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On Lehmer's question for integer-valued polynomials

par Berend RINGELING

Résumé. Nous répondons à une question de type Lehmer sur la mesure de Mahler des polynômes à valeurs entières.

Abstract. We solve a Lehmer-type question about the Mahler measure of integer-valued polynomials.

1. Introduction

In the 1930s Lehmer asks, for a monic polynomial $P(x) = \prod_{j=1}^{d}(x-\alpha_j) \in \mathbb{Z}[x]$, whether the real quantity $\prod_{j=1}^{d} \max\{1, |\alpha_j|\}$ can be made arbitrarily close to but larger than 1. This quantity is called the *Mahler measure of* $P(x)$ [1]. More generally, for $P(x) = c\prod_{j=1}^{d}(x-\alpha_j) \in \mathbb{C}[x]$, the Mahler measure $M(P(x))$ is defined as $|c|\prod_{j=1}^{d} \max\{1, |\alpha_j|\}$. Conjecturally, the answer to Lehmer's question is negative and the suspected lower bound is given by $\alpha = 1.176280818\ldots$, the unique real zero outside the unit circle of *Lehmer's polynomial* $x^{10}+x^9-x^7-x^6-x^5-x^4-x^3+x+1$. Here we want to extend the original question to a bigger class of polynomials, *integer-valued* polynomials, that is, polynomials $P(x) \in \mathbb{Q}[x]$ such that $P(k) \in \mathbb{Z}$ for all $k \in \mathbb{Z}$. These polynomials often occur in counting problems; basic examples include binomial coefficients,

$$\binom{x}{n} = \frac{x(x-1)(x-2)\ldots(x-n+1)}{n!} \in \mathbb{Q}[x]$$

for $n \in \mathbb{N}$. In fact, any integer-valued polynomial is an integer linear combination of such polynomials (see [4, Part 8, Chapter 2, Section 1]).

Question 1.1. Can $M(P(x))$ be made arbitrarily close to but larger than 1, when $P(x)$ is an irreducible integer-valued polynomial?

The irreducibility condition is essential here, since for a reducible integer-valued polynomial $P(x)$ the bound $M(P(x)) \geq 1$ may be violated. This is

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seen from the example

$$P(x) = \frac{x^p - x}{p}$$

for primes p. By Fermat's little theorem $P(x)$ is integer-valued and the Mahler measure $M(P(x)) = 1/p$ tends to 0 as p increases.

In fact, using the polynomials $\frac{a}{p}(x^p - x)$ for p a prime and a an integer, it is clear that the Mahler measure of a (not necessarily irreducible) integer-valued polynomial can be arbitrarily close to but larger than 1. Moreover, it shows that the range of Mahler measures of these polynomials is dense in the interval $[0, \infty)$. The following statement demonstrates that the bound $M(P(x)) \geq 1$ in Question 1.1 is then best possible.

Lemma 1.2. *If $P(x)$ is irreducible and integer-valued, then $M(P(x)) \geq 1$.*

In fact, one can easily construct infinitely many (non-cyclotomic) irreducible integer-valued polynomials $P(x)$ with $M(P(x)) = 1$, this is demonstrated in Example 2.1 below. A goal of this note is to answer Question 1.1 in the affirmative. To accomplish the task, we consider the family of polynomials

$$f_p(x) = \frac{x^p - x}{p} + x^{(p+1)/2} + 1,$$

for odd integers p. For primes p, the polynomials $f_p(x)$ are integer-valued. We prove the following two statements for them.

Theorem 1.3. *For primes $p \equiv 3 \mod 4$, $f_p(x)$ is irreducible.*

Theorem 1.4. *We have the following asymptotics for $M(f_p(x))$:*

$$M(f_p(x)) \sim \frac{1 + \sqrt{1 + 4/p^2}}{2},$$

to all orders in p, as $p \to \infty$. In particular, $\lim_{p\to\infty} M(f_p(x)) = 1$. Moreover, the sequence $M(f_p(x))$ is strictly decreasing in p.

Thus $M(f_p(x)) > 1$. Hence, the affirmative answer to Question 1.1 is given by the family f_p when $p \equiv 3 \mod 4$. Here we tabulate a few values of $M(f_p(x))$ for small primes p:

p	$M(f_p(x))$
3	1.17503...
7	1.02169...
11	1.00821...
19	1.00276...

2. Properties for the Mahler measure of integer-valued polynomials

Proof of Lemma 1.2. For polynomials of degree 1, the statement is trivial. We will assume that the degree is strictly bigger than 1. The Mahler measure of any polynomial $P(x) \in \mathbb{Q}[x]$ is bounded from below by the absolute value of the constant term. Indeed if

$$P(x) = a_d x^d + \cdots + a_0 = a_d \prod_{j=1}^{d} (x - \alpha_j),$$

then

$$M(P(x)) = |a_d| \prod_{j=1}^{d} \max\{1, |\alpha_j|\} \geq |a_d| \prod_{j=1}^{d} |\alpha_j| = |a_d| \frac{|a_0|}{|a_d|} = |a_0|.$$

Moreover, if $P(x)$ is *integer-valued* and *irreducible*, $P(0)$ is guaranteed to be a non-zero integer. Hence $M(P(x)) \geq 1$. □

The following example shows that one can find infinitely many non-cyclotomic irreducible integer-valued polynomials with Mahler measure exactly 1.

Example 2.1. Consider the integer-valued polynomial

$$g_p(x) = \frac{x^p - x}{p} + 1$$

for primes $p > 2$. The zeros of $g_p(x)$ all lie outside the complex unit circle, as otherwise for any zeros α of g_p inside or on the unit circle, we would have the contradictary inequality

$$0 = |g_p(\alpha)| = \left| 1 + \frac{\alpha^p}{p} - \frac{\alpha}{p} \right| \geq 1 - \frac{2}{p} > 0.$$

As a consequence of this, we find $M(g_p(x)) = 1$.

We want to show that the polynomial $pg_p(x)$ is irreducible. If it were reducible, then at least one of the irreducible factors would have constant term 1; this is impossible since g_p has all the zeros outside the unit circle. Thus, we have found an infinite family of (non-cyclotomic) irreducible integer-valued polynomials.

3. Irreducibility

Proof of Theorem 1.3. For a polynomial $P(x)$ of degree d, write $\widetilde{P}(x) = x^d P(1/x)$ for its reciprocal. We prove the irreducibility of the polynomials f_p for primes $p \equiv 3 \mod 4$ following the method first used by Ljunggren in [3], also see the expository notes [2]. The irreducibility of f_3 and f_7 is immediate, so we deal with $p > 7$ from now on.

Write

$$f_p^*(x) = pf_p(x) = x^p + px^{\frac{p+1}{2}} - x + p$$

and

$$\widetilde{f_p^*}(x) = x^p f_p^*(1/x) = px^p - x^{p-1} + px^{\frac{p-1}{2}} + 1$$

for its reciprocal.

Lemma 3.1. *The polynomials f_p^* and $\widetilde{f_p^*}$ have no zeros in common.*

Proof. Suppose α is a zero of both f_p^* and $\widetilde{f_p^*}$, so that

$$\alpha^p - \alpha + p\alpha^{\frac{p+1}{2}} + p = 0 \quad \text{and} \quad \alpha - \alpha^p + p\alpha^{\frac{p+1}{2}} + p\alpha^{p+1} = 0.$$

The equations imply

$$(\alpha^{\frac{p+1}{2}} + 1)^2 = 0,$$

hence $\alpha^{\frac{p+1}{2}} = -1$. Substituting this in the first equation we find that $\alpha = \pm 1$. This is impossible if $p \equiv 3 \mod 4$. □

Suppose $f_p^*(x)$ is reducible, i.e. $f_p^* = gh$ for $g, h \in \mathbb{Z}[x]$ of positive degree. Define an auxiliary polynomial

$$k = g\widetilde{h} = b_p x^p + \cdots + b_0;$$

then $k\widetilde{k} = f_p^*\widetilde{f_p^*}$. Note that $k \neq \pm f_p^*$ or $\pm\widetilde{f_p^*}$, as otherwise $k = g\widetilde{h}$ and $f_p^* = gh$ are equal, up to sign, hence $\widetilde{h}$ and h share a common zero, which is impossible by Lemma 3.1.

We next compute the coefficients of k by comparing the coefficients in

$$\text{(3.1)} \quad k\widetilde{k} = f_p^*\widetilde{f_p^*} = px^{2p} - x^{2p-1} + p^2 x^{\frac{3p+1}{2}} - px^{p+1} + 2(p^2+1)x^p - px^{p-1} + p^2 x^{\frac{p-1}{2}} - x + p.$$

Reading off the x^{2p}-coefficient we have $b_0 b_p = p$. We can assume that $b_0 = \pm p$ and $b_p = \pm 1$, possibly by interchanging k and $\widetilde{k}$. Further we may assume $b_0 = p$ and $b_p = 1$ by possibly replacing k with $-k$ (by replacing f_p^* with $-f_p^*$).

Comparing the x^p-coefficient of $k\widetilde{k}$ in (3.1), we find

$$2(p^2+1) = b_0^2 + \cdots + b_p^2,$$

therefore

$$\text{(3.2)} \qquad p^2 + 1 = b_1^2 + \cdots + b_{p-1}^2.$$

Comparing the x-coefficient in (3.1) we conclude that

$$-1 = b_0 b_{p-1} + b_1 b_p$$

implying $b_1 = -1 - b_{p-1}p$. The latter equality is only possible if either $b_{p-1} = 0$ and $b_1 = -1$, or $b_{p-1} = -1$ and $b_1 = p-1$, as otherwise (3.2) fails. Consider the two cases separately.

Case $b_{p-1} = -1$ *and* $b_1 = p - 1$. We obtain from (3.2)

$$2p - 1 = b_2^2 + \cdots + b_{p-2}^2. \tag{3.3}$$

Compare the x^2-coefficient to find that

$$b_0 b_{p-2} + b_1 b_{p-1} + b_2 b_p = 0,$$

so that $b_2 = -1 - p(b_{p-2} - 1)$. According to (3.3), the equality is only possible if $b_{p-2} = 1$ and $b_2 = -1$. Next compare the x^3-coefficient to find that

$$b_0 b_{p-3} + b_1 b_{p-2} + b_2 b_{p-1} + b_3 b_p = 0,$$

hence $p(b_{p-3} + 1) + b_3 = 0$. Again, from (3.3) we conclude that $b_{p-3} = -1$ and $b_3 = 0$. We claim that $b_{p-j} = (-1)^j$ and $b_j = 0$ for $2 < j < \frac{p-1}{2}$. Comparing the x^j-coefficient for such j gives

$$b_0 b_{p-j} + b_1 b_{p-j+1} + b_2 b_{p-j+2} + \cdots + b_j b_p = 0.$$

By induction all the terms b_i vanish for $2 < i < j$, so that $b_j = -p(b_{p-j} - (-1)^j)$. From (3.3) and the fact that p divides b_j, we conclude that $b_j = 0$ and $b_{p-j} = (-1)^j$. Finally, compare the coefficient of $x^{\frac{p-1}{2}}$ in (3.1):

$$p^2 = b_0 b_{\frac{p+1}{2}} + b_1 b_{\frac{p+1}{2}+1} + b_2 b_{\frac{p+1}{2}+2} + \cdots + b_{\frac{p-1}{2}} b_p.$$

This translates into

$$p^2 = p(b_{\frac{p+1}{2}} - (-1)^{\frac{p-3}{2}}) + b_{\frac{p-1}{2}}.$$

Therefore, $b_{\frac{p-1}{2}}$ is divisible by p, hence $b_{\frac{p-1}{2}} = 0$ from (3.3) implying $b_{\frac{p+1}{2}} = (-1)^{\frac{p-1}{2}} + p = p - 1$. This calculation contradicts (3.3).

Case $b_{p-1} = 0$ *and* $b_1 = -1$. In this case we have

$$p^2 = b_2^2 + \cdots + b_{p-2}^2. \tag{3.4}$$

We claim that $b_j = 0$ for $1 < j < \frac{p-1}{2}$. Suppose otherwise, let $1 < j' < \frac{p-1}{2}$ be the smallest integer such that $b_{j'} \neq 0$. Comparing the x^i-coefficient for $1 < i < j'$ in (3.1) results in

$$b_0 b_{p-i} + b_1 b_{p-i+1} \cdots + b_i b_p = 0;$$

it follows by induction that $b_{p-i} = 0$ for all such i as well.

Comparing the $x^{j'}$-coefficient in (3.1) we find out that

$$b_0 b_{p-j'} + b_1 b_{p-j'+1} + \cdots + b_{j'} b_p = 0,$$

hence $pb_{p-j'} + b_{j'} = 0$. Since $b_{j'} \neq 0$ by our assumption, we have $|b_{j'}| \geq p$ and $|b_{p-j'}| \geq 1$. Comparing this with (3.4) we find this impossible. The contradiction implies that $b_j = 0$ and $b_{p-j} = 0$ for $1 < j < \frac{p-1}{2}$.

Finally, consider the $x^{\frac{p+1}{2}}$-coefficient in (3.1):

$$b_0 b_{p-\frac{p-1}{2}} + b_1 b_{p-\frac{p-1}{2}+1} + \cdots + b_{\frac{p-1}{2}} b_p = p^2;$$

this simplifies to

$$pb_{\frac{p+1}{2}} + b_{\frac{p-1}{2}} = p^2.$$

Comparing with (3.4), the only solution to this equation is $b_{\frac{p+1}{2}} = p$ and $b_{\frac{p-1}{2}} = 0$. We conclude that $k = f_p^*$, which gives a contradiction.

Thus, f_p^* is irreducible. This proves Theorem 1.3. □

4. Asymptotics

For this part, it is more convenient to work with the *logarithmic Mahler measure* $\mathrm{m}(P(x)) = \log(M(P(x)))$. Jensen's formula allows one to write it as

$$\mathrm{m}(P(x)) = \frac{1}{2\pi \mathrm{i}} \oint_{|z|=1} \log|P(z)| \, \frac{\mathrm{d}z}{z}. \tag{4.1}$$

Denote $N = (p-1)/2$ and $Q_p(x) = (x^2-1)/p + x$ and define

$$m_p = \mathrm{m}\left(\frac{x^p - x}{p} + x^{\frac{p+1}{2}} + 1\right) = \mathrm{m}(xQ_p(x^N) + 1).$$

We will show that, for all integers N,

$$m_p \sim \mathrm{m}(xQ_p(x^N)) = \mathrm{m}(Q_p(x)) = \log\frac{1+\sqrt{1+4/p^2}}{2}$$

to all orders in p, i.e. the difference of m_p and $\mathrm{m}(Q_p(x))$ is $\mathcal{O}(p^n)$ for all $n \in \mathbb{Z}$.

We have

$$m_p - \mathrm{m}\Big(xQ_p(x^N)\Big) = \mathrm{m}\left(1 + \frac{1}{xQ_p(x^N)}\right) = \frac{1}{N}\cdot \mathrm{m}\left(1 + \frac{(-1)^{N+1}}{xQ_p(x)^N}\right), \tag{4.2}$$

where the last equality follows from the more general observation:

Lemma 4.1. *If $P(x)$ is a polynomial and N an integer, then*

$$\mathrm{m}\left(1 + \frac{(-1)^{N+1}}{xP(x)^N}\right) = N\cdot \mathrm{m}\left(1 + \frac{1}{xP(x^N)}\right)$$

Proof. Indeed, Jensen's formula implies that

$$\begin{aligned} \mathrm{m}\left(1+\frac{(-1)^{N+1}}{xP(x)^N}\right) &= \mathrm{m}\left(1+\frac{(-1)^{N+1}}{x^N P(x^N)^N}\right) = \mathrm{m}\left(1-\left(\frac{-1}{xP(x^N)}\right)^N\right) \\ &= \sum_{\xi:\, \xi^N=1} \mathrm{m}\left(1+\frac{\xi}{xP(x^N)}\right), \end{aligned}$$

where the sum is over all roots of unity of degree N. The required identity follows from noticing that

$$\mathrm{m}\left(1+\frac{\xi}{xP(x^N)}\right) = \mathrm{m}\left(1+\frac{1}{xP(x^N)}\right),$$

by substituting ξx for x in the integral (4.1) for the corresponding Mahler measure. □

Since

$$\frac{1}{|Q_p(z)|^2} = \left|\frac{p}{z^2+pz-1}\right|^2 = \frac{p^2}{2+p^2-2\,\mathrm{Re}(z^2)} < 1$$

for $z \in \mathbb{C} \setminus \{\pm 1\}, |z|=1$, we get the convergent expansion

$$\log\left(1+\frac{(-1)^{N+1}}{zQ_p(z)^N}\right) = \sum_{\ell=1}^{\infty} \frac{(-1)^{\ell N-1}}{\ell z^\ell Q_p(z)^{\ell N}}.$$

for all such z. From this we find out that

$$\begin{aligned} \mathrm{m}\left(1+\frac{(-1)^{N+1}}{xQ_p(x)^N}\right) &= \mathrm{Re}\,\frac{1}{2\pi \mathrm{i}}\oint_{|z|=1} \log\left(1+\frac{(-1)^{N+1}}{zQ_p(z)^N}\right)\frac{\mathrm{d}z}{z} \\ &= \mathrm{Re}\sum_{\ell=1}^{\infty}\frac{(-1)^{\ell N-1}}{\ell}F_\ell, \end{aligned} \tag{4.3}$$

where

$$F_\ell = \frac{1}{2\pi\mathrm{i}}\oint_{|z|=1}\frac{\mathrm{d}z}{z^{\ell+1}Q_p(z)^{\ell N}} = \frac{p^{\ell N}}{2\pi\mathrm{i}}\oint_{|z|=1}\frac{\mathrm{d}z}{z^{\ell+1}(z-\alpha_1)^{\ell N}(z-\alpha_2)^{\ell N}}$$

and α_1, α_2 are the zeros of $Q_p(x)$ ordered by $|\alpha_2| > 1 > |\alpha_1|$. We will examine the asymptotics of F_ℓ for $\ell \geq 1$ as $p \to \infty$. We can explicitly compute these integrals.

Lemma 4.2. *For $\ell \geq 1$, we have*

$$F_\ell = (-1)^\ell p^{\ell N}\sum_{j=0}^{\ell N-1}\binom{2\ell N-2-j}{\ell N-1}\binom{\ell+j}{j}\frac{1}{(\alpha_2-\alpha_1)^{2\ell N-1-j}\alpha_2^{\ell+1+j}}.$$

Proof. This follows from Cauchy's integral theorem. The integrand has precisely one singularity outside the unit circle. Therefore, the value of the integral is given by

$$-\operatorname{Res}_{z=\alpha_2} \frac{1}{z^{\ell+1} Q_p(z)^{\ell N}}.$$

The formula follows by expanding $1/z^{\ell+1}$ into a series in $z-\alpha_2$:

$$\frac{1}{z^{\ell+1}} = \sum_{j=0}^{\infty} (-1)^j \binom{\ell+j}{j} \frac{1}{\alpha_2^{\ell+1+j}} (z-\alpha_2)^j \tag{4.4}$$

and extracting the nonpositive powers of $z-\alpha_2$ in the Laurent expansion of $1/Q_p(z)^{\ell N}$:

$$\begin{aligned} &\frac{1}{(z-\alpha_1)^{\ell N}(z-\alpha_2)^{\ell N}} \\ &= \sum_{j=0}^{\ell N} (-1)^j \binom{\ell N + j - 1}{j} \frac{1}{(\alpha_2-\alpha_1)^{\ell N + j}} (z-\alpha_2)^{j-\ell N} + \mathcal{O}(z-\alpha_2). \end{aligned} \tag{4.5}$$

Taking the product of (4.4) and (4.5) we conclude with the formula

$$(-1)^{\ell-1} p^{\ell N} \sum_{j=0}^{\ell N-1} \binom{2\ell N - 2 - j}{\ell N - 1} \binom{\ell + j}{j} \frac{1}{(\alpha_2-\alpha_1)^{2\ell N - 1 - j} \alpha_2^{\ell+1+j}}$$

for the coefficient of $1/(z-\alpha_2)$. □

Using Lemma 4.2, we will estimate $|F_\ell|$ from above.

Lemma 4.3. *For $\ell \geq 1$, we have*

$$|F_\ell| \leq \frac{1}{p^{\ell(N+1)}} \binom{2\ell N + \ell - 1}{\ell N}.$$

Proof. The estimates $|\alpha_2 - \alpha_1| \geq p$ and $|\alpha_2| \geq p$ imply

$$\begin{aligned} |F_\ell| &\leq \frac{1}{p^{\ell(N+1)}} \sum_{j=0}^{\ell N-1} \binom{2\ell N - 2 - j}{\ell N - 1} \binom{\ell+j}{j} \\ &= \frac{p-1}{p+1} \frac{1}{p^{\ell(N+1)}} \binom{2\ell N + \ell - 1}{\ell N} \leq \frac{1}{p^{\ell(N+1)}} \binom{2\ell N + \ell - 1}{\ell N}. \end{aligned}$$ □

It follows from Lemma 4.3 that F_ℓ decays exponentially in ℓN.

Proof of Theorem 1.4. Using Equations (4.2), (4.3) and Lemma 4.3, we conclude that

$$|m_p - \mathrm{m}(Q_p(x))| \leq \frac{1}{p^{N+1}} \binom{p-1}{N} =: \epsilon_p$$

meaning that the difference of the Mahler measures decays exponentially[1] as $p \to \infty$. This proves the first part of Theorem 1.4. To show that the sequence m_p for odd p is decreasing, it suffices to prove the inequality

$$\mathrm{m}(Q_p(x)) - \epsilon_p > \mathrm{m}(Q_{p+2}(x)) + \epsilon_{p+2}, \tag{4.6}$$

where ϵ_p is defined in the proof of Theorem 1.4. We can estimate $\mathrm{m}(Q_p(x)) - \mathrm{m}(Q_{p+2}(x))$ from below using that $\log(x) > 1 - \frac{1}{x}$ for $x > 1$. Indeed, for $p \geq 5$ we have

$$\begin{aligned}\mathrm{m}(Q_p(x)) - \mathrm{m}(Q_{p+2}(x)) &= \log \frac{1+\sqrt{1+4/p^2}}{1+\sqrt{1+4/(p+2)^2}} \\ &> \frac{\sqrt{1+4/p^2} - \sqrt{1+4/(p+2)^2}}{1+\sqrt{1+4/p^2}} \\ &\geq \frac{1}{2p^2} - \frac{1}{2(p+2)^2} \geq \frac{1}{p^3}.\end{aligned}$$

On the other hand, using $\binom{2n}{n} \leq 4^n$ for $n \geq 1$, we can estimate $\epsilon_p + \epsilon_{p+2}$ from above: for $p \geq 7$ we obtain

$$\epsilon_p + \epsilon_{p+2} \leq \frac{1}{p^{N+1}}4^N + \frac{1}{(p+2)^{N+2}}4^{N+1} \leq \left(\frac{4}{p}\right)^{N+1} \leq \left(\frac{3}{4}\right)^p \leq \frac{1}{p^3}.$$

This implies inequality (4.6) for $p \geq 7$. Together with

$$m_3 = 0.16129\ldots, \qquad m_5 = 0.04920\ldots, \qquad m_7 = 0.02145\ldots,$$

it concludes our proof of Theorem 1.4. □

5. Discussion

The choice for the family of polynomials $f_p(x)$ is far from optimal: among integer-valued polynomials of prime degree $p \equiv 3 \mod 4$, it is not the one with smallest Mahler measure larger than 1. This can already be seen when $p = 3$: an integer-valued polynomial with the smallest Mahler measure is

$$Q_3(x) = \frac{2}{3}x^3 - \frac{1}{2}x^2 - \frac{1}{6}x - 1,$$

with the Mahler measure $1.02833\ldots$ much smaller than $M(f_3(x)) = 1.17503\ldots$.

For $d = 2, 3, \ldots$, define $Q_d(x)$ to be an irreducible integer-valued polynomial of degree d with smallest Mahler measure larger than 1. Then the following questions arise.

[1] In fact, the decay is much faster than exponentially as the leading term of the asymptotics in ϵ_p is given by

$$\sqrt{\frac{2}{\pi}} \cdot \frac{2^{p-1}}{p^{\frac{p+1}{2}}\sqrt{p-1}} \qquad \text{as} \quad p \to \infty.$$

Question 5.1. How to (efficiently) compute these polynomials $Q_d(x)$?

Question 5.2. What can be said about the asymptotics of $M(Q_d(x))$ for $d \to \infty$?

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