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Semi-local units modulo cyclotomic units in the cyclotomic $\mathbb{Z}_2$-extensions

par Takae TSUJI

Résumé. Fixons un corps abélien k dont le conducteur n'est pas divisible par 8 et notons k_∞/k la $\mathbb{Z}_2$-extension cyclotomique avec le n-ième corps intermédiaire k_n. Soit $\mathcal{U}$ (resp. $\mathcal{C}$) la limite projective des groupes des unités semi-locales (resp. des unités cyclotomiques) en 2 de k_n. Pour un caractère pair non-trivial ψ de $\mathrm{Gal}(k/\mathbb{Q})$, nous étudions la structure galoisienne de la ψ-partie $\mathcal{U}^\psi/\mathcal{C}^\psi$ et du ψ-quotient $(\mathcal{U}/\mathcal{C})_\psi$ de $\mathcal{U}/\mathcal{C}$ y compris dans le cas $2 \mid [k:\mathbb{Q}]$.

Abstract. Fix an abelian field k whose conductor is not divisible by 8 and denote by k_∞/k the cyclotomic $\mathbb{Z}_2$-extension with n-th layer k_n. Let $\mathcal{U}$ (resp. $\mathcal{C}$) be the projective limit of the semi-local units at 2 (resp. of the cyclotomic units) of k_n. For a non-trivial even character ψ of $\mathrm{Gal}(k/\mathbb{Q})$, we study the Galois module structure of the ψ-part $\mathcal{U}^\psi/\mathcal{C}^\psi$ and ψ-quotient $(\mathcal{U}/\mathcal{C})_\psi$ of $\mathcal{U}/\mathcal{C}$, taking into account the case $2 \mid [k:\mathbb{Q}]$.

1. Introduction

Let p be any prime number and k an abelian field. We denote by k_∞/k the cyclotomic $\mathbb{Z}_p$-extension with n-th layer k_n for $n \geqq 0$. Let $\mathcal{U}_{k_n}$ be the semi-local units of k_n at p and $\mathcal{C}_{k_n}$ a group of cyclotomic units of k_n defined in Section 5. Put $\mathcal{U} = \mathcal{U}_{k_\infty} = \varprojlim \mathcal{U}_{k_n}$ and $\mathcal{C} = \mathcal{C}_{k_\infty} = \varprojlim \mathcal{C}_{k_n}$ where the projective limits are taken with respect to the relative norm maps. In this paper, we study the Galois module structure of $\mathcal{U}/\mathcal{C}$ for $p = 2$.

We still assume that p is an arbitrary prime number. We may assume that k is of the first kind, that is, the conductor of k is not divisible by 8 or p^2 if $p = 2$ or not respectively. Then $k \cap \mathbb{Q}_\infty = \mathbb{Q}$ where $\mathbb{Q}_\infty$ is the cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}$ and $\mathrm{Gal}(k_\infty/\mathbb{Q}) = G \times \Gamma$ with $G = \mathrm{Gal}(k/\mathbb{Q})$ and $\Gamma = \mathrm{Gal}(k_\infty/k)$. We regard $\mathcal{U}/\mathcal{C}$ as a module over the completed group ring $\mathbb{Z}_p[G][\![\Gamma]\!]$. We decompose $\mathcal{U}/\mathcal{C}$ by the action of G. Let ψ be a non-trivial even character of G with values in $\overline{\mathbb{Q}_p}^\times$ and e_ψ the idempotent of $\mathbb{Q}_p[G]$ corresponding to ψ. If $[k:\mathbb{Q}] = |G|$ is not divisible by p, then e_ψ is in $\mathbb{Z}_p[G]$ and $e_\psi(\mathcal{U}/\mathcal{C})$ becomes a modules over $\mathbb{Z}_p[\psi][\![\Gamma]\!]$ where $\mathbb{Z}_p[\psi]$ denotes the ring generated by the values of ψ over $\mathbb{Z}_p$. As usual, we regard any

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$\mathbb{Z}_p[\psi][\![\Gamma]\!]$-module as a module over $\Lambda = \mathbb{Z}_p[\psi][\![T]\!]$, by fixing a topological generator of Γ. When $p \mid [k:\mathbb{Q}]$, one cannot define a ψ-component as a direct summand. However we can define two Λ-modules, the ψ-part $\mathcal{U}^\psi/\mathcal{C}^\psi$ and the ψ-quotient $(\mathcal{U}/\mathcal{C})_\psi$. If $p \nmid [k:\mathbb{Q}]$, both $\mathcal{U}^\psi/\mathcal{C}^\psi$ and $(\mathcal{U}/\mathcal{C})_\psi$ coincide with $e_\psi(\mathcal{U}/\mathcal{C})$ and, generally, after tensoring with $\mathbb{Q}_p$, these coincide with $e_\psi((\mathcal{U}/\mathcal{C})\otimes\mathbb{Q}_p)$.

We recall the former results on the structure of the Λ-modules $\mathcal{U}^\psi/\mathcal{C}^\psi$ and $(\mathcal{U}/\mathcal{C})_\psi$. For any prime p and any k with $p \nmid [k:\mathbb{Q}]$, the structure of the Λ-module $e_\psi(\mathcal{U}/\mathcal{C}) = \mathcal{U}^\psi/\mathcal{C}^\psi = (\mathcal{U}/\mathcal{C})_\psi$ was determined by Iwasawa [9] and Gillard [5], which is described in terms of the power series $g_\psi(T)$ of Λ associated to the Kubota–Leopoldt p-adic L-function. For odd prime p and any k without assumption $p \nmid [k:\mathbb{Q}]$, the author [12] determined the structure of the Λ-modules $\mathcal{U}^\psi/\mathcal{C}^\psi$ and $(\mathcal{U}/\mathcal{C})_\psi$. She showed that Coleman's homomorphism induces two Λ-homomorphisms

$$\Psi^\psi : \mathcal{U}^\psi/\mathcal{C}^\psi \longrightarrow \Lambda/(g_\psi(T)/2), \qquad \Psi_\psi : (\mathcal{U}/\mathcal{C})_\psi \longrightarrow \Lambda/(g_\psi(T)/2)$$

and determined the kernels and the cokernels of Ψ^ψ and Ψ_ψ respectively. (In [12], Ψ^ψ and Ψ_ψ was denoted by Col^ψ and Col_ψ respectively.) In particular, she showed that

$$\mathrm{char}_\Lambda(\mathcal{U}^\psi/\mathcal{C}^\psi) = (g_\psi(T)/2), \qquad \mathrm{char}_\Lambda((\mathcal{U}/\mathcal{C})_\psi) = (g_\psi(T)/2)$$

where $\mathrm{char}_\Lambda(M)$ denotes the characteristic ideal of a Λ-module M. We note that $(g_\psi(T)/2) = (g_\psi(T))$ holds as an ideal of Λ since p is odd. For the μ-invariants of $\mathcal{U}^\psi/\mathcal{C}^\psi$ and $(\mathcal{U}/\mathcal{C})_\psi$, we can deduce

$$\mu(\mathcal{U}^\psi/\mathcal{C}^\psi) = 0, \qquad \mu((\mathcal{U}/\mathcal{C})_\psi) = 0$$

from our results and the Ferrero–Washington theorem [3, 4]. Under the assumption $p \nmid [k:\mathbb{Q}]$, the main results of [12] coincide with the results of Iwasawa and Gillard for odd prime p. For any prime p and any k without the assumption $p \nmid [k:\mathbb{Q}]$, Greither [8] determined the structure of the $\Lambda\otimes\mathbb{Q}_p$-module $e_\psi((\mathcal{U}/\mathcal{C})\otimes\mathbb{Q}_p) = (\mathcal{U}^\psi/\mathcal{C}^\psi)\otimes\mathbb{Q}_p = (\mathcal{U}/\mathcal{C})_\psi\otimes\mathbb{Q}_p$. Only in the case where $p = 2$ and $2 \mid [k:\mathbb{Q}]$, the structure of the Λ-modules $\mathcal{U}^\psi/\mathcal{C}^\psi$ and $(\mathcal{U}/\mathcal{C})_\psi$ have not been determined yet. In this paper, we determine those structure in the remaining case, that is, $p = 2$ and $2 \mid [k:\mathbb{Q}]$.

Let $p = 2$ and k be any abelian field of the first kind including the case where $2 \mid [k:\mathbb{Q}]$. We study both cases where k is real and imaginary and let ψ be a non-trivial even character of $\mathrm{Gal}(k/\mathbb{Q})$.

In the main results of this paper, Theorems 3.1, 3.2 and 3.4, we define two Λ-homomorphisms

$$\Psi^\psi : \mathcal{U}^\psi/\mathcal{C}^\psi \longrightarrow \Lambda/(g_\psi(T)/2), \qquad \Psi_\psi : (\mathcal{U}/\mathcal{C})_\psi \longrightarrow \Lambda/(g_\psi(T)/2)$$

and determine the kernels and the cokernels of Ψ^ψ and Ψ_ψ respectively for $p = 2$. We show that the kernel of Ψ_ψ has the μ-invariant 1 when k is imaginary. In particular, we can deduce the following

$$\mathrm{char}_\Lambda(\mathcal{U}^\psi/\mathcal{C}^\psi) = (g_\psi(T)/2)$$

and

$$\mathrm{char}_\Lambda((\mathcal{U}/\mathcal{C})_\psi) = \begin{cases} (g_\psi(T)/2) & \text{if } k \text{ is real,} \\ (g_\psi(T)) & \text{if } k \text{ is imaginary.} \end{cases}$$

Therefore, by using the Ferrero–Washington theorem [3, 4], we obtain the following

$$\mu(\mathcal{U}^\psi/\mathcal{C}^\psi) = 0$$

and

$$\mu((\mathcal{U}/\mathcal{C})_\psi) = \begin{cases} 0 & \text{if } k \text{ is real,} \\ 1 & \text{if } k \text{ is imaginary.} \end{cases}$$

Our results in the cases where $2 \nmid [k:\mathbb{Q}]$ (and $p = 2$) coincide with the results of Iwasawa and Gillard. If k is imaginary, that is, the infinite places ramified in k, then $[k:\mathbb{Q}]$ is divisible by $p = 2$. Therefore Iwasawa and Gillard did not treat the cases where k is imaginary. We show that the structure of $(\mathcal{U}/\mathcal{C})_\psi$ depends on whether 2 is ramified in k or not. Actually, if $p = 2$ is ramified in k, the kernel of Ψ_ψ has a finite Λ-submodule $T_{k,\psi}$ defined in Section 3. When p is odd, whether p is ramified in k does not affect the structure of $(\mathcal{U}/\mathcal{C})_\psi$ and the Λ-module $T_{k,\psi}$ does not appear in the kernel of Ψ_ψ. We remark that the ramification index of p in k is a divisor of p or $p-1$ if $p = 2$ or not since k is of the first kind. We further remark that Iwasawa and Gillard did not treat the cases where 2 is ramified in k.

In this paper, we study a relation between the Iwasawa main conjecture and our main theorems. Let $\mathcal{M}$ be the maximal abelian pro 2-extension of k_∞ unramified outside all primes over 2 and put

$$\mathfrak{X} = \mathrm{Gal}(\mathcal{M}/k_\infty).$$

Then Λ-modules $\mathfrak{X}^\psi$ and $\mathfrak{X}_\psi$ are defined. By our main theorems, we can show that

$$\mu(\mathfrak{X}^\psi) = \mu(\mathcal{U}^\psi/\mathcal{C}^\psi), \quad \mu(\mathfrak{X}_\psi) = \mu((\mathcal{U}/\mathcal{C})_\psi)$$

in both cases where k is real and imaginary. Therefore, by the Iwasawa main conjecture proved by Wiles [14] and our results, we obtain

$$\mathrm{char}_\Lambda(\mathfrak{X}^\psi) = \mathrm{char}_\Lambda(\mathcal{U}^\psi/\mathcal{C}^\psi), \quad \mathrm{char}_\Lambda(\mathfrak{X}_\psi) = \mathrm{char}_\Lambda((\mathcal{U}/\mathcal{C})_\psi)$$

including the μ-invariants.

The content of this paper is as follows: In Section 2, we recall the definition of the ψ-part and the ψ-quotient and their basic properties. In Section 3, we state the main results. In Section 4, we define Λ-homomorphisms $\Psi^\psi : \mathcal{U}^\psi \to \Lambda$ and $\Psi_\psi : \mathcal{U}_\psi \to \Lambda$ and determine their kernels and cokernels. In Section 5, we determine generators of the ψ-part and the ψ-quotient of the cyclotomic units group $\mathcal{C}$. In Section 6, we calculate the images of the generators of the ψ-part and the ψ-quotient of $\mathcal{C}$ via Ψ^{ψ} and Ψ_{ψ} respectively. This completes the proof of the main results. Finally, in Section 7 we mention a relation between the Iwasawa main conjecture and the main theorems.

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2. χ-parts and χ-quotients

In this section, let p be any prime number, Δ any finite abelian group and $\chi : \Delta \to \overline{\mathbb{Q}_p}^{\times}$ any character. We define χ-parts and χ-quotients of $\mathbb{Z}_p[\Delta]$-modules and recall some basic facts. For further properties, see [8, 11, 12].

We denote by $\mathbb{Z}_p[\chi]$ the ring generated by the values of χ over $\mathbb{Z}_p$ and by $\underline{\mathbb{Z}_p[\chi]}$ a free $\mathbb{Z}_p[\chi]$-module of rank one on which Δ acts via χ. For a $\mathbb{Z}_p[\Delta]$-module M, we define the following $\mathbb{Z}_p[\chi]$-modules:

$$M^{\chi} = \mathrm{Hom}_{\mathbb{Z}_p[\Delta]}(\underline{\mathbb{Z}_p[\chi]}, M), \; M_{\chi} = M \otimes_{\mathbb{Z}_p[\Delta]} \underline{\mathbb{Z}_p[\chi]},$$

which we call the χ-part and the χ-quotient of M respectively.

Let I_{χ} denote the ideal of $\mathbb{Z}_p[\chi][\Delta]$ generated by all elements of the form $\delta - \chi(\delta), \delta \in \Delta$. We have isomorphisms of $\mathbb{Z}_p[\chi]$-modules

$$M^{\chi} \cong \{m \in M \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[\chi] \mid \delta m = \chi(\delta)m, \forall\, \delta \in \Delta\}$$

and

$$M_{\chi} \cong (M \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[\chi])/I_{\chi}(M \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[\chi]).$$

Then M^{χ} (resp. M_{χ}) is isomorphic to the largest submodule (resp. quotient module) of $M \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[\chi]$ on which Δ acts via χ.

Let $\xi_{\Delta,\chi} = \sum_{\delta\in\Delta} \chi(\delta)\delta^{-1} \in \mathbb{Z}_p[\chi][\Delta]$. Multiplication of $\xi_{\Delta,\chi}$ defines an endomorphism of $M \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[\chi]$, which induces a $\mathbb{Z}_p[\chi]$-homomorphism

$$\xi^*_{\Delta,\chi} : M_{\chi} \longrightarrow M^{\chi}.$$

We define a quotient module $\widetilde{M}^{\chi}$ of M^{χ} and a submodule $\widetilde{M}_{\chi}$ of M_{χ} by

$$\widetilde{M}^{\chi} \cong M^{\chi}/\mathrm{Im}(\xi_{\Delta,\chi}) = \mathrm{coker}(\xi^*_{\Delta,\chi})$$

and

$$\widetilde{M}_{\chi} \cong \ker(\xi_{\Delta,\chi})/I_{\chi}(M \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[\chi]) = \ker(\xi^*_{\Delta,\chi}).$$

The following lemmas can be proved easily.

Lemma 2.1. *Assume that Δ is a cyclic group and*

$$0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0$$

is an exact sequence of $\mathbb{Z}_p[\Delta]$-modules. We have an exact sequence

$$0 \longrightarrow M_1^{\chi} \longrightarrow M_2^{\chi} \longrightarrow M_3^{\chi} \longrightarrow M_{1,\chi} \longrightarrow M_{2,\chi} \longrightarrow M_{3,\chi} \longrightarrow 0.$$

Furthermore

$$0 \longrightarrow M_1^\chi \longrightarrow M_2^\chi \longrightarrow M_3^\chi \longrightarrow \widetilde{M}_{1,\chi} \longrightarrow \widetilde{M}_{2,\chi} \longrightarrow \widetilde{M}_{3,\chi}$$

and

$$\widetilde{M}_1^\chi \longrightarrow \widetilde{M}_2^\chi \longrightarrow \widetilde{M}_3^\chi \longrightarrow M_{1,\chi} \longrightarrow M_{2,\chi} \longrightarrow M_{3,\chi} \longrightarrow 0$$

are also exact.

Lemma 2.2 ([11, Lemma II.2])**.** *Assume χ to be a faithful character of a cyclic group Δ of p-power order. We denote by C the subgroup of the order p in Δ, and N_C its norm in $\mathbb{Z}_p[\Delta]$. For any $\mathbb{Z}_p[\Delta]$-module M, there are $\mathbb{Z}_p[\Delta]$-isomorphisms*:

$$M^\chi \cong \ker(N_C : M \longrightarrow M) \quad \textit{and} \quad M_\chi \cong \operatorname{coker}(N_C : M \longrightarrow M).$$

Furthermore, we have $\mathbb{Z}_p[\Delta]$-isomorphisms:

$$\widetilde{M}^\chi \cong \widehat{H}^{-1}(C, M) \quad \textit{and} \quad \widetilde{M}_\chi \cong \widehat{H}^0(C, M).$$

Corollary 2.3. *Under the same assumption as in Lemma 2.2, if*

$$0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0$$

is an exact sequence of $\mathbb{Z}_p[\Delta]$-modules, then

$$0 \longrightarrow M_1^\chi \longrightarrow M_2^\chi \longrightarrow M_3^\chi \longrightarrow \widehat{H}^0(C, M_1) \longrightarrow \widehat{H}^0(C, M_2) \longrightarrow \widehat{H}^0(C, M_3)$$

and

$$\widehat{H}^{-1}(C, M_1) \to \widehat{H}^{-1}(C, M_2) \to \widehat{H}^{-1}(C, M_3) \to M_{1,\chi} \to M_{2,\chi} \to M_{3,\chi} \to 0$$

are exact.

3. The main results

For natural number t, let ζ_t be a primitive t-th root of unity with the property that $\zeta_{st}^s = \zeta_t$ for all $s \geq 1$, and we denote by $\boldsymbol{\mu}_t$ the group of t-th roots of unity. Put $\boldsymbol{\mu}_{2^\infty} = \bigcup \boldsymbol{\mu}_{2^n}$. We shall often denote $\mathbb{Q}(\zeta_t)$ by $\mathbb{Q}(t)$.

Put $\mathbb{Q}_n = \mathbb{Q}(\zeta_{2^{n+2}} + \zeta_{2^{n+2}}^{-1})$ for $n \geq 0$ and $\mathbb{Q}_\infty = \bigcup_n \mathbb{Q}_n$. Then $\mathbb{Q}_\infty$ is the cyclotomic $\mathbb{Z}_2$-extension of $\mathbb{Q}$. Let k be a finite abelian extension of $\mathbb{Q}$ of the first kind, that is, the conductor of k is not divisible by 8. We study both cases where k is real and imaginary. Put $k_n = k\mathbb{Q}_n$ for $0 \leq n \leq \infty$, hence k_∞ is the cyclotomic $\mathbb{Z}_2$-extension of k with n-th layer k_n. We can see that for any finite abelian extension k' of $\mathbb{Q}$, there exists an abelian field k of the first kind such that $k'_\infty = k'\mathbb{Q}_\infty = k_\infty$.

Let $\wp$ be a prime ideal of k lying above 2, and $\wp_n$ the unique prime ideal of k_n lying above $\wp$. We denote $U_{k_n,\wp}$ the principal units in the completion $k_{n,\wp}$ of k_n at $\wp_n$. Put

$$\mathcal{U}_n = \mathcal{U}_{k_n} = \prod_{\wp | 2} U_{k_n,\wp}$$

where $\wp$ runs over all prime ideals of k lying above 2, which is called the group of semi-local units of k_n at 2. Then $\mathcal{U}_{k_n}$ is a $\mathbb{Z}_2[\mathrm{Gal}(k_n/\mathbb{Q})]$-module. Let C_{k_n} be a group of cyclotomic units of k_n defined in Section 5. We identify C_{k_n} with its image under the diagonal embedding $k_n^\times \to \prod k_{n,\wp}^\times = (k_n \otimes \mathbb{Q}_2)^\times$. Let $\mathcal{C}_{k_n}$ be the closure of the intersection $\mathcal{U}_{k_n} \cap C_{k_n}$ in $\mathcal{U}_{k_n}$. Then $\mathcal{C}_{k_n}$ is a closed $\mathbb{Z}_2[\mathrm{Gal}(k_n/\mathbb{Q})]$-submodule of $\mathcal{U}_{k_n}$. Put

$$\mathcal{U} = \mathcal{U}_{k_\infty} = \varprojlim \mathcal{U}_{k_n}, \quad \mathcal{C} = \mathcal{C}_{k_\infty} = \varprojlim \mathcal{C}_{k_n},$$

where the projective limits are taken with respect to the relative norms. Put $G = \mathrm{Gal}(k/\mathbb{Q})$ and $\Gamma = \mathrm{Gal}(k_\infty/k)$. Since we assume that k is of the first kind, we have isomorphisms $G \cong \mathrm{Gal}(k_n/\mathbb{Q}_n)$ $(0 \le n \le \infty)$, $\Gamma \cong \mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q})$ and $\mathrm{Gal}(k_\infty/\mathbb{Q}) \cong G \times \Gamma$. Therefore $\mathcal{U}$ and $\mathcal{C}$ are modules over the completed group ring $\mathbb{Z}_2[G][\![\Gamma]\!]$.

Let ψ be a non-trivial, even character of G whose values are in $\overline{\mathbb{Q}_2}^\times$. We define $\mathbb{Z}_2[\psi][\![\Gamma]\!]$-modules $\mathcal{U}^\psi/\mathcal{C}^\psi$ and $(\mathcal{U}/\mathcal{C})_\psi$ as in Section 2. Fixing a topological generator γ of Γ, we identify, as usual, the completed group ring $\mathbb{Z}_2[\psi][\![\Gamma]\!]$ with the formal power series ring $\Lambda = \mathbb{Z}_2[\psi][\![T]\!]$ by $\gamma = 1 + T$. We will investigate the structures of the Λ-modules $\mathcal{U}^\psi/\mathcal{C}^\psi$ and $(\mathcal{U}/\mathcal{C})_\psi$.

We regard ψ as a primitive Dirichlet character. Let $L_2(\psi, s)$ denote the Kubota–Leopoldt 2-adic L-function associated to ψ. We write the cyclotomic character by $\kappa : \mathrm{Gal}(\mathbb{Q}(\boldsymbol{\mu}_{2^\infty})/\mathbb{Q}) \to \mathbb{Z}_2^\times$ and the Teichmüller character by ω. We often regard ω as a character of $\mathrm{Gal}(\mathbb{Q}(\boldsymbol{\mu}_{2^{n+2}})/\mathbb{Q}_n)$. By the isomorphism $\Gamma \cong \mathrm{Gal}(\mathbb{Q}(\boldsymbol{\mu}_{2^\infty})/\mathbb{Q}(\zeta_4))$, we can regard κ as a character of Γ. It is known that there exists a unique power series $g_\psi(T)$ in 2Λ such that

$$g_\psi(\kappa(\gamma)^s - 1) = L_2(\psi, 1 - s)$$

for all $s \in \mathbb{Z}_2$. Let π be a uniformizing parameter for $\mathbb{Q}_2(\psi)$. For a power series $f(T) \neq 0$ in Λ, we can uniquely write

$$f(T) = \pi^{\mu'(f(T))} P(T) U(T)$$

where $\mu'(f(T))$ is a non-negative integer, $P(T)$ is a distinguished polynomial and $U(T)$ is a unit in Λ. We put $\mu(f(T)) = \mu'(f(T))/\mu'(2)$, which we call the μ-invariant of $f(T)$. By the Ferrero–Washington Theorem [3, 4], we know that

$$\mu(g_\psi(T)) = 1 \quad \text{or eauivalently} \quad \mu(g_\psi(T)/2) = 0.$$

For every Λ-module M, we write $\mathrm{char}_\Lambda(M)$ for the characteristic ideal of M and put $\mu(M) = \mu(\mathrm{char}_\Lambda(M))$, the μ-invariant of M. Put $\dot{T} = \kappa(\gamma)(1+T)^{-1} - 1 \in \Lambda$.

In Section 4, we will define Λ-homomorphisms $\Psi_{k_\infty}^\psi : \mathcal{U}_{k_\infty}^\psi \to \Lambda$ and $\Psi_{k_\infty,\psi} : \mathcal{U}_{k_\infty,\psi} \to \Lambda$. In our main theorem, we describe the structure of Λ-module $\mathcal{U}^\psi/\mathcal{C}^\psi$ (resp. $(\mathcal{U}/\mathcal{C})_\psi$) by using $\Psi_{k_\infty}^\psi$ (resp. $\Psi_{k_\infty,\psi}$) in terms of

2-adic L-function $g_\psi(T)$. Our main theorem about the structure of the Λ-module $\mathcal{U}^\psi/\mathcal{C}^\psi$ as follows:

Theorem 3.1.

(i) *If $\psi\omega^{-1}(2) \neq 1$, the Λ-homomorphism $\Psi^\psi_{k_\infty}$ gives an isomorphism:*

$$\mathcal{U}^\psi/\mathcal{C}^\psi \cong \Lambda/(g_\psi(T)/2).$$

(ii) *If $\psi\omega^{-1}(2) = 1$, we have an exact sequence of Λ-modules*

$$0 \longrightarrow \Lambda/(\dot{T}) \longrightarrow \mathcal{U}^\psi/\mathcal{C}^\psi \xrightarrow{\Psi^\psi_{k_\infty}} \Lambda/(g_\psi(T)/2\dot{T}) \longrightarrow 0.$$

In particular, we have

$$\mathrm{char}_\Lambda(\mathcal{U}^\psi/\mathcal{C}^\psi) = (g_\psi(T)/2), \quad \mu(\mathcal{U}^\psi/\mathcal{C}^\psi) = 0.$$

As in the case where p is odd prime, we prepare some notation which we need to state our results about the structure of $(\mathcal{U}/\mathcal{C})_\psi$. Let m (resp. f) be the odd part of the conductor of k (resp. ψ). Clearly $f \mid m$ and, by the assumption that k is of the first kind, the conductor of k (resp. ψ) is m or $4m$ (resp. f or $4f$). Furthermore, since ψ is non-trivial and even, we see that $f \neq 1$ and also $m \neq 1$. We define a finite set $\mathcal{L}$ of prime numbers as follows:

$$\mathcal{L} = \mathcal{L}_{k,\psi} = \{l : \text{prime number} \mid l \mid m, l \nmid f\}.$$

For a subset I of $\mathcal{L}$, we put $m_I = f\prod_{l\in I} l$ and

$$d_I = [\mathbb{Q}(\zeta_{4m_\mathcal{L}}) \cap k(\zeta_4) : \mathbb{Q}(\zeta_{4m_I}) \cap k(\zeta_4)].$$

If k is the cyclic extension of $\mathbb{Q}$ associated to ψ, then $\mathcal{L} = \varnothing$. For $x \in \mathbb{Z}_2^\times$, we denote by t_x the unique element in $\mathbb{Z}_2$ such that $x = \omega(x)\kappa(\gamma)^{t_x}$. We define an ideal of finite index in Λ as follows:

$$\mathfrak{A} = \mathfrak{A}_{k,\psi} = \left\langle d_I \prod_{l\in I}(1 - \psi(l)(1+T)^{t_l}) \;\middle|\; I \subset \mathcal{L} \right\rangle.$$

The quotient $\Lambda/\mathfrak{A}$ is finite since the generator for $I = \varnothing$ is a constant, but for $I = \mathcal{L}$ is not divisible by 2. We note that $\mathfrak{A} = \Lambda$ if $\mathcal{L} = \varnothing$. We put

$$T_{k,\psi} = \Lambda/(\dot{T}, e_{k,2}, \psi(2) + \psi\omega^{-1}(2) - 1)$$

where $e_{k,2}$ is the ramification index of 2 in k. Since k is of the first kind, $e_{k,2}$ is 1 or 2 and either $\psi(2) = 0$ or $\psi\omega^{-1}(2) = 0$ holds. In particular if 2 is unramified in k, then $T_{k,\psi}$ is trivial. The structure of Λ-module $(\mathcal{U}/\mathcal{C})_\psi$ depends on k being real or imaginary. Our main theorems about the structure of the Λ-module $(\mathcal{U}/\mathcal{C})_\psi$ are as follows:

Theorem 3.2. *Assume that k is a real abelian field. Then there is an ideal $\mathfrak{A}'$ of Λ satisfying that $\mathfrak{A} \supset \mathfrak{A}' \supset (\dot{T}, 2)\mathfrak{A}$ and the natural surjection $\Lambda \to T_{k,\psi}$ induces a surjection $s : \Lambda/\mathfrak{A}'(g_\psi(T)/2) \to T_{k,\psi}$. Furthermore, the following hold:*

(i) *If* $\psi\omega^{-1}(2) \neq 1$, *we have an exact sequence of* Λ-*modules*

$$0 \longrightarrow T_{k,\psi} \longrightarrow (\mathcal{U}/\mathcal{C})_\psi \xrightarrow{\Psi_{k_\infty,\psi}} \Lambda/\mathfrak{A}'(g_\psi(T)/2) \xrightarrow{s} T_{k,\psi} \longrightarrow 0.$$

(ii) *If* $\psi\omega^{-1}(2) = 1$, *we have an exact sequence of* Λ-*modules*

$$0 \longrightarrow \Lambda/(\dot{T}) \oplus T_{k,\psi} \longrightarrow (\mathcal{U}/\mathcal{C})_\psi \xrightarrow{\Psi_{k_\infty,\psi}} \Lambda/\mathfrak{A}'(g_\psi(T)/2\dot{T}) \longrightarrow 0.$$

In particular, we have

$$\mathrm{char}_\Lambda((\mathcal{U}/\mathcal{C})_\psi) = (g_\psi(T)/2), \quad \mu((\mathcal{U}/\mathcal{C})_\psi) = 0.$$

Remark 3.3. We will give a sufficient condition for $\mathfrak{A}' = \mathfrak{A}$ in Lemma 5.5 and Remark 6.1. In particular, if 2 is unramified in k, we can show that $\mathfrak{A}' = \mathfrak{A}$. See also Lemma 5.1.

Theorem 3.4. *Assume that* k *is an imaginary abelian field with maximal subfield* k^+. *Then there is an ideal* $\mathfrak{A}''$ *of* Λ *such that*

$$\mathfrak{A}_{k^+,\psi} \supset \mathfrak{A}'' \supset \mathfrak{A}_{k,\psi} + (\dot{T}, 2)\mathfrak{A}_{k^+,\psi}$$

and the natural surjection $\Lambda \to T_{k,\psi}$ *induces a surjection* $s : \Lambda/\mathfrak{A}''g_\psi(T) \to T_{k,\psi}$ *and the following hold:*

(i) *If* $\psi\omega^{-1}(2) \neq 1$, *we have an exact sequence of* Λ-*modules*

$$0 \longrightarrow T_{k,\psi} \longrightarrow (\mathcal{U}/\mathcal{C})_\psi \xrightarrow{\Psi_{k_\infty,\psi}} \Lambda/\mathfrak{A}''g_\psi(T) \xrightarrow{s} T_{k,\psi} \longrightarrow 0.$$

(ii) *If* $\psi\omega^{-1}(2) = 1$, *we have an exact sequence of* Λ-*modules*

$$0 \longrightarrow \Lambda/(\dot{T}) \oplus T_{k,\psi} \longrightarrow (\mathcal{U}/\mathcal{C})_\psi \xrightarrow{\Psi_{k_\infty,\psi}} \Lambda/\mathfrak{A}''(g_\psi(T)/\dot{T}) \longrightarrow 0.$$

In particular, we have

$$\mathrm{char}_\Lambda((\mathcal{U}/\mathcal{C})_\psi) = (g_\psi(T)), \quad \mu((\mathcal{U}/\mathcal{C})_\psi) = 1.$$

For both cases where k is real or imaginary, we have

$$\mathrm{char}_\Lambda((\mathcal{U}/\mathcal{C})_\psi) = ([k : k^+]g_\psi(T)/2)$$

where k^+ is the maximal real subfield of k. Composing $\Psi_{k_\infty,\psi}$ in Theorem 3.2 (resp. Theorem 3.4) with the canonical surjection

$$\Lambda/\mathfrak{A}'(g_\psi(T)/2) \longrightarrow \Lambda/(g_\psi(T)/2) \ (\mathrm{resp.}\Lambda/\mathfrak{A}''g_\psi(T) \longrightarrow \Lambda/(g_\psi(T)/2)),$$

we get a Λ-homomorphism

$$(\mathcal{U}/\mathcal{C})_\psi \longrightarrow \Lambda/(g_\psi(T)/2),$$

which we mentioned in the introduction. Theorem 3.2 and Theorem 3.4 determine the kernels and the cokernels of those homomorphisms. In particular, the μ-invariant of the kernel is 0 or 1 according to k is real or imaginary.

Let be $\mathcal{C}' = \mathcal{C}'_{k_\infty}$ the projective limit of cyclotomic unit groups of k_n in the sense of Sinnott defined in Section 5. We will also determine the structure of $(\mathcal{U}/\mathcal{C}')_\psi$ as follows:

Theorem 3.5. *Let k be an abelian field of the first kind. Then the natural surjection $\Lambda \to T_{k,\psi}$ induces a surjection $s : \Lambda/\mathfrak{A}g_\psi(T) \to T_{k,\psi}$ and the following hold:*

(i) *If $\psi\omega^{-1}(2) \neq 1$, we have an exact sequence of Λ-modules*

$$0 \longrightarrow T_{k,\psi} \longrightarrow (\mathcal{U}/\mathcal{C}')_\psi \xrightarrow{\Psi_{k_\infty,\psi}} \Lambda/\mathfrak{A}g_\psi(T) \xrightarrow{s} T_{k,\psi} \longrightarrow 0.$$

(ii) *If $\psi\omega^{-1}(2) = 1$, we have an exact sequence of Λ-modules*

$$0 \longrightarrow \Lambda/(\dot{T}) \oplus T_{k,\psi} \longrightarrow (\mathcal{U}/\mathcal{C}')_\psi \xrightarrow{\Psi_{k_\infty,\psi}} \Lambda/\mathfrak{A}(g_\psi(T)/\dot{T}) \longrightarrow 0.$$

In particular, we have

$$\mathrm{char}_\Lambda((\mathcal{U}/\mathcal{C}')_\psi) = (g_\psi(T)), \quad \mu((\mathcal{U}/\mathcal{C}')_\psi) = 1.$$

In [12], we proved the following for odd prime p.

Theorem 3.6 ([12, Theorem 3.3])**.** *Let p be an odd prime number. If $\psi\omega^{-1}(p) = 1$, there exists an exact sequence of Λ-modules*

$$0 \longrightarrow \Lambda/(\dot{T}) \longrightarrow (\mathcal{U}/\mathcal{C})_\psi \longrightarrow (\Lambda \oplus \Lambda/(d, \dot{T}))/\mathfrak{A}x_\psi \longrightarrow 0.$$

Here d is the order of the decomposition group of $\mathrm{Gal}(k/\mathbb{Q})$ and x_ψ the element $(g_\psi(T)/\dot{T}, -B_{1,\psi\omega^{-1}})$ of $\Lambda \oplus \Lambda/(d, \dot{T})$.

As the same method in the proof of Theorem 3.5, we can prove a modified version of this theorem as follows.

Theorem 3.7. *Let p be an odd prime number. If $\psi\omega^{-1}(p) = 1$, we have an exact sequence of Λ-modules*

$$0 \longrightarrow \Lambda/(\dot{T}) \longrightarrow (\mathcal{U}/\mathcal{C})_\psi \xrightarrow{\Psi_{k_\infty,\psi}} \Lambda/\mathfrak{A}(g_\psi(T)/\dot{T}) \longrightarrow 0.$$

4. Semi-local units

Recall that k is a finite abelian extension of $\mathbb{Q}$ of the first kind and m is the odd part of the conductor of k, hence the conductor of k is m or $4m$. Put $F = \mathbb{Q}(\zeta_m) \cap k(\zeta_4)$, which is an abelian extension of $\mathbb{Q}$ unramified at 2. Since the conductor of $k(\zeta_4)$ is $4m$, we have $k(\zeta_4)\mathbb{Q}(\zeta_m) = \mathbb{Q}(\zeta_{4m})$, so $[k(\zeta_4) : F] = [\mathbb{Q}(\zeta_{4m}) : \mathbb{Q}(\zeta_m)] = 2$. We see that $k(\zeta_4) \supset F(\zeta_4)$ and $[F(\zeta_4) : F] = 2$, hence $F(\zeta_4) = k(\zeta_4)$. If the conductor of k is m, that is, k is unramified at 2, we see that $k(\zeta_4) \supset F \supset k$ and $[k(\zeta_4) : k] = 2$, hence $k = F$. Put $K_n = F(\boldsymbol{\mu}_{2^{n+2}}) = k(\boldsymbol{\mu}_{2^{n+2}})$ for $n \geq 0$ and $K_\infty = F(\boldsymbol{\mu}_{2^\infty}) = k(\boldsymbol{\mu}_{2^\infty})$ Hence K_∞/K_0 is the cyclotomic $\mathbb{Z}_2$-extension. Put

$$\Delta = \mathrm{Gal}(F/\mathbb{Q}), \quad G_0 = \mathrm{Gal}(K_0/F), \quad G_\infty = \mathrm{Gal}(K_\infty/F).$$

Therefore we have

$$\mathrm{Gal}(K_\infty/\mathbb{Q}) \cong \Delta \times G_\infty, \quad G_\infty \cong G_0 \times \Gamma.$$

Recall that ψ is a non-trivial even character of $G = \mathrm{Gal}(k/\mathbb{Q})$. The Teichmüller character ω is the unique non-trivial character of G_0. We will regard ψ as a character of $\mathrm{Gal}(K_0/\mathbb{Q}) \cong \Delta \times G_0$ and let χ be the restriction of ψ to Δ. We can write

$$\psi = \chi\omega^i$$

with $i = 0$ or 1. Let D be the decomposition group of 2 in Δ and $\sigma \in \Delta$ the Frobenius element of 2, thus $D = \langle\sigma\rangle$. For any prime ideal $\wp$ of F lying above 2, let $F_\wp$ denote the completion of F at $\wp$. Let $\mathcal{O}_F$ (resp. $\mathcal{O}_{F_\wp}$) denote the integer ring of F (resp. $F_\wp$).

$$\widehat{\mathcal{O}}_F := \prod_{\wp|2} \mathcal{O}_{F_\wp} \cong \mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_2,$$

where $\wp$ runs over all prime ideals F lying above 2.

We recall the results of Coleman [1, 2]. For details, see [8, §7] and [13, §13.7–8]. We denote $\mathbb{Z}_2(1) = \varprojlim \boldsymbol{\mu}_{2^{n+2}}$ where the projective limit is taken with respect to the map $\boldsymbol{\mu}_{2^{n+2}} \to \boldsymbol{\mu}_{2^{n+1}}$ defined by $\zeta \mapsto \zeta^2$ for $\zeta \in \boldsymbol{\mu}_{2^{n+2}}$. We fix a generator $(\zeta_{2^{n+2}})_{n\geq 0}$ of $\mathbb{Z}_2(1)$, so $\zeta_{2^{n+2}}^2 = \zeta_{2^{n+1}}$ for $n \geq 1$. For a $\mathbb{Z}_2[\![G_\infty]\!]$-module M, we put $M(1) = M \otimes_{\mathbb{Z}_2} \mathbb{Z}_2(1)$. For $u = (u_n) \in \mathcal{U}_{K_\infty} = \varprojlim \mathcal{U}_{K_n}$, there exists a unique power series $f_u(X) \in \widehat{\mathcal{O}}_F[\![X]\!]$ satisfying

$$f_u(1 - \zeta_{2^{n+2}}) = (u_n)^{\sigma^n},$$

which is called Coleman's power series associated to u. Let

$$D = (1 - X)\frac{\mathrm{d}}{\mathrm{d}X}$$

be the derivative operator on $\widehat{\mathcal{O}}_F[\![X]\!]$. Define the endomorphism φ of $\widehat{\mathcal{O}}_F[\![X]\!]$ by

$$(\varphi f)(X) = \sigma(f(1 - (1 - X)^2))$$

where σ acts on $\widehat{\mathcal{O}}_F[\![X]\!]$ via the coefficients. We can extend a power of the cyclotomic character $\kappa^k : G_\infty \to \mathbb{Z}_2^\times$ to a ring homomorphism $\widehat{\mathcal{O}}_F[\![G_\infty]\!] \to \widehat{\mathcal{O}}_F$ linearly for $k \in \mathbb{N}$. For $u \in \mathcal{U}_{K_\infty}$, there exists a unique element $\Psi_{K_\infty}(u)$ in $\widehat{\mathcal{O}}_F[\![G_\infty]\!]$ satisfying

$$D^k \left(1 - \frac{\varphi}{2}\right) \log f_u(X)\bigg|_{X=0} = (-\kappa)^k(\Psi_{K_\infty}(u)), \tag{4.1}$$

which defines a $\mathbb{Z}_2[\Delta][\![G_\infty]\!]$-homomorphism $\Psi_{K_\infty} : \mathcal{U}_{K_\infty} \to \widehat{\mathcal{O}}_F[\![G_\infty]\!]$. $U_{K_n,\wp}$ contains $\boldsymbol{\mu}_{2^{n+2}}$ and $\varprojlim U_{K_n,\wp}$ contains $\varprojlim \boldsymbol{\mu}_{2^{n+2}}$ where the projective limit is taken with respect to the norm map $N_{n,n-1}$ from $U_{K_n,\wp}$ to $U_{K_{n-1},\wp}$. We see that $N_{n,n-1}(-\zeta_{2^{n+2}}) = (-\zeta_{2^{n+2}})(\zeta_{2^{n+2}}) = -\zeta_{2^{n+2}}^2 = -\zeta_{2^{n+1}}$. Then the following diagram is commutative

$$\begin{array}{ccc} \boldsymbol{\mu}_{2^{n+2}} & \xrightarrow{f_n} & \boldsymbol{\mu}_{2^{n+2}} \\ \downarrow 2 & & \downarrow N_{n,n-1} \\ \boldsymbol{\mu}_{2^{n+1}} & \xrightarrow{f_{n-1}} & \boldsymbol{\mu}_{2^{n+1}} \end{array}$$

where $f_n(\zeta_{2^{n+2}}^a) = (-\zeta_{2^{n+2}})^a$ for $a \in \mathbb{Z}$. Hence the corresponding

$$(\zeta_{2^{n+2}})_{n\geq 0} \longmapsto (-\zeta_{2^{n+2}})_{n\geq 0}$$

defines an injection $\mathbb{Z}_2(1) \to \varprojlim U_{K_{n,\wp}}$, which induces a homomorphism

$$\iota : \mathbb{Z}_2[\Delta/D](1) = \prod_{\wp | 2} \mathbb{Z}_2(1) \longrightarrow \prod_{\wp | 2} \varprojlim U_{K_{n,\wp}} = \mathcal{U}_{K_\infty}$$

of $\mathbb{Z}_2[\Delta][\![G_\infty]\!]$-modules. The cyclotomic character κ induces a $\mathbb{Z}_2[\Delta][\![G_\infty]\!]$-homomorphism $-\kappa : \widehat{\mathcal{O}}_F[\![G_\infty]\!] \to (\widehat{\mathcal{O}}_F/(\sigma-1)\widehat{\mathcal{O}}_F)(1)$. The following is known (cf. [2, Theorem 4] and [8, Theorem 2.8, Proposition 2.10]):

Theorem 4.1 (Coleman)**.** *Let F be a finite abelian extension of $\mathbb{Q}$ unramified at 2 and put $K_\infty = F(\boldsymbol{\mu}_{2^\infty})$. There is an exact sequence of $\mathbb{Z}_2[\Delta][\![G_\infty]\!]$-modules*

$$0 \longrightarrow \mathbb{Z}_2[\Delta/D](1) \xrightarrow{\iota} \mathcal{U}_{K_\infty} \xrightarrow{\Psi_{K_\infty}} \widehat{\mathcal{O}}_F[\![G_\infty]\!] \xrightarrow{-\kappa} (\widehat{\mathcal{O}}_F/(\sigma-1)\widehat{\mathcal{O}}_F)(1) \longrightarrow 0.$$

For $y \in \widehat{\mathcal{O}}_F$, we denote by $y_{F,\chi}$ or y_χ its image under the natural surjection $\widehat{\mathcal{O}}_F \twoheadrightarrow \widehat{\mathcal{O}}_{F,\chi}$. We shall often consider an element of y of $\widehat{\mathcal{O}}_F$ as an element of $\widehat{\mathcal{O}}_F \otimes_{\mathbb{Z}_2} \mathbb{Z}_2[\chi]$. Since 2 is unramified in $F/\mathbb{Q}$, we have $\widehat{\mathcal{O}}_F \cong \mathbb{Z}_2[\Delta]$ as $\mathbb{Z}_2[\Delta]$-modules. Therefore $\widehat{\mathcal{O}}_F^\chi \cong \mathbb{Z}_2[\chi]$ and $\widehat{\mathcal{O}}_{F,\chi} \cong \mathbb{Z}_2[\chi]$. We fix these isomorphisms as follows:

Lemma 4.2.

(a) *The additive group $\widehat{\mathcal{O}}_F^\chi$ is a free $\mathbb{Z}_2[\chi]$-module of rank one generated by $z^\chi = \xi_\chi(\mathrm{Tr}_{\mathbb{Q}(f)/\mathbb{Q}(f)\cap F}(\zeta_f))$ where $\xi_\chi = \sum_\delta \chi(\delta)\delta^{-1}$, δ running over all elements in* $\mathrm{Gal}(\mathbb{Q}(f) \cap F/\mathbb{Q})$. *Further, for all $a \in \mathbb{N}$, we have $\xi_\chi(\mathrm{Tr}_{\mathbb{Q}(f)/\mathbb{Q}(f)\cap F}(\zeta_f^a)) = \chi(a)z^\chi$.*

(b) *The additive group $\widehat{\mathcal{O}}_{F,\chi}$ is a free $\mathbb{Z}_2[\chi]$-module of rank one generated by $z_{F,\chi} = [\mathbb{Q}(m) : \mathbb{Q}(m_\mathcal{L})]^{-1}(\prod_{l\in\mathcal{L}} -\chi(l))(\mathrm{Tr}_{\mathbb{Q}(m)/F}(\zeta_{m_\mathcal{L}}))_{F,\chi}$. Further, if the conductor of F is f, we have $(\mathrm{Tr}_{\mathbb{Q}(f)/F}(\zeta_f^a))_{F,\chi} = \chi(a)z_{F,\chi}$ for all $a \in \mathbb{N}$.*

Proof. The statement (a) is exactly [12, Lemma 5.1 (a)] for $p = 2$, which can be proved similarly. Although (b) can be induced by [12, Lemma 5.1 (b)] for $p = 2$, we prove this directly. Since $\widehat{\mathcal{O}}_F \cong \mathbb{Z}_2[\Delta]$, the map $\xi_{\Delta,\chi}^* : \widehat{\mathcal{O}}_{F,\chi} \to \widehat{\mathcal{O}}_F^\chi$ gives an isomorphism where $\xi_{\Delta,\chi} = \sum_{\delta\in\Delta} \chi(\delta)\delta^{-1} \in \mathbb{Z}_p[\chi][\Delta]$ and $\xi_{\Delta,\chi}^*$ is the $\mathbb{Z}_2[\chi]$-homomorphism induced by $\xi_{\Delta,\chi}$ (see Section 2). Since $\xi_{\Delta,\chi} = \xi_\chi \operatorname{Tr}_{F/\mathbb{Q}(f)\cap F}$, we have

$$\xi^*_{\Delta,\chi}(z_{F,\chi}) = \xi_{\Delta,\chi}\left([\mathbb{Q}(m):\mathbb{Q}(m_{\mathcal{L}})]^{-1}\left(\prod_{l\in\mathcal{L}} -\chi(l)\right)\operatorname{Tr}_{\mathbb{Q}(m)/F}(\zeta_{m_{\mathcal{L}}})\right)$$
$$= [\mathbb{Q}(m):\mathbb{Q}(m_{\mathcal{L}})]^{-1}\left(\prod_{l\in\mathcal{L}} -\chi(l)\right)\xi_\chi(\operatorname{Tr}_{\mathbb{Q}(m)/\mathbb{Q}(f)\cap F}(\zeta_{m_{\mathcal{L}}}))$$
$$= [\mathbb{Q}(m):\mathbb{Q}(m_{\mathcal{L}})]^{-1}\left(\prod_{l\in\mathcal{L}} -\chi(l)\right)\xi_\chi(\operatorname{Tr}_{\mathbb{Q}(m_{\mathcal{L}})/\mathbb{Q}(f)\cap F}(\operatorname{Tr}_{\mathbb{Q}(m)/\mathbb{Q}(m_{\mathcal{L}})}(\zeta_{m_{\mathcal{L}}})))$$
$$= \left(\prod_{l\in\mathcal{L}} -\chi(l)\right)\xi_\chi(\operatorname{Tr}_{\mathbb{Q}(m_{\mathcal{L}})/\mathbb{Q}(f)\cap F}(\zeta_{m_{\mathcal{L}}}))$$
$$= \left(\prod_{l\in\mathcal{L}} -\chi(l)\right)\xi_\chi(\operatorname{Tr}_{\mathbb{Q}(f)/\mathbb{Q}(f)\cap F}\operatorname{Tr}_{\mathbb{Q}(m_{\mathcal{L}})/\mathbb{Q}(f)}(\zeta_{m_{\mathcal{L}}}))$$
$$= \left(\prod_{l\in\mathcal{L}} -\chi(l)\right)\xi_\chi\left(\operatorname{Tr}_{\mathbb{Q}(f)/\mathbb{Q}(f)\cap F}\left(\prod_{l\in\mathcal{L}} -\sigma_l^{-1}\right)(\zeta_f)\right)$$
$$= \xi_\chi(\operatorname{Tr}_{\mathbb{Q}(f)/\mathbb{Q}(f)\cap F}(\zeta_f)) = z^\chi.$$

Here σ_l is the Frobenius element of l in $\operatorname{Gal}(\mathbb{Q}(f)/\mathbb{Q})$. Therefore (b) follows from (a). □

Remark 4.3. Lemma 4.2 holds also for odd prime p.

We can extend the character $\omega^i : G_0 \to \mathbb{Z}_2^\times$ to ring homomorphisms $\omega^i : \widehat{\mathcal{O}}^\chi_F[G_0][\![\Gamma]\!] \to \widehat{\mathcal{O}}^\chi_F[\![\Gamma]\!]$ and $\omega^i : \widehat{\mathcal{O}}_{F,\chi}[G_0][\![\Gamma]\!] \to \widehat{\mathcal{O}}_{F,\chi}[\![\Gamma]\!]$ linearly. For $y \in \widehat{\mathcal{O}}_F[G_0][\![\Gamma]\!]$, we also denote by $y_{F,\chi}$ or y_χ its image under the natural surjection $\widehat{\mathcal{O}}_F[G_0][\![\Gamma]\!] \twoheadrightarrow \widehat{\mathcal{O}}_{F,\chi}[G_0][\![\Gamma]\!]$.

Proposition 4.4. *We write $\psi = \chi\omega^i$ as above.*

(a) *We can define Λ-homomorphism $\Psi^\psi_{k_\infty} : \mathcal{U}^\psi_{k_\infty} \to \Lambda$ by*

$$\Psi^\psi_{k_\infty}(u^\psi)z^\chi = \frac{1}{2}\omega^i(\Psi_{K_\infty}(u^\psi))$$

for $u^\psi \in \mathcal{U}^\psi_{k_\infty}$ where we regard u^ψ as an element of $\mathcal{U}_{K_\infty}$.

(b) *Put $K_\infty = k_\infty(\zeta_4) = k(\boldsymbol{\mu}_{2^\infty})$. We can define Λ-homomorphism $\Psi_{k_\infty,\psi} : \mathcal{U}_{k_\infty,\psi} \to \Lambda$ by*

$$\Psi_{k_\infty,\psi}(u)z_{F,\chi} = \frac{1}{[K_\infty : k_\infty]}\omega^i(\Psi_{K_\infty}(\widetilde{u})_{F,\chi})$$

for $u \in \mathcal{U}_{k_\infty,\psi}$ *where* $\widetilde{u}$ *denotes a representative of* u *in* $\mathcal{U}_{k_\infty}$ *and we regard* $\widetilde{u}$ *as an element of* $\mathcal{U}_{K_\infty}$.

Proof. We simply write Ψ for Ψ_{K_∞}. Put $K_0 = F(\zeta_4) = k(\zeta_4)$ and let τ be a generator of $G_0 = \mathrm{Gal}(K_0/F)$.

Since $\Psi(u^\psi) \in (\widehat{\mathcal{O}}_F[G_0][\![\Gamma]\!])^\psi = (\widehat{\mathcal{O}}_F^\chi[G_0][\![\Gamma]\!])^{\omega^i} = (1+\tau\omega^i(\tau))\widehat{\mathcal{O}}_F^\chi[\![\Gamma]\!]$, we have $\omega^i(\Psi(u^\psi)) \in 2\widehat{\mathcal{O}}_F^\chi[\![\Gamma]\!]$. By Lemma 4.2 (a), there exists an element $\Psi_{k_\infty}^\psi(u^\psi) \in \mathbb{Z}_2[\chi][\![\Gamma]\!] = \Lambda$ such that $\Psi_{k_\infty}^\psi(u^\psi)z^\chi = \frac{1}{2}\omega^i(\Psi(u^\psi))$, which proves (a).

Assume that $k_\infty \neq K_\infty$ i.e. $\zeta_4 \notin k$. Let g denote a generator of $\mathrm{Gal}(K_0/k)$. The restriction $(\tau g)|_{\mathbb{Q}(\zeta_4)}$ of $\tau g \in \mathrm{Gal}(K_0/\mathbb{Q})$ to $\mathbb{Q}(\zeta_4)$ is trivial, since $\tau g(\zeta_4) = \tau(g(\zeta_4)) = \tau(-\zeta_4) = \zeta_4$. For $y \in \widehat{\mathcal{O}}_F[G_0][\![\Gamma]\!]$ and $h \in \mathrm{Gal}(K_0/\mathbb{Q}) = G_0 \times \Delta$, we see that $h(y_\chi) = (hy)_\chi = h|_{\mathbb{Q}(\zeta_4)}\chi(h)y_\chi$. Therefore we have

$$\begin{aligned}\tau(\Psi(\widetilde{u})_\chi) &= (\tau\Psi(\widetilde{u}))_\chi = \Psi(\widetilde{u}^\tau)_\chi = \Psi(\widetilde{u}^{\tau g})_\chi = ((\tau g)\Psi(\widetilde{u}))_\chi \\ &= (\tau g)|_{\mathbb{Q}(\zeta_4)}\chi(\tau g)\Psi(\widetilde{u})_\chi \ = \chi(\tau g)\Psi(\widetilde{u})_\chi\end{aligned}$$

since $\widetilde{u} \in \mathcal{U}_{k_\infty}$. On the other hand, regarding χ, ω^i and ψ as characters of $\mathrm{Gal}(K_0/\mathbb{Q})$, we have

$$\chi(\tau g) = \chi(\tau g)\omega^i(\tau g) = \psi(\tau g) = \psi(\tau)\psi(g) = \psi(\tau) = \chi(\tau)\omega^i(\tau) = \omega^i(\tau).$$

Then we have

$$\tau(\Psi(\widetilde{u})_\chi) = \omega^i(\tau)(\Psi(\widetilde{u})_\chi)$$

and $\Psi(\widetilde{u})_\chi \in \widehat{\mathcal{O}}_{F,\chi}[G_0][\![\Gamma]\!]$ is in $(\widehat{\mathcal{O}}_{F,\chi}[G_0][\![\Gamma]\!])^{\omega^i} = (1+\tau\omega^i(\tau))\widehat{\mathcal{O}}_{F,\chi}[\![\Gamma]\!]$. Hence $\omega^i(\Psi(\widetilde{u})_\chi) \in 2\widehat{\mathcal{O}}_{F,\chi}[\![\Gamma]\!] = [K_\infty : k_\infty]\widehat{\mathcal{O}}_{F,\chi}[\![\Gamma]\!]$. By Lemma 4.2 (b), there exists an element $\Psi_{k_\infty,\psi}(u) \in \mathbb{Z}_2[\chi][\![\Gamma]\!] = \Lambda$ such that $\Psi_{k_\infty,\psi}(u)z_{F,\chi} = \frac{1}{[K_\infty:k_\infty]}\omega^i(\Psi(\widetilde{u})_\chi)$, which proves (b) if $k_\infty \neq K_\infty$. The statement (b) in the case where $k_\infty = K_\infty$ is clear. □

If p is odd, $[k_\infty(\zeta_p) : k_\infty]$ is a divisor of $p-1$ and $\mathbb{Z}_p[G_0] = \oplus_{i=0}^{p-2} e_i\mathbb{Z}_p[G_0] \cong \oplus_{i=0}^{p-2}\mathbb{Z}_p$ where $e_i \in \mathbb{Z}_p[G_0]$ is the idempotent of ω^i, we can also prove the following similarly:

Proposition 4.5. *Assume* p *is an odd prime. We write* $\psi = \chi\omega^i$ *where* $0 \le i \le p-2$ *and the conductor of* χ *is prime to* p.

(a) *We can define* Λ*-homomorphism* $\Psi_{k_\infty}^\psi : \mathcal{U}_{k_\infty}^\psi \to \Lambda$ *by*

$$\Psi_{k_\infty}^\psi(u^\psi)z^\chi = \omega^i(\Psi_{K_\infty}(u^\psi))$$

for $u^\psi \in \mathcal{U}_{k_\infty}^\psi$ *where we regard* u^ψ *as an element of* $\mathcal{U}_{K_\infty}$.

(b) *We can define* Λ*-homomorphism* $\Psi_{k_\infty,\psi} : \mathcal{U}_{k_\infty,\psi} \to \Lambda$ *by*

$$\Psi_{k_\infty,\psi}(u)z_{F,\chi} = \omega^i(\Psi_{K_\infty}(\widetilde{u})_{F,\chi})$$

for $u \in \mathcal{U}_{k_\infty,\psi}$ *where* $\widetilde{u}$ *denotes a representative of* u *in* $\mathcal{U}_{k_\infty}$ *and we regard* $\widetilde{u}$ *as an element of* $\mathcal{U}_{K_\infty}$.

We return the case $p = 2$. By the formula (4.1) and Proposition 4.4, we have

$$(4.2)\qquad (-\kappa)^k(\Psi^{\psi}_{k_\infty}(u^{\psi}))z^{\chi} = \frac{1}{2}D^k\left(1-\frac{\varphi}{2}\right)\log(f_{u^{\psi}}(X))\Big|_{X=0},$$

$$(4.3)\quad (-\kappa)^k(\Psi_{k_\infty,\psi}(u))z_{F,\chi} = \frac{1}{[K_\infty : k_\infty]}\left(D^k\left(1-\frac{\varphi}{2}\right)\log f_{\tilde{u}}(X)\Big|_{X=0}\right)_{F,\chi}$$

for $k \equiv i \mod 2$.

We often write $\Psi = \Psi_{K_\infty}$, $\Psi^{\psi} = \Psi^{\psi}_{k_\infty}$ and $\Psi_{\psi} = \Psi_{k_\infty,\psi}$ simply. We will prove the following propositions.

Proposition 4.6.

(i) *If* $\psi\omega^{-1}(2) \neq 1$,

$$\Psi^{\psi}_{k_\infty} : \mathcal{U}^{\psi}_{k_\infty} \xrightarrow{\sim} \Lambda$$

is an isomorphism.

(ii) *If* $\psi\omega^{-1}(2) = 1$, *we have an exact sequence of* Λ*-modules*

$$0 \longrightarrow \Lambda/(\dot{T}) \longrightarrow \mathcal{U}^{\psi}_{k_\infty} \xrightarrow{\Psi^{\psi}_{k_\infty}} \dot{T}\Lambda \longrightarrow 0$$

where the first map induced by ι.

Proposition 4.7. *Let* $T_{k,\psi}$ *be* $\Lambda/(\dot{T}, e_{k,2}, \psi(2) + \psi\omega^{-1}(2) - 1)$ *where* $e_{k,2}$ *is the ramification index of* 2 *in* k.

(i) *If* $\psi\omega^{-1}(2) \neq 1$, *we have an exact sequence of* Λ*-modules*

$$0 \longrightarrow T_{k,\psi} \longrightarrow \mathcal{U}_{k_\infty,\psi} \xrightarrow{\Psi_{k_\infty,\psi}} \Lambda \longrightarrow T_{k,\psi} \longrightarrow 0,$$

where the first map induced by ι *and the last map is a natural surjection.*

(ii) *If* $\psi\omega^{-1}(2) = 1$, *we have an exact sequence of* Λ*-modules*

$$0 \longrightarrow \Lambda/(\dot{T}) \oplus T_{k,\psi} \longrightarrow \mathcal{U}_{k_\infty,\psi} \xrightarrow{\Psi_{k_\infty,\psi}} \dot{T}\Lambda \longrightarrow 0$$

where the first map induced by ι.

Proofs of Propositions 4.6 and 4.7. Recall that $F = \mathbb{Q}(\zeta_m) \cap k(\zeta_4)$ where m is the odd part of the conductor of k and $K_n = F(\boldsymbol{\mu}_{2^{n+2}}) = k(\boldsymbol{\mu}_{2^{n+2}})$ for $0 \leq n \leq \infty$. Let τ be a generator of $G_0 = \mathrm{Gal}(K_0/F)$.

The case $k_\infty = K_\infty$. Suppose that $k_\infty = K_\infty$, equivalently $\zeta_4 \in k$ or $k = F(\zeta_4)$. By Theorem 4.1, we have a Λ-homomorphism $\Psi^* : \mathcal{U}^{\psi}_{K_\infty} \to (\widehat{\mathcal{O}}_F[\![G_\infty]\!])^{\psi}$ by restricting Ψ to $\mathcal{U}^{\psi}_{K_\infty}$. By Lemma 4.2, we have

$$(\widehat{\mathcal{O}}_F[\![G_\infty]\!])^{\psi} = (\widehat{\mathcal{O}}^{\chi}_F[G_0])^{\omega^i}[\![\Gamma]\!] = \Lambda(1 + \tau\omega^i(\tau))z^{\chi}.$$

Then the Λ-homomorphism $\Psi^{\psi}_{K_\infty} : \mathcal{U}^{\psi}_{K_\infty} \to \Lambda$ in Proposition 4.4 coincides with the composition map of $\Psi^* : \mathcal{U}^{\psi}_{K_\infty} \to (\widehat{\mathcal{O}}_F[\![G_\infty]\!])^{\psi}$ and the isomorphism

$(\widehat{\mathcal{O}}_F[\![G_\infty]\!])^\psi \xrightarrow{\sim} \Lambda$ given by $(1+\tau\omega^i(\tau))z^\chi \to 1$. Hence, to prove Proposition 4.6, we have to decide the kernel and the cokernel of Ψ^*. The homomorphism Ψ in Theorem 4.1 induces a Λ-homomorphism $\Psi_* : \mathcal{U}_{K_\infty,\psi} \to (\widehat{\mathcal{O}}_F[\![G_\infty]\!])_\psi$ naturally and the surjection map $\omega^i : \widehat{\mathcal{O}}_{F,\chi}[G_0] \to \widehat{\mathcal{O}}_{F,\chi}$ induces an isomorphism $(\widehat{\mathcal{O}}_{F,\chi}[G_0])_{\omega^i} \xrightarrow{\sim} \widehat{\mathcal{O}}_{F,\chi}$. By Lemma 4.2, we have $\widehat{\mathcal{O}}_{F,\chi}[\![\Gamma]\!] = \Lambda z_{F,\chi}$. Then the Λ-homomorphism $\Psi_{K_\infty,\psi} : \mathcal{U}_{K_\infty,\psi} \to \Lambda$ in Proposition 4.4 coincides with the composition map of $\Psi_* : \mathcal{U}_{K_\infty,\psi} \to (\widehat{\mathcal{O}}_F[\![G_\infty]\!])_\psi$, the isomorphism $(\widehat{\mathcal{O}}_F[\![G_\infty]\!])_\psi \xrightarrow{\sim} \widehat{\mathcal{O}}_{F,\chi}[\![\Gamma]\!]$ induced by ω^i and the isomorphism $\widehat{\mathcal{O}}_{F,\chi}[\![\Gamma]\!] \xrightarrow{\sim} \Lambda$ given by $z_{F,\chi} \mapsto 1$. Hence, to prove Proposition 4.7, we have to decide the kernel and the cokernel of Ψ_*.

Let H be the kernel of $\chi : \Delta \to \overline{\mathbb{Q}_2}^\times$ and M the fixed field of H. Put $L_n = M(\boldsymbol{\mu}_{2^{n+2}})$ for $n \geq 0$ and $L_\infty = M(\boldsymbol{\mu}_{2^\infty})$. Then we have $\mathcal{U}_{K_\infty}^\psi = (\mathcal{U}_{K_\infty}^H)^\psi = \mathcal{U}_{L_\infty}^\psi$ and $\widehat{\mathcal{O}}_F^\chi = (\widehat{\mathcal{O}}_F^H)^\chi = \widehat{\mathcal{O}}_M^\chi$. Furthermore the generators of $\widehat{\mathcal{O}}_F^\chi$ and $\widehat{\mathcal{O}}_M^\chi$ in Lemma 4.2 coincide. Therefore we have $\Psi_{L_\infty}^\psi = \Psi_{K_\infty}^\psi$. Hence, to prove Proposition 4.6, we may assume that $K_\infty = L_\infty$, i.e. χ is a faithful character of Δ.

To prove Proposition 4.7, we consider the case where χ is not faithful, i.e. H is not trivial. The kernel and the cokernel of a map $\mathcal{U}_{K_n,H} \to \mathcal{U}_{L_n}$ induced by the norm map $N_H : K_n \to L_n$ are $\widehat{H}^{-1}(H,\mathcal{U}_{K_n})$ and $\widehat{H}^0(H,\mathcal{U}_{K_n})$ respectively. Since K_n/L_n is unramified extension at the prime ideals above 2, we have $\widehat{H}^{-1}(H,\mathcal{U}_{K_n}) = \widehat{H}^0(H,\mathcal{U}_{K_n}) = 0$. Hence we have $\mathcal{U}_{K_n,H} \xrightarrow{\sim} \mathcal{U}_{L_n}$. By using $\mathcal{U}_{K_n,\psi} = (\mathcal{U}_{K_n,H})_\psi$, we have an isomorphism $N_H^* : \mathcal{U}_{K_\infty,\psi} \xrightarrow{\sim} \mathcal{U}_{L_\infty,\psi}$. Similarly the trace map $\mathrm{Tr}_H : F \to M$ induces an isomorphism $\widehat{\mathcal{O}}_{F,H} \xrightarrow{\sim} \widehat{\mathcal{O}}_M$ since $\widehat{H}^{-1}(H,\widehat{\mathcal{O}}_F) = \widehat{H}^0(H,\widehat{\mathcal{O}}_F) = 0$. Then we have an isomorphism $\mathrm{Tr}_H^* : \widehat{\mathcal{O}}_{F,\chi} \xrightarrow{\sim} \widehat{\mathcal{O}}_{M,\chi}$ and we see that $\mathrm{Tr}_H(y)_{M,\chi} = \mathrm{Tr}_H^*(y_{F,\chi})$ for $y \in \widehat{\mathcal{O}}_F$. Recall that

$$\mathcal{L}_{k,\psi} = \{l : \text{prime number} \mid l \mid m, l \nmid f\}$$

where m (resp. f) is the odd part of the conductor of k (resp. ψ). Since the conductor of M is f, we have $\mathcal{L}_{M(\zeta_4),\psi} = \varnothing$. The generetors of the additive groups $\widehat{\mathcal{O}}_{F,\chi}$ and $\widehat{\mathcal{O}}_{M,\chi}$ in Lemma 4.2 are

$$z_{F,\chi} = [\mathbb{Q}(m) : \mathbb{Q}(m_{\mathcal{L}})]^{-1}\left(\prod_{l\in\mathcal{L}} -\chi(l)\right)(\mathrm{Tr}_{\mathbb{Q}(m)/F}(\zeta_{m_{\mathcal{L}}}))_{F,\chi}.$$

and

$$z_{M,\chi} = (\mathrm{Tr}_{\mathbb{Q}(f)/M}(\zeta_f))_{M,\chi}$$

respectively. Here we put $\mathcal{L} = \mathcal{L}_{k,\psi}$. We can see that

$$\begin{aligned}\mathrm{Tr}_H(\mathrm{Tr}_{\mathbb{Q}(m)/F}(\zeta_{m_\mathcal{L}})) &= \mathrm{Tr}_{\mathbb{Q}(m)/M}(\zeta_{m_\mathcal{L}}) \\ &= [\mathbb{Q}(m):\mathbb{Q}(m_\mathcal{L})]\,\mathrm{Tr}_{\mathbb{Q}(f)/M}(\mathrm{Tr}_{\mathbb{Q}(m_\mathcal{L})/\mathbb{Q}(f)}(\zeta_{m_\mathcal{L}})) \\ &= [\mathbb{Q}(m):\mathbb{Q}(m_\mathcal{L})]\,\mathrm{Tr}_{\mathbb{Q}(f)/M}\left(\left(\prod_{l\in\mathcal{L}} -\sigma_l^{-1}\right)(\zeta_f)\right)\end{aligned}$$

where σ_l is a Frobenius of l in $\mathrm{Gal}(\mathbb{Q}(\zeta_f)/\mathbb{Q})$, so we have

$$\begin{aligned}\mathrm{Tr}_H^*(z_{F,\chi}) &= [\mathbb{Q}(m):\mathbb{Q}(m_\mathcal{L})]^{-1}\left(\prod_{l\in\mathcal{L}} -\chi(l)\right)(\mathrm{Tr}_H(\mathrm{Tr}_{\mathbb{Q}(m)/F}(\zeta_{m_\mathcal{L}})))_{M,\chi} \\ &= (\mathrm{Tr}_{\mathbb{Q}(f)/M}(\zeta_f))_{M,\chi} = z_{M,\chi}.\end{aligned}$$

Then, for $u \in \mathcal{U}_{K_\infty,\psi}$, we have

$$\begin{aligned}\Psi_{K_\infty,\psi}(u)z_{M,\chi} &= \Psi_{K_\infty,\psi}(u)\,\mathrm{Tr}_H^*(z_{F,\chi}) = \mathrm{Tr}_H^*(\Psi_{K_\infty,\psi}(u)z_{F,\chi}) \\ &= \mathrm{Tr}_H^*(\omega^i(\Psi(\widetilde{u})_{F,\chi})) = \omega^i(\mathrm{Tr}_H^*(\Psi(\widetilde{u})_{F,\chi})) \\ &= \omega^i(\mathrm{Tr}_H(\Psi(\widetilde{u}))_{M,\chi}) = \omega^i(\Psi(N_H(\widetilde{u}))_{M,\chi}) \\ &= \omega^i(\Psi(\widetilde{N_H^*(u)})_{M,\chi}) = \Psi_{L_\infty,\psi}(N_H^*(u_\psi))z_{M,\chi},\end{aligned}$$

and therefore

$$\Psi_{K_\infty,\psi}(u) = \Psi_{L_\infty,\psi}(N_H^*(u)). \tag{4.4}$$

Hence, to prove Proposition 4.7, we may assume $K_\infty = L_\infty$.

In the rest of the proof in the case $k_\infty = K_\infty$, we assume that χ is a faithful character of Δ, that is, $F = M$, the fixed field of χ and $K_\infty = L_\infty = M(\boldsymbol{\mu}_{2^\infty})$.

We fix a prime ideal $\wp$ of F over 2, and put $U = U_{K_\infty,\wp} = \varprojlim U_{K_n,\wp}$. Then U is a $\mathbb{Z}_2[D][\![G_\infty]\!]$-module and we have $\mathbb{Z}_2[\Delta][\![G_\infty]\!]$-isomorphisms

$$\mathcal{U} \cong U \otimes_{\mathbb{Z}_2[D]} \mathbb{Z}_2[\Delta] \cong \mathrm{Hom}_{\mathbb{Z}_2[D]}(\mathbb{Z}_2[\Delta], U),$$

where D is the decomposition group of 2 in Δ. We put $\chi_D = \chi|_D$ and $\psi_D = \chi_D\omega^i$. Then we can define $\mathbb{Z}_2[\psi_D][\![\Gamma]\!]$-modules U^{ψ_D} and U_{ψ_D}, and the above isomorphisms induce Λ-isomorphisms

$$\mathcal{U}^\psi \cong U^{\psi_D} \otimes_{\mathbb{Z}_2[\psi_D]} \mathbb{Z}_2[\psi] \text{ and } \mathcal{U}_\psi \cong U_{\psi_D} \otimes_{\mathbb{Z}_2[\psi_D]} \mathbb{Z}_2[\psi]. \tag{4.5}$$

Theorem 4.1 is equivalent to the assertion that there is an exact sequence of $\mathbb{Z}_2[D][\![G_\infty]\!]$-modules

$$0 \longrightarrow \mathbb{Z}_2(1) \xrightarrow{\iota} U \xrightarrow{\Psi} \mathcal{O}_{F_\wp}[\![G_\infty]\!] \xrightarrow{-\kappa} (\mathcal{O}_{F_\wp}/(\sigma-1)\mathcal{O}_{F_\wp})(1) \longrightarrow 0. \tag{4.6}$$

We will consider the kernel and the cokernel of the homomorphisms $\Psi^* : U^{\psi_D} \to (\mathcal{O}_{F_\wp}[\![G_\infty]\!])^{\psi_D}$ and $\Psi_* : U_{\psi_D} \to (\mathcal{O}_{F_\wp}[\![G_\infty]\!])_{\psi_D}$ induced by the map Ψ in (4.6). We put $V = \ker(-\kappa : \mathcal{O}_{F_\wp}[\![G_\infty]\!] \to (\mathcal{O}_{F_\wp}/(\sigma-1)\mathcal{O}_{F_\wp})(1))$.

First we assume $\chi(2) \notin \boldsymbol{\mu}_{2^\infty}$ i.e. the order of χ_D is not 2-power. Then $\mathbb{Z}_2(1)^{\chi_D} = \mathbb{Z}_2(1)_{\chi_D} = 1$. By the exact sequence (4.6) and Lemma 2.1, we obtain $\Psi^* : U^{\psi_D} \xrightarrow{\sim} (\mathcal{O}_{F_\wp}[\![G_\infty]\!])^{\psi_D}$ and $\Psi_* : U_{\psi_D} \xrightarrow{\sim} (\mathcal{O}_{F_\wp}[\![G_\infty]\!])_{\psi_D}$. In this case, $\psi(2)+\psi\omega^{-1}(2)-1 = \chi(2)-1$ is unit in $\mathbb{Z}_2[\psi]$, so $T_{k,\psi}$ is trivial. Hence the assertion follows from (4.5).

Next, we assume $\chi(2) \in \boldsymbol{\mu}_{2^\infty}$ and $\chi(2) \neq 1$ i.e. χ_D is non-trivial and of 2-power order. In this case, $\mathbb{Z}_2(1)^{\chi_D} = 1$ and $\mathbb{Z}_2(1)_{\chi_D} = \boldsymbol{\mu}_2$. Let C be the subgroup of order 2 in D. Since $F_\wp/\mathbb{Q}_2$ is an unramified extension, we have $\widehat{H}^0(C, U) = 1$. Then, by the exact sequence (4.6) and Corollary 2.3, we have an exact sequence

$$0 \longrightarrow U^{\chi_D} \longrightarrow V^{\chi_D} \longrightarrow \boldsymbol{\mu}_2 \longrightarrow 0.$$

Furthermore, by Lemma 2.1, we have $U_{\chi_D} \cong V_{\chi_D}$. On the other hand, by the exact sequence (4.6) and Lemma 2.1, we have $V^{\chi_D} = \mathcal{O}_{F_\wp}^{\chi_D}[\![G_\infty]\!]$ and an exact sequence

$$0 \longrightarrow V_{\chi_D} \longrightarrow \mathcal{O}_{F_\wp,\chi_D}[\![G_\infty]\!] \longrightarrow \boldsymbol{\mu}_2 \longrightarrow 0.$$

Therefore, we have exact sequences

$$0 \longrightarrow U^{\chi_D} \longrightarrow \mathcal{O}_{F_\wp}^{\chi_D}[\![G_\infty]\!] \longrightarrow \boldsymbol{\mu}_2 \longrightarrow 0, \tag{4.7}$$

$$0 \longrightarrow U_{\chi_D} \longrightarrow \mathcal{O}_{F_\wp,\chi_D}[\![G_\infty]\!] \longrightarrow \boldsymbol{\mu}_2 \longrightarrow 0. \tag{4.8}$$

By these exact sequences, we have $\widehat{H}^j(G_0, U^{\chi_D}) = \widehat{H}^j(G_0, U_{\chi_D}) = \boldsymbol{\mu}_2$ for $j = -1, 0$. Taking ω^i-parts of the exact sequence (4.7) and using Lemma 2.1, we have an exact sequence

$$0 \longrightarrow U^{\psi_D} \longrightarrow (\mathcal{O}_{F_\wp}[\![G_\infty]\!])^{\psi_D} \longrightarrow \boldsymbol{\mu}_2 \longrightarrow \boldsymbol{\mu}_2 \longrightarrow 0.$$

Hence we have an isomorphism $\Psi^* : U^{\psi_D} \xrightarrow{\sim} (\mathcal{O}_{F_\wp}[\![G_\infty]\!])^{\psi_D}$ and, in this case, Proposition 4.6 follows from (4.5). Also taking ω^i-quotients of the exact sequence (4.8) and using Lemma 2.1, we have an exact sequence

$$0 \longrightarrow \boldsymbol{\mu}_2 \longrightarrow U_{\psi_D} \xrightarrow{\Psi_*} (\mathcal{O}_{F_\wp}[\![G_\infty]\!])_{\psi_D} \longrightarrow \boldsymbol{\mu}_2 \longrightarrow 0.$$

We recall

$$T_{k,\psi} = \Lambda/(\dot{T}, e_{k,2}, \psi(2) + \psi\omega^{-1}(2) - 1) = \Lambda/(\dot{T}, e_{k,2}, \chi(2) - 1)$$

where $e_{k,2}$ is the ramification index of 2 in k. Since assuming $\zeta_4 \in k$, we have $e_{k,2} = 2$. In the case where $\chi(2) \in \boldsymbol{\mu}_{2^\infty}$ and $\chi(2) \neq 1$, we see that $\chi(2) - 1$ divides $e_{k,2}$ and

$$\boldsymbol{\mu}_2 \otimes_{\mathbb{Z}_2[\psi_D]} \mathbb{Z}_2[\psi] = (\mathbb{Z}_2[\psi_D]/(\chi(2) - 1)) \otimes_{\mathbb{Z}_2[\psi_D]} \mathbb{Z}_2[\psi] = T_{k,\psi}.$$

Hence, in this case, Proposition 4.7 follows from (4.5).

In the case where $\chi(2)=1$, i.e. χ_D is trivial, we have $F_\wp=\mathbb{Q}_2$ and $D=1$, so $U^{\psi_D}=U^{\omega^i}$ and $U_{\psi_D}=U_{\omega^i}$. In this case, we note that $\mathbb{Z}_2[\psi_D]=\mathbb{Z}_2$ and

$$\boldsymbol{\mu}_2\otimes_{\mathbb{Z}_2[\psi_D]}\mathbb{Z}_2[\psi]=\mathbb{Z}_2[\psi]/(2)=\Lambda/(\dot{T},e_{k,2},\chi(2)-1)=T_{k,\psi}. \tag{4.9}$$

Here $e_{k,2}=2$ since we assume that $\zeta_4\in k$.

We first assume that $\chi(2)=1$ and $\psi=\chi\omega^0=\chi$. By the exact sequence (4.6) and Lemma 2.1, we have an exact sequence

$$0\longrightarrow U^{\omega^0}\xrightarrow{\Psi^*}V^{\omega^0}\longrightarrow\boldsymbol{\mu}_2\longrightarrow U_{\omega^0}\xrightarrow{\Psi_*}V_{\omega^0}\longrightarrow 0.$$

Also we have $V^{\omega^0}=(\mathbb{Z}_2[\![G_\infty]\!])^{\omega^0}$ and an exact sequence

$$0\longrightarrow V_{\omega^0}\longrightarrow(\mathbb{Z}_2[\![G_\infty]\!])_{\omega^0}\longrightarrow\boldsymbol{\mu}_2\longrightarrow 0.$$

In [5], Gillard proved that $U_{\mathbb{Q}_{2,\infty}}\cong\mathbb{Z}_2[\![\Gamma]\!]$, where $\mathbb{Q}_{2,\infty}$ is the cyclotomic $\mathbb{Z}_2$-extension of $\mathbb{Q}_2$. Since $U^{\omega^0}=U^{G_0}=U_{\mathbb{Q}_{2,\infty}}$, we have an isomorphism $\Psi^*:U^{\omega^0}\xrightarrow{\sim}(\mathbb{Z}_2[\![G_\infty]\!])^{\omega^0}$, and hence

$$0\longrightarrow\boldsymbol{\mu}_2\longrightarrow U_{\omega^0}\xrightarrow{\Psi_*}(\mathbb{Z}_2[\![G_\infty]\!])_{\omega^0}\longrightarrow\boldsymbol{\mu}_2\longrightarrow 0$$

is exact. Hence, in this case, Propositions 4.6 and 4.7 follows from (4.5) and (4.9).

We finally assume that $\chi(2)=1$ and $\psi=\chi\omega$. Recall that τ be a generator of G_0. By the exact sequence (4.6), Corollary 2.3 and $\widehat{H}^0(G_0,\mathbb{Z}_2(1))=1$, we have an exact sequence

$$0\longrightarrow\mathbb{Z}_2(1)\longrightarrow U^\omega\xrightarrow{\Psi^*}V^\omega\longrightarrow 0.$$

Since $V=(1+\tau)\mathbb{Z}_2[\![G_\infty]\!]+\dot{T}\mathbb{Z}_2[\![G_\infty]\!]$, we see that

$$V^\omega=\dot{T}(1-\tau)\mathbb{Z}_2[\![\Gamma]\!]=\dot{T}(\mathbb{Z}_2[\![G_\infty]\!])^\omega.$$

Hence, in the case where $\chi(2)=1$ and $\psi=\chi\omega$, Proposition 4.6 can be proved by using (4.5). We will decide the image and the kernel of the homomorphism

$$\Psi_*:U_\omega\longrightarrow(\mathbb{Z}_2[\![G_\infty]\!])_\omega.$$

Here note that $U_\omega=U/(1+\tau)U$ and

$$(\mathbb{Z}_2[\![G_\infty]\!])_\omega=\mathbb{Z}_2[\![G_\infty]\!]/(1+\tau)\mathbb{Z}_2[\![G_\infty]\!].$$

Since the image of Ψ is $V=(1+\tau)\mathbb{Z}_2[\![G_\infty]\!]+\dot{T}\mathbb{Z}_2[\![G_\infty]\!]$, the image of Ψ_* is

$$\begin{aligned}(V+(1+\tau)&\mathbb{Z}_2[\![G_\infty]\!])/(1+\tau)\mathbb{Z}_2[\![G_\infty]\!]\\&=((1+\tau)\mathbb{Z}_2[\![G_\infty]\!]+\dot{T}\mathbb{Z}_2[\![G_\infty]\!])/(1+\tau)\mathbb{Z}_2[\![G_\infty]\!]\\&=\dot{T}(\mathbb{Z}_2[\![G_\infty]\!]/(1+\tau)\mathbb{Z}_2[\![G_\infty]\!])\\&=\dot{T}(\mathbb{Z}_2[\![G_\infty]\!])_\omega=\dot{T}\mathbb{Z}_2[\![\Gamma]\!].\end{aligned}$$

For $u \in U$, assume that $u \bmod (1+\tau)U \in \ker(\Psi_*)$. Then we have $\Psi(u) \in (1+\tau)\mathbb{Z}_2[\![G_\infty]\!]$. Since $(1+\tau)\mathbb{Z}_2[\![G_\infty]\!] = \mathbb{Z}_2[\![G_\infty]\!]^{\omega^0}$ and we proved that $\Psi(U^{\omega^0}) = \mathbb{Z}_2[\![G_\infty]\!]^{\omega^0}$ in the above. Hence we have $u \in \ker(\Psi) + U^{\omega^0}$. Conversely, we can also prove that if $u \in \ker(\Psi) + U^{\omega^0}$ then $u \bmod (1+\tau)U \in \ker(\Psi_*)$. Therefore we obtain

$$\ker(\Psi_*) = (\ker(\Psi) + U^{\omega^0})/(1+\tau)U.$$

We can see that $\ker(\Psi) \cong \mathbb{Z}_2(1)$, $U^{\omega^0} \cap \ker(\Psi) = 1$ and $U^{\omega^0}/(1+\tau)U = \widehat{H}^0(G_0, U)$. By using the definition of V, we have

$$\widehat{H}^0(G_0, V) = \widehat{H}^{-1}(G_0, \mathbb{Z}_2(1)) = \boldsymbol{\mu}_2, \quad \widehat{H}^{-1}(G_0, V) = \widehat{H}^0(G_0, \mathbb{Z}_2(1)) = 1$$

and an exact sequence

$$1 \longrightarrow \widehat{H}^0(G_0, U) \longrightarrow \boldsymbol{\mu}_2 \longrightarrow \boldsymbol{\mu}_2 \longrightarrow \widehat{H}^{-1}(G_0, U) \longrightarrow 1$$

If $\widehat{H}^0(G_0, U) = 1$, then we have $\widehat{H}^{-1}(G_0, U) = 1$, and hence $U^{\omega^0} \cong U_{\omega^0}$. This is a contradiction to the above results in the case where $\chi(2) = 1$ and $\psi = \chi$. Hence $\widehat{H}^0(G_0, U)$ is nontrivial, so $\widehat{H}^0(G_0, U) = \boldsymbol{\mu}_2$. Summarizing the above, we obtain an isomorphism

$$\ker(\Psi_*) \cong \mathbb{Z}_2(1) \oplus \boldsymbol{\mu}_2.$$

Hence we have an exact sequence

$$0 \longrightarrow \mathbb{Z}_2(1) \oplus \boldsymbol{\mu}_2 \longrightarrow U_\omega \longrightarrow \dot{T}\mathbb{Z}_2[\![\Gamma]\!] \longrightarrow 0.$$

Proposition 4.7 can be proved by using (4.5) and (4.9).

The case $k_\infty \neq K_\infty$. Suppose that $k_\infty \neq K_\infty$, i.e. $\zeta_4 \notin k$. Then K_∞/k_∞ is a quadratic extension and put $\mathcal{G} = \mathrm{Gal}(K_\infty/k_\infty) \cong \mathrm{Gal}(k(\zeta_4)/k)$.

Since $\mathcal{U}_{k_\infty} = \mathcal{U}_{K_\infty}^{\mathcal{G}}$, we have $\mathcal{U}_{k_\infty}^{\psi} = (\mathcal{U}_{K_\infty}^{\mathcal{G}})^{\psi} = \mathcal{U}_{K_\infty}^{\psi}$. Therefore Proposition 4.6 in this case is reduced to the case $k_\infty = K_\infty$.

We will prove Proposition 4.7. First, we assume that $k/\mathbb{Q}$ is an unramified extension at 2. In this case, $k = F$, $\psi = \chi\omega^0 = \chi$ and $G = \Delta$. Let k^ψ be the fixed field of $\ker\psi = H$. As in the case where $k_\infty = K_\infty$, we can show that $N_H^* : \mathcal{U}_{k_\infty, H} \xrightarrow{\sim} \mathcal{U}_{k_\infty}^H = \mathcal{U}_{k_\infty^\psi}$. Assume that the order of Δ/H is even. Let C be the subgroup of order 2 in Δ/H. Since $k^\psi/\mathbb{Q}$ is an unramified extension at 2, we have $\widehat{H}^j(C, \mathcal{U}_{k_\infty^\psi}) = 1$ for $j = -1, 0$. By Lemma 2.2, $\xi_{\chi,\Delta/H} = \sum_{\delta \in \Delta/H} \chi(\delta)^{-1}\delta$ gives an isomorphism $\mathcal{U}_{k_\infty^\psi, \psi} \xrightarrow{\sim} \mathcal{U}_{k_\infty^\psi}^{\psi}$. If the order of Δ/H is odd, we also have $\mathcal{U}_{k_\infty^\psi, \psi} \xrightarrow{\sim} \mathcal{U}_{k_\infty^\psi}^{\psi}$. We see that $\xi_{\chi,\Delta} = \sum_{\delta \in \Delta} \chi(\delta)^{-1}\delta = N_H \xi_{\chi,\Delta/H}$. Therefore, for $u_\psi \in \mathcal{U}_{k_\infty,\psi}$, the correspondence $u_\psi \mapsto \xi_{\chi,\Delta}(\widetilde{u}_\psi)$ gives the isomorphism $\mathcal{U}_{k_\infty,\psi} \xrightarrow{\sim} \mathcal{U}_{k_\infty}^{\psi}$ where $\widetilde{u}_\psi$ is a representative of u_ψ in $\mathcal{U}_{k_\infty}$. We define $\Psi'_{k_\infty,\psi}(u_\psi) = \Psi_{k_\infty}^{\psi}(\xi_{\chi,\Delta}(\widetilde{u}_\psi))$.

Then we have an isomorphism $\Psi'_{k_\infty,\psi}:\mathcal{U}_{k_\infty,\psi}\xrightarrow{\sim}\Lambda$ by using Proposition 4.6 for the isomorphism $\Psi^{\psi}_{k_\infty}:\mathcal{U}^{\psi}_{k_\infty}\xrightarrow{\sim}\Lambda$. By the definition of $\Psi^{\psi}_{k_\infty}$, we have

$$\begin{aligned}\Psi'_{k_\infty,\psi}(u_\psi)z^{\chi} &= \Psi^{\psi}_{k_\infty}(\xi_{\chi,\Delta}(\widetilde{u}_\psi))z^{\chi}\\ &= \frac{1}{2}\omega^0(\Psi_{K_\infty}(\xi_{\chi,\Delta}(\widetilde{u}_\psi)))\\ &= \frac{1}{2}\omega^0(\xi_{\chi,\Delta}(\Psi_{K_\infty}(\widetilde{u}_\psi))).\end{aligned}$$

Then $\xi_{\chi,\Delta}(\Psi_{K_\infty}(\widetilde{u}_\psi))$ maps to $\Psi_{K_\infty}(\widetilde{u}_\psi)_{F,\chi}$, by the isomorphism

$$\widehat{\mathcal{O}}^{\chi}_{F}[\![G_\infty]\!]\xrightarrow{\sim}\widehat{\mathcal{O}}_{F,\chi}[\![G_\infty]\!]$$

given by $z^{\chi}\mapsto z_{F,\chi}$. Hence $\Psi'_{k_\infty,\psi}(u_\psi)z_{F,\chi}=\frac{1}{2}\omega^0(\Psi_{K_\infty}(\widetilde{u}_\psi)_{F,\chi})$, so $\Psi'_{k_\infty,\psi}$ coincides with $\Psi_{k_\infty,\psi}$ in Proposition 4.4. Then we have an isomorphism $\Psi_{k_\infty,\psi}:\mathcal{U}_{k_\infty,\psi}\xrightarrow{\sim}\Lambda$. Since we assume that 2 is unramified in k, i.e. $e_{k,2}=1$, T_ψ is trivial. Hence, in this case, we prove Proposition 4.7.

Next, we assume that $k/\mathbb{Q}$ is a ramified extension at 2. In this case the conductor of k is 4m, thus K/k is an unramified extension at 2. Then norm map $N_{\mathcal{G}}$ induces an isomorphism $N^*_{\mathcal{G}}:\mathcal{U}_{K_\infty,\mathcal{G}}\xrightarrow{\sim}\mathcal{U}^{\mathcal{G}}_{K_\infty}=\mathcal{U}_{k_\infty}$. Therefore $\mathcal{U}_{k_\infty,\psi}=(\mathcal{U}^{\mathcal{G}}_{K_\infty})_\psi\cong(\mathcal{U}_{K_\infty,\mathcal{G}})_\psi=\mathcal{U}_{K_\infty,\psi}$. We define a homomorphism $\Psi'_{k_\infty,\psi}:\mathcal{U}_{k_\infty,\psi}\to\Lambda$ to the composition of this isomorphism $\mathcal{U}_{k_\infty,\psi}\cong\mathcal{U}_{K_\infty,\psi}$ and $\Psi_{K_\infty,\psi}:\mathcal{U}_{K_\infty,\psi}\to\Lambda$. The kernel and the cokernel of $\Psi'_{k_\infty,\psi}$ coincide with these of $\Psi_{K_\infty,\psi}$. Furthermore, we see that $e_{k,2}=e_{K,2}=2$ and

$$T_{k,\psi}=\Lambda/(\dot{T},2,\psi(2)+\psi\omega^{-1}(2)-1)=T_{K,\psi}.$$

Hence, it is enough to show that $\Psi'_{k_\infty,\psi}$ coincides with $\Psi_{k_\infty,\psi}$ in Proposition 4.4. Let u_ψ be an element of $\mathcal{U}_{k_\infty,\psi}$. We take a representative $\widetilde{u}_\psi\in\mathcal{U}_{k_\infty}$ of u_ψ. Since $\mathcal{U}_{k_\infty}=\mathcal{U}^{\mathcal{G}}_{K_\infty}=N_{\mathcal{G}}(\mathcal{U}_{K_\infty})$, there exists $u'\in\mathcal{U}_{K_\infty}$ such that $N_{\mathcal{G}}(u')=\widetilde{u}_\psi$. Denote by $[u']$ the residue class of u' in $\mathcal{U}_{K_\infty,\psi}$. Then we have $\Psi'_{k_\infty,\psi}(u_\psi)=\Psi_{K_\infty,\psi}([u'])$. By the definition of $\Psi_{K_\infty,\psi}$ in Proposition 4.4, we have $\Psi'_{k_\infty,\psi}(u_\psi)z_{F,\chi}=\omega^i(\Psi_{K_\infty}(u')_{F,\chi})$. By regarding $\widetilde{u}_\psi$ as an element of $\mathcal{U}_{K_\infty}$, we have

$$\begin{aligned}\omega^i(\Psi_{K_\infty}(\widetilde{u}_\psi)_{F,\chi}) &= \omega^i(\Psi_{K_\infty}(N_{\mathcal{G}}(u'))_{F,\chi})\\ &= \omega^i(((1+g)\Psi_{K_\infty}(u'))_{F,\chi})\\ &= (1+\psi(g))\omega^i(\Psi_{K_\infty}(u')_{F,\chi})\\ &= 2\omega^i(\Psi_{K_\infty}(u')_{F,\chi})\end{aligned}$$

where g is a generator of $\mathcal{G}$. Therefore we obtain

$$\Psi'_{k_\infty,\psi}(u_\psi)z_{F,\chi}=\frac{1}{2}\omega^i(\Psi_{K_\infty}(\widetilde{u}_\psi)_{F,\chi}).$$

Thus $\Psi'_{k_\infty,\psi}$ coincides with $\Psi_{k_\infty,\psi}$ in Proposition 4.4. □

In [12], we proved that if p is odd and $\psi\omega^{-1}(p) = 1$, there exists an exact sequence of Λ-modules

$$0 \longrightarrow \Lambda/(\dot{T}) \longrightarrow \mathcal{U}_{k_\infty,\psi} \longrightarrow \dot{T}\Lambda \oplus \Lambda/(d, \dot{T}) \longrightarrow 0$$

where d is the order of the decomposition group of $\mathrm{Gal}(k/\mathbb{Q})$. In the same way as the proof of Proposition 4.7, we can also prove the following:

Proposition 4.8. *If p is odd prime number and $\psi\omega^{-1}(p) = 1$, we have an exact sequence of Λ-modules*

$$0 \longrightarrow \Lambda/(\dot{T}) \longrightarrow \mathcal{U}_{k_\infty,\psi} \xrightarrow{\Psi_{k_\infty,\psi}} \dot{T}\Lambda \longrightarrow 0.$$

5. cyclotomic units

In this section we recall the definition of the cyclotomic units in the sense of Sinnott [10] and we define two cyclotomic units groups $\mathcal{C}_{k_\infty}$ and $\mathcal{C}'_{k_\infty}$. We will determine generators of the ψ-part of $\mathcal{C}_{k_\infty}$ and the ψ-quotients of $\mathcal{C}_{k_\infty}$ and $\mathcal{C}'_{k_\infty}$. For any abelian field L, let D_L denote the subgroup of the multiplicative group $L^\times$ generated by

$$\{\pm 1, N_{\mathbb{Q}(t)/\mathbb{Q}(t)\cap L}(1 - \zeta_t^a) \mid t, a \in \mathbb{Z}, t > 1, (a, t) = 1\}.$$

Denote E_L by the group of units in L. The cyclotomic units C'_L in L in the sense of Sinnott is defined by $D_L \cap E_L$. For a real abelian field L, let $C_{1,L}$ be the group of units in L whose squares lie in C'_L. We define a group of cyclotomic units C_L in L by $C'_L \cdot C_{1,L^+}$ where L^+ is the maximal real subfield of L. If L is real, then $C'_L \subset C_{1,L}$, and hence $C_L = C_{1,L}$.

Recall that $F = \mathbb{Q}(m) \cap k(\zeta_4)$ where m is the odd part of the conductor of k and $K_n = F(\boldsymbol{\mu}_{2^{n+2}}) = k(\boldsymbol{\mu}_{2^{n+2}})$. We define

$$\eta_t = (N_{\mathbb{Q}(2^{n+2}t)/\mathbb{Q}(2^{n+2}t)\cap K_n}(1 - \zeta_{2^{n+2}}\zeta_t^{\sigma^{-n}}))_{n\geq 0} \in \varprojlim C_{K_n}$$

for $t \mid m,\ t \neq 1$ and

$$\eta_1 = ((-\zeta_{2^{n+2}})^{\frac{\kappa(\gamma)-1}{2}}(1 - \zeta_{2^{n+2}})^{\gamma-1})_{n\geq 0} \in \varprojlim C_{K_n}.$$

For an abelian field L, we identify C_L and C'_L with their images under the diagonal embedding $E_L \to (\mathcal{O}_L \otimes_{\mathbb{Z}} \mathbb{Z}_2)^\times$. Since $(\mathcal{O}_L \otimes_{\mathbb{Z}} \mathbb{Z}_2)^\times$ is decomposed into a product of the principal units $\mathcal{U}_L$ and a finite group of odd order, there is the projection $(\mathcal{O}_L \otimes \mathbb{Z}_2)^\times \to \mathcal{U}_L$. Let $\mathcal{C}_L$ and $\mathcal{C}'_L$ be the closure of the intersections $\mathcal{U}_L \cap C_L$ and $\mathcal{U}_L \cap C'_L$ in $\mathcal{U}_L$. Hence $\mathcal{C}_L$ and $\mathcal{C}'_L$ are the closure of the image of C_L and C'_L under the projection $(\mathcal{O}_L \otimes \mathbb{Z}_2)^\times \to \mathcal{U}_L$ respectively. Put

$$\mathcal{C} = \mathcal{C}_{k_\infty} = \varprojlim \mathcal{C}_{k_n}, \quad \mathcal{C}' = \mathcal{C}'_{k_\infty} = \varprojlim \mathcal{C}'_{k_n}.$$

For $\eta \in \varprojlim C_{K_n}$ or $\eta \in \varprojlim C_{k_n}$, we shall also denote by η its image under the projection $\varprojlim(\mathcal{O}_{K_n} \otimes \mathbb{Z}_2)^\times \to \mathcal{U}_{K_\infty}$ or $\varprojlim(\mathcal{O}_{k_n} \otimes \mathbb{Z}_2)^\times \to \mathcal{U}_{k_\infty}$ so η_t is in $\mathcal{C}_{K_\infty}$. For any t with $t \mid m$, identifying $\mathrm{Gal}(\mathbb{Q}(2^{n+2}t) \cap K_n/\mathbb{Q}(2^{n+2}t) \cap k_n)$

with $\mathcal{G} = \mathrm{Gal}(K_0/k)$, we have $N_{\mathcal{G}}(\eta_t) \in \mathcal{C}'_{k_\infty}$. Put $\epsilon = (-\zeta_{2^{n+2}})_{n\geq 0} \in \mathcal{C}_{K_\infty}$. First, we prove the following:

Lemma 5.1.

(a) $\mathcal{C}'_{k_\infty}$ *is generated by*

$$\{N_{\mathcal{G}}(\epsilon),\ N_{\mathcal{G}}(\eta_t) \mid t \mid m\}$$

as $\mathbb{Z}_2[G][\![\Gamma]\!]$*-module.*

(b) *Assume that k is a real abelian field. Then $\mathcal{C}_{k_\infty}$ has a submodule of finite index generated by*

$$\{N_{\mathcal{G}}(\eta_t),\ \eta_t^{\gamma-\kappa(\gamma)},\ \eta_1 \mid t \mid m, t \neq 1\}$$

as $\mathbb{Z}_2[G][\![\Gamma]\!]$-module and $\mathcal{C}'_{k_\infty}$ has a submodule $\mathcal{C}^2_{k_\infty}$ of finite index. Furthermore if $h_t = [\mathbb{Q}(t) : \mathbb{Q}(t) \cap F]$ is even for $t \mid m, t \neq 1$, then $\mathcal{C}_{k_\infty}$ is generated by

$$\left\{\epsilon^{-h_t/2}\eta_t,\ \eta_1 \,\middle|\, t \mid m, t \neq 1\right\}$$

as $\mathbb{Z}_2[G][\![\Gamma]\!]$-modules.

Proof. In [8], it is proved that $\mathcal{C}'_{K_\infty}$ is generated by $\{\epsilon,\ \eta_t \mid t \mid m\}$. Since $(\mathbb{Q}(2^{n+2}t) \cap k_n)(\zeta_4) = \mathbb{Q}(2^{n+2}t) \cap K_n$, we have $(\mathbb{Q}(2^{n+2}t) \cap K_n) \cdot k_n = K_n$, so $\mathcal{G} \cong \mathrm{Gal}(\mathbb{Q}(2^{n+2}t) \cap K_n/\mathbb{Q}(2^{n+2}t) \cap k_n)$. Therefore $N_{\mathcal{G}}(\mathcal{C}'_{K_\infty}) = \mathcal{C}'_{k_\infty}$ and the claim (a) is proved.

Assume that k is real. Fix t with $1 \neq t \mid m$. Let $\widetilde{g}_n$ be the element of $\mathrm{Gal}(\mathbb{Q}(2^{n+2}t)/\mathbb{Q})$ such that $(\zeta_{2^{n+2}}\zeta_t)^{\widetilde{g}_n} = \zeta_{2^{n+2}}^{-1}\zeta_t^{-1}$ and g_n the restriction of $\widetilde{g}_n$ to $\mathbb{Q}(2^{n+2}t) \cap K_n$. Since $\mathbb{Q}(2^{n+2}t) \cap k_n$ is the maximal real subfield of $\mathbb{Q}(2^{n+2}t) \cap K_n$, the element g_n is a generator of $\mathrm{Gal}(\mathbb{Q}(2^{n+2}t) \cap K_n/\mathbb{Q}(2^{n+2}t) \cap k_n)$. We see that

$$\begin{aligned}
&N_{\mathbb{Q}(2^{n+2}t)/\mathbb{Q}(2^{n+2}t)\cap K_n}(1-\zeta_{2^{n+2}}\zeta_t^{\sigma^{-n}})^{(\gamma-\kappa(\gamma))g_n}\\
&\quad= N_{\mathbb{Q}(2^{n+2}t)/\mathbb{Q}(2^{n+2}t)\cap K_n}\left(\frac{(1-\zeta_{2^{n+2}}^{\kappa(\gamma)}\zeta_t^{\sigma^{-n}})^{g_n}}{(1-\zeta_{2^{n+2}}\zeta_t^{\sigma^{-n}})^{\kappa(\gamma)g_n}}\right)\\
&\quad= N_{\mathbb{Q}(2^{n+2}t)/\mathbb{Q}(2^{n+2}t)\cap K_n}\left(\frac{1-\zeta_{2^{n+2}}^{-\kappa(\gamma)}\zeta_t^{-\sigma^{-n}}}{(1-\zeta_{2^{n+2}}^{-1}\zeta_t^{-\sigma^{-n}})^{\kappa(\gamma)}}\right)\\
&\quad= N_{\mathbb{Q}(2^{n+2}t)/\mathbb{Q}(2^{n+2}t)\cap K_n}\left(\frac{(-\zeta_{2^{n+2}}^{-\kappa(\gamma)}\zeta_t^{-\sigma^{-n}})(1-\zeta_{2^{n+2}}^{\kappa(\gamma)}\zeta_t^{\sigma^{-n}})}{(-\zeta_{2^{n+2}}^{-1}\zeta_t^{-\sigma^{-n}})^{\kappa(\gamma)}(1-\zeta_{2^{n+2}}\zeta_t^{\sigma^{-n}})^{\kappa(\gamma)}}\right)\\
&\quad= N_{\mathbb{Q}(2^{n+2}t)/\mathbb{Q}(2^{n+2}t)\cap K_n}\left(\frac{1-\zeta_{2^{n+2}}^{\kappa(\gamma)}\zeta_t^{\sigma^{-n}}}{(1-\zeta_{2^{n+2}}\zeta_t^{\sigma^{-n}})^{\kappa(\gamma)}}\right)\\
&\quad= N_{\mathbb{Q}(2^{n+2}t)/\mathbb{Q}(2^{n+2}t)\cap K_n}(1-\zeta_{2^{n+2}}\zeta_t^{\sigma^{-n}})^{\gamma-\kappa(\gamma)}.
\end{aligned}$$

We used that $N_{\mathbb{Q}(2^{n+2}t)/\mathbb{Q}(2^{n+2}t)\cap K_n}(\zeta_t) = 1$ in $\mathcal{U}_{K_n}$. Therefore

$$N_{\mathbb{Q}(2^{n+2}t)/\mathbb{Q}(2^{n+2}t)\cap K_n}(1-\zeta_{2^{n+2}}\zeta_t^{\sigma^{-n}})^{\gamma-\kappa(\gamma)}$$

is an element of $\mathbb{Q}(2^{n+2}t)\cap k_n$. Then $(\eta_t^{\gamma-\kappa(\gamma)})^2 = N_{\mathcal{G}}(\eta_t)^{\gamma-\kappa(\gamma)}$. Therefore by (a), we have

$$\begin{aligned}\left\langle\left\{N_{\mathcal{G}}(\eta_t),\ \eta_t^{\gamma-\kappa(\gamma)} \;\middle|\; t \mid m\right\}\right\rangle^2 &= \left\langle\left\{N_{\mathcal{G}}(\eta_t)^2,\ N_{\mathcal{G}}(\eta_t)^{\gamma-\kappa(\gamma)} \;\middle|\; t \mid m\right\}\right\rangle \\ &= (\mathcal{C}'_{k_\infty})^{(2,\dot{T})}.\end{aligned}$$

In particular, we have $\langle\{N_{\mathcal{G}}(\eta_t),\ \eta_t^{\gamma-\kappa(\gamma)} \mid t \mid m\}\rangle^2 \subset \mathcal{C}'_{k_\infty}$. Since $\mathcal{C}_{k_\infty} = \{\eta\in\mathcal{E}_{k_\infty} \mid \eta^2\in\mathcal{C}'_{k_\infty}\}$, we have

$$\left\langle\left\{N_{\mathcal{G}}(\eta_t),\ \eta_t^{\gamma-\kappa(\gamma)} \;\middle|\; t \mid m\right\}\right\rangle \subset \mathcal{C}_{k_\infty}.$$

Therefore we have

$$\mathcal{C}'_{k_\infty} \supset \mathcal{C}^2_{k_\infty} \supset \left\langle\left\{N_{\mathcal{G}}(\eta_t),\ \eta_t^{\gamma-\kappa(\gamma)} \;\middle|\; t \mid m\right\}\right\rangle^2 = (\mathcal{C}'_{k_\infty})^{(2,\dot{T})}.$$

Since $\mathcal{C}'_{k_\infty}$ is a finitely genereted $\mathbb{Z}_2[\![T]\!]$-module, $\mathcal{C}'_{k_\infty}/(\mathcal{C}'_{k_\infty})^{(2,\dot{T})}$ is finite. Then $\mathcal{C}'_{k_\infty}\supset\mathcal{C}^2_{k_\infty}$ and $\mathcal{C}^2_{k_\infty}\supset\langle\{N_{\mathcal{G}}(\eta_t),\ \eta_t^{\gamma-\kappa(\gamma)} \mid t\mid m\}\rangle^2$ are of finite index. The module $\mathcal{U}_{k_\infty}$ has no nontrivial element killed by 2 by Theorem 4.1. Therefore $\mathcal{C}_{k_\infty}\supset\langle\{N_{\mathcal{G}}(\eta_t),\ \eta_t^{\gamma-\kappa(\gamma)} \mid t\mid m\}\rangle$ is also of finite index.

Assume that h_t is even, i.e. $\frac{h_t}{2}\in\mathbb{Z}$. We also see that

$$\begin{aligned}&\left((-\zeta_{2^{n+2}})^{-\frac{h_t}{2}} N_{\mathbb{Q}(2^{n+2}t)/\mathbb{Q}(2^{n+2}t)\cap K_n}(1-\zeta_{2^{n+2}}\zeta_t^{\sigma^{-n}})\right)^{g_n} \\ &= (-\zeta_{2^{n+2}})^{\frac{h_t}{2}} N_{\mathbb{Q}(2^{n+2}t)/\mathbb{Q}(2^{n+2}t)\cap K_n}(1-\zeta_{2^{n+2}}^{-1}\zeta_t^{-\sigma^{-n}}) \\ &= (-\zeta_{2^{n+2}})^{\frac{h_t}{2}} N_{\mathbb{Q}(2^{n+2}t)/\mathbb{Q}(2^{n+2}t)\cap K_n}\left((-\zeta_{2^{n+2}}^{-1}\zeta_t^{-\sigma^{-n}})(1-\zeta_{2^{n+2}}\zeta_t^{\sigma^{-n}})\right) \\ &= (-\zeta_{2^{n+2}})^{-\frac{h_t}{2}} N_{\mathbb{Q}(2^{n+2}t)/\mathbb{Q}(2^{n+2}t)\cap K_n}(1-\zeta_{2^{n+2}}\zeta_t^{\sigma^{-n}}))\end{aligned}$$

where we used $[\mathbb{Q}(2^{n+2}t):\mathbb{Q}(2^{n+2}t)\cap K_n] = [\mathbb{Q}(t):\mathbb{Q}(t)\cap F] = h_t$ and $N_{\mathbb{Q}(2^{n+2}t)/\mathbb{Q}(2^{n+2}t)\cap K_n}(\zeta_t) = 1$ in $\mathcal{U}_{K_n}$. Hence

$$(-\zeta_{2^{n+2}})^{-\frac{h_t}{2}} N_{\mathbb{Q}(2^{n+2}t)/\mathbb{Q}(2^{n+2}t)\cap K_n}(1-\zeta_{2^{n+2}}\zeta_t^{\sigma^{-n}})$$

is a unit in k_n. Then $(\epsilon^{-h_t/2}\eta_t)^2 = N_{\mathcal{G}}(\epsilon^{h_t/2}\eta_t) = N_{\mathcal{G}}(\eta_t)\in N_{\mathcal{G}}(\mathcal{C}_{K_\infty}) = \mathcal{C}'_{k_\infty}$ since $N_{\mathcal{G}}(\epsilon)=1$, so $\epsilon^{-h_t/2}\eta_t$ is in $\mathcal{C}_{k_\infty}$. Similarly we see that $\eta_1\in\mathcal{C}_{k_\infty}$. Furthermore, we see that

$$\mathcal{C}'_{k_\infty}\supset\mathcal{C}^2_{k_\infty}\supset\langle\{\epsilon^{-h_t/2}\eta_t,\ \eta_1 \mid t\mid m, t\neq 1\}\rangle^2$$

and

$$\begin{aligned}\mathcal{C}'_{k_\infty} &= N_{\mathcal{G}}(\mathcal{C}_{K_\infty}) = \langle \{N_{\mathcal{G}}(\epsilon),\ N_{\mathcal{G}}(\eta_t) \mid t \mid m\} \rangle \\ &= \langle \{(\epsilon^{-h/2}\eta_t)^2, \eta_1^2 \mid t \mid m, t \neq 1\} \rangle \\ &= \langle \{\epsilon^{-h_t/2}\eta_t,\ \eta_1 \mid t \mid m, t \neq 1\} \rangle^2.\end{aligned}$$

Therefore $\mathcal{C}^2_{k_\infty} = \langle \{\epsilon^{-h_t/2}\eta_t,\ \eta_1 \mid t \mid m, t \neq 1\} \rangle^2$. The module $\mathcal{U}_{k_\infty}$ has no nontrivial element killed by 2 by Theorem 4.1. Therefore, we obtain

$$\mathcal{C}_{k_\infty} = \langle \{\epsilon^{-h_t/2}\eta_t,\ \eta_1 \mid t \mid m, t \neq 1\} \rangle.$$

□

Remark 5.2. The condition that $2 \mid h_t = [\mathbb{Q}(t) : \mathbb{Q}(t) \cap F]$ in Lemma 5.1 (b) holds, if 2 is unramified in k. Indeed, if k is real and 2 is unramified in k, then $F = k$, so F is real. Hence, $\mathbb{Q}(t) \cap F$ is also real, so $2 \mid [\mathbb{Q}(t) : \mathbb{Q}(t) \cap F]$.

Lemma 5.3. *Assume that $\psi = \chi\omega^i$ is a non-trivial, even character of G. Put $\xi_\chi = \sum_\delta \chi(\delta)\delta^{-1}$, δ running over all elements in* $\mathrm{Gal}(F \cap \mathbb{Q}(f)/\mathbb{Q})$. *Then the Λ-module $\mathcal{C}^\psi_{k_\infty}$ is generated by $\eta_f^{\xi_\chi}$.*

Proof. We regard χ as a character of $\mathrm{Gal}(\mathbb{Q}(f)/\mathbb{Q})$. Since $\chi(\delta) = 1$ for $\delta \in \mathrm{Gal}(\mathbb{Q}(f)/\mathbb{Q}(f) \cap F)$, we have

$$\eta_f^{\xi_\chi} = \left(\prod_{\delta \in \mathrm{Gal}(\mathbb{Q}(f)/\mathbb{Q})} (1 - \zeta_{2^{n+2}} \zeta_f^{\sigma^{-n}\delta^{-1}})^{\chi(\delta)} \right)_{n \geq 0}$$

Let k^ψ be the fixed field of ψ and $F^\chi = \mathbb{Q}(f) \cap k^\psi(\zeta_4)$. Then $F^\chi(\zeta_4) = k^\psi(\zeta_4)$. Put $k^\psi_n = k^\psi \mathbb{Q}_n = k^\psi(\zeta_{2^{n+2}} + \zeta_{2^{n+2}}^{-1})$, $k^\psi_\infty = \bigcup k^\psi_n$, $K^\psi_n = k^\psi(\zeta_{2^{n+2}}) = F^\chi(\zeta_{2^{n+2}})$ and $K^\psi_\infty = \bigcup K^\psi_n$. Let $\widetilde{g}_n$ be the element of $\mathrm{Gal}(\mathbb{Q}(2^{n+2}f)/\mathbb{Q})$ such that $(\zeta_{2^{n+2}}\zeta_f)^{\widetilde{g}_n} = \zeta_{2^{n+2}}^{-1}\zeta_f^{-1}$ and g_n the restriction of $\widetilde{g}_n$ to K^ψ_n. Since k^ψ_n is a real abelian field and $[K^\psi_n : k^\psi_n] = 2$, the element g_n is a generator of $\mathrm{Gal}(K^\psi_n / k^\psi_n)$. We see that

$$\begin{aligned}&\left(\prod_\delta (1 - \zeta_{2^{n+2}} \zeta_f^{\sigma^{-n}\delta^{-1}})^{\chi(\delta)} \right)^{g_n} \\ &\qquad = \prod_\delta (1 - \zeta_{2^{n+2}}^{-1} \zeta_f^{-\sigma^{-n}\delta^{-1}})^{\chi(\delta)} \\ &\qquad = \prod_\delta (-\zeta_{2^{n+2}}^{-1} \zeta_f^{-\sigma^{-n}\delta^{-1}})^{\chi(\delta)} (1 - \zeta_{2^{n+2}} \zeta_f^{\sigma^{-n}\delta^{-1}})^{\chi(\delta)} \\ &\qquad = \prod_\delta (-\zeta_{2^{n+2}}^{-\chi(\delta)} \zeta_f^{-\sigma^{-n}\chi(\delta)\delta^{-1}}) (1 - \zeta_{2^{n+2}} \zeta_f^{\sigma^{-n}\delta^{-1}})^{\chi(\delta)} \\ &\qquad = \prod_\delta (\zeta_f^{-\sigma^{-n}\chi(\delta)\delta^{-1}}) (1 - \zeta_{2^{n+2}} \zeta_f^{\sigma^{-n}\delta^{-1}})^{\chi(\delta)}\end{aligned}$$

where δ runs over all elements of $\mathrm{Gal}(\mathbb{Q}(f)/\mathbb{Q})$. The image of

$$\Big(\prod\Big(\zeta_f^{-\sigma^{-n}\chi(\delta)\delta^{-1}}\Big)\Big)_{n\geq 0}$$

under the projection $\varprojlim(\mathcal{O}_{K_n}\otimes\mathbb{Z}_2)^\times \to \mathcal{U}_{K_\infty}$ is 1. Then we obtain $(\eta_f^{\xi_\chi})^g = \eta_f^{\xi_\chi}$ where $g=(g_n)_{n\geq 0}\in \mathrm{Gal}(K_\infty^\psi/k_\infty^\psi)$. Therefore $\eta_f^{\xi_\chi}$ is an element in $\mathcal{E}_{K_\infty^\psi}$ on which g acts trivially hence is in $\mathcal{E}_{k_\infty^\psi}$. We see that

$$\begin{aligned}(\eta_f^{\xi_\chi})^2 &= N_{K_\infty^\psi/k_\infty^\psi}(\eta_f^{\xi_\chi})\\ &= \left(\prod_{\delta\in\mathrm{Gal}(\mathbb{Q}(f)\cap F^\chi/\mathbb{Q})} N_{\mathbb{Q}(2^{n+2}f)/\mathbb{Q}(2^{n+2}f)\cap k_n^\psi}(1-\zeta_{2^{n+2}}\zeta_f^{\sigma^{-n}\delta^{-1}})^{\chi(\delta)}\right)_{n\geq 0}\end{aligned}$$

is in $\mathcal{C}'_{k_\infty^\psi}$, so in $\mathcal{C}'_{k_\infty^+}$. Hence $\eta_f^{\xi_\chi}\in\mathcal{C}_{k_\infty^+}\subset\mathcal{C}_{k_\infty}$. Let τ be the generator of $\mathrm{Gal}(K_0^\psi/F^\chi)$. As in the proof of Proposition 4.4, by the isomorphism $\mathrm{Gal}(K_0^\psi/\mathbb{Q})\cong\mathrm{Gal}(K_0^\psi/F^\chi)\times\mathrm{Gal}(F^\chi/\mathbb{Q})$, τg maps to $(1,(\tau g)|_{F^\chi})$ and $\chi(\tau g)=\omega^i(\tau)$. Hence we see that

$$(\eta_f^{\xi_\chi})^\tau = (\eta_f^{\xi_\chi})^{\tau g} = (\eta_f^{\xi_\chi})^{\chi(\tau g)} = (\eta_f^{\xi_\chi})^{\omega^i(\tau)}.$$

Clearly any element of Δ acts on $\eta_f^{\xi_\chi}$ via χ. Therefore any element of G acts on $\eta_f^{\xi_\chi}$ via ψ, so $\eta_f^{\xi_\chi}$ is in $\mathcal{C}_{k_\infty}^\psi$. Then $\mathcal{C}_{k_\infty}^\psi$ contains the submodule generated by $\eta_f^{\xi_\chi}$. We can show that both modules are coincide as in the proof of odd prime version [12, Lemma 6.2 (a)]. □

For $\eta\in\mathcal{U}_{k_\infty}$, we denote by $\eta_{k_\infty,\psi}=\eta_\psi$ its image under the surjection $\mathcal{U}_{k_\infty}\to\mathcal{U}_{k_\infty,\psi}$. Let $\widetilde{\mathcal{C}}_{k_\infty,\psi}=\widetilde{\mathcal{C}}_\psi$ and $\widetilde{\mathcal{C}}'_{k_\infty,\psi}=\widetilde{\mathcal{C}}'_\psi$ denote the images of $\mathcal{C}_{k_\infty}$ and $\mathcal{C}'_{k_\infty}$ under the surjection $\mathcal{U}_{k_\infty}\to\mathcal{U}_{k_\infty,\psi}$ respectively. Then we have isomorphisms $(\mathcal{U}/\mathcal{C})_\psi\cong\mathcal{U}_\psi/\widetilde{\mathcal{C}}_\psi$ and $(\mathcal{U}/\mathcal{C}')_\psi\cong\mathcal{U}_\psi/\widetilde{\mathcal{C}}'_\psi$.

Lemma 5.4. *Assume that* k *is an abelian field of the first kind and* $\psi=\chi\omega^i$ *is a non-trivial, even character of* $G=\mathrm{Gal}(k/\mathbb{Q})$. *The* Λ*-module* $\widetilde{\mathcal{C}}'_{k_\infty,\psi}$ *is generated by* $\{N_{\mathcal{G}}(\eta_{m_I})_{k_\infty,\psi}\mid I\subset\mathcal{L}\}$.

Proof. By Lemma 5.1 (a), $\widetilde{\mathcal{C}}'_\psi$ is generated by $\{N_{\mathcal{G}}(\epsilon)_\psi, N_{\mathcal{G}}(\eta_t)_\psi\mid t\mid m\}$. If $f\nmid t$, then there exists $\delta\in\Delta$ such that $\delta\notin\ker\chi$ and $\delta\in\mathrm{Gal}(F/\mathbb{Q}(t)\cap F)$, hence $\chi(\delta)\neq 1$ and $\eta_t^{\delta-1}=1$. Let s_χ be the surjection map from $\mathcal{U}_{K_\infty}$ to $\mathcal{U}_{K_\infty,\chi}$, the χ-quotient as a $\mathbb{Z}_2[\Delta]$-module $\mathcal{U}_{K_\infty}$. Since $s_\chi(\eta_t^{\delta-1})=s_\chi(\eta_t)^{\chi(\delta)-1}$, we have $s_\chi(\eta_t)^{\chi(\delta)-1}=1$, i.e. $s_\chi(\eta_t)$ is a torsion element of $\mathcal{U}_{K_\infty,\chi}$. In the proof of Proposition 4.7 we see that U_{χ_D} has no torsion element and $\mathcal{U}_{K_\infty,\chi}=U_{\chi_D}\otimes_{\mathbb{Z}_2[\chi_D]}\mathbb{Z}_2[\chi]$, then $s_\chi(\eta_t)=1$. The image $(\eta_t)_{K_\infty,\psi}$ of $s_\chi(\eta_t)$ under the surjection $\mathcal{U}_{K_\infty,\chi}\to\mathcal{U}_{K_\infty,\psi}$ is trivial, where $\mathcal{U}_{K_\infty,\psi}$ is the

ψ-quotient of the $\mathbb{Z}_2[\mathrm{Gal}(K_0/\mathbb{Q})]$-module $\mathcal{U}_{K_\infty}$. If $k_\infty = K_\infty$, i.e. $\mathcal{G} = \{1\}$, then $(N_{\mathcal{G}}(\eta_t))_\psi = (\eta_t)_{K_\infty,\psi} = 1$. In the case where $k_\infty \neq K_\infty$ and 2 is ramified in k, we see that $\mathcal{U}_{k_\infty,\psi} \cong \mathcal{U}_{K_\infty,\psi}$ and $N_{\mathcal{G}}(\eta_t)_\psi$ maps to $(\eta_t)_{K_\infty,\psi}$ by this isomorphism in the proof of Proposition 4.7, hence $N_{\mathcal{G}}(\eta_t)_\psi = 1$. In the case where $k_\infty \neq K_\infty$ and 2 is unramified in k, $\mathcal{U}_{k_\infty,\psi}$ has no torsion element, hence $N_{\mathcal{G}}(\eta_t)_\psi = 1$ by the same method as in another case. For any $\delta \in \mathrm{Gal}(F/\mathbb{Q})$, we have $\epsilon^{\delta-1} = 1$. Hence we can prove that $N_{\mathcal{G}}(\epsilon)_\psi = 1$ similarly. Therefore $\widetilde{\mathcal{C}}'_\psi$ is generated by $\{N_{\mathcal{G}}(\eta_t)_\psi \mid t \mid m,\ f \mid t\}$. The rest of the proof, we can prove in the same way as in the proof of odd prime version [12, Lemma 6.2 (b)]. □

Lemma 5.5. *Assume that k is a real abelian field of the first kind and $\psi = \chi\omega^i$ is a non-trivial, even character of $G = \mathrm{Gal}(k/\mathbb{Q})$. The Λ-module $\widetilde{\mathcal{C}}_{k_\infty,\psi}$ has a submodule of finite index generated by*

$$\{N_{\mathcal{G}}(\eta_{m_I})_{k_\infty,\psi}, (\eta_{m_I}^{\gamma-\kappa(\gamma)})_{k_\infty,\psi} \mid I \subset \mathcal{L}\}$$

and the Λ-module $\widetilde{\mathcal{C}}'_{k_\infty,\psi}$ has a submodule $\widetilde{\mathcal{C}}^2_{k_\infty,\psi}$ of finite index. Furthermore if $h_I = [\mathbb{Q}(m_I) : \mathbb{Q}(m_I) \cap F]$ is even for $I \subset \mathcal{L}$ then $\widetilde{\mathcal{C}}_{k_\infty,\psi}$ is generated by

$$\{(\epsilon^{-h_I/2}\eta_{m_I})_{k_\infty,\psi} \mid I \subset \mathcal{L}\}.$$

Proof. By Lemma 5.1 (b), we can show that the Λ-module $\widetilde{\mathcal{C}}_{k_\infty,\psi}$ has a submodule of finite index generated by

$$\{N_{\mathcal{G}}(\eta_{m_I})_{k_\infty,\psi}, (\eta_{m_I}^{\gamma-\kappa(\gamma)})_{k_\infty,\psi} \mid I \subset \mathcal{L}\}$$

similarly as the proof of the previous lemma. Similarly we can show that if h_I is even for $I \subset \mathcal{L}$ then $\widetilde{\mathcal{C}}_{k_\infty,\psi}$ is generated by

$$\{(\epsilon^{-h_I/2}\eta_{m_I})_{k_\infty,\psi} \mid I \subset \mathcal{L}\}$$

by Lemma 5.1 (b). Also we can see that $\widetilde{\mathcal{C}}'_{k_\infty,\psi}$ has a submodule $\widetilde{\mathcal{C}}^2_{k_\infty,\psi}$ of finite index by Lemma 5.1 (b). □

6. The proof of main theorems

Proof of Theorem 3.1. Coleman's power series of $\eta_f^{\xi_\chi}$ is

$$\prod_{\delta \in \mathrm{Gal}(\mathbb{Q}(f)/\mathbb{Q})} (1 - (1 - X)\zeta_f^{\delta^{-1}})^{\chi(\delta)}.$$

By using the formula (4.2) and Lemma 4.2 (a), for $k \equiv i \mod 2$,

$$
\begin{aligned}
&(-\kappa)^k(\Psi^{\psi}_{k_\infty}(\eta_f^{\xi_\chi}))z^\chi \\
&= \frac{1}{2}D^k\left(1-\frac{\varphi}{2}\right)\log\left(\prod_\delta(1-(1-X)\zeta_f^{\delta^{-1}})^{\chi(\delta)}\right)\Bigg|_{X=0} \\
&= \frac{1}{2}D^{k-1}(1-\varphi)D\log\left(\prod_\delta(1-(1-X)\zeta_f^{\delta^{-1}})^{\chi(\delta)}\right)\Bigg|_{X=0} \\
&= \frac{1}{2}\sum_\delta \delta^{-1}\chi(\delta)(D^{k-1}(1-\varphi)D\log(1-(1-X)\zeta_f))|_{X=0} \\
&= \frac{1}{2}\sum_\delta \delta^{-1}\chi(\delta)D^{k-1}\left(\frac{(1-X)\zeta_f}{1-(1-X)\zeta_f}-\frac{(1-X)^2\zeta_f^2}{1-(1-X)^2\zeta_f^2}\right)\Bigg|_{X=0} \\
&= \frac{1}{2}\sum_\delta \delta^{-1}\chi(\delta)D^{k-1}\sum_{a=1}^{f}\left(\frac{(1-X)^a\zeta_f^a}{1-(1-X)^f}-\frac{(1-X)^{2a}\zeta_f^{2a}}{1-(1-X)^{2f}}\right)\Bigg|_{X=0} \\
&= \frac{1}{2}D^{k-1}\sum_{a=1}^{f}\left(\frac{(1-X)^a\xi_\chi(\mathrm{Tr}(\zeta_f^a))}{1-(1-X)^f}-\frac{(1-X)^{2a}\xi_\chi(\mathrm{Tr}(\zeta_f^{2a}))}{1-(1-X)^{2f}}\right)\Bigg|_{X=0} \\
&= \frac{1}{2}D^{k-1}\sum_{a=1}^{f}\left(\frac{\chi(a)(1-X)^a}{1-(1-X)^f}-\frac{\chi(2a)(1-X)^{2a}}{1-(1-X)^{2f}}\right)z^\chi\Bigg|_{X=0} \\
&= \frac{1}{2}\left(-\frac{\mathrm{d}}{\mathrm{d}Z}\right)^{k-1}\sum_{a=1}^{f}\left(\frac{\chi(a)e^{aZ}}{1-e^{fZ}}-\frac{\chi(2a)e^{2aZ}}{1-e^{2fZ}}\right)z^\chi\Bigg|_{Z=0} \\
&= \frac{1}{2}\left(-\frac{\mathrm{d}}{\mathrm{d}Z}\right)^{k-1}\sum_{n=1}^{\infty}\left(-\frac{B_{n,\chi}}{n!}Z^{n-1}+\chi(2)\frac{B_{n,\chi}}{n!}(2Z)^{n-1}\right)z^\chi\Bigg|_{Z=0} \\
&= \frac{1}{2}(-1)^k(1-\chi(2)2^{k-1})\frac{B_{k,\chi}}{k}z^\chi \\
&= \frac{1}{2}(-1)^{k-1}g_\psi(\kappa(\gamma)^k-1)z^\chi \\
&= \frac{1}{2}(-\kappa)^k(-g_\psi(\gamma-1))z^\chi
\end{aligned}
$$

where δ runs over all elements of $\mathrm{Gal}(\mathbb{Q}(f)/\mathbb{Q})$, $\mathrm{Tr} = \mathrm{Tr}_{\mathbb{Q}(f)/\mathbb{Q}(f)\cap F}$ and $1-X = e^Z$, so $D = (1-X)d/dX = -d/dZ$. Therefore we obtain

$$\Psi^{\psi}_{k_\infty}(\eta_f^{\xi_\chi}) = -\frac{1}{2}g_\psi(T).$$

By Lemma 5.3, we have

$$\Psi^{\psi}_{k_\infty}(\mathcal{C}^\psi) = (g_\psi(T)/2) \subset \Lambda.$$

Therefore Theorem 3.1 follows from Proposition 4.6. □

Proof of Theorems 3.2, 3.4 and 3.5. We first calculate $\Psi_{K_\infty,\psi}((\eta_{m_I})_{K_\infty,\psi})$. Let M be the fixed field of $\ker\chi$ and $L_n = M(\boldsymbol{\mu}_{2^{n+2}})$ for $0 \le n \le \infty$. We see that

$$\begin{aligned}
&N_{K_\infty/L_\infty}(\eta_{m_I})\\
&\quad = (N_{K_n/L_n} N_{\mathbb{Q}(2^{n+2}m_I)/\mathbb{Q}(2^{n+2}m_I)\cap K_n}(1-\zeta_{2^{n+2}}\zeta_{m_I}^{\sigma^{-n}}))_{n\ge 0}\\
&\quad = (N_{\mathbb{Q}(2^{n+2}m_I)/L_n}(1-\zeta_{2^{n+2}}\zeta_{m_I}^{\sigma^{-n}})^{[K_n:\mathbb{Q}(2^{n+2}m_I)\cap K_n]})_{n\ge 0}\\
&\quad = (N_{\mathbb{Q}(2^{n+2}f)/L_n} N_{\mathbb{Q}(2^{n+2}m_I)/\mathbb{Q}(2^{n+2}f)}(1-\zeta_{2^{n+2}}\zeta_{m_I}^{\sigma^{-n}})^{[F:\mathbb{Q}(m_I)\cap F]})_{n\ge 0}\\
&\quad = (\eta'_f)^{[F:\mathbb{Q}(m_I)\cap F]\prod_{l\in I}(\gamma_l-\sigma_l^{-1})}
\end{aligned}$$

where γ_l (resp. σ_l) is the Frobenius element of l in G_∞ (resp. $\mathrm{Gal}(M/\mathbb{Q})$) and we put $\eta'_f = (N_{\mathbb{Q}(2^{n+2}f)/L_n}(1-\zeta_{2^{n+2}}\zeta_f^{\sigma^{-n}}))_{n\ge 0}$. We have, by (4.4)

$$\begin{aligned}
&\Psi_{K_\infty,\psi}((\eta_{m_I})_{K_\infty,\psi})\\
&= \Psi_{L_\infty,\psi}(N_{K_\infty/L_\infty}(\eta_{m_I})_{L_\infty,\psi})\\
&= \Psi_{L_\infty,\psi}(((\eta'_f)^{[F:\mathbb{Q}(m_I)\cap F]\prod_{l\in I}(\gamma_l-\sigma_l^{-1})})_\psi)\\
&= [F:\mathbb{Q}(m_I)\cap F]\left(\prod_{l\in I}(\omega^i(l)(1+T)^{t_l}-\chi(l)^{-1})\right)\Psi_{L_\infty,\psi}((\eta'_f)_{L_\infty,\psi})\\
&= [F:\mathbb{Q}(m_{\mathcal{L}})\cap F]d_I\left(\prod_{l\in I}-\chi(l)^{-1}(1-\psi(l)(1+T)^{t_l})\right)\Psi_{L_\infty,\psi}((\eta'_f)_{L_\infty,\psi}).
\end{aligned}$$

For $k\equiv i \mod 2$, by the formula (4.3) and Lemma 4.2 (b), we have

$$\begin{aligned}
&(-\kappa)^k\Psi_{L_\infty,\psi}((\eta'_f)_{L_\infty,\psi})z_{M,\chi}\\
&\quad = (D^k(1-\frac{\varphi}{2})\log N(1-(1-X)\zeta_f)|_{X=0})_{M,\chi}\\
&\quad = (D^{k-1}(1-\varphi)\,\mathrm{Tr}(D\log(1-(1-X)\zeta_f))|_{X=0})_{M,\chi}\\
&\quad = \left(D^{k-1}\sum_{a=1}^{f}\left(\frac{(1-X)^a\,\mathrm{Tr}(\zeta_f^a)}{1-(1-X)^f}-\frac{(1-X)^{2a}\,\mathrm{Tr}(\zeta_f^{2a})}{1-(1-X)^{2f}}\right)\Bigg|_{X=0}\right)_{M,\chi}\\
&\quad = D^{k-1}\sum_{a=1}^{f}\left(\frac{(1-X)^a\,\mathrm{Tr}(\zeta_f^a)_{M,\chi}}{1-(1-X)^f}-\frac{(1-X)^{2a}\,\mathrm{Tr}(\zeta_f^{2a})_{M,\chi}}{1-(1-X)^{2f}}\right)\Bigg|_{X=0}\\
&\quad = D^{k-1}\sum_{a=1}^{f}\left(\frac{\chi(a)(1-X)^a}{1-(1-X)^f}-\frac{\chi(2a)(1-X)^{2a}}{1-(1-X)^{2f}}\right)z_{M,\chi}\Bigg|_{X=0}
\end{aligned}$$

where $N = N_{\mathbb{Q}(f)/M}$ and $\mathrm{Tr} = \mathrm{Tr}_{\mathbb{Q}(f)/M}$. Then we have

$$\Psi_{L_\infty,\psi}((\eta'_f)_{L_\infty,\psi}) = -g_\psi(T),$$

and hence

$$\Psi_{K_\infty,\psi}((\eta_{m_I})_{K_\infty,\psi}) = v_I d_I \left(\prod_{l\in I}(1-\psi(l)(1+T)^{t_l})\right) g_\psi(T) \tag{6.1}$$

where we put $v_I = -[F:\mathbb{Q}(m_{\mathcal{L}})\cap F]\prod_{l\in I}(-\chi(l)^{-1})$ and this is a unit in Λ.

Assume $k_\infty = K_\infty$. In this case, the Galois group $\mathcal{G}$ is trivial, and then we have

$$\begin{aligned}\Psi_{k_\infty,\psi}(N_{\mathcal{G}}(\eta_{m_I})_{k_\infty,\psi}) &= \Psi_{K_\infty,\psi}((\eta_{m_I})_{K_\infty,\psi})\\ &= v_I d_I \prod_{l\in I}(1-\psi(l)(1+T)^{t_l}) g_\psi(T).\end{aligned}$$

Assume $k_\infty \neq K_\infty$. By Proposition 4.4 (b), we have

$$\begin{aligned}&\Psi_{k_\infty,\psi}(N_{\mathcal{G}}(\eta_{m_I})_{k_\infty,\psi}) z_{F,\chi}\\ &\qquad= \frac{1}{2}\omega^i(\Psi_{K_\infty}(N_{\mathcal{G}}(\eta_{m_I}))_\chi)\\ &\qquad= \frac{1}{2}(\omega^i(\Psi_{K_\infty}(\eta_{m_I})_\chi) + \omega^i(\Psi_{K_\infty}(\eta_{m_I}^g)_\chi))\\ &\qquad= \frac{1}{2}(\Psi_{K_\infty,\psi}(\eta_{m_I})_{K_\infty,\psi}) + \Psi_{K_\infty,\psi}((\eta_{m_I}^g)_{K_\infty,\psi}))z_{F,\chi}\\ &\qquad= \frac{1}{2}(\Psi_{K_\infty,\psi}((\eta_{m_I})_{K_\infty,\psi}) + \Psi_{K_\infty,\psi}((\eta_{m_I})_{K_\infty,\psi}^{\psi(g)}))z_{F,\chi}\\ &\qquad= \frac{1}{2}(\Psi_{K_\infty,\psi}((\eta_{m_I})_{K_\infty,\psi}) + \Psi_{K_\infty,\psi}((\eta_{m_I})_{K_\infty,\psi}))z_{F,\chi}\\ &\qquad= \Psi_{K_\infty,\psi}((\eta_{m_I})_{K_\infty,\psi})z_{F,\chi},\end{aligned}$$

and hence, by (6.1),

$$\Psi_{k_\infty,\psi}(N_{\mathcal{G}}(\eta_{m_I})_{k_\infty,\psi}) = v_I d_I \left(\prod_{l\in I}(1-\psi(l)(1+T)^{t_l})\right) g_\psi(T).$$

By Lemma 5.4, we have

$$\begin{aligned}\Psi_{k_\infty,\psi}(\widetilde{\mathcal{C}}'_{k_\infty,\psi}) &= \left\langle d_I \prod_{l\in I}(1-\psi(l)(1+T)^{t_l}) g_\psi(T) \;\middle|\; I\subset\mathcal{L}\right\rangle\\ &= \mathfrak{A} g_\psi(T) \subset \Lambda.\end{aligned}$$

We can see that $\ker(\Psi_{k_\infty,\psi}) \cap \widetilde{\mathcal{C}}'_{k_\infty,\psi} = \{1\}$. Therefore Theorem 3.5 follows from Proposition 4.7.

Assume that k is real. We will calculate the values $\Psi_{k_\infty,\psi}((\eta_{m_I}^{\gamma-\kappa(\gamma)})_{k_\infty,\psi})$ and $\Psi_{k_\infty,\psi}((\epsilon^{-h_I/2}\eta_{m_I})_{k_\infty,\psi})$. We can see that

$$\begin{aligned}\Psi_{k_\infty,\psi}((\eta_{m_I}^{\gamma-\kappa(\gamma)})_{k_\infty,\psi})z_{F,\chi} &= \frac{1}{2}\omega^i(\Psi_{K_\infty}(\eta_{m_I}^{\gamma-\kappa(\gamma)})_\chi)\\ &= \frac{1}{2}(\gamma-\kappa(\gamma))\omega^i(\Psi_{K_\infty}(\eta_{m_I})_\chi)\\ &= \frac{1}{2}\dot{T}\Psi_{K_\infty,\psi}((\eta_{m_I})_{K_\infty,\psi})z_{F,\chi}\end{aligned}$$

and hence, by (6.1),

$$\Psi_{k_\infty,\psi}((\eta_{m_I}^{\gamma-\kappa(\gamma)})_{k_\infty,\psi}) = v_I d_I\left(\prod_{l\in I}(1-\psi(l)(1+T)^{t_l})\right)\frac{1}{2}\dot{T}g_\psi(T).$$

Furthermore, we can see that

$$\begin{aligned}\Psi_{k_\infty,\psi}((\epsilon^{-h_I/2}\eta_{m_I})_{k_\infty,\psi})z_{F,\chi} &= \frac{1}{2}\omega^i(\Psi_{K_\infty}(\epsilon^{-h_I/2}\eta_{m_I})_\chi)\\ &= \frac{1}{2}\omega^i(\Psi_{K_\infty}(\epsilon^{-h_I/2})_\chi + \Psi_{K_\infty}(\eta_{m_I})_\chi)\\ &= \frac{1}{2}\omega^i(\Psi_{K_\infty}(\eta_{m_I})_\chi)\\ &= \frac{1}{2}\Psi_{K_\infty,\psi}((\eta_{m_I})_{K_\infty,\psi})z_{F,\chi}\end{aligned}$$

and hence, by (6.1),

$$\Psi_{k_\infty,\psi}((\epsilon^{-h_I/2}\eta_{m_I})_{k_\infty,\psi}) = v_I d_I\left(\prod_{l\in I}(1-\psi(l)(1+T)^{t_l})\right)\frac{1}{2}g_\psi(T).$$

Let $\widetilde{\mathcal{C}''}_{k_\infty,\psi}$ be the submodule of $\widetilde{\mathcal{C}}_{k_\infty,\psi}$ generated by

$$\{N_{\mathcal{G}}(\eta_{m_I})_{k_\infty,\psi}, (\eta_{m_I}^{\gamma-\kappa(\gamma)})_{k_\infty,\psi} \mid I\subset\mathcal{L}\}.$$

By the above calculation, we have

$$\begin{aligned}&\Psi_{k_\infty,\psi}(\widetilde{\mathcal{C}''}_{k_\infty,\psi})\\ &= \langle 2d_I\prod_{l\in I}(1-\psi(l)(1+T)^{t_l}), \dot{T}d_I\prod_{l\in I}(1-\psi(l)(1+T)^{t_l}) \mid I\subset\mathcal{L}\rangle g_\psi(T)/2\\ &= (2,\dot{T})\mathfrak{A}(g_\psi(T)/2).\end{aligned}$$

By Lemma 5.5, $\widetilde{\mathcal{C}}_{k_\infty,\psi} \supset \widetilde{\mathcal{C}''}_{k_\infty,\psi}$ and $\langle N_{\mathcal{G}}(\eta_{m_I})_{k_\infty,\psi} \mid I \subset \mathcal{L}\rangle = \widetilde{\mathcal{C}'}_{k_\infty,\psi} \supset \widetilde{\mathcal{C}}^2_{k_\infty,\psi}$, so we have

$$\Psi_{k_\infty,\psi}(\widetilde{\mathcal{C}}_{k_\infty,\psi}) \supset (2,\dot{T})\mathfrak{A}(g_\psi(T)/2),$$
$$\Psi_{k_\infty,\psi}(\widetilde{\mathcal{C}'}_{k_\infty,\psi}) = \mathfrak{A}g_\psi(T) \supset 2\Psi_{k_\infty,\psi}(\widetilde{\mathcal{C}}_{k_\infty,\psi}).$$

Since Λ is an integral domain, we have

$$\mathfrak{A}g_\psi(T)/2 \supset \Psi_{k_\infty,\psi}(\widetilde{\mathcal{C}}_{k_\infty,\psi}) \supset (2,\dot{T})\mathfrak{A}(g_\psi(T)/2).$$

Therefore there is an ideal $\mathfrak{A}'$ of Λ such that $\mathfrak{A} \supset \mathfrak{A}' \supset (\dot{T},2)\mathfrak{A}$ and $\Psi_{k_\infty,\psi}(\widetilde{\mathcal{C}}_{k_\infty,\psi}) = \mathfrak{A}'(g_\psi(T)/2)$. This completes the proof of Theorem 3.2.

Assume that k is imaginary. We put

$$\eta_t^+ = (N_{\mathbb{Q}(2^{n+2}t)/\mathbb{Q}(2^{n+2}t)\cap k_n^+(\zeta_4)}(1-\zeta_{2^{n+2}}\zeta_t^{\sigma^{-n}}))_{n\geq 0} \in \varprojlim C_{k_n^+(\zeta_4)}$$

for $t \mid m,\ t \neq 1$. Recall $\mathcal{C}_{k_\infty} = \mathcal{C}'_{k_\infty} \cdot \mathcal{C}_{k_\infty^+}$. Then $\widetilde{\mathcal{C}}_{k_\infty,\psi}$ has a submodule of finite index generated by

$$\{N_{\mathcal{G}}(\eta_{m_I})_{k_\infty,\psi},\ N_{k_\infty^+(\zeta_4)/k_\infty^+}(\eta_{m_I}^+)_{k_\infty,\psi},\ ((\eta_{m_I}^+)^{\gamma-\kappa(\gamma)})_{k_\infty,\psi} \mid I \subset \mathcal{L}\}$$

and if $2 \mid h_I^+ = [\mathbb{Q}(4m_I) : \mathbb{Q}(4m_I) \cap k^+(\zeta_4)]$ for $I \subset \mathcal{L}$, then $\mathcal{C}_{k_\infty,\psi}$ is generated by

$$\{(\epsilon^{-h_I^+/2}\eta_{m_I}^+)_{k_\infty,\psi}, N_{\mathcal{G}}(\eta_{m_I})_{k_\infty,\psi} \mid I \subset \mathcal{L}\}$$

by Lemma 5.1. We put $\alpha_I = [\mathbb{Q}(4m_I) \cap K_0 : \mathbb{Q}(4m_I) \cap k^+(\zeta_4)]$, which is 1 or 2. Then $\eta_{m_I}^+ = \eta_{m_I}$ or $N_{K_\infty/k_\infty^+(\zeta_4)}(\eta_{m_I})$ if $\alpha_I = 1$ or 2 respectively. Since $k_\infty^+(\zeta_4) \supset k_\infty^\psi$, we have $(N_{K_\infty/k_\infty^+(\zeta_4)}(\eta_{m_I}))_{K_\infty,\psi} = (\eta_{m_I})^2_{K_\infty,\psi}$. Therefore

$$(\eta_{m_I}^+)_{K_\infty,\psi} = (\eta_{m_I})^{\alpha_I}_{K_\infty,\psi}$$

By Proposition 4.4, we compute the following:

$$\begin{aligned}\Psi_{k_\infty,\psi}((\epsilon^{-h_I^+/2}\eta_{m_I}^+)_{k_\infty,\psi})z_{F,\chi} &= \frac{1}{[K_0:k]}\omega^i(\Psi_{K_\infty}(\epsilon^{-h_I/2}\eta_{m_I}^+)_\chi)\\ &= \frac{1}{[K_0:k]}\omega^i(\Psi_{K_\infty}(\eta_{m_I}^+)_\chi)\\ &= \frac{1}{[K_0:k]}\Psi_{K_\infty,\psi}((\eta_{m_I}^+)_{K_\infty,\psi})z_{F,\chi}\\ &= \frac{\alpha_I}{[K_0:k]}\Psi_{K_\infty,\psi}((\eta_{m_I})_{K_\infty,\psi})z_{F,\chi},\end{aligned}$$

$$\begin{aligned}
\Psi_{k_\infty,\psi}(N_{k_\infty^+(\zeta_4)/k_\infty^+}(\eta_{m_I}^+)_{k_\infty,\psi})z_{F,\chi}
&= \frac{1}{[K_0:k]}\omega^i(\Psi_{K_\infty}(N_{k_\infty^+(\zeta_4)/k_\infty^+}(\eta_{m_I}^+))_\chi) \\
&= \frac{1}{[K_0:k]}\Psi_{K_\infty,\psi}((N_{k_\infty^+(\zeta_4)/k_\infty^+}(\eta_{m_I}^+))_{K_\infty,\psi})z_{F,\chi} \\
&= \frac{1}{[K_0:k]}\Psi_{K_\infty,\psi}((\eta_{m_I}^+)^2_{K_\infty,\psi})z_{F,\chi} \\
&= \frac{2\alpha_I}{[K_0:k]}\Psi_{K_\infty,\psi}((\eta_{m_I})_{K_\infty,\psi})z_{F,\chi},
\end{aligned}$$

and

$$\begin{aligned}
\Psi_{k_\infty,\psi}(((\eta_{m_I}^+)^{\gamma-\kappa(\gamma)})_{k_\infty,\psi})z_{F,\chi} &= \frac{1}{[K_0:k]}\omega^i(\Psi_{K_\infty}((\eta_{m_I}^+)^{\gamma-\kappa(\gamma)})_\chi) \\
&= \frac{1}{[K_0:k]}(\gamma-\kappa(\gamma))\omega^i(\Psi_{K_\infty}(\eta_{m_I}^+)_\chi) \\
&= \frac{1}{[K_0:k]}\dot{T}\Psi_{K_\infty,\psi}((\eta_{m_I}^+)_{K_\infty,\psi})z_{F,\chi} \\
&= \frac{\alpha_I}{[K_0:k]}\dot{T}\Psi_{K_\infty,\psi}((\eta_{m_I})_{K_\infty,\psi})z_{F,\chi}.
\end{aligned}$$

Therefore, by (6.1), we have

$$\begin{aligned}
\Psi_{k_\infty,\psi}((\epsilon^{-h_I^+/2}\eta_{m_I}^+)_{k_\infty,\psi}) &= \frac{v_I d_I \alpha_I}{[K_0:k]}\left(\prod_{l\in I}(1-\psi(l)(1+T)^{t_l})\right)g_\psi(T), \\
\Psi_{k_\infty,\psi}(N_{k_\infty^+(\zeta_4)/k_\infty^+}(\eta_{m_I}^+)_{k_\infty,\psi}) &= \frac{2v_I d_I \alpha_I}{[K_0:k]}\left(\prod_{l\in I}(1-\psi(l)(1+T)^{t_l})\right)g_\psi(T), \\
\Psi_{k_\infty,\psi}(((\eta_{m_I}^+)^{\gamma-\kappa(\gamma)})_{k_\infty,\psi}) &= \frac{v_I d_I \alpha_I}{[K_0:k]}\left(\prod_{l\in I}(1-\psi(l)(1+T)^{t_l})\right)\dot{T}g_\psi(T).
\end{aligned}$$

Since k is imaginary, we have $[k:k^+]=[k^+(\zeta_4):k^+]=2$, and hence $[K_0:k]=[K_0:k^+(\zeta_4)]$. Recall that $v_I=-[F:\mathbb{Q}(m_{\mathcal{L}})\cap F]\prod_{l\in I}(-\chi(l)^{-1})=-[K_0:\mathbb{Q}(4m_{\mathcal{L}})\cap K_0]\prod_{l\in I}(-\chi(l)^{-1})$ and $d_I=[\mathbb{Q}(4m_{\mathcal{L}})\cap K_0:\mathbb{Q}(4m_I)\cap K_0]$. Then we have

$$\begin{aligned}
\frac{v_I d_I \alpha_I}{[K_0:k]} &= -\frac{[K_0:\mathbb{Q}(4m_I)\cap k^+(\zeta_4)]}{[K_0:k^+(\zeta_4)]}\left(\prod_{l\in I}-\chi(l)\right) \\
&= -[k^+(\zeta_4):\mathbb{Q}(4m_I)\cap k^+(\zeta_4)]\left(\prod_{l\in I}-\chi(l)\right).
\end{aligned}$$

We can show that

$$\mathfrak{A}_{k^+,\psi} = \left\langle \frac{\alpha_I d_I}{[K_0 : k]} \left(\prod_{l \in I} (1 - \psi(l)(1+T)^{t_l}) \right) \,\middle|\, I \subset \mathcal{L} \right\rangle \supset \mathfrak{A}_{k,\psi}.$$

Using $\langle N_{k_\infty^+(\zeta_4)/k_\infty^+}(\eta_{m_I}^+)_{k_\infty,\psi} \mid I \subset \mathcal{L}\rangle = \widetilde{\mathcal{C}'}_{k_\infty^+,\psi} \supset \widetilde{\mathcal{C}}^2_{k_\infty^+,\psi}$, we have

$$\Psi_{k_\infty,\psi}(\widetilde{\mathcal{C}}'_{k_\infty^+,\psi}) = 2\mathfrak{A}_{k^+,\psi} g_\psi(T)$$

and hence

$$\mathfrak{A}_{k^+,\psi} g_\psi(T) \supset \Psi_{k_\infty,\psi}(\widetilde{\mathcal{C}}_{k_\infty,\psi}) \supset (\mathfrak{A}_{k,\psi} + (2,\dot{T})\mathfrak{A}_{k^+,\psi}) g_\psi(T).$$

If $2 \mid h_I^+$ for $I \subset \mathcal{L}$, then $\Psi_{k_\infty,\psi}(\widetilde{\mathcal{C}}_{k_\infty,\psi}) = \mathfrak{A}_{k^+,\psi} g_\psi(T)$. We note that if $\zeta_4 \in k$, then $\mathfrak{A}_{k^+,\psi} = \mathfrak{A}_{k,\psi}$. This completes the proof of Theorem 3.4. □

Remark 6.1. By Lemma 5.5, if $[\mathbb{Q}(t) : \mathbb{Q}(t) \cap F]$ is even for $t \mid m, t \neq 1$ then $\widetilde{\mathcal{C}}_{k_\infty,\psi}$ is generated by

$$\{(\epsilon^{-h_I/2}\eta_{m_I})_{k_\infty,\psi} \mid I \subset \mathcal{L}\}$$

where $h_I = [\mathbb{Q}(m_I) : \mathbb{Q}(m_I) \cap F]$. In this case, using the above calculation, we have

$$\Psi_{k_\infty,\psi}(\widetilde{\mathcal{C}}_{k_\infty,\psi}) = \mathfrak{A}(g_\psi(T)/2).$$

Then under the condition that $[\mathbb{Q}(t) : \mathbb{Q}(t) \cap F]$ is even for $t \mid m, t \neq 1$, we have $\mathfrak{A}' = \mathfrak{A}$. In Remark 5.2, we mentioned that if 2 is unramified in k then this condition holds. Therefore, if 2 is unramified in k, it holds that $\mathfrak{A}' = \mathfrak{A}$ in Theorem 3.2.

7. μ-invariants and the Iwasawa main conjecture

Let $\mathcal{M}$ be the maximal abelian pro 2-extension of k_∞ unramified outside all primes over 2 and put

$$\mathfrak{X} = \mathrm{Gal}(\mathcal{M}/k_\infty).$$

As usual, $\mathfrak{X}$ is a module over $\mathbb{Z}_2[G][\![\Gamma]\!]$ so Λ-modules $\mathfrak{X}^\psi$ and $\mathfrak{X}_\psi$ are defined. We will consider the μ-invariants of $\mathfrak{X}^\psi$ and $\mathfrak{X}_\psi$.

Assume that k is real. In this case, it is known that $\mathfrak{X}$ is a finite generated torsion $\mathbb{Z}_2[\![\Gamma]\!]$-module. Furthermore, in [7], it is shown that the μ-invariant of $\mathfrak{X}$ is zero by using Ferrero's result [3]. Therefore we have $\mu(\mathfrak{X}^\psi) = 0$ and $\mu(\mathfrak{X}_\psi) = 0$.

Assume that k is imaginary. In this case, the $\mathbb{Z}_2[\![\Gamma]\!]$-rank of $\mathfrak{X}$ is equal to $[k : \mathbb{Q}]/2$. Let k^+ be the maximal real subfield of k and J the generator of $\mathrm{Gal}(k/k^+) \cong \mathrm{Gal}(k_\infty/k_\infty^+)$, i.e. J is the complex conjugation. We put

$$\mathfrak{X}^+ = \{x \in \mathfrak{X} \mid Jx = x\}, \quad \mathfrak{X}_+ = \mathfrak{X}/(J-1)\mathfrak{X}.$$

Since ψ is even, we regard ψ as a character of $\mathrm{Gal}(k^+/\mathbb{Q})$ and we have

$$\mathfrak{X}^\psi \cong (\mathfrak{X}^+)^\psi, \quad \mathfrak{X}_\psi \cong (\mathfrak{X}_+)_\psi.$$

Let $\mathcal{M}^+$ be the maximal abelian pro 2-extension of k_∞^+ unramified outside all primes over 2. We can show that $\mathfrak{X}^+$ is pseudo-isomorphic to $\mathrm{Gal}(\mathcal{M}^+/k_\infty^+)$. Therefore the μ-invariant of $\mathfrak{X}^\psi$ is zero. Let $\mathcal{M}'/k_\infty^+$ be the maximal abelian subfield of $\mathcal{M}/k_\infty^+$, then $\mathcal{M}'$ is the fixed field of $(J-1)\mathfrak{X}$, i.e. $\mathrm{Gal}(\mathcal{M}'/k_\infty) \cong \mathfrak{X}_+$. Let $\widetilde{\mathcal{M}}^+$ be the maximal abelian pro 2-extension of k_∞^+ unramified outside all primes over 2 and all infinite primes. Then $k_\infty^+ \subset k_\infty \subset \widetilde{\mathcal{M}}^+ \subset \mathcal{M}'$. Since all infinite primes are totally ramified in k_∞/k_∞^+ and the number of finite primes of k_∞^+ which ramified in $\mathcal{M}'$ is finite, the degree $[\mathcal{M}' : \widetilde{\mathcal{M}}^+]$ is finite. Therefore the kernel and the cokernel of the restriction map

$$\mathfrak{X}_+ \longrightarrow \mathrm{Gal}(\widetilde{\mathcal{M}}^+/k_\infty^+)$$

are finite. By [6, Proposition 8], the torsion submodule of $\mathrm{Gal}(\widetilde{\mathcal{M}}^+/k_\infty^+)$ is pseudo-isomorphic to $(\mathbb{Z}_2[\![\Gamma]\!]/(2))[\mathrm{Gal}(k^+/\mathbb{Q})]$. Therefore $\mu(\mathfrak{X}_\psi) = 1$.

Summarizing the above, we have

$$\mu(\mathfrak{X}^\psi) = 0 \tag{7.1}$$

and

$$\mu(\mathfrak{X}_\psi) = \begin{cases} 0 & \text{if } k \text{ is real,} \\ 1 & \text{if } k \text{ is imaginary.} \end{cases} \tag{7.2}$$

By Theorem 3.1, 3.2 and 3.4, the μ-invariant of $\mathfrak{X}^\psi$ and $\mathfrak{X}_\psi$ coincide with that of $\mathcal{U}^\psi/\mathcal{C}^\psi$ and $(\mathcal{U}/\mathcal{C})_\psi$ respectively, that is, we obtain the following:

Theorem 7.1. *For an abelian field k of the first kind and an even character ψ of* $\mathrm{Gal}(k/\mathbb{Q})$*,*

$$\mu(\mathfrak{X}^\psi) = \mu(\mathcal{U}^\psi/\mathcal{C}^\psi), \qquad \mu(\mathfrak{X}_\psi) = \mu((\mathcal{U}/\mathcal{C})_\psi).$$

Put

$$W = \mathfrak{X} \otimes_{\mathbb{Z}_2} \overline{\mathbb{Q}_2}$$

and define $W^{(\psi)}$ the eigenspace of W corresponding to the action of G via ψ and $\mathrm{char}_\Lambda(W^{(\psi)})$ the characteristic polynomial of T acting on this space. We see that

$$W^{(\psi)} \cong \mathfrak{X}^\psi \otimes_{\mathbb{Z}_2} \overline{\mathbb{Q}_2} \cong \mathfrak{X}_\psi \otimes_{\mathbb{Z}_2} \overline{\mathbb{Q}_2}.$$

The Iwasawa main conjecture proved by Wiles [14] is that

$$\mathrm{char}_\Lambda(W^{(\psi)}) = \frac{1}{2} g_\psi(T).$$

It is known that $\mu(\frac{1}{2} g_\psi(T)) = 0$ by Ferrero–Washington [3, 4]. By (7.1) and (7.2), the Iwasawa main conjecture is the following:

Theorem 7.2. *For an abelian field k of the first kind and an even character ψ,*

$$\mathrm{char}_\Lambda(\mathfrak{X}^\psi) = g_\psi(T)/2$$

and

$$\mathrm{char}_\Lambda(\mathfrak{X}_\psi) = \begin{cases} g_\psi(T)/2 & \text{if } k \text{ is real,} \\ g_\psi(T) & \text{if } k \text{ is imaginary.} \end{cases}$$

By Theorems 3.1, 3.2 and 3.4, we can show the following:

Theorem 7.3. *For an abelian field k of the first kind and an even character ψ of* $\mathrm{Gal}(k/\mathbb{Q})$*,*

$$\mathrm{char}_\Lambda(\mathfrak{X}^\psi) = \mathrm{char}_\Lambda(\mathcal{U}^\psi/\mathcal{C}^\psi), \quad \mathrm{char}_\Lambda(\mathfrak{X}_\psi) = \mathrm{char}_\Lambda((\mathcal{U}/\mathcal{C})_\psi).$$

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