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Non-totally real number fields and toroidal groups

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Résumé. Dans cet article, nous étudions la relation entre les corps de nombres non totalement réels K et les groupes toroïdaux $\mathcal{T}$, ainsi que les fonctions périodiques méromorphes, en exploitant une représentation de $\mathcal{T}$ en termes de la jacobienne généralisée $\mathfrak{J}_L(\mathcal{C})$ d'une courbe elliptique appropriée $\mathcal{C}$. Nous considérons en détail les cas cubique et quartique.

Dans ces cas, nous écrivons les relations entre le polynôme minimal d'un élément primitif convenable de K et les paramètres définissant la jacobienne généralisée $\mathfrak{J}_L(\mathcal{C})$ correspondant au groupe toroïdal associé à l'anneau des entiers. En outre, pour un tel groupe toroïdal, nous décrivons explicitement les représentations analytique et rationnelle de son anneau d'endomorphismes, le premier donnant une nouvelle représentation (complexe) de l'anneau des entiers de K.

De plus, dans le cas cubique, nous donnons une description explicite de la m-torsion de $\mathcal{T}$ en termes de la correspondance géométrique entre $\mathcal{T}$ et $\mathfrak{J}_L(\mathcal{C})$ comme l'image d'un idéal fractionnaire de K.

Abstract. In this paper we study the relationship between non-totally real number fields K and toroidal groups $\mathcal{T}$, as well as meromorphic periodic functions, exploiting a representation of $\mathcal{T}$ as the generalized Jacobian $\mathfrak{J}_L(\mathcal{C})$ of a suitable elliptic curve $\mathcal{C}$. We consider in detail the cubic and quartic cases.

In these cases, we write down the relations between the minimal polynomial of a suitable primitive element of K and the parameters defining the generalized Jacobian $\mathfrak{J}_L(\mathcal{C})$ corresponding to the toroidal group associated with the ring of integers. Furthermore, for such a toroidal group we explicitly show the analytic and rational representations of its ring of endomorphisms, the former giving in turn a new (complex) representation of the ring of integers of K.

Moreover, for the cubic case, we give an explicit description of the m-torsion of $\mathcal{T}$ in the geometric correspondence of $\mathcal{T}$ with $\mathfrak{J}_L(\mathcal{C})$, as image of a fractional ideal of K.

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Mots-clefs. Toroidal groups; non-totally real number fields; generalized Jacobians.

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Introduction

In the present paper we highlight the relationship among apparently unrelated objects, i.e. non-totally real number fields (with one complex embedding), generalized Jacobians (of an elliptic curve), and n-variate meromorphic periodic functions (with $n+1$ $\mathbb{R}$-independent periods), through their connection with toroidal groups (of dimension n and real rank $n+1$). This relationship allows us to achieve the results introduced in the Abstract.

A *toroidal group* is a complex Lie group $\mathbb{C}^n/\Lambda$, with Λ a lattice of $\mathbb{C}^n$, which admits no non-constant holomorphic function. A quotient group $\mathbb{C}^n/\Lambda$, with $\mathrm{rk}_{\mathbb{C}}\,\Lambda = n$, is toroidal if, and only if, the irrationality condition holds, that is, there exists no non-zero $\mathbb{C}$-linear homomorphism $\varphi : \mathbb{C}^n \to \mathbb{C}$ such that $\varphi(\Lambda) \subseteq \mathbb{Z}$ (see [2, Thm. 1.1.4 (2) p. 5]).

Hence, the first connection we recall is the one between toroidal groups $\mathbb{C}^n/\Lambda$, with real rank $n+1$, and n-variate meromorphic functions with $n+1$ $\mathbb{R}$-independent periods, which we can assume, up to a change of basis, to be in the standard form $(1,0,\ldots,0),\ldots,(0,0,\ldots,1)$, and $(\tau, s_1,\ldots,s_{n-1})$, with $\tau \notin \mathbb{R}$.

Note that, if one can write, for some $(a_0,\ldots,a_{n-1}) \in \mathbb{Z}^n \setminus (0,\ldots,0)$,

$$a_0\tau + a_1 s_1 + \cdots + a_{n-1}s_{n-1} \in \mathbb{Z},$$

then the function

$$G(z_0, z_1, \ldots, z_{n-1}) = \exp\left(2\pi\, i(a_0 z_0 + a_1 z_1 + \cdots + a_{n-1}z_{n-1})\right),$$

is, trivially, holomorphic, non-constant and periodic on

$$\Lambda = \langle (1,0,\ldots,0),\ldots,(0,0,\ldots,1),(\tau, s_1,\ldots,s_{n-1})\rangle_{\mathbb{Z}},$$

the corresponding quotient group $\mathbb{C}^n/\Lambda$ hence being, by definition, not toroidal.

The above remark allows us to confine ourselves to the case of toroidal groups, for which, on the contrary, we describe a method for representing non-trivial n-variate meromorphic (non-holomorphic) functions with the $n+1$ $\mathbb{R}$-independent periods $(1,0,\ldots,0),\ldots,(0,0,\ldots,1),(\tau,s_1,\ldots,s_{n-1})$, by means of the elliptic curve $\mathcal{C}$ having fundamental parallelogram generated by $\{1,\tau\}$.

The connection between non-totally real number fields of degree $\delta = n+1$ and isogeny classes of toroidal groups $\mathcal{T}$ of dimension n with extra multiplications (that is, such that the multiplications by an integer do not exhaust the whole ring $\mathrm{End}(\mathcal{T})$) was introduced in 1973 by Andreotti and Gherardelli in [4] and investigated in 2012, in the case where $\delta = r_1 + 2r_2$ with $r_2 = 1$ (hence $n = r_1+1$), i.e. the case where K has only two conjugate complex embeddings, by Vallières in [16]: let K be a non-totally real number field with $r_2 = 1$, $\sigma_1,\ldots,\sigma_{r_1}\colon K \to \mathbb{R} \subset \mathbb{C}$ the real embeddings, and $\sigma_n : K \to \mathbb{C}$ one of the two conjugate complex embeddings of K. For any fractional ideal $\frac{1}{\nu}\mathcal{J}$ of K, where $\nu \in \mathcal{O}_K$ and $\mathcal{J}$ is an ideal of $\mathcal{O}_K$, the Minkowski embedding $\mu : K \to \mathbb{C}^n$, defined by the map

$$z \longmapsto (\sigma_1(z), \ldots, \sigma_{r_1}(z), \sigma_n(z)),$$

gives in turn the lattice $\mu(\frac{1}{\nu}\mathcal{J})$ of $\mathbb{C}^n$, which has real rank equal to the rank of $\frac{1}{\nu}\mathcal{J}$, that is, $n+1$. The quotient group $\mathcal{T} = \mathbb{C}^n/\mu(\frac{1}{\nu}\mathcal{J})$ is proved to be a toroidal group ([4], see also [1]) with extra multiplications, and, in this situation, $\mathrm{End}_0(\mathcal{T}) = \mathrm{End}(\mathcal{T}) \otimes_{\mathbb{Z}} \mathbb{Q}$ is a non-totally real number field which is isomorphic to K (independently of the given fractional ideal) and $\mathrm{End}(\mathcal{T})$ turns out to be isomorphic, not only to an order of K, but to the whole ring of integers $\mathcal{O}_K$ (see [16]).

Finally, the fact that, in the cases under consideration, where the toroidal group comes from the field K, all the entries in the last column of its period matrix in standard coordinates are not real numbers (cf. Remarks 2.7, 2.14) distinguishes these cases from the general ones of arbitrary toroidal groups.

Conversely, if we are given an n-dimensional toroidal group $\mathcal{T}$ with real rank $n+1$, admitting extra multiplications, then (see [4]) $\mathrm{End}_0(\mathcal{T}) = \mathrm{End}(\mathcal{T}) \otimes_{\mathbb{Z}} \mathbb{Q}$ is a non-totally real number field with one complex embedding. In the case where the field $K = \mathrm{End}_0(\mathcal{T})$ has degree $n+1$ and admits an *essential polynomial*, we conjecture that $\mathbb{C}^n/\mu(\mathcal{O}_K)$ is isomorphic to $\mathcal{T}$, as we can prove this in the cases of small dimension $n = 2, 3$, which are sufficient to illustrate the richness of this matter. In these two cases, we write down the relations between the essential polynomial and the lattice of $\mathcal{T}$, and we give the analytic and rational representations of $\mathrm{End}(\mathcal{T})$.

In fact, we conjecture that $\mathbb{C}^n/\mu(\mathcal{O}_K)$ is isogenous to $\mathcal{T}$ also in the case where K (necessarily, in this conjecture, of degree $n+1$) does not admit an essential polynomial, a fact which would confirm the close relationship between toroidal groups and non-totally real number fields.

Finally, it is easy to see (cf. Theorem 1.2) that the toroidal group $\mathcal{T}$, having lattice Λ, which, as above, can be taken in the form

$$\Lambda = \langle (1, 0, \ldots, 0), \ldots, (0, 0, \ldots, 1), (\tau, s_1, \ldots, s_{n-1}) \rangle_{\mathbb{Z}},$$

with $\tau \notin \mathbb{R}$, determines a linear subtorus $\mathcal{L}$, isomorphic to $(\mathbb{C}^*)^{n-1}$, with the corresponding quotient isomorphic to the Jacobian $\mathfrak{J}(\mathcal{C})$ of the elliptic curve $\mathcal{C}$ with fundamental parallelogram generated by $\{1, \tau\}$. Thus we have the non-splitting extension

$$1 \longrightarrow (\mathbb{C}^*)^{n-1} \longrightarrow \mathcal{T} \longrightarrow \mathfrak{J}(\mathcal{C}) \longrightarrow \Omega, \tag{0.1}$$

and we can represent an element of $\mathcal{T}$ with a n-tuple

$$(P, k_1, \ldots, k_{n-1}) \in \mathcal{C} \times (\mathbb{C}^*)^{n-1},$$

in such a way that

$$\begin{aligned}(P_1, k_1, \dots, k_{n-1})&(P_2, h_1, \dots, h_{n-1})\\ &= (P_1 + P_2, k_1 h_1 c_{L_1}(P_1, P_2), \dots, k_{n-1} h_{n-1} c_{L_{n-1}}(P_1, P_2)),\end{aligned}$$

where the factor systems c_{L_i} are defined (almost everywhere) in (1.1). Note that, since the quotient group $\mathcal{T}/\mathcal{L}$ is an Abelian variety, being a complex torus of dimension one, in this case, by T. Umeno [15] $\mathcal{T}$ is a *quasi-Abelian* variety, in the sense of [5] (cf. also [2]).

In the case where $n = 2$, this representation of $\mathcal{T}$ as a generalized Jacobian of $\mathcal{C}$ gives us in turn an explicit formula for the m-torsion points of $\mathcal{T}$. In the above correspondence with a non-totally real cubic number field K, these torsion points correspond under the Minkowski map μ to the fractional ideal $\frac{1}{m}\mathcal{O}_K$.

We end this Introduction by shortly reporting on the historical background, which, in our opinion, are worth to be included, but which can skipped at a first reading.

Historical background. As early as 1722, when De Moivre published his celebrated formula, and even better when Euler rewrote it in terms of the exponential map $\theta \mapsto e^{2\pi i\theta}$, it was clear that the fractional ideals of $\mathbb{Q}$ are related to the torsion subgroups of the unit circle $\mathbb{S}^1 \leq \mathbb{C}^*$.

It was Gauss who first described in 1799 a continuous doubly periodic function on $\mathbb{C}$ with two $\mathbb{R}$-independent periods. This gives a correspondence between fractional ideals of a non-totally real quadratic number field and the torsion subgroups of the Jacobian $\mathfrak{J}(\mathcal{C})$ of an elliptic curve $\mathcal{C}$ (on the contrary, when a quadratic field is totally real any fractional ideal is dense in $\mathbb{R}$, reflecting the fact that we cannot have a continuous doubly periodic function on $\mathbb{R}$).

More precisely, and in order to introduce the notation, let us consider such a field $K = \mathbb{Q}(\omega)$, with $\omega^2 = d$, for a square-free negative integer d, and its ring of integers $\mathcal{O}_K = \langle 1, \rho\rangle_{\mathbb{Z}}$. A fixed complex embedding $K \to \mathbb{C}$ coincides with the corresponding Minkowski map μ and we have that $\mu(\mathcal{O}_K) = \langle 1, \tau\rangle_{\mathbb{Z}}$ (with τ the embedding of ρ) is a lattice of $\mathbb{C}$ of real rank 2, with the resulting quotient group $\mathbb{C}/\mu(\mathcal{O}_K)$ being a 1-dimensional complex torus, hence isomorphic to the Jacobian of an elliptic curve $\mathcal{C}$. Let us consider the periodic Weierstrass functions $\wp$ and $\wp'$ with period lattice $\Lambda := \mu(\mathcal{O}_K)$. The map

$$z \longmapsto (\wp(z), \wp'(z)),$$

which can be projectively extended to

$$\mathrm{P}\colon z \longmapsto \mathrm{P}(z) := [z^3 : z^3\wp(z) : z^3\wp'(z)], \tag{0.2}$$

(including in this way the point at infinity $\Omega := \mathrm{P}(0) = [0 : 0 : 1]$ of $\mathcal{C}$) gives in turn an isomorphism $z + \Lambda \mapsto \big(\mathrm{P}(z)\big) - (\Omega)$ between the quotient group $\mathbb{C}/\Lambda$ and the Jacobian of the elliptic curve $\mathcal{C}$.

Every element of a given fractional ideal $\frac{1}{a+\rho b}\mathcal{J}$, where $\mathcal{J}$ is an ideal of $\mathcal{O}_K$, $a, b \in \mathbb{Z}$ not both zero, is mapped onto a point $P = \mathrm{P}(\frac{1}{a+\tau b}c)$ (with $c \in \mu(\mathcal{J})$), such that, if we put $m = \|a + \tau b\|$, then

$$mP = \|a+\tau b\|\,\mathrm{P}\left(\frac{1}{a+\tau b}c\right) = \|a+\tau b\|\,\mathrm{P}\left(\frac{\overline{a+\tau b}}{\|a+\tau b\|}c\right) = \mathrm{P}\Big((\overline{a+\tau b})c\Big) = \Omega,$$

because $(\overline{a+\tau b})c \in \mu(\mathcal{O}_K)$, mirroring the fact that evaluating the map P on the elements

$$\frac{1}{m}(d_1 + \tau d_2), \quad d_1, d_2 = 0, \ldots, m-1,$$

with $\frac{d_1}{m}, \frac{d_2}{m}$ dotting the fundamental parallelogram, we get points of order dividing m.

In 1889, a prize, sponsored by King Oscar II of Sweden, was offered to investigate two-variate complex functions with four real-independent periods. The prize was won by Poincaré, but Appell gave, as well, a solution which gained the second place, and was inspiring for further research: the solutions left the question of a two-variate complex function with three real-independent periods unexplored, and it was Cousin, a student of Poincaré, who proved the existence of such a function, and showed its main properties [7]. With the restrictions mentioned in the Introduction, such a periodic function defines a toroidal group, given by the quotient of $\mathbb{C}^2$ by the lattice spanned by the three periods. Periodic meromorphic functions and toroidal groups are the subject of a paper dated 1991 [6] by Capocasa and Catanese, where the authors prove that, for a lattice $\Lambda \subset \mathbb{C}^n$, there exists a non-constant Λ-periodic meromorphic function on $\mathbb{C}^n$ if and only if there exists a positive definite Hermitian form H on $\mathbb{C}^n$ such that the imaginary part of H takes integer values on $\Lambda \times \Lambda$, that is, if and only if $\mathbb{C}^n/\Lambda$ is *quasi-Abelian* (see [5], cf. also [2]). Notice indeed that, as mentioned in the Introduction, when the real rank of Λ is $n+1$, the latter conditions are manifestly fulfilled, because the quotient variety in (0.1) is a one-dimensional complex torus, hence an Abelian variety, and by the cited result by Umeno, $\mathbb{C}^n/\Lambda$ is quasi-Abelian. Quasi-Abelian varieties were introduced in 1947 by Severi ([14]) exactly in the context of periodic functions, and were later studied by Rosenlicht ([12]) as generalizations of the classic Jacobians of a curve (see Remark 1.5). Rosenlicht was awarded the Frank Nelson Cole Prize in Algebra 1960 for this construction, which gives an explicit example of a connected commutative Lie (resp. algebraic) group which is neither linear, nor compact (resp. complete), nor the direct product of a linear group and a compact (resp. complete) group. As such, generalized Jacobians are a central subject of Serre's first book [13].

Coming full circle of the connection between toroidal groups and non-totally real number fields, in 1973 Andreotti and Gherardelli associated a toroidal group of dimension n and real rank $n+1$ with any non-totally real number field of degree $\delta = n+1$, and they showed, conversely, that tensoring with $\mathbb{Q}$ the endomorphism ring of a toroidal group of dimension n and real rank $n+1$, one obtains a non-totally real number field of degree $\delta \leq n+1$ ([4], see also [3]). More generally, Y. Abe in [1] showed that, given a non-totally real number field K of degree $\delta = r_1 + 2r_2$ and a Minkowski map $\mu : K \to \mathbb{C}^{r_1+r_2}$, the image $\mu(\mathcal{O}_K)$ is a lattice Λ of complex rank $r_1 + r_2$ and real rank δ, such that the quotient group $\mathbb{C}^{r_1+r_2}/\Lambda$ is a toroidal group, and furthermore proved that for any toroidal group $\mathcal{T}$ which does not contains toroidal subgroups, the ring $\mathrm{End}_0(\mathcal{T})$ is a division algebra.

Finally, Vallières in [16] proved, in the case where $\mathcal{T}$ is $\mathbb{C}^2/\mu(\mathcal{O}_K)$, that the non-totally real number field $\mathrm{End}_0(\mathcal{T})$ is isomorphic to K, and, conversely, that every 2-dimensional toroidal group with real rank 3, having extra multiplications, is isomorphic to $\mathbb{C}^2/\mu(\mathfrak{a})$ for a suitable fractional ideal $\mathfrak{a}$ of the non-totally real number field $\mathrm{End}_0(\mathcal{T})$, hence isogenous to $\mathbb{C}^2/\mu(\mathcal{O}_K)$ (cf. Theorem 2.1).

1. Periodic functions, toroidal groups, and generalized Jacobians

Lattices $\Lambda \subset \mathbb{C}^n$ of real rank $n+1$ arise naturally when one considers n-variate meromorphic functions with $n+1$ $\mathbb{R}$-independent periods. Up to a change of basis, we can assume the lattice to be in *standard coordinates*

$$\Lambda = \langle (1,0,\dots,0), (0,1,\dots,0), \dots, (0,\dots,1), (\tau, s_1, \dots, s_{n-1}) \rangle_{\mathbb{Z}}.$$

As we pointed out in the Introduction, we can confine ourselves to the case where the irrationality condition holds, that is, the case where the only n-tuple $(a_0, \dots, a_{n-1}) \in \mathbb{Z}^n$ such that

$$a_0\tau + a_1 s_1 + \cdots + a_{n-1}s_{n-1} \in \mathbb{Z}$$

is $(0,\dots,0)$, ruling out elementary periodic functions such as

$$G(z_0, z_1, \dots, z_{n-1}) = \exp\Big(2\pi i(a_0 z_0 + a_1 z_1 + \cdots + a_{n-1}z_{n-1})\Big)$$

with $a_0\tau + a_1 s_1 + \cdots + a_{n-1}s_{n-1} \in \mathbb{Z}$.

In particular, we can assume that $\tau \notin \mathbb{R}$, and consequently we denote the first variable by ζ and the other ones by $z_1, \dots, z_{n-1}$. In the following theorem, which is simply a recap of known facts, we give a useful representation of the toroidal group $\mathcal{T} = \mathbb{C}^n/\Lambda$ as a group extension of a linear subtorus, isomorphic to $(\mathbb{C}^*)^{n-1}$, with the Jacobian $\mathfrak{J}(\mathcal{C})$ of the elliptic curve $\mathcal{C}$ with fundamental parallelogram generated by $\{1, \tau\}$. This description is in fact central in the paper. For our convenience, we define the function c_L here:

Definition 1.1. Let the elliptic curve $\mathcal{C}$ be defined by the lattice $\langle 1, \tau \rangle$, and let Ω be its point at infinity. Let $L = (M) + (N)$ (with $M, N \in \mathcal{C}$) be a *modulus* (that is, an effective divisor), and let $\ell_{P,Q}(X) = 0$ be (up to an uninfluent scalar) the equation of the line through the points P and Q, or the tangent line in P to $\mathcal{C}$ in the case where $Q = P$. We define the map

$$c_L(P_1, P_2) = \frac{\ell_{P_1,P_2}(M)}{\ell_{P_1+P_2,\Omega}(M)} \frac{\ell_{P_1+P_2,\Omega}(N)}{\ell_{P_1,P_2}(N)}, \tag{1.1}$$

for any two points P_1, P_2 such that $(P_1) + (P_2) - 2(\Omega)$ is not linearly equivalent to either $-(N) + (\Omega)$ or $-(M) + (\Omega)$.

Theorem 1.2. *Let* $\Lambda = \langle (1, 0, \ldots, 0), \ldots, (0, \ldots, 0, 1), (\tau, s_1, \ldots, s_{n-1}) \rangle_{\mathbb{Z}}$ *with* $\tau \notin \mathbb{R}$*, and let* $\mathcal{T} = \mathbb{C}^n/\Lambda$*. If* $\mathcal{V}(\zeta) = \{(\zeta, z_1, \ldots, z_{n-1}) \in \mathbb{C}^n : \zeta = 0\}$*, then the subgroup* $\mathcal{L} := (\mathcal{V}(\zeta) + \Lambda)/\Lambda$ *of* $\mathcal{T}$ *is a linear subtorus, isomorphic to* $(\mathbb{C}^*)^{n-1}$*, with the quotient group* $\mathcal{T}/\mathcal{L}$ *isomorphic to the Jacobian* $\mathfrak{J}(\mathcal{C})$ *of the elliptic curve* $\mathcal{C}$ *with fundamental parallelogram generated by* $\{1, \tau\}$*. Moreover, in the corresponding non-splitting extension*

$$1 \longrightarrow \mathcal{L} \longrightarrow \mathcal{T} \longrightarrow \mathfrak{J}(\mathcal{C}) \longrightarrow \Omega,$$

we can represent an element of $\mathcal{T}$ *with a* n*-tuple* $(P, k_1, \ldots, k_{n-1}) \in \mathcal{C} \times (\mathbb{C}^*)^{n-1}$*, in such a way that* $(\Omega, 1, \ldots, 1)$ *is the zero element, and*

$$\begin{aligned} &(P_1, k_1, \ldots, k_{n-1})(P_2, h_1, \ldots, h_{n-1}) \\ &\qquad = (P_1 + P_2, k_1 h_1 c_{L_1}(P_1, P_2), \ldots, k_{n-1} h_{n-1} c_{L_{n-1}}(P_1, P_2)), \end{aligned} \tag{1.2}$$

where c_{L_j} *is defined as in* (1.1) *for* $j = 1, \ldots, n-1$*, by* $n-1$ *given moduli* $L_j = (T_j) + (T_{j+1})$*, such that* $(T_j) - (T_{j+1})$ *is linearly equivalent to* $(P_j) - (\Omega)$*, where* $P_j = \mathrm{P}(s_j)$*, and* P *is the map defined in* (0.2) *(where, formally,* $P_1 + P_2$ *denotes for short the point* P_3 *in* $\mathcal{C}$ *such that the divisor* $(P_3) + (\Omega)$ *is linearly equivalent to* $(P_1) + (P_2)$*).*

Proof. Since $\mathcal{V}(\zeta) \cap \Lambda = \langle (0, 1, \ldots, 0), \ldots, (0, \ldots, 1) \rangle_{\mathbb{Z}}$, the subgroup $\mathcal{L}$ is a linear subtorus, and, by [9, Prop. 2.2.3], the quotient group $\mathcal{T}/\mathcal{L}$ is isomorphic to $\mathfrak{J}(\mathcal{C})$. Since the functor Ext is additive, we can reduce the argument to the case $n = 2$, for which the situation is described in [8, Introduction]. □

Remark 1.3. Indeed, one can choose L_j to be defined by any two points M_j, N_j such that $(M_j) - (N_j)$ is linearly equivalent to $(P_j) - (\Omega)$, however we choose $L_j = (T_j) + (T_{j+1})$ just in order to simplify the function G we will exhibit in (1.3).

Remark 1.4. Throughout the paper we denote, as usual, the Weierstrass elliptic functions by $\wp$ and σ.

Note that the non-trivial factor system c_{L_j} induces a factor system $c_{L_j}\mathrm{P} : \mathbb{C} \times \mathbb{C} \to \mathbb{C}^*$, as well, with $(\zeta_1, \zeta_2) \mapsto c_{L_j}(\mathrm{P}(\zeta_1), \mathrm{P}(\zeta_2))$. But, as every commutative Lie group extension of $\mathbb{C}^*$ by $\mathbb{C}$ is splitting, $c_{L_j}\mathrm{P}$ turns out indeed to be a trivial factor system, that is, a coboundary $\delta^1(g_j) : \mathbb{C}\times\mathbb{C} \to \mathbb{C}^*$, where $g_j : \mathbb{C} \to \mathbb{C}^*$ is a section (cf. [8, Thm., p. 128]) defined by

$$g_j(\zeta) = \exp(-2\eta_1 s_j \zeta)\frac{\sigma(t_j)}{\sigma(t_{j+1})}\frac{\sigma(\zeta - t_{j+1})}{\sigma(\zeta - t_j)},$$

where $t_j \in \mathbb{C}$ is such that $\mathrm{P}(t_j) = T_j$ and η_1 is a constant.[1]

This unveils the cocycle $c_{L_j}\big(\mathrm{P}(\zeta_1), \mathrm{P}(\zeta_2)\big)$ as the coboundary

$$(c_{L_j}\mathrm{P})(\zeta_1, \zeta_2) = \delta^1(g_j)(\zeta_1, \zeta_2).$$

Turning back to n-variate meromorphic functions with $n+1$ $\mathbb{R}$-independent periods, we recall from the Introduction that in this case the toroidal group $\mathbb{C}^n/\Lambda$ is quasi-Abelian, hence by [6] such functions do exist, as pointed out in the Historical background.

In order to find out how such a function $G(\zeta, z_1, \ldots, z_{n-1})$ is built, we stress its geometric meaning. Note in passing that, whereas such a function determines a unique period lattice, a lattice $\Lambda \subset \mathbb{C}^n$ of real rank $n+1$ determines a whole family of periodic functions.

Any such a function G defines a further function

$$\widehat{G}\big((\zeta, z_1, \ldots, z_{n-1}) + \Lambda\big) := G(\zeta, z_1, \ldots, z_{n-1}),$$

from $\mathcal{T}$ to $\mathbb{C}$, giving in turn the commutative diagram

$$\begin{tikzcd}
1 \arrow[r] & (\mathbb{C}^*)^{n-1} \arrow[r] & \mathcal{T} \arrow[r, "\pi"] \arrow[dr, "\hat{G}"] & \mathfrak{J}(\mathcal{C}) \arrow[r] & 0 \\
 & & \mathbb{C}^n \arrow[u, "\varpi"] \arrow[r, "G"'] & \mathbb{C} &
\end{tikzcd}$$

Whilst ϖ is independent of the elliptic curve $\mathcal{C}$, the role played by the choice of $\mathcal{C}$ comes in here: by the Theorem in [8] and the above Theorem 1.2, if we let again $t_j \in \mathbb{C}$ be such that $\mathrm{P}(t_j) = T_j$, then

$$\varpi(\zeta, z_1, \ldots, z_{n-1}) \equiv \big(\mathrm{P}(\zeta), G_1(\zeta, z_1), \ldots, G_{n-1}(\zeta, z_{n-1})\big)$$

with $G_j(\zeta, z_j) := g_j(\zeta)\exp(2\pi i z_j)$, and $g_j(\zeta)$ as in Remark 1.4. The map ϖ is an epimorphism from the additive group of $\mathbb{C}^n$ to (a representation of) the toroidal group $\mathcal{T}$ with Λ as the kernel (cf. Remark 1.5 below). As mentioned in passing above, the period lattice of any function of $G_1, \ldots, G_{n-1}$ is contained in Λ. As a prototype of such a n-variate meromorphic function

[1] More precisely $\eta_1 = -\frac{\pi^2}{6}\frac{\theta_1'''(0)}{\theta_1'(0)}$, where $\theta_1(z) = -\exp\big(\frac{1}{4}\pi i\tau + \pi i\big(z + \frac{1}{2}\big)\big)\vartheta\big(z + \frac{1}{2}\tau + \frac{1}{2}; \tau\big)$ and ϑ is the Jacobi theta function $\vartheta(z, \tau) = \sum_{n=-\infty}^{+\infty} \exp\big(\pi i n^2\tau + 2\pi i n z\big)$.

with period lattice Λ, we simply take here their product, which, with the above choice of $L_i = (T_i) + (T_{i+1})$, reduces to

$$(1.3)\quad \begin{aligned} G(\zeta, z_1, \dots, z_{n-1}) &:= G_1(\zeta, z_1) \cdots G_{n-1}(\zeta, z_{n-1}) \\ &= \exp(-2\eta_1 s\ \zeta) \frac{\sigma(t_1)}{\sigma(t_n)} \frac{\sigma(\zeta - t_n)}{\sigma(\zeta - t_1)}\ \exp(2\pi\, i(z_1 + \cdots + z_{n-1})), \end{aligned}$$

with $s = s_1 + s_2 + \cdots + s_{n-1}$.

Remark 1.5. It is necessary at this point to remark that ϖ induces a rational isomorphism (although non-regular, hence in the sense of [17]) between the toroidal group $\mathbb{C}^n/\Lambda$ and (a representation of) the *generalized Jacobian* $\mathfrak{J}_L(\mathcal{C})$, with modulus $L = (T_1) + \cdots + (T_n) \in \operatorname{div}(\mathcal{C})$ defined by n pairwise distinct points $T_j \in \mathcal{C}$, that is, the quotient group of divisors of degree zero on $\mathcal{C}$ modulo the subgroup of principal divisors of functions g such that $v_{T_j}(1-g) \geq 1$ with $j = 1, \dots, n$, where v_X is the discrete valuation of the local ring $\mathcal{O}_X$ of the rational functions of $\mathcal{C}$ that are regular in a point X. Note, indeed, that a divisor class D of degree 0 in the generalized Jacobian determines the element

$$(1.4)\qquad (P, k_1, \dots, k_{n-1}) \in \mathcal{C} \times (\mathbb{C}^*)^{n-1},$$

where $P \in \mathcal{C}$ is such that D is linearly equivalent to $(P) - (\Omega)$, that is, such that $D = (P) - (\Omega) + \operatorname{div}(f)$, for a suitable function f, and $k_j = \frac{f(T_{j+1})}{f(T_j)} \in \mathbb{C}^*$ (see [13, Ch. V, §3], see also [8]).

2. Toroidal groups and non-totally real number fields

In this section we study the relationship between non-totally real number fields of degree $\delta = r_1 + 2r_2$ with $r_2 = 1$ and isogeny classes of toroidal groups of dimension $n = \delta - 1$ and real rank δ with extra multiplications, introduced in 1973 by Andreotti and Gherardelli in [4] and investigated in 2012 by Vallières in [16], and in 2013 by Abe in [1]. From now on, we consider non-totally real number fields with precisely one pair of complex embeddings.

As introduced above, let K be a non-totally real number field, $\sigma_1, \dots, \sigma_{r_1}$ the real embeddings $K \to \mathbb{R} \subset \mathbb{C}$, and σ_n one of the two conjugate complex embeddings $K \to \mathbb{C}$. For any fractional ideal $\frac{1}{\nu}\mathcal{J}$ of K, where $\nu \in \mathcal{O}_K$ and $\mathcal{J}$ is an ideal of $\mathcal{O}_K$, the Minkowski embedding $\mu : K \to \mathbb{C}^n$, defined by the map

$$z \longmapsto (\sigma_1(z), \dots, \sigma_{r_1}(z), \sigma_n(z)),$$

gives in turn the lattice $\mu(\frac{1}{\nu}\mathcal{J})$ of $\mathbb{C}^n$, which has real rank equal to the rank of $\frac{1}{\nu}\mathcal{J}$, that is, $\delta = n + 1$.

Now we note that replacing the lattice $\mu(\frac{1}{\nu}\mathcal{J})$ with $\mu(\mathcal{O}_K)$ corresponds to applying an isogeny.

Theorem 2.1. *Let K be a number field, $\mu\colon K \to \mathbb{C}^n$ a Minkowski map, $\mathcal{J}$ an ideal of $\mathcal{O}_K$ and $\nu \in \mathcal{O}_K$. The quotient groups $\mathbb{C}^n/\mu(\frac{1}{\nu}\mathcal{J})$, $\mathbb{C}^n/\mu(\mathcal{J})$ and $\mathbb{C}^n/\mu(\mathcal{O}_K)$ are isogeneous.*

Proof. Let us consider the maps

$$\mathbb{C}^n/\mu(\tfrac{1}{\nu}\mathcal{J}) \xrightarrow{\iota} \mathbb{C}^n/\mu(\mathcal{J}) \xrightarrow{\pi} \mathbb{C}^n/\mu(\mathcal{O}_K),$$

where π is the canonical projectionand ι is induced by the following linear isomorphism $\mathbb{C}^n \to \mathbb{C}^n$:

$$(x_1,\dots,x_n) \longmapsto (\sigma_1(\nu)x_1,\dots,\sigma_n(\nu)x_n).$$

The kernel of π is $\mu(\mathcal{O}_K)/\mu(\mathcal{J})$, that is finite because $\mu(\mathcal{J})$ is a free $\mathbb{Z}$-submodule of $\mu(\mathcal{O}_K)$ of the same rank. The map ι is an isogeny by construction. □

The quotient group $\mathcal{T} = \mathbb{C}^n/\mu(\mathcal{O}_K)$ is proved to be a toroidal group ([1], see also [4]) with extra multiplications, and, in this situation, $\mathrm{End}_0(\mathcal{T}) = \mathrm{End}(\mathcal{T}) \otimes_{\mathbb{Z}} \mathbb{Q}$ is a non-totally real number field which is isomorphic to K and $\mathrm{End}(\mathcal{T})$ turns out to be isomorphic, not only to an order of K, but to the whole ring of integers $\mathcal{O}_K$ (see [16]).

Using Theorem 2.1, we can generalize this result.

Corollary 2.2. *The group $\mathbb{C}^n/\mu(\mathfrak{a})$ is toroidal for every fractional ideal $\mathfrak{a} = \frac{1}{\nu}\mathcal{J}$ of K.*

Proof. According to the proof of Theorem 2.1, given a fractional ideal $\mathfrak{a}$ of K, we know that there exists an isogeny from the group $\mathcal{T}_1 = \mathbb{C}^n/\mu(\mathfrak{a})$ to $\mathcal{T}_2 = \mathbb{C}^n/\mu(\mathcal{O}_K)$, hence we have a further isogeny $\mathcal{T}_2 \to \mathcal{T}_1$ (see, e.g., [9, Prop. 2.2.1]). The claim follows from the well-known fact that every epimorphic image of a toroidal group is again toroidal. □

On the other hand, for those non-totally real number fields K admitting an *essential polynomial*, that is, an irreducible polynomial $f(x) = a_0 + a_1x + \cdots + a_{n+1}x^{n+1} \in \mathbb{Z}[x]$ such that $K = \mathbb{Q}(\omega)$ with $f(\omega) = 0$ and the discriminant of $f(x)$ is equal to the discriminant of the field K, by [10] an integral basis of the ring of integers $\mathcal{O}_K$ is given by $\{\rho_0, \rho_1, \dots, \rho_n\}$ where

$$\rho_0 = 1 \text{ and } \rho_j = \sum_{k=0}^{j-1} a_{n+1-k}\omega^{j-k}, \text{ for } j = 1,\dots,n. \tag{2.1}$$

In this case, let $\alpha_1,\dots,\alpha_{r_1}$ be the real roots of $f(x)$ and let $\gamma, \overline{\gamma}$ be the two non-real roots. Let $\Phi = \{\sigma_1,\dots,\sigma_{r_1},\sigma_n\}$ be the type defined by

$$\sigma_1\colon \omega \longmapsto \alpha_1, \quad \dots \quad \sigma_{r_1}\colon \omega \longmapsto \alpha_{r_1}, \quad \sigma_n\colon \omega \longmapsto \gamma.$$

The lattice $\mu_\Phi(\mathcal{O}_K)$ is generated by the vectors

$$\begin{aligned}
\mu_\Phi(1) &= (1,\dots,1,1),\\
\mu_\Phi(\rho_1) &= (a_{n+1}\alpha_1,\dots,a_{n+1}\alpha_{r_1},a_{n+1}\gamma),\\
\mu_\Phi(\rho_2) &= (a_{n+1}\alpha_1^2+a_n\alpha_1,\dots,a_{n+1}\alpha_{r_1}^2+a_n\alpha_{r_1},a_{n+1}\gamma^2+a_n\gamma),\\
&\ \ \vdots\\
\mu_\Phi(\rho_n) &= (a_{n+1}\alpha_1^n+\dots+a_2\alpha_1,\dots,\\
&\qquad a_{n+1}\alpha_{r_1}^n+\dots+a_2\alpha_{r_1},a_{n+1}\gamma^n+\dots+a_2\gamma),
\end{aligned}$$

therefore its period lattice Λ is generated by the columns of the matrix

$$\begin{pmatrix}
1 & a_{n+1}\alpha_1 & a_{n+1}\alpha_1^2+a_n\alpha_1 & \dots & a_{n+1}\alpha_1^n+\dots+a_2\alpha_1\\
\vdots & \vdots & \vdots & \dots & \vdots\\
1 & a_{n+1}\alpha_{r_1} & a_{n+1}\alpha_{r_1}^2+a_n\alpha_{r_1} & \dots & a_{n+1}\alpha_{r_1}^n+\dots+a_2\alpha_{r_1}\\
1 & a_{n+1}\gamma & a_{n+1}\gamma^2+a_n\gamma & \dots & a_{n+1}\gamma^n+\dots+a_2\gamma
\end{pmatrix}.$$

Being isomorphic to $\mathcal{O}_K$, $\mathrm{End}(\mathcal{T})$ consists of the endomorphisms induced by the multiplications by algebraic integers. In the following we write down, in the case of low dimension (that is, $n=2,3$), which illustrate the general case, the relations between the essential polynomial and the parameters in the last column of the period matrix of $\mathcal{T}$ in standard form, and we give the analytic and rational representations of $\mathrm{End}(\mathcal{T})$.

2.1. Toroidal groups and non-totally real cubic number fields. Let now K be a non-totally real cubic number field, $\sigma_1\colon K\to\mathbb{R}\subset\mathbb{C}$ the real embedding, and $\sigma_2: K\to\mathbb{C}$ one of the two conjugate complex embeddings of K. The Minkowski embedding $\mu: K\to\mathbb{C}^2$, defined by the map

$$z\longmapsto\big(\sigma_1(z),\sigma_2(z)\big),$$

gives in turn the lattice $\mu(\mathcal{O}_K)$ of $\mathbb{C}^2$, which has real rank equal to the rank of $\mathcal{O}_K$, that is three, and the quotient group $\mathcal{T}=\mathbb{C}^2/\mu(\mathcal{O}_K)$ is a toroidal group with extra multiplications (see [16]).

On the other hand, for any non-totally real cubic number field K, there always exists an essential polynomial, hence it is possible to choose an element $\omega\in K$ such that $K=\mathbb{Q}(\omega)$, with $a\omega^3+b\omega^2+c\omega+d=0$ ($a,b,c,d\in\mathbb{Z}$) and

$$\mathcal{O}_K=\langle 1,\rho_1,\rho_2\rangle,$$

where $\rho_1=a\omega$, $\rho_2=a\omega^2+b\omega$ (see, e.g., [11, §1.3]). Note that $a\rho_2=\rho_1^2+b\rho_1$.

In this case, the embeddings σ_1 and σ_2 are definied by $\sigma_1(\omega)=\alpha$, and $\sigma_2(\omega)=\beta$, where

$$ax^3+bx^2+cx+d=a(x-\alpha)(x-\beta)(x-\overline{\beta}).$$

As claimed above, we write down the period matrix in standard form of $\mathcal{T}$, and we give the analytic and rational representations of $\mathrm{End}(\mathcal{T})$.

Theorem 2.3. *Let $K = \mathbb{Q}(\omega)$ be any non-totally real cubic number field, with $a\omega^3 + b\omega^2 + c\omega + d = 0$, and let $\mu : K \to \mathbb{C}^2$ be a Minkowski embedding, as above. If $\mathcal{T} = \mathbb{C}^2/\mu(\mathcal{O}_K)$ is the toroidal group associated with $\mathcal{O}_K$, then the parameters defining the lattice in standard form are $\tau = -a\alpha\beta$ and $s_1 = -\overline{\beta}$ (hence $\tau s_1 = -d$).*

Moreover, $\mathcal{T}$ has extra multiplications, and the analytic and rational representations of $\mathrm{End}(\mathcal{T})$ *yield the free $\mathbb{Z}$-modules generated, respectively, by the matrices $I_2, (\rho_1)_a, (\rho_2)_a$ and $I_3, (\rho_1)_r, (\rho_2)_r$, where*

$$(\rho_1)_a = \begin{pmatrix} 0 & a\tau \\ 1 & -b + as_1 \end{pmatrix}, \quad (\rho_2)_a = \begin{pmatrix} \tau & -ad \\ s_1 & -c \end{pmatrix}, \tag{2.2}$$

and

$$(\rho_1)_r = \begin{pmatrix} 0 & 0 & -ad \\ 1 & -b & -c \\ 0 & a & 0 \end{pmatrix}, \quad (\rho_2)_r = \begin{pmatrix} 0 & -ad & -bd \\ 0 & -c & -d \\ 1 & 0 & -c \end{pmatrix}. \tag{2.3}$$

Proof. We can assume that the Minkowski embedding $\mu : K \to \mathbb{C}^2$ is defined, as above, by $\mu(z) = \big(\sigma_1(z), \sigma_2(z)\big)$, with $\sigma_1(\omega) = \alpha$, and $\sigma_2(\omega) = \beta$. Since we have chosen the primitive element $\omega \in K$ such that $\mathcal{O}_K = \langle 1, \rho_1, \rho_2 \rangle$, with $\rho_1 = a\omega$, $\rho_2 = a\omega^2 + b\omega$, the Minkowski embedding μ produces the lattice

$$\mu(\mathcal{O}_K) = \langle (1,1), (a\alpha, a\beta), (a\alpha^2 + b\alpha, a\beta^2 + b\beta) \rangle.$$

It is clear that the set $\{\mu(1) = (1,1), \mu(\rho_1) = (a\alpha, a\beta)\}$ is a basis of $\mathbb{C}^2$ because $\alpha \neq \beta$. Replacing this basis with the canonical basis of $\mathbb{C}^2$ we obtain that the lattice can be rewritten as

$$\langle (1,0), (0,1), (\tau, s_1) \rangle,$$

with

$$\begin{pmatrix} \tau \\ s_1 \end{pmatrix} = \begin{pmatrix} 1 & a\alpha \\ 1 & a\beta \end{pmatrix}^{-1} \cdot \begin{pmatrix} a\alpha^2 + b\alpha \\ a\beta^2 + b\beta \end{pmatrix} = \begin{pmatrix} -a\alpha\beta \\ -\overline{\beta} \end{pmatrix},$$

and the first assertion follows.

By [16] we know that the toroidal group $\mathcal{T}$ has extra multiplications, which in our setting are induced by multiplications by ρ_1 and ρ_2. Since $\{\mu(1), \mu(\rho_1)\}$ is a basis of $\mathbb{C}^2$, we have to check how the multiplications by ρ_1 and ρ_2 induce endomorphisms of $\mathbb{C}^2$. Seeing that

$$\mu(\rho_1^2) = \mu(-b\rho_1 + a\rho_2) = a\tau\mu(1) + (-b + as_1)\mu(\rho_1),$$

and that

$$\mu(\rho_2) = \tau\mu(1) + s_1\mu(\rho_1),$$
$$\mu(\rho_2\rho_1) = \mu(a^2\omega^3 + ab\omega^2) = \mu(-ad - ac\omega) = -ad\mu(1) - c\mu(\rho_1),$$

the analytic representation of ρ_1 and ρ_2 is given by the matrices in (2.2).

Since $\{\mu(1), \mu(\rho_1), \mu(\rho_2)\}$ is an integral basis of the lattice $\mu(\mathcal{O}_K)$, the same relations, together with

$$\begin{aligned}\mu(\rho_2^2) &= \mu(a^2\omega^4 + 2ab\omega^3 + b^2\omega^2)\\ &= \mu\big((-ab\omega^3 - ac\omega^2 - ad\omega) + 2ab\omega^3 + b^2\omega^2\big)\\ &= \mu\big((-b^2\omega^2 - bc\omega - bd) - ac\omega^2 - ad\omega + b^2\omega^2\big)\\ &= \mu(-bc\omega - bd - ac\omega^2 - ad\omega)\\ &= -bd\mu(1) - d\mu(\rho_1) - c\mu(\rho_2)\end{aligned}$$

(where we applied twice the equality $a\omega^3 + b\omega^2 + c\omega + d = 0$), prove that the rational representations of ρ_1 and ρ_2 are the ones given in (2.3). $\square$

Remark 2.4. As a matter of fact, these matrix representations satisfy the following Hurwitz relations:

$$(\rho_j)_a \begin{pmatrix} 1 & 0 & \tau \\ 0 & 1 & s_1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & \tau \\ 0 & 1 & s_1 \end{pmatrix} (\rho_j)_r, \quad j = 1, 2.$$

Under the necessary conditions seen before, the following theorem in [16] characterizes, conversely, a general toroidal group $\mathcal{T}$ as determined by the lattice $\mu(\frac{1}{\nu}\mathcal{J}) \subset \mathbb{C}^2$, for some fractional ideal $\frac{1}{\nu}\mathcal{J}$ of the non-totally real cubic number field $K = \mathrm{End}_0(\mathcal{T})$.

Theorem 2.5. *Given a toroidal group $\mathcal{T}$ of complex rank $\mathrm{rk}_{\mathbb{C}}\,\mathcal{T} = 2$ and real rank $\mathrm{rk}_{\mathbb{R}}\,\mathcal{T} = 3$ with extra multiplications, the field $K = \mathrm{End}_0(\mathcal{T})$ is a non-totally real cubic number field, and if the ring $\mathrm{End}(\mathcal{T})$ coincides with the ring of algebraic integers $\mathcal{O}_K$, then there exists a fractional ideal $\frac{1}{\nu}\mathcal{J}$, with $\nu \in \mathcal{O}_K$ and $\mathcal{J}$ an ideal of $\mathcal{O}_K$, such that $\mathcal{T}$ is isomorphic to the quotient*

$$\mathbb{C}^2/\mu(\tfrac{1}{\nu}\mathcal{J}),$$

for a suitable Minkowski embedding $\mu : K \to \mathbb{C}^2$.

2.2. Fractional ideals and torsion points, in the geometric correspondence with a generalized Jacobian. In this section we represent the torsion points of a toroidal group $\mathbb{C}^2/\Lambda$ in the geometric correspondence with a generalized Jacobian with modulus $L = (M)+(N)$, with $M = \mathrm{P}(t_M)$ and $N = \mathrm{P}(t_N)$.

Remark 2.6. Recall that any fractional ideal $\frac{1}{\nu}\mathcal{J}$ of K defines the isogeny $\pi\iota$ as in Theorem 2.1. This allows us to consider just toroidal groups of the form $\mathcal{T} = \mathbb{C}^2/\mu(\mathcal{O}_K)$. In this case $\mathcal{T}$ is birationally isomorphic to the generalized Jacobian $\mathfrak{J}_L(\mathcal{C})$ of the elliptic curve $\mathcal{C}$ having periods 1 and $\tau = -a\alpha\beta$, with $L = (M) + (N)$ such that $(M) - (N) = (\mathrm{P}(s_1)) - (\Omega)$, $s_1 = -\overline{\beta}$, where a, α and β are given as in Section 2.1.

Remark 2.7. As remarked in the Introduction, the fact that, in the case where the toroidal group comes from the ring $\mathcal{O}_K$, the entry s_1 is not a real number distinguishes this from the general case of an arbitrary toroidal group. Note moreover that, in such a given case, we could make the same construction, starting from the elliptic curve defined by the lattice $\langle 1, s_1\rangle_{\mathbb{Z}}$, obtaining a different meromorphic function $G(\zeta, z)$ with the same lattice Λ.

It is worthwhile to point out that, even if the m-torsion subgroup $\mathcal{T}[m]$ of $\mathcal{T}$ is isomorphic to $(\mathbb{Z}/m\mathbb{Z})^3$, its elements are parametrized only by the complex variable ζ of the pair $(\zeta, z) \in \mathbb{C}^2$, the complex variable z playing, virtually, no role (cf. Remark 2.9 below).

Theorem 2.8. *In the above notation, if $m > 0$ is a fixed integer, then the torsion subgroup*

$$\mathcal{T}[m] = \mu(\tfrac{1}{m}\mathcal{O}_K)/\mu(\mathcal{O}_K)$$

is isomorphic to $(\mathbb{Z}/m\mathbb{Z})^3$. Furthermore, an element of $\mathfrak{J}_L(\mathcal{C})$, represented as in (1.4) by the pair $(P, k) \in \mathcal{C} \times \mathbb{C}^*$, with $P = \mathrm{P}(t)$, belongs to the m-torsion subgroup if and only if there exists $\zeta \in \langle 1, \tau\rangle$ such that

$$\begin{cases} t = \frac{1}{m}\zeta, \\ k = \frac{\sigma\left(\frac{1}{m}\zeta - t_N\right)}{\sigma\left(\frac{1}{m}\zeta - t_M\right)} \sqrt[m]{\frac{\sigma(\zeta - t_M)}{\sigma(\zeta - t_N)} \frac{\sigma(t_M)^{m-1}}{\sigma(t_N)^{m-1}}} \end{cases}$$

Proof. The first assertion follows from the trivial fact that

$$n\big((\zeta, z) + \mu(\mathcal{O}_K)\big) = \mu(\mathcal{O}_K) \iff (\zeta, z) \in \tfrac{1}{m}\mu(\mathcal{O}_K) = \mu\left(\tfrac{1}{m}\mathcal{O}_K\right),$$

so

$$\mathcal{T}[m] = \left\{\tfrac{d_1}{m}(1,0) + \tfrac{d_2}{m}(0,1) + \tfrac{d_3}{m}(\tau, s_1) + \mu(\mathcal{O}_K) \;:\; 0 \le d_j \le m-1\right\} \simeq (\mathbb{Z}/m\mathbb{Z})^3.$$

As the sum in the generalized Jacobian is given by

$$\begin{aligned}\big(\mathrm{P}(\zeta_1), G(\zeta_1, z_1)\big) &+ \big(\mathrm{P}(\zeta_2), G(\zeta_2, z_2)\big) \\ &= \Big(\mathrm{P}(\zeta_1 + \zeta_2), G(\zeta_1, z_1)G(\zeta_2, z_2)(c_L\mathrm{P})(\zeta_1, \zeta_2)\Big),\end{aligned}$$

where

$$G(\zeta, z) = \exp(-2\eta_1 s_1 \zeta)\frac{\sigma(t_M)}{\sigma(t_N)}\frac{\sigma(\zeta - t_N)}{\sigma(\zeta - t_M)}\ \exp(2\pi\, iz), \tag{2.4}$$

recursively we find

$$G(m\zeta, mz) = G(\zeta, z)^m \prod_{j=1}^{m-1} (c_L\mathrm{P})(\zeta, j\zeta).$$

Therefore, if $(\zeta, z) \in \Lambda$, then

$$1 = G(\zeta, z) = G(\tfrac{1}{m}\zeta, \tfrac{1}{m}z)^m \prod_{j=1}^{m-1} (c_L \mathrm{P})(\tfrac{1}{m}\zeta, j\tfrac{1}{m}\zeta), \tag{2.5}$$

whence we obtain

$$G(\tfrac{1}{m}\zeta, \tfrac{1}{m}z)^m = \prod_{j=1}^{m-1} \frac{1}{(c_L \mathrm{P})(\frac{1}{m}\zeta, j\frac{1}{m}\zeta)}.$$

By equations (2.4) and (2.5), we can write now

$$\frac{\sigma(t_M)}{\sigma(t_N)} \frac{\sigma(\zeta - t_N)}{\sigma(\zeta - t_M)} = \frac{\sigma(t_M)^m}{\sigma(t_N)^m} \frac{\sigma(\frac{1}{m}\zeta - t_N)^m}{\sigma(\frac{1}{m}\zeta - t_M)^m} \prod_{j=1}^{m-1} (c_L \mathrm{P})(\tfrac{1}{m}\zeta, j\tfrac{1}{m}\zeta),$$

and the second assertion follows. □

Remark 2.9. Apparently, z plays no role in the parametrization of the elements of $\mathcal{T}[m]$, but if $(\zeta, z) \in \Lambda$, then

$$\left(\frac{1}{m}\zeta, \frac{1}{m}z\right) = \left(\frac{d_1 + d_3\tau}{m}, \frac{d_2 + d_3 s_1}{m}\right),$$

so the role of the integer parameter d_2 in z is linked to that of the m-th root.

2.3. Toroidal groups arising from quartic fields. Now we want to exhibit the construction of a toroidal group arising from a non-totally real quartic number field with one pair of complex embeddings. Let K be such a number field, $\Phi = \{\sigma_1, \sigma_2, \sigma_3\}$ a type of K and μ_Φ the corresponding Minkowski map. Let us consider the connected Abelian complex Lie group $\mathcal{T} = \mathbb{C}^3/\mu_\Phi(\mathcal{O}_K)$. By [10], we know that if there exist an essential polynomial $f(x) = ax^4 + bx^3 + cx^2 + dx + e \in \mathbb{Z}[x]$ for the quartic number field K, then $\mathcal{O}_K$ admits an integral basis of the form

$$\{\rho_0 = 1,\ \rho_1 = a\omega,\ \rho_2 = a\omega^2 + b\omega,\ \rho_3 = a\omega^3 + b\omega^2 + c\omega\}, \tag{2.6}$$

where ω is a primitive element of K with $f(\omega) = 0$.

Remark 2.10. We note that the fact that K has precisely one pair of complex embeddings is necessary in order to have a toroidal group of complex dimension 3. Indeed, if we suppose that it admits 2 pairs of complex embeddings then, for every type Φ and fractional ideal $\mathfrak{a}$, we would have a lattice $\Lambda = \mu_\Phi(\mathfrak{a}) \subset \mathbb{C}^2$.

Theorem 2.11. *Let K be a quartic field admitting an essential polynomial $f(x)$ with real roots α, β and complex roots $\gamma, \overline{\gamma}$, and let us fix the type Φ corresponding to the set of roots $\{\alpha, \beta, \gamma\}$. The group $\mathcal{T} = \mathbb{C}^3/\mu_\Phi(\mathcal{O}_K)$ is toroidal and its period matrix in standard coordinates*

$$\Pi = \left(\begin{array}{ccc|c} 1 & 0 & 0 & \tau \\ 0 & 1 & 0 & s_1 \\ 0 & 0 & 1 & s_2 \end{array}\right)$$

is given by

$$\tau = \frac{e}{\overline{\gamma}}, \quad s_1 = -\overline{\gamma}^2, \quad s_2 = -\overline{\gamma}. \tag{2.7}$$

Proof. The proof proceeds as in the case where $n = 3$. Let $K = \mathbb{Q}(\omega)$ be a quartic field admitting an essential polynomial $f(x)$ with real roots α, β and complex roots $\gamma, \overline{\gamma}$. Let $\Phi = \{\sigma_1, \sigma_2, \sigma_3\}$ be the type defined by

$$\sigma_1 \colon \omega \longmapsto \alpha, \quad \sigma_2 \colon \omega \longmapsto \beta, \quad \sigma_3 \colon \omega \longmapsto \gamma.$$

The lattice $\mu_\Phi(\mathcal{O}_K)$ is generated by the vectors

$$\begin{aligned} \mu_\Phi(1) &= (1, 1, 1), \\ \mu_\Phi(\rho_1) &= (a\alpha, a\beta, a\gamma), \\ \mu_\Phi(\rho_2) &= (a\alpha^2 + b\alpha, a\beta^2 + b\beta, a\gamma^2 + b\gamma), \\ \mu_\Phi(\rho_3) &= (a\alpha^3 + b\alpha^2 + c\alpha, a\beta^3 + b\beta^2 + c\beta, a\gamma^3 + b\gamma^2 + c\gamma), \end{aligned}$$

therefore its period matrix is given by

$$\begin{pmatrix} 1 & a\alpha & a\alpha^2 + b\alpha & a\alpha^3 + b\alpha^2 + c\alpha \\ 1 & a\beta & a\beta^2 + b\beta & a\beta^3 + b\beta^2 + c\beta \\ 1 & a\gamma & a\gamma^2 + b\gamma & a\gamma^3 + b\gamma^2 + c\gamma \end{pmatrix},$$

which, changing basis, gives in turn the period matrix in standard coordinates $\Pi = (I_3 | T)$ where T is the column

$$T = \begin{pmatrix} a\alpha\beta\gamma \\ -\frac{1}{a^2}\left(b^2 - ac + a^2(\alpha\beta + \alpha\gamma + \beta\gamma) + ab(\alpha + \beta + \gamma)\right) \\ \frac{b}{a} + \alpha + \beta + \gamma \end{pmatrix}.$$

Using the following relations occuring among the roots and the coefficients of a polynomial,

$$\begin{gathered} \alpha\beta\gamma\overline{\gamma} = \frac{e}{a}, \\ \alpha + \beta + \gamma + \overline{\gamma} = -\frac{b}{a}, \\ \alpha\beta + \alpha\gamma + \alpha\overline{\gamma} + \beta\gamma + \beta\overline{\gamma} + \gamma\overline{\gamma} = \frac{c}{a}, \end{gathered}$$

we obtain that

$$\tau = a\alpha\beta\gamma = \frac{a\alpha\beta\gamma\overline{\gamma}}{\overline{\gamma}} = \frac{e}{\overline{\gamma}},$$

$$\begin{aligned} s_1 &= -\frac{b^2 - ac + a^2(\alpha\beta + \alpha\gamma + \beta\gamma) + ab(\alpha + \beta + \gamma)}{a^2} \\ &= -\frac{b^2 - ac + ac + a\overline{\gamma}(b + a\overline{\gamma}) + b(-b - a\overline{\gamma})}{a^2} = -\frac{a^2\overline{\gamma}^2}{a^2} = -\overline{\gamma}^2, \end{aligned}$$

$$s_2 = \frac{b}{a} + \alpha + \beta + \gamma = \frac{b}{a} - \frac{b + a\overline{\gamma}}{a} = -\overline{\gamma}.$$

Now we show that this group is toroidal. The irrationality condition in standard coordinates reduces to the fact that the only vector $(l_1, l_2, l_3) \in \mathbb{Z}^3$ that satisfies $(l_1, l_2, l_3) \cdot (\tau, s_1, s_2) \in \mathbb{Z}$ is the zero vector. Let us suppose that $(l_1, l_2, l_3) \cdot (\tau, s_1, s_2) = l \in \mathbb{Z}$. Therefore we have

$$l_1 \frac{e}{\overline{\gamma}} - l_2\overline{\gamma}^2 - l_3\overline{\gamma} = l,$$

that implies

$$l_2\overline{\gamma}^3 + l_3\overline{\gamma}^2 + l\overline{\gamma} - l_1 e = 0.$$

This gives $(l_1, l_2, l_3) = (0, 0, 0)$, otherwise we would have a contradiction with the fact that $f(x)$ is a times the minimal polynomial of $\overline{\gamma}$ over $\mathbb{Q}$. $\square$

Remark 2.12. As mentioned in the Historical background, the fact that $\mathbb{C}^3/\mu_\Phi(\mathcal{O}_K)$ is toroidal is already in [2], but the above proof shows that, in the case where K admits an essential polynomial, it can be proved more directly. Finally, using Corollary 2.2, $\mathbb{C}^3/\mu_\Phi(\mathfrak{a})$ is, as well, toroidal for every fractional ideal $\mathfrak{a}$ of K.

Now we can generalize a theorem of [16], the proof of which is similar.

Theorem 2.13. *If $\Phi = \{\sigma_1, \sigma_2, \sigma_3\}$ is a type of a quartic field K admitting an essential polynomial, and if $\mathfrak{a}$ is a fractional ideal of K, then the toroidal group $\mathcal{T} = \mathbb{C}^3/\mu_\Phi(\mathfrak{a})$ has extra multiplications. Moreover, the ring $\mathcal{O}_K$ is isomorphic to* $\mathrm{End}(\mathcal{T})$ *and the field K is isomorphic to* $\mathrm{End}_0(\mathcal{T})$.

Proof. For every $\nu \in \mathcal{O}_K$, the matrix

$$D(\nu) = \begin{pmatrix} \sigma_1(\nu) & 0 & 0 \\ 0 & \sigma_2(\nu) & 0 \\ 0 & 0 & \sigma_3(\nu) \end{pmatrix} \tag{2.8}$$

induces an endomorphism of $\mathcal{T}$, that we keep denoting with the same symbol. In fact, if $(\sigma_1(\varepsilon), \sigma_2(\varepsilon), \sigma_3(\varepsilon)) \in \mu_\Phi(\mathfrak{a})$ then

$$D(\nu)\begin{pmatrix} \sigma_1(\varepsilon) \\ \sigma_2(\varepsilon) \\ \sigma_3(\varepsilon) \end{pmatrix} = \begin{pmatrix} \sigma_1(\nu) & 0 & 0 \\ 0 & \sigma_2(\nu) & 0 \\ 0 & 0 & \sigma_3(\nu) \end{pmatrix}\begin{pmatrix} \sigma_1(\varepsilon) \\ \sigma_2(\varepsilon) \\ \sigma_3(\varepsilon) \end{pmatrix} = \begin{pmatrix} \sigma_1(\nu\varepsilon) \\ \sigma_2(\nu\varepsilon) \\ \sigma_3(\nu\varepsilon) \end{pmatrix},$$

and $(\sigma_1(\nu\varepsilon), \sigma_2(\nu\varepsilon), \sigma_3(\nu\varepsilon)) \in \mu_\Phi(\mathfrak{a})$. Since $\mathbb{Z}$ is strictly contained in $\mathcal{O}_K$, we have that $\mathcal{T}$ has extra multiplications. Tensoring with $\mathbb{Q}$, the injective map

$$\begin{aligned} D\colon \mathcal{O}_K &\longrightarrow \mathrm{End}(\mathcal{T}) \\ \nu &\longmapsto D(\nu) \end{aligned}$$

induces a further map

$$K \longrightarrow \mathrm{End}_0(\mathcal{T}). \tag{2.9}$$

Since by [4] we know that $\mathrm{End}_0(\mathcal{T})$ is a non-totally real number field, and its dimension must divide 4, the map in (2.9) turns out to be an isomorphism. We also conclude that $\mathrm{End}(\mathcal{T})$ is an order in $\mathrm{End}_0(\mathcal{T}) \simeq K$, and since it contains (up to isomorphism) the maximal order $\mathcal{O}_K$, we obtain that the map D is an isomorphism. □

Remark 2.14. While by [4] we know that for all toroidal groups $\mathbb{C}^3/\Lambda$ of real rank 4, the ring $\mathrm{End}_0(\mathbb{C}^3/\Lambda)$ is actually a non-totally real number field, Theorem 2.11 shows that the period lattice of a toroidal group must fulfill very strict conditions in order to be a group arising from a non-totally real quartic field admitting an essential polynomial.

This can be seen as a further evidence that the class of fields with essential polynomial is very small compared to all the non-totally real number fields.

2.4. Representations of the endomorphisms. Coming back to the 3-dimensional general case, let $\mathcal{T} = \mathbb{C}^3/\Lambda$ be a toroidal group of complex dimension $\mathrm{rk}_\mathbb{C}\, \mathcal{T} = 3$ and real rank $\mathrm{rk}_\mathbb{R}\, \mathcal{T} = 4$ having extra multiplications. We recall again that $\mathrm{End}_0(\mathcal{T})$ is a non-totally real number field. Since

$$\dim_\mathbb{Q} \mathrm{End}_0(\mathcal{T}) \mid \dim_\mathbb{Q} \mathrm{End}_\mathbb{Q}(\mathrm{span}_\mathbb{Q}(\Lambda)) = \mathrm{rk}_\mathbb{R}\, \Lambda = 4,$$

we have that $\dim_\mathbb{Q} \mathrm{End}_0(\mathcal{T})$ can be equal to 2 or 4, that is, $\mathrm{End}_0(\mathcal{T})$ can be respectively a quadratic or a quartic number field. The natural map $\mathrm{End}(\mathcal{T}) \to \mathrm{End}(\mathcal{T}) \otimes_\mathbb{Z} \mathbb{Q} = \mathrm{End}_0(\mathcal{T})$ is injective because $\mathrm{End}(\mathcal{T})$ is torsion-free. We conclude that $\mathrm{End}(\mathcal{T})$ is an order of the number field $\mathrm{End}_0(\mathcal{T})$ since $\mathrm{rk}_\mathbb{R} \mathrm{End}(\mathcal{T}) = [\mathrm{End}_0(\mathcal{T}) : \mathbb{Q}]$. Let us suppose from now on that $\mathrm{End}_0(\mathcal{T})$ is a quartic number field K admitting an essential polynomial $f(x) = ax^4 + bx^3 + cx^2 + dx + e \in \mathbb{Z}[x]$ and that the order $\mathrm{End}(\mathcal{T})$ is maximal, so it coincides with the ring of integers $\mathcal{O}_K$. Again by [10], we have an integral basis of the form (2.6) and hence a matrix representation of $\mathcal{O}_K$ into $\mathrm{M}_4(\mathbb{Z})$

given by

$$\rho_0 = 1 \mapsto I_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \rho_1 \mapsto A_1 = \begin{pmatrix} 0 & 0 & 0 & -ae \\ 1 & -b & -c & -d \\ 0 & a & 0 & 0 \\ 0 & 0 & a & 0 \end{pmatrix}, \tag{2.10}$$

$$\rho_2 \mapsto A_2 = \begin{pmatrix} 0 & 0 & -ae & -be \\ 0 & -c & -d & -e \\ 1 & 0 & -c & -d \\ 0 & a & b & 0 \end{pmatrix}, \rho_3 \mapsto A_3 = \begin{pmatrix} 0 & -ae & -be & -ce \\ 0 & -d & -e & 0 \\ 0 & 0 & -d & -e \\ 1 & 0 & 0 & -d \end{pmatrix}.$$

Theorem 2.15. *Let $\mathcal{T}$ be a toroidal group of complex dimension 3 and real rank 4 having extra multiplications and period matrix given by*

$$\Pi = \left(\begin{array}{ccc|c} 1 & 0 & 0 & \tau \\ 0 & 1 & 0 & s_1 \\ 0 & 0 & 1 & s_2 \end{array}\right). \tag{2.11}$$

Suppose that the non-totally real number field $K = \mathrm{End}_0(\mathcal{T})$ is a quartic field admitting an essential polynomial $f(x) = ax^4 + bx^3 + cx^2 + dx + e$ and that $\mathrm{End}(\mathcal{T}) = \mathcal{O}_K$. Suppose (2.10) defines the rational representation of $\mathrm{End}(\mathcal{T})$. Then $\mathcal{T}$ is isomorphic to $\mathbb{C}^3/\mu_\Phi(\mathcal{O}_K)$,for a suitable Minkowski map μ_Φ. Furthermore the analytic representation in standard coordinates is given by

$$\rho_0 = 1 \mapsto I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \rho_1 \mapsto M_1 = \begin{pmatrix} 0 & 0 & a\tau \\ 1 & -b & -c + as_1 \\ 0 & a & as_2 \end{pmatrix}, \tag{2.12}$$

$$\rho_2 \mapsto M_2 = \begin{pmatrix} 0 & a\tau & -ae + b\tau \\ 0 & -c + as_1 & -d + bs_1 \\ 1 & as_2 & -c + bs_2 \end{pmatrix}, \quad \rho_3 \mapsto M_3 = \begin{pmatrix} \tau & -ae & -be \\ s_1 & -d & -e \\ s_2 & 0 & -d \end{pmatrix}.$$

Proof. Suppose that the matrix Π in (2.11) is the period matrix of $\mathcal{T}$ in standard coordinates. With the embedding in (2.10) as the rational representation of the ring $\mathrm{End}(\mathcal{T})$, the Hurwitz relations

$$M_i \Pi = \Pi A_i, \quad i = 1, 2, 3, \tag{2.13}$$

must hold, where $M_i \in \mathrm{M}_3(\mathbb{C})$ is the analytic representation of ρ_i. Direct computations give in turn the matrices M_i, that are

$$M_1 = \begin{pmatrix} 0 & 0 & a\tau \\ 1 & -b & -c + as_1 \\ 0 & a & as_2 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0 & a\tau & -ae + b\tau \\ 0 & -c + as_1 & -d + bs_1 \\ 1 & as_2 & -c + bs_2 \end{pmatrix},$$

$$M_3 = \begin{pmatrix} \tau & -ae & -be \\ s_1 & -d & -e \\ s_2 & 0 & -d \end{pmatrix}.$$

Comparing the last columns of equation (2.13) for $i = 1$, we obtain that τ, s_1 and s_2 must satisfy the following conditions:

$$\begin{cases} \tau = -\frac{e}{s_2}, \\ s_1 = -s_2^2, \\ f(-s_2) = 0. \end{cases}$$

Equation (2.13) with $i = 2, 3$ does not give any further information. Hence we have that $-s_2$ is a root of the polynomial $f(x)$ which cannot be real (otherwise the complex rank of Λ would be 3). Let α and β be the real roots of $f(x)$. If we consider the type $\Phi = \{\sigma_1, \sigma_2, \sigma_3\}$ defined by

$$\sigma_1 \colon \omega \longmapsto \alpha, \quad \sigma_2 \colon \omega \longmapsto \beta, \quad \sigma_3 \colon \omega \longmapsto -\overline{s_2},$$

we have that Λ coincides with the lattice $\mu_\Phi(\mathcal{O}_K)$ described in Theorem 2.11. $\square$

Remark 2.16. The above representation in $\mathrm{M}_4(\mathbb{Z})$ of the ring of integers $\mathcal{O}_K$ of a non-totally real quartic field K with essential polynomial is already in [10]. We remark here that the representation of $\mathrm{End}(\mathcal{T})$ in (2.12) gives in turn, as well, a representation of smaller degree into $\mathrm{M}_3(\mathbb{C})$ of $\mathcal{O}_K$.

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