this completes the proof. □

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