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Comparing direct limit and inverse limit of even K-groups in multiple $\mathbb{Z}_p$-extensions

par Meng Fai LIM

Résumé. Iwasawa a établi une dualité entre la limite directe et la limite inverse des groupes des classes dans une $\mathbb{Z}_p$-extension. Récemment, ce résultat a été généralisé au cas de $\mathbb{Z}_p$-extensions multiples par de nombreux auteurs. Dans cet article, nous établissons une dualité analogue entre la limite directe et la limite inverse des K-groupes en degrés pairs dans une $\mathbb{Z}_p^d$-extension. Nous donnons ensuite quelques exemples où la limite directe peut être nulle ou pas.

Abstract. Iwasawa first established a duality relating the direct limit and the inverse limit of class groups in a $\mathbb{Z}_p$-extension, and this result has recently been extended to multiple $\mathbb{Z}_p$-extensions by many authors. In this paper, we establish an analogous duality for the direct limit and the inverse limit of higher even K-groups in a $\mathbb{Z}_p^d$-extension. We then give some examples where the direct limit may or may not vanish.

1. Introduction

Let F be a number field and let K be a $\mathbb{Z}_p$-extension of F. For every finite extension L of F contained in K, we let A_L denote the p-primary part of the ideal class group of L. The ring of integer of L is then denoted by $\mathcal{O}_L$. Suppose that L' is an extension of L contained in K. Then there is a natural map $A_L \to A_{L'}$ induced by the natural inclusion $\mathcal{O}_L \subseteq \mathcal{O}_{L'}$. On the other hand, we have a map $A_{L'} \to A_L$ going the other way which is induced by the (ideal) norm. Then one has

$$\varinjlim_L A_L \quad \text{and} \quad \varprojlim_L A_L,$$

where the direct limit (resp., the inverse limit) is taken with respect to the inclusion maps (resp., the norm maps). These two limit modules come naturally equipped with $\mathbb{Z}_p[\![\Gamma]\!]$-module structures, where $\Gamma = \mathrm{Gal}(K/F) \cong$

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$\mathbb{Z}_p$. For a $\mathbb{Z}_p[\![\Gamma]\!]$-module M, we write M^ι for the $\mathbb{Z}_p[\![\Gamma]\!]$-module which is the same underlying $\mathbb{Z}_p$-module M but whose Γ-action is given by

$$\gamma \cdot_\iota x = \gamma^{-1}x, \quad \gamma \in \Gamma, x \in M.$$

A classical theorem of Iwasawa [6] then asserts that there is a pseudo-isomorphism

$$\left(\varprojlim_L A_L\right)^\iota \sim \left(\varinjlim_L A_L\right)^\vee$$

of $\mathbb{Z}_p[\![\Gamma]\!]$-modules. Here $(-)^\vee$ is the Pontryagin dual. This result of Iwasawa has been generalized to the context of a $\mathbb{Z}_p^d$-extension (see the works of Nekovář [15], Vauclair [26] and, more recently, that of Lai and Tan [9]).

In this paper, we will consider the situation of the higher even K-groups. As before, let p be a prime, and let F be a number field. In the event that $p = 2$, we shall assume further that the number field F has no real primes. Let F_∞ be a $\mathbb{Z}_p^d$-extension of F. Fix an integer $i \geq 2$. For each finite intermediate extension L of F_∞/F, the works of Quillen [20] and Borel [3] tell us that the higher even K-group $K_{2i-2}(\mathcal{O}_L)$ is finite. As is standard in Iwasawa theory, we are interested in the Sylow p-subgroup $K_{2i-2}(\mathcal{O}_L)[p^\infty]$ of $K_{2i-2}(\mathcal{O}_L)$. Now, for two finite subextensions $L \subseteq L'$, the inclusion $\mathcal{O}_L \to \mathcal{O}_{L'}$ induces a map $\jmath_{L/L'} : K_{2i-2}(\mathcal{O}_L)[p^\infty] \to K_{2i-2}(\mathcal{O}_{L'})[p^\infty]$ by functoriality. (Note that the induced map may not be injective in general.) In the other direction, there is the norm map (also called the transfer map) $\mathrm{Tr}_{L'/L} : K_{2i-2}(\mathcal{O}_{L'})[p^\infty] \to K_{2i-2}(\mathcal{O}_L)[p^\infty]$. Similar to the class groups situation, we consider the following direct limit and inverse limit

$$\varinjlim_L K_{2i-2}(\mathcal{O}_L)[p^\infty] \quad \text{and} \quad \varprojlim_L K_{2i-2}(\mathcal{O}_L)[p^\infty],$$

whose transition maps are given by the maps $\jmath_{L/L'}$ and $\mathrm{Tr}_{L'/L}$ respectively. Again, these limit modules come equipped with natural $\mathbb{Z}_p[\![G]\!]$-module structures, where $G = \mathrm{Gal}(F_\infty/F) \cong \mathbb{Z}_p^d$. For a $\mathbb{Z}_p[\![G]\!]$-module M, the module M^ι is defined similarly as before. The main result of this paper is then as follows.

Theorem (Theorem 3.1)**.** *Retain the notation as above. Then there is a pseudo-isomorphism*

$$\left(\varprojlim_L K_{2i-2}(\mathcal{O}_L)[p^\infty]\right)^\iota \sim \left(\varinjlim_L K_{2i-2}(\mathcal{O}_L)[p^\infty]\right)^\vee$$

of $\mathbb{Z}_p[\![G]\!]$*-modules.*

Since $K_0(\mathcal{O}_L)[p^\infty] = A_L$, our result may therefore be interpreted as a generalization of the previous results of Iwasawa et al. to the higher even K-groups. For the remainder of the introductional section, we shall sketch the ideas of our proof, leaving the details to the body of the paper.

Let M be a finitely generated $\mathbb{Z}_p[\![G]\!]$-module. If U is an open subgroup of G, we write M_U for the largest quotient of M on which U acts trivially. For two open subgroups $V \subseteq U$ of G, the norm map $N_{U/V}$ on M_V factors through $M_U = (M_V)_{U/V}$ to yield a map $M_U \to M_V$, which by abuse of notation is also denoted by $N_{U/V}$. Then $\{M_U\}$ forms a direct system. We then say that the $\mathbb{Z}_p[\![G]\!]$-module M is *systematically coinvariant-finite* if M_U is finite for every open subgroup U of G. For such a systematically coinvariant-finite module M, the direct limit $\varinjlim_U M_U$ is naturally a discrete $\mathbb{Z}_p[\![G]\!]$-module. The following algebraic result is an important ingredient for the proof of our main theorem.

Theorem (Theorem 2.3)**.** *Let M be a finitely generated $\mathbb{Z}_p[\![G]\!]$-module which is systematically coinvariant-finite. Then we have an isomorphism*

$$\mathrm{Ext}^1_{\mathbb{Z}_p[\![G]\!]}(M, \mathbb{Z}_p[\![G]\!]) \cong \left(\varinjlim_U M_U \right)^\vee \tag{1.1}$$

of $\mathbb{Z}_p[\![G]\!]$-modules.

The above result turns out to be a consequence of an exact sequence of Jannsen ([7, Theorem 2.1] or [16, Theorem 5.4.13]). We emphasize that in deducing the above said theorem from Jannsen's result, the systematically coinvariant-finite property and the commutativity of G are crucially used. We deviate for a while, and say a bit more on the exact sequence of Jannsen. In the original work of Jannsen, the exact sequence is obtained via a spectral sequence argument. More recently, Vauclair [26, Lemme 5.4] has also obtained this said exact sequence via a derived categorical approach. In an Appendix, we shall give a direct proof of Theorem 2.3 without invoking the exact sequence of Jannsen which might be of independent interest.

We now turn back to the focus of this paper. Since a systematically coinvariant-finite $\mathbb{Z}_p[\![G]\!]$-module is automatically torsion over $\mathbb{Z}_p[\![G]\!]$ (see Lemma 2.2 below), the left hand of the isomorphism (1.1) is pseudo-isomorphic to M^ι (for instance, see [18, Proposition 8]). In view of this, to prove our Theorem 3.1, it suffices to take $M = \varprojlim_L K_{2i-2}(\mathcal{O}_L)[p^\infty]$, and show that this module satisfies the following properties:

(I) For every open subgroup U of G, we have an isomorphism

$$t_U : \left(\varprojlim_L K_{2i-2}(\mathcal{O}_L)[p^\infty] \right)_U \cong K_{2i-2}(\mathcal{O}_{L_U})[p^\infty],$$

where L_U is the fixed field of U. In particular, the $\mathbb{Z}_p[\![G]\!]$-module $\varprojlim_L K_{2i-2}(\mathcal{O}_L)[p^\infty]$ is systematically coinvariant-finite.

(II) For open subgroups $V \subseteq U$ of G, the following diagram

$$\begin{array}{ccc} \left(\varprojlim_L K_{2i-2}(\mathcal{O}_L)[p^\infty]\right)_U & \xrightarrow[\sim]{} & K_{2i-2}(\mathcal{O}_{L_U})[p^\infty] \\ \downarrow{\scriptstyle N_{U/V}} & & \downarrow{\scriptstyle \jmath_{L_U/L_V}} \\ \left(\varprojlim_L K_{2i-2}(\mathcal{O}_L)[p^\infty]\right)_V & \xrightarrow[\sim]{} & K_{2i-2}(\mathcal{O}_{L_V})[p^\infty] \end{array} \tag{1.2}$$

commutes.

The verification of the above two properties will be dealt with in Section 3 (in particular, see Proposition 3.2). As a corollary, we have the following result which is reminiscent to that for the class groups (see [2, Proposition 2], [9, Corollary 1.2.1] and [10, Théorème 4.1 and Théorème 4.4]).

Corollary (Corollary 3.3)**.** *Retain the above notation. Then*

$$\varprojlim_L K_{2i-2}(\mathcal{O}_L)[p^\infty]$$

is pseudo-null over $\mathbb{Z}_p[\![G]\!]$ *if and only if* $\varinjlim_L K_{2i-2}(\mathcal{O}_L)[p^\infty] = 0$.

We end the introductory section giving an outline of the paper. The details of the above discussion will be given in Sections 2 and 3. In Section 4, we mention briefly how the above discussion may be applied to study the cohomology group of $\mathbb{Z}_p(i)$ for $i \leq 0$. In Section 5, we then give some examples where the limit $\varinjlim_L K_{2i-2}(\mathcal{O}_L)[p^\infty]$ can be zero (or not). Finally, in Section 6, we shall supply a direct proof of Theorem 2.3.

2. Systematically coinvariant-finite modules

Throughout this section, p will always denote a fixed prime. Let $d \geq 1$. We shall denote by G the group isomorphic to the d-copies of the additive group $\mathbb{Z}_p$. The completed group algebra $\mathbb{Z}_p[\![G]\!]$ is defined by

$$\varprojlim_U \mathbb{Z}_p[G/U],$$

where U runs through all open subgroups of G and the transition maps are given by the natural projection $\mathbb{Z}_p[G/V] \twoheadrightarrow \mathbb{Z}_p[G/U]$ for $V \subseteq U$. It is well-known that the ring $\mathbb{Z}_p[\![G]\!]$ can be identified with the power series ring in d variables over $\mathbb{Z}_p$. In particular, it is a local ring.

Definition 2.1. Let M be a $\mathbb{Z}_p[\![G]\!]$-module. For an open subgroup U of G, we write M_U for the largest quotient of M on which U acts trivially. The $\mathbb{Z}_p[\![G]\!]$-module M is then said to be *systematically coinvariant-finite* if M_U is finite for every open subgroup U of G. As seen in the introduction, $\{M_U\}_U$ forms a direct system of finite modules with transition maps given by

$$N_{U/V} : M_U \longrightarrow M_V$$

for $V \subseteq U$. In particular, $\varinjlim_U M_U$ is a discrete $\mathbb{Z}_p[\![G]\!]$-module.

Lemma 2.2. *A systematically coinvariant-finite $\mathbb{Z}_p[\![G]\!]$-module M is finitely generated torsion over $\mathbb{Z}_p[\![G]\!]$.*

Proof. Let $\mathfrak{m}$ be the (unique) maximal ideal of $\mathbb{Z}_p[\![G]\!]$. Then it contains the augmentation ideal I_G (for instances, see [16, Proposition 5.2.16]). Thus, $M/\mathfrak{m}M$ is a quotient of M_G and hence finite. By the Nakayama lemma, this in turn implies that M is finitely generated over $\mathbb{Z}_p[\![G]\!]$. Finally, since $G \cong \mathbb{Z}_p^d$, it is in particular a solvable p-adic Lie group. Therefore, we may apply the main result of [1] to conclude that M is torsion over $\mathbb{Z}_p[\![G]\!]$. $\square$

We can now state the main theorem of this section.

Theorem 2.3. *Let M be a systematically coinvariant-finite $\mathbb{Z}_p[\![G]\!]$-module. Then we have an isomorphism*

$$\mathrm{Ext}^1_{\mathbb{Z}_p[\![G]\!]}(M, \mathbb{Z}_p[\![G]\!]) \cong \left(\varinjlim_U M_U\right)^\vee.$$

Proof. By Jannsen's result (cf. [7, Theorem 2.1] or [16, Theorem 5.4.13]), there is a short exact sequence

$$0 \longrightarrow \left(\varinjlim_U (M_U[p^\infty])\right)^\vee \longrightarrow \mathrm{Ext}^1_{\mathbb{Z}_p[\![G]\!]}(M, \mathbb{Z}_p[\![G]\!]) \longrightarrow \left(\varinjlim_U H_1(U, M) \otimes \mathbb{Q}_p/\mathbb{Z}_p\right)^\vee \longrightarrow 0.$$

By the systematically coinvariant-finite property, the leftmost term in the above exact sequence is precisely $\left(\varinjlim_U M_U\right)^\vee$. It therefore remains to show that the rightmost term in the above exact sequence is zero which reduces us to showing that $H_1(U, M)$ is finite for every open subgroup U of G. Since $U \cong \mathbb{Z}_p^d$ and taking the systematically coinvariant-finite assumption into account, it is enough to show that $H_1(G, M)$ is finite whenever M_G is finite. This latter assertion essentially follows from a rather terse discussion in [22, p. 56-57], where it is also shown that $H_j(G, M)$ is finite for every $j \geq 1$. For the convenience of the readers, we supply a slightly more detailed argument here.

Throughout our discussion, we shall identify $H_j(G, M)$ with $\mathrm{Tor}^j_{\mathbb{Z}_p[\![G]\!]}(\mathbb{Z}_p, M)$ (see [16, Proposition 5.2.6]). For a $\mathbb{Z}_p[\![G]\!]$-module N, write $\mathrm{Ann}_{\mathbb{Z}_p[\![G]\!]}(N)$ for its annihilator. As the ring $\mathbb{Z}_p[\![G]\!]$ is commutative, the group $\mathrm{Tor}^j_{\mathbb{Z}_p[\![G]\!]}(\mathbb{Z}_p, M)$ can be endowed with a $\mathbb{Z}_p[\![G]\!]$-module structure, and therefore, it makes sense to speak of $\mathrm{Ann}_{\mathbb{Z}_p[\![G]\!]}\big(\mathrm{Tor}^j_{\mathbb{Z}_p[\![G]\!]}(\mathbb{Z}_p, M)\big)$. By [22, p. 56, two lines after Proposition 4], one has

$$\mathrm{Ann}_{\mathbb{Z}_p[\![G]\!]}(\mathbb{Z}_p) + \mathrm{Ann}_{\mathbb{Z}_p[\![G]\!]}(M) \subseteq \mathrm{Ann}_{\mathbb{Z}_p[\![G]\!]}\Big(\mathrm{Tor}^j_{\mathbb{Z}_p[\![G]\!]}(\mathbb{Z}_p, M)\Big). \tag{2.1}$$

Note that $\mathrm{Ann}_{\mathbb{Z}_p[\![G]\!]}(\mathbb{Z}_p)$ is the augmentation ideal I_G of $\mathbb{Z}_p[\![G]\!]$ which is a prime ideal of $\mathbb{Z}_p[\![G]\!]$. For an ideal J of $\mathbb{Z}_p[\![G]\!]$, we write $\mathrm{rad}(J)$ for the (prime) radical of J, i.e., $\mathrm{rad}(J)$ consists of elements $x \in \mathbb{Z}_p[\![G]\!]$ such that $x^{n(x)} \in J$ for some positive integer $n(x)$. We claim that $I_G + \mathrm{Ann}_{\mathbb{Z}_p[\![G]\!]}(M) \neq I_G$ under the assumption that $M_G = M/I_GM$ is finite. Suppose on the contrary that $I_G + \mathrm{Ann}_{\mathbb{Z}_p[\![G]\!]}(M) = I_G$. Then we have

$$\begin{aligned}\mathrm{rad}\Big(\mathrm{Ann}_{\mathbb{Z}_p[\![G]\!]}(M/I_GM)\Big) &= \mathrm{rad}\Big(I_G + \mathrm{Ann}_{\mathbb{Z}_p[\![G]\!]}(M)\Big)\\ &= \mathrm{rad}(I_G) = I_G,\end{aligned} \tag{2.2}$$

where the first equality follows from [14, p. 13, Exercise 2.2], the second by our supposition and the final equality follows from the fact that I_G is a prime ideal. It then follows from the finiteness of M/I_GM and (2.2) that $p^t \in I_G$ for some positive integer t, and hence $p \in I_G$ by the primeness of I_G, which yields the required contradiction. Hence we must have $I_G + \mathrm{Ann}_{\mathbb{Z}_p[\![G]\!]}(M) \neq I_G$. In other words, $\big(I_G + \mathrm{Ann}_{\mathbb{Z}_p[\![G]\!]}(M)\big)/I_G$ is a nonzero ideal of $\mathbb{Z}_p[\![G]\!]/I_G = \mathbb{Z}_p$. Consequently, there exists a nonnegative integer k such that $I_G + \mathrm{Ann}_{\mathbb{Z}_p[\![G]\!]}(M) = I_G + p^k\mathbb{Z}_p[\![G]\!]$. As the latter is an ideal of $\mathbb{Z}_p[\![G]\!]$ of finite index, the inclusion (2.1) then implies that $\mathrm{Ann}_{\mathbb{Z}_p[\![G]\!]}\big(\mathrm{Tor}^j_{\mathbb{Z}_p[\![G]\!]}(\mathbb{Z}_p, M)\big)$ is also an ideal of $\mathbb{Z}_p[\![G]\!]$ of finite index. Hence it follows that $H_j(G, M) = \mathrm{Tor}^j_{\mathbb{Z}_p[\![G]\!]}(\mathbb{Z}_p, M)$ is finite. The proof of the theorem is now complete. □

Remark 2.4. It is instructive to specialize our theorem to the context of $G = \Gamma \cong \mathbb{Z}_p$. In this context, $\mathbb{Z}_p[\![\Gamma]\!]$ identifies with the power series ring $\mathbb{Z}_p[\![T]\!]$ in one variable. Denote by $w_n := w_n(T)$ the polynomial $(1 + T)^{p^n} - 1$. For a $\mathbb{Z}_p[\![T]\!]$-module, one has a natural identification $M/w_nM \cong M_{\Gamma_n}$, where Γ_n is the subgroup of Γ of index p^n. If M is systematically coinvariant-finite, then the prime ideals dividing w_n, $n \geq 0$, are disjoint to the set of prime ideals of height one in the support of M. The asserted isomorphism in Theorem 2.3 therefore coincides with the isomorphism

$$\mathrm{Ext}^1_{\mathbb{Z}_p[\![\Gamma]\!]}(M, \mathbb{Z}_p[\![\Gamma]\!]) \cong \Big(\varinjlim_n M/w_n\Big)^\vee$$

obtained via the theory of Iwasawa adjoint as in [16, Proposition 5.5.6].

We end this section with two useful corollaries. Recall that for a $\mathbb{Z}_p[\![G]\!]$-module M, M^ι denotes the $\mathbb{Z}_p[\![G]\!]$-module which is the same underlying

$\mathbb{Z}_p$-module M with G-action given by

$$g \cdot_\iota x = g^{-1}x, \quad g \in G,\ x \in M.$$

Corollary 2.5. *Let M be a systematically coinvariant-finite $\mathbb{Z}_p[\![G]\!]$-module. Then one has a pseudo-isomorphism*

$$M^\iota \sim \left(\varinjlim_U M_U\right)^\vee$$

of $\mathbb{Z}_p[\![G]\!]$-modules.

Proof. Since $\mathrm{Ext}^1_{\mathbb{Z}_p[\![G]\!]}(M, \mathbb{Z}_p[\![G]\!]) \sim M^\iota$ (cf. [18, Proposition 8]), the conclusion of the corollary follows from this and Theorem 2.3. □

Corollary 2.6. *Let M be a systematically coinvariant-finite $\mathbb{Z}_p[\![G]\!]$-module. Then M is pseudo-null over $\mathbb{Z}_p[\![G]\!]$ if and only if $\varinjlim_U M_U = 0$.*

Proof. By virtue of Lemma 2.2, we already know that M is finitely generated torsion over $\mathbb{Z}_p[\![G]\!]$. Therefore, for M to be pseudo-null over $\mathbb{Z}_p[\![G]\!]$, it is equivalent to having $\mathrm{Ext}^1_{\mathbb{Z}_p[\![G]\!]}(M, \mathbb{Z}_p[\![G]\!]) = 0$. In view of Theorem 2.3, this is the same as saying that $\varinjlim_U M_U = 0$. □

3. Arithmetic

Let F be a number field. In the event $p = 2$, we assume further that F has no real places. For a ring R with identity, write $K_n(R)$ for the algebraic K-groups of R in the sense of Quillen [19] (also see [8, 30]). Let $i \geq 2$. We let $\mathcal{O}_F$ denote the ring of integers of F. By the fundamental results of Garland [4], Quillen [20] and Borel [3], the group $K_{2i-2}(\mathcal{O}_F)$ is finite for each $i \geq 2$. For a finite extension L of F, we have a map

$$\jmath_{F/L} : K_{2i-2}(\mathcal{O}_F) \longrightarrow K_{2i-2}(\mathcal{O}_L)$$

induced by the inclusion $\mathcal{O}_F \to \mathcal{O}_L$ via functoriality. On the other hand, we have the transfer map

$$\mathrm{Tr}_{L/F} : K_{2i-2}(\mathcal{O}_L) \longrightarrow K_{2i-2}(\mathcal{O}_F).$$

Let F_∞ be a $\mathbb{Z}_p^d$-extension of F, where $d \geq 1$. The Galois group $\mathrm{Gal}(F_\infty/F)$ will always be denoted by G. We then consider the following direct limit and inverse limit

$$\varinjlim_L K_{2i-2}(\mathcal{O}_L)[p^\infty] \quad \text{and} \quad \varprojlim_L K_{2i-2}(\mathcal{O}_L)[p^\infty],$$

where the transition maps for the direct limit (resp., the inverse limit) are given by $\jmath_{L/L'}$ (resp., $\mathrm{Tr}_{L/L'}$). For the direct limit, we shall sometimes write

$$K_{2i-2}(\mathcal{O}_{F_\infty})_p := \varinjlim_L K_{2i-2}(\mathcal{O}_L)[p^\infty].$$

The following is the main theorem of this paper.

Theorem 3.1. *For $i \geq 2$, there is a pseudo-isomorphism*

$$\left(\varprojlim_L K_{2i-2}(\mathcal{O}_L)[p^\infty]\right)^\iota \sim \left(K_{2i-2}(\mathcal{O}_{F_\infty})_p\right)^\vee.$$

of $\mathbb{Z}_p[\![G]\!]$-modules.

As seen from the discussion in the introduction, it suffices to show the following assertion.

Proposition 3.2. *The $\mathbb{Z}_p[\![G]\!]$-module $\varprojlim_L K_{2i-2}(\mathcal{O}_L)[p^\infty]$ is systematically coinvariant-finite such that for every open subgroup U of G, there is an isomorphism*

$$t_U : \left(\varprojlim_L K_{2i-2}(\mathcal{O}_L)[p^\infty]\right)_U \cong K_{2i-2}(\mathcal{O}_{L_U})[p^\infty],$$

where L_U is the fixed field of U. Furthermore, if V is an open subgroup of G contained in U with fixed field L_V, we then have the following commutative diagram

$$\begin{array}{ccc} \left(\varprojlim_L K_{2i-2}(\mathcal{O}_L)[p^\infty]\right)_U & \xrightarrow[\sim]{} & K_{2i-2}(\mathcal{O}_{L_V})[p^\infty] \\ \big\downarrow N_{U/V} & & \big\downarrow \jmath_{L_U/L_V} \\ \left(\varprojlim_L K_{2i-2}(\mathcal{O}_L)[p^\infty]\right)_V & \xrightarrow[\sim]{} & K_{2i-2}(\mathcal{O}_{L_U})[p^\infty] \end{array} \tag{3.1}$$

Theorem 3.1 will then follow immediately from a combination of Corollary 2.5 and Proposition 3.2. Furthermore, combining Corollary 2.6 with Proposition 3.2 yields the following corollary.

Corollary 3.3. *Retain the above notation. Then $\varprojlim_L K_{2i-2}(\mathcal{O}_L)[p^\infty]$ is pseudo-null over $\mathbb{Z}_p[\![G]\!]$ if and only $\varinjlim_L K_{2i-2}(\mathcal{O}_L)[p^\infty] = 0$.*

The remainder of the section will be devoted to the verification of Proposition 3.2. Throughout, we shall let S denote the set of primes of F consisting of those above p and the infinite primes. Write F_S for the maximal algebraic extension of F unramified outside S. Denoting by μ_{p^n} the cyclic group generated by a primitive $p^{n\text{-th}}$-root of unity, we then write μ_{p^∞} for the direct limit of the groups μ_{p^n}. These have natural $G_S(F)$-module structures. The action of $G_S(F)$ on μ_{p^∞} induces the cyclotomic character

$$\chi : G_S(F) \longrightarrow \mathrm{Aut}(\mu_{p^\infty}) \cong \mathbb{Z}_p^\times.$$

For a discrete or compact $G_S(F)$-module X, we shall write $X(i)$ for i-fold Tate twist of X. More precisely, $X(i)$ is the $G_S(F)$-module which is X as a $\mathbb{Z}_p$-module but with a $G_S(F)$-action given by

$$\sigma \cdot x = \chi(\sigma)^i \sigma x,$$

where the action on the right is the original action of $G_S(F)$ on X. Plainly, we have $X(0) = X$ and $\mu_{p^\infty} \cong \mathbb{Q}_p/\mathbb{Z}_p(1)$. One can also check directly that

$$X(i+j) \cong \big(X(i)\big)(j).$$

The key approach towards proving Proposition 3.2 is via cohomology. For this, we need to recall the works of Soulé [25], Rost and Voevodsky [28] which allow us to translate our problem into a cohomological one. In [25], Soulé connected the K-groups with continuous cohomology groups via the p-adic Chern class maps

$$\mathrm{ch}_i^F : K_{2i-2}(\mathcal{O}_F)[p^\infty] \cong K_{2i-2}(\mathcal{O}_F)\otimes \mathbb{Z}_p \longrightarrow H^2\left(G_S(F), \mathbb{Z}_p(i)\right)$$

for $i \geq 2$. For the precise definition of these maps, we refer readers to loc. cit. Soulé has proved that these maps are surjective (see [25, Théorème 6 (iii)]). Thanks to the deep work of Rost and Voevodsky [28] (also see [29]), we now know that these maps are isomorphisms.

Now, if L is a finite extension of F contained in F_S, we shall write $G_S(L)$ for the Galois group $\mathrm{Gal}(F_S/L)$. Then one has the following commutative diagrams (see [25, Chapter III])

$$\begin{array}{ccc} K_{2i-2}(\mathcal{O}_F)[p^\infty] & \xrightarrow{\mathrm{ch}_i^F} & H^2\left(G_S(F), \mathbb{Z}_p(i)\right) \\ \Big\downarrow{\scriptstyle \jmath_{L/F}} & & \Big\downarrow{\scriptstyle \mathrm{res}} \\ K_{2i-2}(\mathcal{O}_L)[p^\infty] & \xrightarrow{\mathrm{ch}_i^L} & H^2\left(G_S(L), \mathbb{Z}_p(i)\right) \end{array} \tag{3.2}$$

$$\begin{array}{ccc} K_{2i-2}(\mathcal{O}_L)[p^\infty] & \xrightarrow{\mathrm{ch}_i^L} & H^2\left(G_S(L), \mathbb{Z}_p(i)\right) \\ \Big\downarrow{\scriptstyle \mathrm{Tr}_{L/F}} & & \Big\downarrow{\scriptstyle \mathrm{cor}} \\ K_{2i-2}(\mathcal{O}_F)[p^\infty] & \xrightarrow{\mathrm{ch}_i^F} & H^2\left(G_S(F), \mathbb{Z}_p(i)\right) \end{array} \tag{3.3}$$

We turn back to our $\mathbb{Z}_p^d$-extension F_∞. Note that by [6, Theorem 1], the extension F_∞ is contained in F_S. Hence it makes sense to speak of $G_S(L) = \mathrm{Gal}(F_S/L)$. We then define the Iwasawa cohomology group $H^k_{\mathrm{Iw}}(F_\infty/F, \mathbb{Z}_p(i))$ to be

$$H^k_{\mathrm{Iw}}(F_\infty/F, \mathbb{Z}_p(i)) := H^k_{\mathrm{Iw},S}(F_\infty/F, \mathbb{Z}_p(i)) = \varprojlim_L H^k(G_S(L), \mathbb{Z}_p(i)),$$

where the inverse limit is taken over all the finite extensions L of F contained in F_∞ and with respect to the corestriction maps. It can be shown that these cohomology groups are finitely generated over $\mathbb{Z}_p[\![G]\!]$ (for instance, see [13, Proposition 4.1.3]). From the commutative diagram (3.3), we obtain the following relation between the inverse limit of K-groups and the second Iwasawa cohomology groups.

Lemma 3.4. *Suppose that $i \geq 2$. Then there is an isomorphism*

$$\varprojlim_L K_{2i-2}(\mathcal{O}_L)[p^\infty] \cong H^2_{\mathrm{Iw},S}(F_\infty/F, \mathbb{Z}_p(i))$$

of $\mathbb{Z}_p[\![G]\!]$-modules.

We now recall the following version of Tate's descent spectral sequence for Iwasawa cohomology groups following Nekovář [15]. This will allow us to relate the coinvariant of the Iwasawa cohomology group with the intermediate cohomology groups.

Proposition 3.5. *Let U be an open normal subgroup of $G = \mathrm{Gal}(F_\infty/F)$ and write L for the fixed field of U. Then we have a homological spectral sequence*

$$H_r\big(U, H^{-s}_{\mathrm{Iw}}(F_\infty/F, \mathbb{Z}_p(i))\big) \Longrightarrow H^{-r-s}\big(G_S(L), \mathbb{Z}_p(i)\big).$$

In particular, we have an isomorphism

$$H^2_{\mathrm{Iw}}\big(F_\infty/F, \mathbb{Z}_p(i)\big)_U \cong H^2\big(G_S(L), \mathbb{Z}_p(i)\big).$$

induced by the corestriction map on cohomology.

Proof. Had F_∞/F being a finite extension, this is essentially the Tate spectral sequence (for instance, see [16, Theorem 2.5.3]). In the general context of the proposition, this follows from the work of Nekovář [15, Proposition 8.4.8.1]. The final isomorphism in the proposition follows from reading off the initial $(0, -2)$-term of the spectral sequence. $\square$

In view of the preceding proposition and commutative diagram (3.2), for the verification of the commutativity of (3.1), one is reduced to proving the following lemma.

Lemma 3.6. *For open subgroups U, V of G such that $V \subseteq U$, let L (resp., K) be the fixed field of U (resp., V). Then we have the following commutative diagram*

$$\begin{array}{ccc} H^2_{\mathrm{Iw}}\big(F_\infty/F, \mathbb{Z}_p(i)\big)_U & \xrightarrow[\sim]{\mathrm{cor}} & H^2\big(G_S(L), \mathbb{Z}_p(i)\big) \\ \Big\downarrow N_{U/V} & & \Big\downarrow \mathrm{res} \\ H^2_{\mathrm{Iw}}\big(F_\infty/F, \mathbb{Z}_p(i)\big)_V & \xrightarrow[\sim]{\mathrm{cor}} & H^2\big(G_S(K), \mathbb{Z}_p(i)\big) \end{array}$$

Proof. Let W be any open subgroup of G contained in V. Write $E = E_W$ for the fixed field of W. By either appealing to the double coset formula (cf. [16, Proposition 1.5.11]) or a direct verification, one has a commutative diagram

$$\begin{array}{ccc} H^2\big(G_S(E),\mathbb{Z}_p(i)\big) & \xrightarrow{\text{cor}} & H^2\big(G_S(L),\mathbb{Z}_p(i)\big) \\ {\scriptstyle N_{U/V}}\big\downarrow & & \big\downarrow{\scriptstyle \text{res}} \\ H^2\big(G_S(E),\mathbb{Z}_p(i)\big) & \xrightarrow{\text{cor}} & H^2\big(G_S(K),\mathbb{Z}_p(i)\big) \end{array}$$

Varying W, we obtain a commutative diagram

$$\begin{array}{ccc} H^2_{\text{Iw}}\big(F_\infty/F,\mathbb{Z}_p(i)\big) & \xrightarrow{\text{cor}} & H^2\big(G_S(L),\mathbb{Z}_p(i)\big) \\ {\scriptstyle N_{U/V}}\big\downarrow & & \big\downarrow{\scriptstyle \text{res}} \\ H^2_{\text{Iw}}\big(F_\infty/F,\mathbb{Z}_p(i)\big) & \xrightarrow{\text{cor}} & H^2\big(G_S(K),\mathbb{Z}_p(i)\big) \end{array}$$

By Proposition 3.5, the horizontal maps factor through $H^2_{\text{Iw}}\big(F_\infty/F,\mathbb{Z}_p(i)\big)_U$ and $H^2_{\text{Iw}}\big(F_\infty/F,\mathbb{Z}_p(i)\big)_V$ respectively. From which, we obtain the required diagram of the lemma. □

We may now conclude the section.

Proof of Proposition 3.2. The isomorphism t_U is a consequence of Lemma 3.4 and Proposition 3.5. On the other hand, the commutativity of diagram (3.1) follows from a combination of Proposition 3.5, Lemma 3.6 and diagram (3.2). □

4. Some further remarks

A conjecture of Schneider (cf. [21, p. 192]) asserted that the cohomology group $H^2(G_S(F),\mathbb{Z}_p(i))$ is finite for $i \leq 0$. (Note that for $i = 0$, the finiteness of $H^2(G_S(F),\mathbb{Z}_p)$ is precisely Leopoldt's conjecture; see [16, Corollary 10.3.10].) Granted this conjecture, the argument in the preceding section carries over to yield a similar sort of result for these cohomology groups. We will just record two situations, where the conjecture of Schneider is known to hold.

Theorem 4.1. *Suppose that we are in either of the following situations.*

(a) *The number field F is totally real (and so $p \geq 3$ by our standing assumption), F_∞ is the cyclotomic $\mathbb{Z}_p$-extension of F and i is a negative odd integer.*
(b) *The number field F is an imaginary quadratic field, F_∞ is a $\mathbb{Z}_p^d$-extension of F (note that $d = 1$ or 2) and $i = 0$.*

Then we have a pseudo-isomorphism

$$\left(\varprojlim_L H^2(G_S(L),\mathbb{Z}_p(i))\right)^\iota \sim \left(\varinjlim_L H^2(G_S(L),\mathbb{Z}_p(i))\right)^\vee$$

of $\mathbb{Z}_p[\![\mathrm{Gal}(F_\infty/F)]\!]$-modules.

Proof. The proof is similar as before. We simply mention that in case (a), the finiteness of $H^2(G_S(L), \mathbb{Z}_p(i))$ is a consequence of [17, Proposition 3.8]. For case (b), since each intermediate extension L of F contained in F_∞ is abelian over F, Leopoldt's conjecture is known to be valid over such an L by Brumer's theorem (cf. [16, Theorem 10.3.16]). □

5. Some classes of examples

We now give some classes of examples of F_∞, where $K_{2i-2}(\mathcal{O}_{F_\infty})_p := \varinjlim_L K_{2i-2}(\mathcal{O}_L)[p^\infty]$ can be either zero or not. We first show that one can construct many examples of nonzero $K_{2i-2}(\mathcal{O}_{F_\infty})_p$. Recall that for every $\mathbb{Z}_p[\![G]\!]$-module M, there is a $\mathbb{Z}_p[\![G]\!]$-homomorphism

$$M[p^\infty] \longrightarrow \bigoplus_{i=1}^{s} \mathbb{Z}_p[\![G]\!]/p^{\alpha_i}$$

with pseudo-null kernel and cokernel (cf. [27, Theorem 3.40]). The μ_G-invariant of M is then defined to be $\sum_{i=1}^{s} \alpha_i$. Note that if M (and hence $M[p^\infty]$) is pseudo-null over $\mathbb{Z}_p[\![G]\!]$, then its μ_G-invariant is trivial.

Proposition 5.1. *Let $i \geq 2$ and let $d \geq 1$. Suppose that p is a prime such that $p > 2d+1$. Then there exist infinitely many pairs (F, F_∞), where F is a finite cyclic extension of $\mathbb{Q}(\mu_p)$ and F_∞ is a $\mathbb{Z}_p^d$-extension of F such that $K_{2i-2}(\mathcal{O}_{F_\infty})_p \neq 0$.*

Proof. Indeed, under the hypothesis of the result, it has been shown that there exist infinitely many pairs (F, F_∞), where F is a finite cyclic extension of $\mathbb{Q}(\mu_p)$ and F_∞ is a $\mathbb{Z}_p^d$-extension of F such that $H^2_{\mathrm{Iw}}(F_\infty/F, \mathbb{Z}_p(i))$ has nontrivial μ_G-invariant (cf. [12, Proposition 5.2.2]). By the remark before the proposition, the module $H^2_{\mathrm{Iw}}(F_\infty/F, \mathbb{Z}_p(i))$ is therefore not pseudo-null over $\mathbb{Z}_p[\![G]\!]$. Consequently, it follows from Lemma 3.4 and Corollary 3.3 that $K^p_{2i-2}(\mathcal{O}_{F_\infty}) \neq 0$. □

The next proposition gives examples, where $K_{2i-2}(\mathcal{O}_{F_\infty})_p$ vanishes.

Proposition 5.2. *Let $F = \mathbb{Q}(\mu_p)$, where p is an irregular prime < 1000. If F_∞ is the compositum of all $\mathbb{Z}_p$-extensions of F, then $K_{2i-2}(\mathcal{O}_{F_\infty})_p = 0$ for every $i \geq 2$.*

Proof. Let L_∞ be the maximal unramified abelian pro-p extension of F_∞, and let L'_∞ be the maximal subextension of L_∞ in which every prime of F_∞ above p splits completely. Sharifi has shown that the Greenberg conjecture (see [5, Conjecture 3.5]) is valid under the hypothesis of the proposition (cf. [23, Theorem 1.3]). In other words, $\mathrm{Gal}(L_\infty/F_\infty)$ is pseudo-null over $\mathbb{Z}_p[\![G]\!]$. Since $\mathrm{Gal}(L'_\infty/F_\infty)$ is a quotient of $\mathrm{Gal}(L_\infty/F_\infty)$, it is also pseudo-null over $\mathbb{Z}_p[\![G]\!]$. Now, by the Poitou–Tate sequence, we have an exact sequence

$$0 \longrightarrow \mathrm{Gal}(L'_\infty/F_\infty) \longrightarrow H^2_{\mathrm{Iw}}(F_\infty/F, \mathbb{Z}_p(1)) \longrightarrow \varprojlim_L \bigoplus_{w_L \in S(L)} H^2(L_{w_L}, \mathbb{Z}_p(1)),$$

where $S(L)$ is the set of primes of L above p. Since the decomposition group of F_∞/F at the prime of F above p has dimension ≥ 2 (cf. [10, Théorème 3.2]), we may apply a similar argument to that in [11, Lemma 5.3] to conclude that $\varprojlim_L \bigoplus_{w_L \in S(L)} H^2(L_{w_L}, \mathbb{Z}_p(1))$ is pseudo-null over $\mathbb{Z}_p[\![G]\!]$, and whence, $H^2_{\mathrm{Iw}}(F_\infty/F, \mathbb{Z}_p(1))$ is pseudo-null over $\mathbb{Z}_p[\![G]\!]$. On the other hand, as F_∞ contains μ_{p^∞}, an application of [24, Lemma 2.5.1 (c)] tells us that

$$H^2_{\mathrm{Iw}}(F_\infty/F, \mathbb{Z}_p(i)) \cong H^2_{\mathrm{Iw}}(F_\infty/F, \mathbb{Z}_p(1)) \otimes \mathbb{Z}_p(i-1)$$

for every $i \geq 2$. Thus, it follows that $H^2_{\mathrm{Iw}}(F_\infty/F, \mathbb{Z}_p(i))$ is pseudo-null over $\mathbb{Z}_p[\![G]\!]$ for every $i \geq 2$. Combining this latter observation with Lemma 3.4 and Corollary 3.3, we obtain the conclusion of the proposition. □

6. Appendix

We retain the notation and settings of Section 2. As Theorem 2.3 is a main tool for our discussion in the paper, we have thought that it might be of interest to give a direct proof of the said theorem without appealing to the exact sequence of Jannsen or the heavy machinery of Vauclair. This is the goal of the present appendix. The proof of the theorem will be split into a few lemmas. The first of which is the following general observation.

Lemma 6.1. *For every finitely generated $\mathbb{Z}_p[\![G]\!]$-module M, there is an isomorphism*

$$\mathrm{Ext}^1_{\mathbb{Z}_p[\![G]\!]}(M, \mathbb{Z}_p[\![G]\!]) \cong \varprojlim_U \mathrm{Ext}^1_{\mathbb{Z}_p[\![G]\!]}(M, \mathbb{Z}_p[G/U]),$$

where the inverse limit is taken with respect to the canonical projection maps.

Proof. The isomorphism

$$\mathrm{Hom}_{\mathbb{Z}_p[\![G]\!]}(M, \mathbb{Z}_p[\![G]\!]) \cong \varprojlim_U \mathrm{Hom}_{\mathbb{Z}_p[\![G]\!]}(M, \mathbb{Z}_p[G/U]).$$

plainly holds when M is free of finite rank. But since the ring $\mathbb{Z}_p[\![G]\!]$ is Noetherian, every finitely generated module M has a resolution consisting of finitely generated free $\mathbb{Z}_p[\![G]\!]$-modules. Applying the above isomorphism to this free resolution and taking homology, we obtain the conclusion of the lemma. □

Before continuing, we make a remark.

Remark 6.2. Let U be an open (normal) subgroup of G and let M be a $\mathbb{Z}_p[\![G]\!]$-module. A $\mathbb{Z}_p[\![G]\!]$-module homomorphism $M \to \mathbb{Z}_p[G/U]$ naturally factors through M_U to yield a $\mathbb{Z}_p[\![G]\!]$-module homomorphism $M_U \to \mathbb{Z}_p[G/U]$. Furthermore, one sees easily that this latter map can be viewed as a $\mathbb{Z}_p[G/U]$-module homomorphism. Conversely, given a $\mathbb{Z}_p[G/U]$-module homomorphism $M_U \to \mathbb{Z}_p[G/U]$, by considering the composition $M \to M_U \to \mathbb{Z}_p[G/U]$, we obtain a $\mathbb{Z}_p[\![G]\!]$-module homomorphism. In conclusion, we have identifications

$$\begin{aligned}\mathrm{Hom}_{\mathbb{Z}_p[\![G]\!]}(M, \mathbb{Z}_p[G/U]) &= \mathrm{Hom}_{\mathbb{Z}_p[\![G]\!]}(M_U, \mathbb{Z}_p[G/U]) \\ &= \mathrm{Hom}_{\mathbb{Z}_p[G/U]}(M_U, \mathbb{Z}_p[G/U]).\end{aligned}$$

These identifications will be frequently utilized in the subsequent discussion without any further mention.

For the next two lemmas, we shall require the module M to be systematically coinvariant-finite. In particular, the next lemma also makes crucial use of the commutativity of G.

Lemma 6.3. *Suppose that M is a systematically coinvariant-finite $\mathbb{Z}_p[\![G]\!]$-module. Then we have an isomorphism*

$$\theta_U : \mathrm{Ext}^1_{\mathbb{Z}_p[\![G]\!]}(M, \mathbb{Z}_p[G/U]) \cong \mathrm{Ext}^1_{\mathbb{Z}_p[G/U]}(M_U, \mathbb{Z}_p[G/U]).$$

Furthermore, if V is another open subgroup of G which is contained in U, then we have the following commutative diagram

$$\begin{array}{ccc} \mathrm{Ext}^1_{\mathbb{Z}_p[\![G]\!]}(M, \mathbb{Z}_p[G/V]) & \xrightarrow[\sim]{\theta_V} & \mathrm{Ext}^1_{\mathbb{Z}_p[G/V]}(M_V, \mathbb{Z}_p[G/V]) \\ \downarrow & & \downarrow \\ \mathrm{Ext}^1_{\mathbb{Z}_p[\![G]\!]}(M, \mathbb{Z}_p[G/U]) & \xrightarrow[\sim]{\theta_U} & \mathrm{Ext}^1_{\mathbb{Z}_p[G/U]}(M_U, \mathbb{Z}_p[G/U]) \end{array} \tag{6.1}$$

where the vertical maps are induced by the projection $\mathbb{Z}_p[G/V] \twoheadrightarrow \mathbb{Z}_p[G/U]$.

Proof. Since M is finitely generated over $\mathbb{Z}_p[\![G]\!]$, we can find a short exact sequence

$$0 \longrightarrow N \longrightarrow \mathbb{Z}_p[\![G]\!]^r \longrightarrow M \longrightarrow 0$$

of finitely generated $\mathbb{Z}_p[\![G]\!]$-modules. Taking U-homology of this short exact sequence, we obtain an exact sequence

$$0 \longrightarrow H_1(U, M) \longrightarrow N_U \longrightarrow \mathbb{Z}_p[G/U]^r \longrightarrow M_U \longrightarrow 0$$

of $\mathbb{Z}_p[G/U]$-modules. Let C denote the kernel of the map $\mathbb{Z}_p[G/U]^r \to M_U$. Consider the following diagram

$$
\begin{array}{ccc}
\mathrm{Hom}_{\mathbb{Z}_p[G/U]}(\mathbb{Z}_p[G/U]^r, \mathbb{Z}_p[G/U]) & = & \mathrm{Hom}_{\mathbb{Z}_p[\![G]\!]}(\mathbb{Z}_p[\![G]\!]^r, \mathbb{Z}_p[G/U]) \\
\downarrow & & \downarrow \\
\mathrm{Hom}_{\mathbb{Z}_p[G/U]}(C, \mathbb{Z}_p[G/U]) & \xrightarrow[\alpha_U]{} & \mathrm{Hom}_{\mathbb{Z}_p[\![G]\!]}(N, \mathbb{Z}_p[G/U]) \\
\downarrow & & \downarrow \\
\mathrm{Ext}^1_{\mathbb{Z}_p[G/U]}(M_U, \mathbb{Z}_p[G/U]) & \overset{}{\underset{\theta_U}{\dashrightarrow}} & \mathrm{Ext}^1_{\mathbb{Z}_p[\![G]\!]}(M, \mathbb{Z}_p[G/U]) \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}
\tag{6.2}
$$

with exact columns. Here the middle horizontal map α_U is given by

$$
\begin{aligned}
\mathrm{Hom}_{\mathbb{Z}_p[G/U]}(C, \mathbb{Z}_p[G/U]) \longrightarrow {} & \mathrm{Hom}_{\mathbb{Z}_p[G/U]}(N_U, \mathbb{Z}_p[G/U]) \\
& = \mathrm{Hom}_{\mathbb{Z}_p[\![G]\!]}(N, \mathbb{Z}_p[G/U]).
\end{aligned}
$$

It is straightforward to check that the topmost square is commutative and this in turn induces the bottommost horizontal map which is our θ_U. Furthermore, the map α_U is injective with cokernel contained in

$$\mathrm{Hom}_{\mathbb{Z}_p[G/U]}\big(H_1(U,M), \mathbb{Z}_p[G/U]\big).$$

In view that M_U is finite, it follows from the argument in the proof of Theorem 2.3 that $H_1(U,M)$ is also finite. Since a $\mathbb{Z}_p[G/U]$-module homomorphism is also a group homomorphism and $\mathbb{Z}_p[G/U]$ has no p-torsion, one must have $\mathrm{Hom}_{\mathbb{Z}_p[G/U]}\big(H_1(U,M), \mathbb{Z}_p[G/U]\big) = 0$. Consequently, the map α_U is an isomorphism. From this and diagram (6.2), we see that θ_U is an isomorphism. Finally, one checks easily that the topmost square in (6.2) is natural in U, which in turn implies that the map θ_U is natural in U. The proof of the lemma is therefore completed. □

Lemma 6.4. *Suppose that M is a systematically coinvariant-finite $\mathbb{Z}_p[\![G]\!]$-module. Then we have an isomorphism*

$$\psi_U : \mathrm{Ext}^1_{\mathbb{Z}_p[G/U]}(M_U, \mathbb{Z}_p[G/U]) \cong (M_U)^\vee.$$

Furthermore, if V is another open subgroup of G which is contained in U, there is a commutative diagram

$$
\begin{array}{ccc}
\mathrm{Ext}^1_{\mathbb{Z}_p[G/V]}(M_V, \mathbb{Z}_p[G/V]) & \xrightarrow[\sim]{\psi_V} & (M_V)^\vee \\
\downarrow & & \downarrow \\
\mathrm{Ext}^1_{\mathbb{Z}_p[G/U]}(M_U, \mathbb{Z}_p[G/U]) & \xrightarrow[\sim]{\psi_U} & (M_U)^\vee
\end{array}
\tag{6.3}
$$

where the vertical map on the left is induced by the projection $\mathbb{Z}_p[G/V] \twoheadrightarrow \mathbb{Z}_p[G/U]$ *and the vertical map on the right is induced by the norm map* $N_{U/V} : M_U \to M_V$.

Proof. Since M_U is finite, it is annihilated by p^t for some large enough t. On the other hand, multiplication by p^t induces an automorphism on $\mathbb{Z}_p[G/U] \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. Hence it follows that

$$\mathrm{Ext}^i_{\mathbb{Z}_p[G/U]}(M_U, \mathbb{Z}_p[G/U] \otimes_{\mathbb{Z}_p} \mathbb{Q}_p) = 0$$

for every $i \geq 0$. Taking this into account, upon applying $\mathrm{Hom}_{\mathbb{Z}_p[G/U]}(M_U, -)$ to the short exact sequence

$$0 \longrightarrow \mathbb{Z}_p[G/U] \longrightarrow \mathbb{Z}_p[G/U] \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \longrightarrow \mathbb{Z}_p[G/U] \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p \longrightarrow 0,$$

we see that the connecting morphism

$$\partial_U : \mathrm{Hom}_{\mathbb{Z}_p[G/U]}(M_U, \mathbb{Z}_p[G/U] \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p) \longrightarrow \mathrm{Ext}^1_{\mathbb{Z}_p[G/U]}(M_U, \mathbb{Z}_p[G/U])$$

is an isomorphism which is natural in U. It therefore remains to show the existence of an isomorphism

$$\eta_U : \mathrm{Hom}_{\mathbb{Z}_p[G/U]}(M_U, \mathbb{Z}_p[G/U] \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p) \cong (M_U)^\vee$$

for every open subgroup U of G with the property that if $V \subseteq U$, there is a commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\mathbb{Z}_p[G/V]}(M_V, \mathbb{Z}_p[G/V] \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p) & \xrightarrow[\sim]{\eta_V} & (M_V)^\vee \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{\mathbb{Z}_p[G/U]}(M_U, \mathbb{Z}_p[G/U] \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p) & \xrightarrow[\sim]{\eta_U} & (M_U)^\vee \end{array} \tag{6.4}$$

where the left vertical map is induced by the projection $\mathbb{Z}_p[G/V] \twoheadrightarrow \mathbb{Z}_p[G/U]$ and the right vertical map is induced by the norm map $N_{U/V} : M_U \to M_V$. To simplify notation, we shall write $\mathbb{Q}_p/\mathbb{Z}_p[G/U] = \mathbb{Z}_p[G/U] \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p$. Now, for each $f \in \mathrm{Hom}_{\mathbb{Z}_p[G/U]}(M_U, \mathbb{Q}_p/\mathbb{Z}_p[G/U])$ and $x \in M_U$, we write

$$f(x) = \sum_{\sigma \in G/U} f_\sigma(x)\sigma,$$

where $f_\sigma(x) \in \mathbb{Q}_p/\mathbb{Z}_p$. We then define the map

$$\eta_U : \mathrm{Hom}_{\mathbb{Z}_p[G/U]}(M_U, \mathbb{Q}_p/\mathbb{Z}_p[G/U]) \longrightarrow (M_U)^\vee$$

by sending $f \mapsto f_1$, where "1" denotes the identity of the group G/U. Let $\tau \in G/U$. Since f is a $\mathbb{Z}_p[G/U]$-homomorphism, we have the identity $\tau f(x) = f(\tau x)$ which in turn yields

$$f_\sigma(\tau x) = f_{\tau^{-1}\sigma}(x)$$

for every $\sigma \in G/U$. In particular, we have

$$f_1(\tau x) = f_{\tau^{-1}}(x)$$

for every $\tau \in G/U$. Hence f is uniquely determined by f_1, and therefore, η_U is an isomorphism. It remains to show that η_U is natural in U. Let V be an open subgroup of G which is contained in U. The projection $\mathbb{Z}_p[G/V] \to \mathbb{Z}_p[G/U]$ induces a map

$$\begin{aligned}\rho_{VU} : \mathrm{Hom}_{\mathbb{Z}_p[G/V]}&(M_V, \mathbb{Q}_p/\mathbb{Z}_p[G/V]) \\ &\longrightarrow \mathrm{Hom}_{\mathbb{Z}_p[G/V]}(M_V, \mathbb{Q}_p/\mathbb{Z}_p[G/U]) \\ &\qquad = \mathrm{Hom}_{\mathbb{Z}_p[G/U]}(M_U, \mathbb{Q}_p/\mathbb{Z}_p[G/U]).\end{aligned}$$

Let $x \in M_U$ and $h \in \mathrm{Hom}_{\mathbb{Z}_p[G/V]}(M_V, \mathbb{Z}_p[G/V] \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p)$. By abuse of notation, we write $x \in M_V$ for a preimage of x under natural surjection $M_V \to (M_V)_{U/V} = M_U$. Let $\sigma_1 = 1, \sigma_2, \ldots, \sigma_r \in G/V$ be a complete set of coset representatives of U/V in G/V. Then one has

$$h(x) = \sum_{\tau \in U/V} \sum_{i=1}^{r} h_{\tau\sigma_i}(x)\tau\sigma_i.$$

From which, we have

$$\rho_{VU}(h)(x) = \rho_{VU}(h(x)) = \sum_{i=1}^{r}\Big(\sum_{\tau \in U/V} h_{\tau\sigma_i}(x)\Big)\sigma_i,$$

which in turn implies that

$$\begin{aligned}\rho_{VU}(h)_1(x) = \sum_{\tau \in U/V} h_\tau(x) &= \sum_{\tau \in U/V} h_1(\tau^{-1}x) \\ &= h_1\left(\sum_{\tau \in U/V} \tau^{-1}x\right) = h_1\big(N_{U/V}(x)\big).\end{aligned}$$

This establishes the commutativity of the diagram (6.4). The proof is therefore completed. □

Theorem 2.3 is now a consequence of a combination of Lemmas 6.1, 6.3 and 6.4. This concludes our alternative and direct proof of Theorem 2.3.

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