Journal de Théorie des Nombres de Bordeaux

anciennement Séminaire de Théorie des Nombres de Bordeaux

Takehiro HASEGAWA

Explicit formulas for the exponential and logarithm of the Carlitz–Tate twist, and applications

Tome 36, nº 1 (2024), p. 251-291.

https://doi.org/10.5802/jtnb.1278

Le Journal de Théorie des Nombres de Bordeaux *est membre du Centre Mersenne pour l'édition scientifique ouverte*
http://www.centre-mersenne.org/
e-ISSN : 2118-8572

Journal de Théorie des Nombres de Bordeaux **36** (2024), 251–291

Explicit formulas for the exponential and logarithm of the Carlitz–Tate twist, and applications

par Takehiro HASEGAWA

Résumé. Nous présentons des formules explicites pour l'exponentielle et le logarithme de la puissance tensorielle n-ième du module de Carlitz, introduit par Anderson et Thakur en 1990. Nous les utilisons pour prouver des résultats de transcendance pour les fonctions hypergéométriques de type log sur les corps de fonctions définies dans notre article précédent [17].

Abstract. We present explicit formulas for the exponential and logarithm of the nth tensor power of the Carlitz module, introduced by Anderson and Thakur in 1990. We use these to prove transcendence results of the log-type hypergeometric functions for function fields defined in our previous paper [17].

1. Introduction

1.1. Notation. We introduce the notation most often used in this paper below:

$$\begin{aligned}
\mathbb{F}_q &:= \text{the finite field of size } q \text{ and characteristic } p.\\
A &:= \mathbb{F}_q[\theta], \quad \text{the polynomial ring in } \theta \text{ over } \mathbb{F}_q.\\
k &:= \mathbb{F}_q(\theta), \quad \text{the quotient field of } A.\\
k_\infty &:= \mathbb{F}_q((1/\theta)), \quad \text{the } \infty\text{-adic completion of } k \text{ at the infinite place.}\\
\overline{k}_\infty &:= \text{an algebraic closure of } k_\infty.\\
\mathbb{C}_\infty &:= \text{the completion of } \overline{k}_\infty.\\
\overline{k} &:= \text{the algebraic closure of } k \text{ in } \mathbb{C}_\infty.\\
\mathbb{C}_\infty\{\tau\} &:= \text{the twisted polynomial ring in the } q\text{th-power Frobenius } \tau,\\
&\qquad \text{where the multiplication law is given by } \tau c = c^q\tau \text{ for } c \in \mathbb{C}_\infty.\\
[0] &:= 0, \quad [i] := \theta^{q^i} - \theta, \quad (\text{for } i \in \mathbb{Z}_{\geq 1}).\\
\Delta_0(f) &:= f(\theta\tau) - \theta f(\tau).
\end{aligned}$$

Manuscrit reçu le 4 octobre 2022, révisé le 31 juillet 2023, accepté le 15 septembre 2023.

2020 *Mathematics Subject Classification.* 11G09, 11J91.

Mots-clefs. Carlitz–Tate twist, Anderson–Thakur exponential, Anderson–Thakur logarithm, Thakur hypergeometric function, log-type hypergeometric function.

This work was supported by JSPS KAKENHI Grant Number 22K03246.

$D_0 := 1, \quad D_i := [i][i-1]^q \cdots [1]^{q^{i-1}}, \quad (\text{for } i \in \mathbb{Z}_{\geq 1}).$

$L_0 := 1, \quad L_i := (-[i])(-[i-1]) \cdots (-[1]), \quad (\text{for } i \in \mathbb{Z}_{\geq 1}).$

$M_{l,m}$ = the (l, m)th entry of a matrix M.

1.2. Introduction. In 1935 Carlitz [4] introduced the notion of the Carlitz module (a rank-one Drinfeld module) $C : A \to \mathbb{C}_\infty\{\tau\}$ given by $C(\theta) = \theta\tau^0 + \tau$, and defined the Carlitz exponential $e_C := \tau^0 + \sum_{i\geq 1} e_i\tau^i$ and the Carlitz logarithm $\log_C := \tau^0 + \sum_{i\geq 1} l_i\tau^i$. The coefficients e_i and l_i can be determined from the functional equations $C \circ e_C = e_C \circ \theta$ and $\log_C \circ C = \theta \circ \log_C$: $e_i = 1/D_i$ and $l_i = 1/L_i$. In 1974 Drinfeld [10, 11] generalized the module C to a rank-r Drinfeld module $\phi : A \to \mathbb{C}_\infty\{\tau\}$ defined by $\phi(\theta) = \theta\tau^0 + \sum_{i=1}^{r} A_i\tau^i$, and gave the Drinfeld exponential $e_\phi := \tau^0 + \sum_{i\geq 1} \alpha_i\tau^i$ and the Drinfeld logarithm $\log_\phi := \tau^0 + \sum_{i\geq 1} \beta_i\tau^i$. The coefficients α_i and β_i can be computed explicitly with functional equations (see, for example, [14, Theorems 3.1 and 3.3]). In 1990 Anderson and Thakur [2] extended the module C to higher dimension $C^{\otimes n} : A \to M_n(\mathbb{C}_\infty\{\tau\})$ given by

$$
\begin{aligned}
C^{\otimes n}(\theta) &= (\theta I_n + N)\tau^0 + E\tau \\
&= \left(\begin{bmatrix} \theta & 0 & 0 & \cdots & 0 \\ 0 & \theta & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & \theta \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \right) \tau^0 + \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \end{bmatrix} \tau,
\end{aligned}
$$

and defined the Anderson–Thakur exponential $\exp_n := I_n\tau^0 + \sum_{i\geq 1} E(i)\tau^i$ and the Anderson–Thakur logarithm $\log_n := I_n\tau^0 + \sum_{i\geq 1} L(i)\tau^i$. We see that $C^{\otimes 1} = C$, $\exp_1 = e_C$, and $\log_1 = \log_C$. It follows from the functional equations $C^{\otimes n} \circ \exp_n = \exp_n \circ (\theta I_n + N)$ and $\log_n \circ C^{\otimes n} = (\theta I_n + N) \circ \log_n$ that we have

$$
\begin{aligned}
E(i) &= \sum_{j=0}^{2(n-1)} \frac{\mathrm{ad}(N)^j (E \cdot E(i-1)^{(q)}/[i])}{[i]^j}, \\
L(i) &= \sum_{j=0}^{2(n-1)} \frac{\mathrm{ad}(N)^j (-L(i-1) \cdot E/[i])}{[i]^j}
\end{aligned}
$$

(see [27, Section 7.6]), which are nowhere close to being explicit. It is important that the explicit formulas for $E(i)$ and $L(i)$ are discussed, and that is the purpose of this study. The key is not to analyze the matrix $E(i)$ (resp. $L(i)$) itself but to analyze relationships between its entries $E(i)_{l,m}$ (resp. $L(i)_{l,m}$), where $E(i) = [E(i)_{l,m}]$ (resp. $L(i) = [L(i)_{l,m}]$).

In Section 2 we prove the following recursions (see Theorems 2.3, 2.6, and Corollaries 2.4, 2.7):

$$\begin{aligned}
&D_i^n \cdot E(i)_{l,m} \\
&= \sum_{j=0}^{m-1} \binom{l-1}{j} [i]^{(l-1)-j} \sum_{s=1}^{m-1-j} (-1) \binom{n}{s} \sum_{t=1}^{i} \left(\frac{D_t^n \cdot E(t)_{1,m-j-s}}{[t]^s} \right)^{q^{i-t}}, \\
&L_i^n \cdot L(i)_{l,m} \\
&= \sum_{j=0}^{n-l} \binom{n-m}{j} (-[i])^{(n-m)-j} \sum_{s=1}^{n-l-j} (-1) \binom{n}{s} \sum_{t=1}^{i} \frac{L_t^n \cdot L(t)_{l+j+s,n}}{(-[t])^s},
\end{aligned}$$

which are the main theorems of this paper. Using these, the explicit formulas for $E(i)$ and $L(i)$ can be obtained. As an application, we present relations between $\exp_n$ and $\mathcal{E}_n := \tau^0 + \sum_{i\geq 1} \tau^i / D_i^n$, and $\log_n$ and $\mathcal{L}_n := \tau^0 + \sum_{i\geq 1} \tau^i / L_i^n$. In particular, when $n = p^r$, a power of the characteristic p of $\mathbb{F}_q$, we show that

$$\exp_{p^r} = [(\exp_{p^r})_{l,m}] = \left[\binom{l-1}{m-1} \Delta_0^{l-m} \mathcal{E}_{p^r} \right],$$

$$\log_{p^r} = [(\log_{p^r})_{l,m}] = \left[\binom{p^r-m}{p^r-l} (-\Delta_0)^{l-m} \mathcal{L}_{p^r} \right],$$

which are lower triangular (see Corollaries 2.4 and 2.10). The main theorem and its application are refinements of results in [2, Sections 2.1 and 2.2], [15, Section 5.10], and [27, Section 7.6].

There has been much discussion about formulas for the exponential Exp_G and the logarithm Log_G. In 2021, for the special point

$$\mathbf{z} = {}^t(\overbrace{0,\dots,0,(-1)^{r-1} z_1 \cdots z_r}^{n_1}, \overbrace{0,\dots,0,(-1)^{r-2} z_2 \cdots z_r}^{n_2}, \dots, \overbrace{0,\dots,0,z_r}^{n_r})$$

in $G(\overline{k})$, Chang, Green, and Mishiba obtained a formula for $\mathrm{Log}_G(\mathbf{z})$, where $G = (\mathbb{G}_a^n, \rho)$ is a t-module, and Log_G is the logarithm of G (see [5, Theorem 3.3.5], [7, Theorem 4.2.3], and [6, Section 3]). A specialization of theirs is that of ours, as we will now explain. If we set the depth r in their setting to $r = 1$, then we have $n = n_1$, $G = C^{\otimes n}$ and $\mathrm{Log}_G = \log_n$, and thus we have a formula for the last column ${}^t((\log_n)_{1,n}, \dots, (\log_n)_{n,n})$ of $\log_n$, which is expressed by means of the t-motivic CMSPL. On the other hand, in this paper, we obtain not only the last column but also all columns.

In the last half of this paper we investigate transcendence problems for entries of $\log_n$. In 1995 and 2000 Thakur [25, 26] introduced the series $F_{C,\exp}(a_1, \dots, a_r; b_1, \dots, b_s; \tau)$ associated to function fields, which is called the Thakur hypergeometric function or the exp-type hypergeometric function. On the other hand, in 2022 Hasegawa [17] defined the other series $F_{C,\log}(a_1,\ldots,a_r;b_1,\ldots,b_s;\tau)$, which is called the log-type hypergeometric function. In [27, Remarks 10.5.4 (2)] Thakur asked

"What are the transcendence results for the hypergeometric functions?"

This question is interesting. In the classical setting, transcendences for the classical hypergeometric functions were well studied, and those are related to the André-Oort conjecture (see, for example, Wolfart [30], Cohen and Wüstholz [9], and Edixhoven and Yafaev [13]). In the function field setting, Thakur–Wen–Yao–Zhao [28] and Harada [16] gave answers for the function $F_{C,\exp}$, respectively, in completely different approaches. In fact, Thakur–Wen–Yao–Zhao proved that when $r < s+1$, the special value $F_{C,\exp}(a_1,\ldots,a_r;b_1,\ldots,b_s;\gamma)$ is transcendental over k for many $\gamma \in \overline{k}$. On the other hand, Harada showed that when $r = s+1$, the special value $F_{C,\exp}(a_1,\ldots,a_{s+1};b_1,\ldots,b_s;\gamma)$ is transcendental over k for many $\gamma \in \overline{k}$. The question for the other function $F_{C,\log}$ remains open, and we study it in this paper. Indeed, we relate the function $F_{C,\log}$ to entries of $\log_n$, and combining this with a result from Yu [32], we show that the special value $F_{C,\log}(-;2^{n-m},1^{m-1};\gamma)$ is transcendental for all $\gamma \in \overline{k}$ whenever $F_{C,\log}(-;2^{n-m},1^{m-1};\gamma)$ is nonzero. Our way is completely different from theirs.

Throughout this paper we adopt the notation of Anderson–Thakur [2], Goss [15], and Thakur [27]. The remainder of this paper is organized as follows. In the second section we prove the main theorems. The first and second subsections focus on the logarithm $\log_n$ and the exponential $\exp_n$, respectively. The main theorems in this paper are Theorems 2.3 and 2.6 and Corollaries 2.4 and 2.7. In the third section we describe applications of these theorems. To be precise, we express entries of $\exp_n$ and $\log_n$ by means of hypergeometric functions, and we investigate transcendence results for them.

Acknowledgments. The author thanks the referee for their valuable comments and careful review.

2. Main theorems

In this section, we give explicit formulas for the two series $\exp_n$ and $\log_n$.

Let $A := \mathbb{F}_q[\theta]$ be a polynomial ring in a variable θ over the finite field $\mathbb{F}_q$ with q elements, and let $k := \mathbb{F}_q(\theta)$ be the rational function field. We denote by $v_\infty : k \to \mathbb{R} \cup \{\infty\}$ the valuation associated to $1/\theta$, normalized so that $v_\infty(1/\theta) = 1$. In this paper, we often use $\deg(x) = -v_\infty(x)$ and $|x|_\infty := q^{-v_\infty(x)}$, so $\deg(\theta) = 1$ and $|\theta|_\infty = q$. Let $k_\infty := \mathbb{F}_q((1/\theta))$ be the associated completion of k, let $\overline{k}_\infty$ be an algebraic closure of k_∞, and let $\mathbb{C}_\infty$ be the completion of $\overline{k}_\infty$ coming from the canonical extension to $\overline{k}_\infty$. We still denote by v_∞ the extended map from k_∞ to $\overline{k}_\infty$, and hence to $\mathbb{C}_\infty$. Let $\overline{k}$ be the algebraic closure of k inside $\mathbb{C}_\infty$. When we refer to a transcendental element in $\mathbb{C}_\infty$, we simply mean an element not in $\overline{k}$. Note that $\overline{k}$ equals the algebraic closure of k in $\overline{k}_\infty$.

We should be aware of the following basic analogy:

$$A \leftrightarrow \mathbb{Z}, \quad k \leftrightarrow \mathbb{Q}, \quad k_\infty \leftrightarrow \mathbb{R}, \quad \overline{k}_\infty \text{ or } \mathbb{C}_\infty \leftrightarrow \mathbb{C}, \quad \overline{k} \leftrightarrow \overline{\mathbb{Q}}.$$

Before diving into Subsection 2.1, we define $\exp_n$ and $\log_n$ and give an overview of previous studies. The Carlitz module is the rank-one Drinfeld module $C : A \to \mathbb{C}_\infty\{\tau\}$ defined by $C(\theta) = \theta\tau^0 + \tau$ (see [15, Definition 3.3.5] and [27, Section 2.1]). We denote by $C^{\otimes n}$ the nth tensor power of C, which is defined by the $\mathbb{F}_q$-algebra homomorphism $C^{\otimes n} : A \to M_n(\mathbb{C}_\infty\{\tau\})$ given by

$$C^{\otimes n}(\theta) = (\theta I_n + N)\tau^0 + E\tau,$$

$$N := \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \qquad \text{and} \qquad E := \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

(see [2, Section 1], [15, Definition 5.8.1], and [27, Examples 7.1.3 (3)]). Although the module $C^{\otimes n}$ is an example of the t-modules introduced by Anderson [1], we do not need the notion of t-motives in this paper. The exponential and logarithm of $C^{\otimes n}$ are denoted by

$$\exp_n := \sum_{i=0}^{\infty} E(i)\tau^i = I_n\tau^0 + \sum_{i=1}^{\infty} E(i)\tau^i, \qquad (\text{for } E(i) \in M_n(k)),$$
$$\log_n := \sum_{i=0}^{\infty} L(i)\tau^i = I_n\tau^0 + \sum_{i=1}^{\infty} L(i)\tau^i, \qquad (\text{for } L(i) \in M_n(k)),$$

respectively (see [2, Section 2], [15, Section 5.10], and [27, Section 7.6]). We denote the (a, b)th entry of an $n \times n$ matrix M by $M_{a,b}$, that is, $M = [M_{a,b}]$. For convenience, we set the empty entry to be zero. For example, $M_{n+1,b} = 0$ and $M_{a,n+1} = 0$. In this paper, we use the following notation:

$$\exp_n = [(\exp_n)_{a,b}], \qquad (\exp_n)_{a,b} = \sum_{i=0}^{\infty} E(i)_{a,b}\tau^i,$$

$$\log_n = [(\log_n)_{a,b}], \qquad (\log_n)_{a,b} = \sum_{i=0}^{\infty} L(i)_{a,b}\tau^i.$$

For each $i \geq 1$, let $[0] = 0$ and $[i] = \theta^{q^i} - \theta \in A$, $D_0 = 1$ and $D_i = [i]D_{i-1}^q \in A$, and $L_0 = 1$ and $L_i = (-[i])L_{i-1} \in A$ (see [15, Definition 3.1.4] and [27, Section 2.5]). For any $n \geq 1$, we define the functions $\mathcal{E}_n(z)$ and

$\mathcal{L}_n(z)$ as

$$\mathcal{E}_n(z) := \sum_{i=0}^{\infty} \frac{1}{D_i^n} z^{q^i} \qquad \text{and} \qquad \mathcal{L}_n(z) := \sum_{i=0}^{\infty} \frac{1}{L_i^n} z^{q^i},$$

respectively. The function $\mathcal{E}_n(z)$ is called the Carlitz poly-exponential (see [15, Section 5.10]), and the function $\mathcal{L}_n(z)$ is called the Carlitz poly-logarithm (see [15, Section 5.10] and [27, Remarks 7.6.2]). Note that $\mathcal{E}_1(z) = e_C(z)$ and $\mathcal{L}_1(z) = \log_C(z)$ are the Carlitz exponential and the Carlitz logarithm, respectively (see [15, Sections 3.2 and 3.4] and [27, Section 2.5]). Fix a $(q-1)$th root of $-[1]$ in $\overline{k}$, and let $\widetilde{\pi}$ denote the fundamental period of the Carlitz module:

$$\begin{aligned}\widetilde{\pi} := \widetilde{\pi}_C &= (-[1])^{1/(q-1)} \prod_{i=1}^{\infty} \left(1 - \frac{[i]}{[i+1]}\right) \\ &= (-\theta)^{1/(q-1)} \theta \prod_{i=1}^{\infty} (1 - \theta^{1-q^i})^{-1}\end{aligned}$$

(see [15, Definition 3.2.7] and [27, Section 2.5]). It is well known that $\mathrm{Ker}(e_C) = A\widetilde{\pi}$ (see [15, Corollary 3.2.9] and [27, Section 2.1]), and hence $e_C(\widetilde{\pi}) = 0$. We define a linear operator Δ_0 given by

$$\Delta_0 \left(\sum_{i=0}^{\infty} a_i \tau^i \right) = \sum_{i=0}^{\infty} [i] a_i \tau^i = \sum_{i=1}^{\infty} [i] a_i \tau^i,$$

or equivalently, $\Delta_0(\sum_{i=0}^{\infty} a_i z^{q^i}) = \sum_{i=0}^{\infty} [i] a_i z^{q^i} = \sum_{i=1}^{\infty} [i] a_i z^{q^i}$, which was defined by Carlitz in 1935 (see [27, Section 4.14]).

Now, we will consider some examples.

Example.

(1) ([27, Remarks 7.6.2]) The series $\exp_2$ is given by

$$\exp_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tau^0 + \sum_{i=1}^{\infty} \begin{bmatrix} \frac{1}{D_i^2} & -2k_i \frac{1}{D_i^2} \\ \frac{[i]}{D_i^2} & (-2[i]k_{i-1}^q - 1) \frac{1}{D_i^2} \end{bmatrix} \tau^i,$$

where $k_i := \frac{1}{[i]} + \frac{1}{[i-1]^q} + \cdots + \frac{1}{[1]^{q^{i-1}}}$. Note that $(\exp_2)_{1,1} = \mathcal{E}_2$ and $(\exp_2)_{2,1} = \Delta_0 \mathcal{E}_2$.

(2) The series $\log_2$ is given by

$$\log_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tau^0 + \sum_{i=1}^{\infty} \begin{bmatrix} (-2[i]k_{i-1} - 1) \frac{1}{L_i^2} & 2k_i \frac{1}{L_i^2} \\ \frac{-[i]}{L_i^2} & \frac{1}{L_i^2} \end{bmatrix} \tau^i,$$

where $k_i := \frac{1}{[i]} + \frac{1}{[i-1]} + \cdots + \frac{1}{[1]}$. Note that $(\log_2)_{2,2} = \mathcal{L}_2$ and $(\log_2)_{2,1} = (-\Delta_0)\mathcal{L}_2$.

It is not difficult to compute the above examples, but we encounter difficulties when increasing the matrix size n. In this section, we give explicit formulas for $\exp_n$ and $\log_n$ for all n.

We write $\exp_n(\mathbf{z})$ and $\log_n(\mathbf{z})$ as

$$\exp_n(\mathbf{z}) = \exp_n\left(\begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}\right) = \begin{bmatrix} e_1(\mathbf{z}) \\ e_2(\mathbf{z}) \\ \vdots \\ e_n(\mathbf{z}) \end{bmatrix} = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix} + \sum_{i=1}^{\infty} E(i) \begin{bmatrix} z_1^q \\ z_2^q \\ \vdots \\ z_n^q \end{bmatrix},$$

$$\log_n(\mathbf{z}) = \log_n\left(\begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}\right) = \begin{bmatrix} l_1(\mathbf{z}) \\ l_2(\mathbf{z}) \\ \vdots \\ l_n(\mathbf{z}) \end{bmatrix} = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix} + \sum_{i=1}^{\infty} L(i) \begin{bmatrix} z_1^q \\ z_2^q \\ \vdots \\ z_n^q \end{bmatrix},$$

respectively. It is shown in [2] that $\exp_n(\mathbf{z})$ converges for all $\mathbf{z} \in \mathbb{C}_\infty^n$, and $\log_n(\mathbf{z})$ converges for all $\mathbf{z} \in \mathbb{C}_\infty^n$ with $|z_i|_\infty < |\theta|_\infty^{i-n+(nq/(q-1))}$ for all i. It is known that $\exp_n(\log_n(\mathbf{z})) = \mathbf{z}$ in this range. See [15, Section 5.10] and [27, Section 7.6].

Note that $\exp_n$ and $\log_n$ are vector-valued functions and they are expressed as matrices with entries in $\mathbb{C}_\infty\{\{\tau\}\}$, which is the non-commutative power series in τ over $\mathbb{C}_\infty$ with the multiplication law $\tau c = c^q \tau$ for $c \in \mathbb{C}_\infty$.

The series $\exp_n$ and $\log_n$ are related to the series $\mathcal{E}_n$ and $\mathcal{L}_n$, respectively.

Fact.

(1) ([2, p. 174], [15, Section 5.10]) For a special point ${}^t(z_1, 0, \ldots, 0)$, we have

$$\exp_n\left(\begin{bmatrix} z_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}\right) = \begin{bmatrix} \mathcal{E}_n(z_1) \\ * \\ \vdots \\ * \end{bmatrix}.$$

(2) ([15, Section 5.10], [27, Remarks 7.6.2 (1)]) For $\mathbf{z} = {}^t(z_1, \ldots, z_n)$, we have

$$\log_n(\mathbf{z}) = \begin{bmatrix} l_1(\mathbf{z}) \\ l_2(\mathbf{z}) \\ \vdots \\ l_n(\mathbf{z}) \end{bmatrix} = \begin{bmatrix} * \\ \vdots \\ * \\ \sum_{m=0}^{n-1} (-\Delta_0)^m \mathcal{L}_n(z_{n-m}) \end{bmatrix}.$$

In the remainder of this section, all of their entries are presented.

2.1. Logarithm. In this subsection, we present an explicit formula for $\log_n$. Recall that $L(0) = I_n$, the identity matrix, and that $\log_n$ is defined by

$$\log_n := I_n\tau^0 + \sum_{i=1}^{\infty} L(i)\tau^i = \sum_{i=0}^{\infty} L(i)\tau^i.$$

For an $n \times n$ matrix M, we inductively define the commutators $\mathrm{ad}(N)^j(M)$ as

$$\begin{aligned}&\mathrm{ad}(N)^0(M) := M,\\ &\mathrm{ad}(N)^j(M) := N \cdot \mathrm{ad}(N)^{j-1}(M) - \mathrm{ad}(N)^{j-1}(M) \cdot N, \quad (\text{for all } j \geq 1).\end{aligned}$$

From the fundamental functional equation $\log_n \circ C^{\otimes n}(\theta) = d[\theta]_n \circ \log_n$, namely, $\log_n \circ ((\theta I_n + N)\tau^0 + E\tau) = (\theta I_n + N) \circ \log_n$, we obtain $(\theta I_n + N)L(i) = L(i)(\theta^{q^i} I_n + N) + L(i-1)E$, and

$$L(i) = \sum_{j=0}^{\infty} \frac{\mathrm{ad}(N)^j(-L(i-1)E/[i])}{[i]^j} = \sum_{j=0}^{2(n-1)} \frac{\mathrm{ad}(N)^j(-L(i-1)E/[i])}{[i]^j}$$

for any $i \geq 1$ (see [15, Section 5.10] and [27, Section 7.6]).

First, we give recursions for the coefficients $L(i)$, which are used in the proof of the theorem.

Lemma 2.1. *Let $1 \leq a \leq n$ and $1 \leq b \leq n$. Let $L(i)_{a,b}$ denote the (a,b)th entry of the matrix $L(i)$:*

$$L(i) = \begin{bmatrix} * & \cdots & * & * & \cdots & * \\ \vdots & & \vdots & \vdots & & \vdots \\ * & \cdots & L(i)_{a,b-1} & L(i)_{a,b} & \cdots & L(i)_{a,n} \\ * & \cdots & * & L(i)_{a+1,b} & \cdots & L(i)_{a+1,n} \\ \vdots & & \vdots & \vdots & & \vdots \\ * & \cdots & * & * & \cdots & L(i)_{n,n} \end{bmatrix}.$$

(1) *Assume that $1 \leq a \leq n-1$ and $2 \leq b \leq n$. Then, for any $i \geq 1$, the following holds:*

$$L(i)_{a,b} = \frac{1}{[i]} L(i)_{a+1,b} - \frac{1}{[i]} L(i)_{a,b-1}.$$

(2) *For any $i \geq 1$, the following holds:*

$$L(i)_{a,b} = (-1)^b \sum_{j=0}^{n-a} \binom{b-1+j}{j} \left(\frac{1}{[i]}\right)^{b+j} L(i-1)_{a+j,n}.$$

Proof. Before we prove items (1) and (2), we must present several facts. Let $M = [M_{a,b}]$ be an $n \times n$ matrix. From the definitions of the nilpotent matrix N and the commutator $\mathrm{ad}(N)^1(M)$, we have

$$\begin{aligned}
&\mathrm{ad}(N)^1(M)_{a,b} = (NM - MN)_{a,b} \\
&= \left(\begin{bmatrix} M_{2,1} & M_{2,2} & \cdots & M_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ M_{n,1} & M_{n,2} & \cdots & M_{n,n} \\ 0 & 0 & \cdots & 0 \end{bmatrix} - \begin{bmatrix} 0 & M_{1,1} & \cdots & M_{1,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & M_{n-1,1} & \cdots & M_{n-1,n-1} \\ 0 & M_{n,1} & \cdots & M_{n,n-1} \end{bmatrix} \right)_{a,b} \\
&= M_{a+1,b} - M_{a,b-1}.
\end{aligned}$$

Using induction, we then obtain $\mathrm{ad}(N)^j(M)_{a,b} = \mathrm{ad}(N)^{j-1}(M)_{a+1,b} - \mathrm{ad}(N)^{j-1}(M)_{a,b-1}$. If $a - b \geq n - j$, then we have $\mathrm{ad}(N)^j(M)_{a,b} = 0$, and $\mathrm{ad}(N)^{2(n-1)}(M)_{a+1,b} = \mathrm{ad}(N)^{2(n-1)}(M)_{a,b-1} = 0$. With these facts in hand, we may proceed with proof.

(1). For simplicity, we set

$$Y := -\frac{1}{[i]} L(i-1)E = -\frac{1}{[i]} \begin{bmatrix} L(i-1)_{1,n} & 0 & \cdots & 0 \\ L(i-1)_{2,n} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ L(i-1)_{n,n} & 0 & \cdots & 0 \end{bmatrix},$$

and so we have $\mathrm{ad}^0(Y)_{a,b} = Y_{a,b} = 0$ for $b \geq 2$. Combining this and the facts above, we obtain

$$\begin{aligned}
L(i)_{a,b} &= \sum_{j=0}^{2(n-1)} \frac{\mathrm{ad}(N)^j(Y)_{a,b}}{[i]^j} = \sum_{j=1}^{2(n-1)} \frac{\mathrm{ad}(N)^j(Y)_{a,b}}{[i]^j} \\
&= \sum_{j=1}^{2(n-1)} \frac{\mathrm{ad}(N)^{j-1}(Y)_{a+1,b} - \mathrm{ad}(N)^{j-1}(Y)_{a,b-1}}{[i]^j} \\
&= \sum_{j=1}^{2(n-1)+1} \frac{\mathrm{ad}(N)^{j-1}(Y)_{a+1,b} - \mathrm{ad}(N)^{j-1}(Y)_{a,b-1}}{[i]^j} \\
&= \frac{1}{[i]} \sum_{j=0}^{2(n-1)} \frac{\mathrm{ad}(N)^j(Y)_{a+1,b}}{[i]^j} - \frac{1}{[i]} \sum_{j=0}^{2(n-1)} \frac{\mathrm{ad}(N)^j(Y)_{a,b-1}}{[i]^j} \\
&= \frac{1}{[i]} L(i)_{a+1,b} - \frac{1}{[i]} L(i)_{a,b-1},
\end{aligned}$$

thus finishing the proof of item (1).

(2). We prove item (2) using induction. Recall that $Y := -L(i-1)E/[i]$. Note that

$$Y_{a,b} = \begin{cases} -L(i-1)_{a,n}/[i] & \text{if } b = 1, \\ 0 & \text{if } b \geq 2. \end{cases}$$

First, we compute $L(i)_{a,1}$. Using the definition of $\mathrm{ad}(N)^j(Y)$, we can write $\mathrm{ad}(N)^j(Y) = N^jY + MN$ for some matrix M. Since $(MN)_{a,1} = 0$, we have $\mathrm{ad}(N)^j(Y)_{a,1} = (N^jY)_{a,1}$, and so

$$\mathrm{ad}(N)^j(Y)_{a,1} = (N^jY)_{a,1} = \begin{cases} -L(i-1)_{a+j,n}/[i] & \text{if } j \leq n-a, \\ 0 & \text{if } j > n-a. \end{cases}$$

Hence, we obtain

$$\begin{aligned} L(i)_{a,1} &= \sum_{j=0}^{\infty} \frac{\mathrm{ad}(N)^j(Y)_{a,1}}{[i]^j} = \sum_{j=0}^{n-a} \frac{-L(i-1)_{a+j,n}/[i]}{[i]^j} \\ &= -\sum_{j=0}^{n-a} \left(\frac{1}{[i]}\right)^{1+j} L(i-1)_{a+j,n}. \end{aligned}$$

Next, we compute $L(i)_{n,b}$. Similarly, we can write $\mathrm{ad}(N)^j(Y) = NM + (-1)^jYN^j$ for some matrix M. Since $(NM)_{n,b} = 0$, we have $\mathrm{ad}(N)^j(Y)_{n,b} = (-1)^j(YN^j)_{n,b}$, and so

$$\mathrm{ad}(N)^j(Y)_{n,b} = (-1)^j(YN^j)_{n,b} = \begin{cases} (-1)^b L(i-1)_{n,n}/[i] & \text{if } j = b-1, \\ 0 & \text{if } j \neq b-1. \end{cases}$$

Hence, we obtain

$$\begin{aligned} L(i)_{n,b} &= \sum_{j=0}^{\infty} \frac{\mathrm{ad}(N)^j(Y)_{n,b}}{[i]^j} = \frac{(-1)^b L(i-1)_{n,n}/[i]}{[i]^{b-1}} \\ &= (-1)^b \left(\frac{1}{[i]}\right)^{b} L(i-1)_{n,n}. \end{aligned}$$

Last, we assume that $1 \leq a < n$ and $1 < b \leq n$. By the inductive hypothesis, we have

$$\begin{aligned} L(i)_{a+1,b} &= (-1)^b \sum_{j=0}^{n-(a+1)} \binom{b-1+j}{j} \left(\frac{1}{[i]}\right)^{b+j} L(i-1)_{(a+1)+j,n}, \\ L(i)_{a,b-1} &= (-1)^{b-1} \sum_{j=0}^{n-a} \binom{(b-1)-1+j}{j} \left(\frac{1}{[i]}\right)^{(b-1)+j} L(i-1)_{a+j,n}. \end{aligned}$$

Using item (1), we obtain

$$\begin{aligned}L(i)_{a,b} &= \frac{1}{[i]}L(i)_{a+1,b} - \frac{1}{[i]}L(i)_{a,b-1} \\ &= (-1)^b \sum_{j=0}^{n-a} \left(\binom{b-2+j}{j-1} + \binom{b-2+j}{j} \right) \left(\frac{1}{[i]}\right)^{b+j} L(i-1)_{a+j,n} \\ &= (-1)^b \sum_{j=0}^{n-a} \binom{b-1+j}{j} \left(\frac{1}{[i]}\right)^{b+j} L(i-1)_{a+j,n}.\end{aligned}$$

Note that $\binom{b-2}{-1} = 0$. Hence, we complete the proof of item (2). □

Remarks. In item (2) of the lemma, $L(i)_{1,n}, L(i)_{2,n}, \ldots, L(i)_{n,n}$, the last column, can be interpreted as generators of $L(i)_{a,b}$, that is, $L(i)_{a,b}$ is spanned by $L(i)_{a,n}, L(i)_{a+1,n}, \ldots, L(i)_{n,n}$.

Now, we consider an elementary identity that is used to prove the theorem.

Lemma 2.2. *For any n and j, the following identity holds:*

$$\sum_{k=0}^{j} (-1)^{j-k} \binom{n}{k} \binom{n-1+j-k}{j-k} = 0.$$

Proof. It is proven using induction on n and j. □

Next, we provide an explicit formula for $\log_n$. For simplicity, we set $C(i) = C_{\log}(i) := L_i^n \cdot L(i)$, which is more essential than $L(i)$ in the proof.

Theorem 2.3 (Logarithm I). *Let $0 \le l \le n-1$ and $0 \le m \le n-1$. Let $(\log_n)_{n-l,n-m}$ denote the $(n-l, n-m)$th entry of the matrix $\log_n$:*

$$\log_n = \begin{bmatrix} * & \cdots & * & \cdots & * \\ \vdots & & \vdots & & \vdots \\ * & \cdots & (\log_n)_{n-l,n-m} & \cdots & (\log_n)_{n-l,n} \\ \vdots & & \vdots & & \vdots \\ * & \cdots & * & \cdots & (\log_n)_{n-1,n} \\ * & \cdots & * & \cdots & (\log_n)_{n,n} \end{bmatrix}.$$

Let $L(i)_{n-l,n-m}$ denote the coefficient of τ^i of the series $(\log_n)_{n-l,n-m}$:

$$(\log_n)_{n-l,n-m} = \sum_{i=0}^{\infty} L(i)_{n-l,n-m} \tau^i.$$

(1) *For any l and m, the following holds:*

$$(\log_n)_{n-l,n-m} = \sum_{j=0}^{l} \binom{m}{j} (-\Delta_0)^{m-j} (\log_n)_{n-l+j,n}.$$

In particular, the entry $(\log_n)_{n,n-m}$ of the last row equals $(\log_n)_{n,n-m} = (-\Delta_0)^m (\log_n)_{n,n}$.

(2) *The entries of the last column are given by*

$$(\log_n)_{n,n} = \sum_{i=0}^{\infty} L(i)_{n,n}\tau^i = \sum_{i=0}^{\infty} \frac{1}{L_i^n}\tau^i \ (= \mathcal{L}_n(\tau)),$$

$$\begin{aligned}(\log_n)_{n-l,n} &= \sum_{i=1}^{\infty} L(i)_{n-l,n}\tau^i = \sum_{i=1}^{\infty} C(i)_{n-l,n}\frac{1}{L_i^n}\tau^i \\ &= \sum_{i=1}^{\infty}\left(\sum_{s=1}^{l}(-1)\binom{n}{s}\sum_{t=1}^{i}\frac{C(t)_{n-(l-s),n}}{(-[t])^s}\right)\frac{1}{L_i^n}\tau^i\end{aligned}$$

for any $1 \le l \le n-1$. *Recall that* $L(0)_{n-l,n} = 0$. *In particular, for* $1 \le l \le n-1$, *the following recursion holds:*

$$C(i)_{n-l,n} = \sum_{s=1}^{l}(-1)\binom{n}{s}\sum_{t=1}^{i}\frac{C(t)_{n-(l-s),n}}{(-[t])^s}.$$

Proof.

(1). First, suppose that $i = 0$. The coefficient $L(0)_{n-l,n-m}$ of τ^0 on the left hand side is

$$L(0)_{n-l,n-m} = \begin{cases} 0 & \text{if } l \neq m, \\ 1 & \text{if } l = m. \end{cases}$$

On the other hand, the coefficient of τ^0 on the right hand side is

$$\sum_{j=0}^{l}\binom{m}{j}(-[0])^{m-j}L(0)_{n-l+j,n} = \binom{m}{l}(-[0])^{m-l}L(0)_{n,n} = \begin{cases} 0 & \text{if } l > m, \\ 0 & \text{if } l < m, \\ 1 & \text{if } l = m. \end{cases}$$

We recall that $\Delta_0(\sum_{i=0}^{\infty} a_i\tau^i) = \sum_{i=0}^{\infty}[i]a_i\tau^i = \sum_{i=1}^{\infty}[i]a_i\tau^i$. We note that $L(0)_{n-l+j,n} = 0$ for $j \neq l$, and that $\binom{m}{l} = 0$ for $l > m$, $(-[0])^{m-l} = 0$ for $l < m$, and $(-[0])^{m-l} = 1$ for $l = m$. Hence, the claim holds for $i = 0$.

Next, assume that $i \ge 1$. We will show that the equality

$$L(i)_{n-l,n-m} = \sum_{j=0}^{l}\binom{m}{j}(-[i])^{m-j}L(i)_{n-l+j,n}$$

holds, and using induction on l and m, we will prove item (1). By the inductive hypothesis, we have

$$L(i)_{n-l,n-(m-1)} = \sum_{j=0}^{l}\binom{m-1}{j}(-[i])^{(m-1)-j}L(i)_{n-l+j,n},$$

$$L(i)_{n-(l-1),n-(m-1)} = \sum_{j=0}^{l-1}\binom{m-1}{j}(-[i])^{(m-1)-j}L(i)_{n-(l-1)+j,n}.$$

Using Lemma 2.1 (1), we obtain

$$\begin{aligned}
L(i)_{n-l,n-m} &= -[i]L(i)_{n-l,n-m+1} + L(i)_{n-l+1,n-m+1} \\
&= \sum_{j=0}^{l} \left(\binom{m-1}{j} + \binom{m-1}{j-1} \right) (-[i])^{m-j} L(i)_{n-l+j,n} \\
&= \sum_{j=0}^{l} \binom{m}{j} (-1)^{m-j} [i]^{m-j} L(i)_{n-l+j,n}.
\end{aligned}$$

Recall that $\binom{m-1}{-1} = 0$. Hence, we finish the proof of item (1).

(2). We prove item (2) using induction on i. First, we compute the diagonal entry $L(i)_{n,n}$. When $i = 0$, we have $L(0)_{n,n} = 1 = 1/L_0^n$. Recall that $L(0) = I_n$ and $L_0 = 1$. Assume that $i \geq 1$. By the inductive hypothesis, we have $L(i-1)_{n,n} = 1/L_{i-1}^n$. It follows from Lemma 2.1(2) that

$$L(i)_{n,n} = (-1)^n \left(\frac{1}{[i]} \right)^n L(i-1)_{n,n} = \frac{1}{L_i^n}.$$

Recall that $L_i = -[i]L_{i-1}$. Hence, the first claim has been proven.

Next, we determine the non-diagonal entry $L(i)_{n-l,n}$ for $1 \leq l \leq n-1$. For simplicity, we set $C(i)_{n-l,n} := L(i)_{n-l,n} L_i^n$. When $i = 0$, we have $C(0)_{n-l,n} = 0$. Suppose that $i \geq 1$. We show that

$$C(i)_{n-l,n} = \sum_{s=1}^{l} (-1) \binom{n}{s} \sum_{t=1}^{i} \frac{C(t)_{n-(l-s),n}}{(-[t])^s}.$$

By the inductive hypothesis, we have

$$C(i-1)_{n-l,n} = \sum_{s=1}^{l} (-1) \binom{n}{s} \sum_{t=1}^{i-1} \frac{C(t)_{n-(l-s),n}}{(-[t])^s}.$$

It follows from Lemma 2.1 (2) and $L_i = -[i]L_{i-1}$ that

$$\begin{aligned}
C(i)_{n-l,n} &= L(i)_{n-l,n} L_i^n \\
&= \left((-1)^n \sum_{j=0}^{l} \binom{n-1+j}{j} \left(\frac{1}{[i]} \right)^{n+j} L(i-1)_{n-l+j,n} \right) L_i^n \\
&= \sum_{j=0}^{l} \binom{n-1+j}{j} \left(\frac{1}{[i]} \right)^{j} L(i-1)_{n-l+j,n} L_{i-1}^n \\
&= \sum_{j=0}^{l} (-1)^j \binom{n-1+j}{j} \frac{C(i-1)_{n-l+j,n}}{(-[i])^j}. \qquad (2.1)
\end{aligned}$$

Using the inductive hypothesis, we obtain

$$\begin{aligned}
&\sum_{j=0}^{l}(-1)^j\binom{n-1+j}{j}\frac{C(i-1)_{n-l+j,n}}{(-[i])^j}\\
&= C(i-1)_{n-l,n}+\sum_{j=1}^{l}(-1)^j\binom{n-1+j}{j}\frac{C(i-1)_{n-l+j,n}}{(-[i])^j}\\
&= \sum_{s=1}^{l}(-1)\binom{n}{s}\sum_{t=1}^{i-1}\frac{C(t)_{n-(l-s),n}}{(-[t])^s}+\sum_{j=1}^{l}(-1)^j\binom{n-1+j}{j}\frac{C(i-1)_{n-l+j,n}}{(-[i])^j}.
\end{aligned}$$

We transform the last equality as follows:

$$\begin{aligned}
&\sum_{s=1}^{l}(-1)\binom{n}{s}\sum_{t=1}^{i-1}\frac{C(t)_{n-(l-s),n}}{(-[t])^s}+\sum_{j=1}^{l}(-1)^j\binom{n-1+j}{j}\frac{C(i-1)_{n-l+j,n}}{(-[i])^j}\\
&= \sum_{s=1}^{l}(-1)\binom{n}{s}\sum_{t=1}^{i}\frac{C(t)_{n-(l-s),n}}{(-[t])^s}+\sum_{s=1}^{l}\binom{n}{s}\frac{C(i)_{n-(l-s),n}}{(-[i])^s}\\
&\qquad+\sum_{j=1}^{l}(-1)^j\binom{n-1+j}{j}\frac{C(i-1)_{n-l+j,n}}{(-[i])^j}.
\end{aligned}$$

Next, we check that the sum of the second and third terms equals zero. It follows from equality (2.1) that

$$\begin{aligned}
&\sum_{s=1}^{l}\binom{n}{s}\frac{C(i)_{n-(l-s),n}}{(-[i])^s}+\sum_{j=1}^{l}(-1)^j\binom{n-1+j}{j}\frac{C(i-1)_{n-l+j,n}}{(-[i])^j}\\
&= \sum_{s=1}^{l}\binom{n}{s}\left(\sum_{j=0}^{l-s}(-1)^j\binom{n-1+j}{j}\frac{C(i-1)_{n-(l-s)+j,n}}{(-[i])^j}\right)\frac{1}{(-[i])^s}\\
&\qquad+\sum_{j=1}^{l}(-1)^j\binom{n-1+j}{j}\frac{C(i-1)_{n-l+j,n}}{(-[i])^j}\\
&= \sum_{s=1}^{l}\sum_{j=0}^{l-s}\left((-1)^j\binom{n}{s}\binom{n-1+j}{j}\right)\frac{C(i-1)_{n-l+(s+j),n}}{(-[i])^{s+j}}\\
&\qquad+\sum_{j=1}^{l}(-1)^j\binom{n-1+j}{j}\frac{C(i-1)_{n-l+j,n}}{(-[i])^j}.
\end{aligned}$$

The next step is key. We transform the last equality as follows:

$$\begin{aligned}
&\sum_{s=1}^{l}\sum_{j=0}^{l-s}\left((-1)^j\binom{n}{s}\binom{n-1+j}{j}\right)\frac{C(i-1)_{n-l+(s+j),n}}{(-[i])^{s+j}}\\
&\qquad+\sum_{j=1}^{l}(-1)^j\binom{n-1+j}{j}\frac{C(i-1)_{n-l+j,n}}{(-[i])^j}\\
&=\sum_{j=1}^{l}\left(\sum_{k=0}^{j}(-1)^{j-k}\binom{n}{k}\binom{n-1+j-k}{j-k}\right)\frac{C(i-1)_{n-l+j,n}}{(-[i])^j}.
\end{aligned}$$

By the identity in Lemma 2.2, we see that the last equality is equal to zero.

Going back to the proof of the theorem, we obtain

$$\begin{aligned}
&\sum_{s=1}^{l}(-1)\binom{n}{s}\sum_{t=1}^{i}\frac{C(t)_{n-(l-s),n}}{(-[t])^s}+\sum_{s=1}^{l}\binom{n}{s}\frac{C(i)_{n-(l-s),n}}{(-[i])^s}\\
&\qquad+\sum_{j=1}^{l}(-1)^j\binom{n-1+j}{j}\frac{C(i-1)_{n-l+j,n}}{(-[i])^j}\\
&=\sum_{s=1}^{l}(-1)\binom{n}{s}\sum_{t=1}^{i}\frac{C(t)_{n-(l-s),n}}{(-[t])^s},
\end{aligned}$$

and hence, the second claim has been proven. □

Remarks.

(1) We can interpret the entries

$$\Delta_0^m(\log_n)_{n-l,n},\Delta_0^{m-1}(\log_n)_{n-l+1,n},\ldots,\Delta_0^{m-l}(\log_n)_{n,n}$$

of the last column as generators of $(\log_n)_{n-l,n-m}$.

(2) Using the theorem, we can explicitly determine all the coefficients $L(i)_{n-l,n-m}$. It is sufficient to show that we can explicitly determine all the coefficients $C(i)_{n-l,n}$. First, we recall that $C(i)_{n,n}=1$ for all i. Next, by using the recursion

$$C(i)_{n-l,n}=\sum_{s=1}^{l}(-1)\binom{n}{s}\sum_{t=1}^{i}\frac{C(t)_{n-(l-s),n}}{(-[t])^s}$$

several times, $C(i)_{n-l,n}$ can be reduced to $C(1)_{n,n},\dots,C(i)_{n,n}$. For example,

$$C(i)_{n-1,n}=(-1)\binom{n}{1}\sum_{a=1}^{i}\frac{C(a)_{n,n}}{-[a]}=(-1)n\sum_{a=1}^{i}\frac{1}{-[a]},$$

$$\begin{aligned}C(i)_{n-2,n}&=(-1)\binom{n}{1}\sum_{b=1}^{i}\frac{C(b)_{n-1,n}}{-[b]}+(-1)\binom{n}{2}\sum_{a=1}^{i}\frac{C(a)_{n,n}}{(-[a])^2}\\&=(-1)n\sum_{b=1}^{i}\frac{(-1)n\sum_{a=1}^{b}\frac{1}{-[a]}}{-[b]}+(-1)\frac{n(n-1)}{2}\sum_{a=1}^{i}\frac{1}{(-[a])^2}.\end{aligned}$$

(3) Comparing [5, Theorem 3.3.5] with our theorem, we see that the specialization of [5, Theorem 3.3.5] corresponds to that of our theorem. If we set $r=1$ in [5, Theorem 3.3.5], then we have $G=C^{\otimes d}$ and $\mathrm{Log}_G=\log_d$. Hence, we obtain $\log_d({}^t(0,0,\dots,u))={}^t(\alpha_{d-1}(u),\alpha_{d-2}(u),\dots,\alpha_0(u))$, where $\alpha_j(u)$ is given by the jth Taylor coefficient of the following series at $t=\theta$:

$$\sum_{i\geq 0}\frac{u^{q^i}}{\Big((t-\theta^q)\cdots(t-\theta^{q^i})\Big)^d}=\sum_{j=0}^{\infty}\alpha_j(u)(t-\theta)^j.$$

It is known that $\alpha_0(u)=\mathcal{L}_d(u)$, the poly-logarithm. To the best of our knowledge, the others $\alpha_{d-1}(u),\dots,\alpha_1(u)$ are not known explicitly, but we can compute them using our theorem:

$$\alpha_{d-l}(u)=\sum_{i=1}^{\infty}L(i)_{d-l,d}u^{q^i}$$

for any $1\leq l\leq d-1$.

If the matrix size n is divided by p and we write n as $n=n_0p^r$, then we can give a more detailed formula for $\log_n$. In fact, every entry of $\log_n$ can be expressed by means of $(\log_n)_{p^r,n},\dots,(\log_n)_{n_0p^r,n}$. Moreover, if $n_0=1$, namely $n=p^r$, then every entry can be expressed only in terms of $\mathcal{L}_n$.

Corollary 2.4 (Logarithm II). *Let $n=n_0p^r$, $r\in\mathbb{Z}_{\geq 1}$, and $\gcd(n_0,p)=1$.*

(1) *Let $1\leq l\leq n-1$. If l is of the form $l=l_0p^r$, where $l_0\geq 1$ and $\gcd(l_0,p)=1$, then $(\log_n)_{n-l_0p^r,n}$ is given by*

$$(\log_n)_{n-l_0p^r,n}=\sum_{i=1}^{\infty}\left(\sum_{s_0=1}^{l_0}(-1)\binom{n_0}{s_0}\sum_{t=1}^{i}\frac{C(t)_{n-(l_0-s_0)p^r,n}}{(-[t])^{s_0p^r}}\right)\frac{1}{L_i^n}\tau^i,$$

where $C(i)=L_i^n\cdot L(i)$ is the modified coefficient of $\log_n$.

On the other hand, when l is not divided by p^r, that is, $l_0p^r<l<(l_0+1)p^r$, where $l_0\geq 0$ and $\gcd(l_0,p)=1$, then $(\log_n)_{n-l,n}$ is equal to 0, namely, $(\log_n)_{n-l,n}=0$.

(2) *Let* $0 \leq l \leq n-1$ *and* $0 \leq m \leq n-1$. *Then* $(\log_n)_{n-l,n-m}$ *is given by*

$$(\log_n)_{n-l,n-m} = \sum_{k=0}^{n_0-1} \binom{m}{l-kp^r} (-\Delta_0)^{m-(l-kp^r)} (\log_n)_{n-kp^r,n},$$

and so

$$\begin{aligned}\log_n &= [(\log_n)_{n-l,n-m}] \\ &= \sum_{k=0}^{n_0-1} \left[\binom{m}{l-kp^r} (-\Delta_0)^{m-(l-kp^r)} (\log_n)_{n-kp^r,n} \right].\end{aligned}$$

(3) *Suppose that* $n_0 = 1$, *that is,* $n = p^r$. *Then* $\log_{p^r}$ *is lower triangular. More precisely,*

$$(\log_{p^r})_{p^r-l,p^r-m} = \binom{m}{l} (-\Delta_0)^{m-l} \mathcal{L}_{p^r},$$

$$\log_{p^r} = \begin{bmatrix} \mathcal{L}_{p^r} & 0 & \cdots & 0 & 0 \\ \binom{p^r-1}{p^r-2}(-\Delta_0)\mathcal{L}_{p^r} & \mathcal{L}_{p^r} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \binom{p^r-1}{1}(-\Delta_0)^{p^r-2}\mathcal{L}_{p^r} & \binom{p^r-2}{1}(-\Delta_0)^{p^r-3}\mathcal{L}_{p^r} & \cdots & \mathcal{L}_{p^r} & 0 \\ (-\Delta_0)^{p^r-1}\mathcal{L}_{p^r} & (-\Delta_0)^{p^r-2}\mathcal{L}_{p^r} & \cdots & (-\Delta_0)\mathcal{L}_{p^r} & \mathcal{L}_{p^r} \end{bmatrix}.$$

Proof. Before we show item (1), we prove the following: If s is of the form $s = s_0 p^r$ for $p \nmid s_0$, then we have $\binom{n}{s} = \binom{n_0}{s_0}$. Otherwise, we have $\binom{n}{s} = 0$. We prove this fact first. Let $b(X) := (1+X)^n$ be a polynomial over $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$. It is well known that $b(X) = \sum_{s=0}^{n} \binom{n}{s} X^s$. On the other hand, it is also well known that $b(X) = (1 + X^{p^r})^{n_0} = \sum_{s_0=0}^{n_0} \binom{n_0}{s_0} X^{s_0 p^r}$ since $(1+X)^{p^r} = 1 + X^{p^r}$. By comparing the coefficients, we obtain $\binom{n}{s} = \binom{n_0}{s_0}$ for $s = s_0 p^r$ and $\binom{n}{s} = 0$ for $s \neq s_0 p^r$.

(1). First, suppose that $0 < l < p^r$. Recall that $\binom{n}{s} = 0$ for $1 \leq s \leq l$. Then, it follows from Theorem 2.3 (2) that we have

$$\begin{aligned}(\log_n)_{n-l,n} &= \sum_{i=1}^{\infty} C(i)_{n-l,n} \frac{1}{L_i^n} \tau^i \\ &= \sum_{i=1}^{\infty} \left(\sum_{s=1}^{l} (-1) \binom{n}{s} \sum_{t=1}^{i} \frac{C(t)_{n-(l-s),n}}{(-[t])^s} \right) \frac{1}{L_i^n} \tau^i \\ &= 0.\end{aligned}$$

In particular, we obtain $C(i)_{n-l,n} = 0$ for any i when $n - p^r < n - l < n$. Second, assume that $l = p^r$. Recall that $\binom{n}{p^r} = \binom{n_0}{1}$, and $\binom{n}{s} = 0$ if $1 \leq s <$

p^r. By Theorem 2.3 (2), we have

$$(\log_n)_{n-p^r,n} = \sum_{i=1}^{\infty}\left(\sum_{s=1}^{p^r}(-1)\binom{n}{s}\sum_{t=1}^{i}\frac{C(t)_{n-(p^r-s),n}}{(-[t])^s}\right)\frac{1}{L_i^n}\tau^i$$
$$= \sum_{i=1}^{\infty}\left((-1)\binom{n_0}{1}\sum_{t=1}^{i}\frac{C(t)_{n,n}}{(-[t])^{p^r}}\right)\frac{1}{L_i^n}\tau^i.$$

Third, suppose that $p^r < l < 2p^r$. By Theorem 2.3 (2), we have

$$(\log_n)_{n-l,n} = \sum_{i=1}^{\infty} C(i)_{n-l,n}\frac{1}{L_i^n}\tau^i$$
$$= \sum_{i=1}^{\infty}\left((-1)\binom{n_0}{1}\sum_{t=1}^{i}\frac{C(t)_{n-(l-p^r),n}}{(-[t])^{p^r}}\right)\frac{1}{L_i^n}\tau^i.$$

We now compute the coefficient $C(t)_{n-(l-p^r),n}$. Since $n-p^r < n-(l-p^r) < n$, we have $C(t)_{n-(l-p^r),n} = 0$, and thus $(\log_n)_{n-l,n} = 0$. In particular, we obtain $C(i)_{n-l,n} = 0$ for any i when $n-2p^r < n-l < n-p^r$. Fourth, assume that $l = 2p^r$. By Theorem 2.3 (2), we have

$$(\log_n)_{n-2p^r,n}$$
$$= \sum_{i=1}^{\infty}\left((-1)\binom{n_0}{1}\sum_{t=1}^{i}\frac{C(t)_{n-p^r,n}}{(-[t])^{p^r}} + (-1)\binom{n_0}{2}\sum_{t=1}^{i}\frac{C(t)_{n,n}}{(-[t])^{2p^r}}\right)\frac{1}{L_i^n}\tau^i.$$

Repeating the above computations, we can show item (1).

(2). Theorem 2.3 (1) can be written as

$$(\log_n)_{n-l,n-m} = \sum_{j=0}^{n-1}\binom{m}{l-j}(-\Delta_0)^{m-(l-j)}(\log_n)_{n-j,n}.$$

Using item (1), we obtain

$$(\log_n)_{n-l,n-m} = \sum_{j=0}^{n-1}\binom{m}{l-j}(-\Delta_0)^{m-(l-j)}(\log_n)_{n-j,n}$$
$$= \sum_{k=0}^{n_0-1}\binom{m}{l-kp^r}(-\Delta_0)^{m-(l-kp^r)}(\log_n)_{n-kp^r,n}.$$

(3). It follows that $(\log_{p^r})_{p^r-l,p^r-m} = \binom{m}{l}(-\Delta_0)^{m-l}(\log_{p^r})_{p^r,p^r}$ from item (2). Recall that $\binom{m}{l} = 0$ for $l > m$. $\square$

Remarks. In the classical setting, it is well known that $\log(z)$ is transcendental if z is algebraic but not 0 or 1 (by the Lindemann's theorem). Little is known about the transcendence of the polylogarithm $\mathrm{Li}_n(z) = \sum_{i\geq 1} z^i/i^n$

for any nonzero algebraic number z. It is known that $\mathrm{Li}_{2m}(1)$ is transcendental, because $\mathrm{Li}_n(1) = \zeta(n)$ for $n \geq 2$. The following is its function-field analogue.

Let $\mathbf{z} = {}^t(z_1, \ldots, z_n)$ be in $\overline{k}^n$ such that $|z_i|_\infty < |\theta|_\infty^{i-n+nq/(q-1)}$ for each i. Then, the last coordinate $l_n(\mathbf{z})$ is either zero or transcendental. In particular, for any $z \in \overline{k}$ with $|z|_\infty < |\theta|_\infty^{nq/(q-1)}$, the poly-logarithm $\mathcal{L}_n(z) = \sum_{i=0}^\infty \frac{z^{q^i}}{L_i^n}$ is either zero or transcendental. The first claim immediately follows from [32, Theorem 2.3]. (The paper [32, Theorem 2.3] is described in terms of $\overline{k}_\infty$, but this also holds for $\mathbb{C}_\infty$ (see [27, p. 337]).) The second claim follows from our Theorem 2.3 and the first claim. In fact, we have $l_n({}^t(0, \ldots, 0, z)) = \mathcal{L}_n(z)$. In the next section, we will study deeper results.

The transcendence of $\mathcal{L}_n(z)$ also follows from [23, Theorem 6.4.2] and [8, Corollary 3.2].

2.2. Exponential. We start by showing an explicit formula for $\exp_n$. Note that $\exp_n$ is given by

$$\exp_n := I_n\tau^0 + \sum_{i=1}^\infty E(i)\tau^i = \sum_{i=0}^\infty E(i)\tau^i,$$

and that $E(0) = I_n$. For an $n \times n$ matrix $M = [M_{a,b}]$, we define the matrix $M^{(q)}$ as $M^{(q)} = [M_{a,b}^q]$. From the fundamental functional equation $\exp_n \circ(\theta I_n + N) = ((\theta I_n + N)\tau^0 + E\tau) \circ \exp_n$, we obtain $E(i)(\theta^{q^i} I_n + N) = (\theta I_n + N)E(i) + E \cdot E(i-1)^{(q)}$, and thus

$$E(i) = \sum_{j=0}^\infty \frac{\mathrm{ad}(N)^j(E \cdot E(i-1)^{(q)}/[i])}{[i]^j} = \sum_{j=0}^{2(n-1)} \frac{\mathrm{ad}(N)^j(E \cdot E(i-1)^{(q)}/[i])}{[i]^j}$$

for any $i \geq 1$ (see [15, Section 5.10] and [27, Section 7.6]).

Similarly to the coefficients $L(i)$, we compute recursions for the coefficients $E(i)$.

Lemma 2.5. *Let $1 \leq a \leq n$ and $1 \leq b \leq n$. Let $E(i)_{a,b}$ denote the (a,b)th entry of the matrix $E(i)$:*

$$E(i) = \begin{bmatrix} E(i)_{1,1} & \cdots & E(i)_{1,b-1} & E(i)_{1,b} & \cdots & * \\ \vdots & & \vdots & \vdots & \vdots & \vdots \\ * & \cdots & E(i)_{a,b-1} & E(i)_{a,b} & \cdots & * \\ * & \cdots & * & E(i)_{a+1,b} & \cdots & * \\ \vdots & & \vdots & \vdots & \vdots & \vdots \\ * & \cdots & * & * & \cdots & * \end{bmatrix}.$$

Let $E(i)_{a,b}^{(q)}$ denote the (a,b)th entry of the matrix $E(i)^{(q)}$.

(1) *Suppose that* $1 \leq a \leq n-1$ *and* $2 \leq b \leq n$. *Then, for any* $i \geq 1$, *we have*

$$E(i)_{a,b} = \frac{1}{[i]} E(i)_{a+1,b} - \frac{1}{[i]} E(i)_{a,b-1}.$$

(2) *For any* $i \geq 1$, *we have*

$$E(i)_{a,b} = \sum_{j=0}^{b-1} (-1)^j \binom{n-a+j}{j} \left(\frac{1}{[i]}\right)^{n-a+1+j} E(i-1)^{(q)}_{1,b-j}.$$

Proof. This can be proven by using the same technique that was used to prove Lemma 2.1. For simplicity, the matrix

$$\frac{1}{[i]} E \cdot E(i-1)^{(q)} = \frac{1}{[i]} \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ E(i-1)^{(q)}_{1,1} & E(i-1)^{(q)}_{1,2} & \cdots & E(i-1)^{(q)}_{1,n} \end{bmatrix}$$

is denoted by Y.

(1). We compute $E(i)_{a,b}$ as follows:

$$\begin{aligned} E(i)_{a,b} &= \sum_{j=0}^{2(n-1)} \frac{\mathrm{ad}(N)^j (Y)_{a,b}}{[i]^j} = \sum_{j=1}^{2(n-1)} \frac{\mathrm{ad}(N)^j (Y)_{a,b}}{[i]^j} \\ &= \sum_{j=1}^{2(n-1)} \frac{\mathrm{ad}(N)^{j-1} (Y)_{a+1,b} - \mathrm{ad}(N)^{j-1} (Y)_{a,b-1}}{[i]^j} \\ &= \sum_{j=1}^{2(n-1)+1} \frac{\mathrm{ad}(N)^{j-1} (Y)_{a+1,b} - \mathrm{ad}(N)^{j-1} (Y)_{a,b-1}}{[i]^j} \\ &= \frac{1}{[i]} \sum_{j=0}^{2(n-1)} \frac{\mathrm{ad}(N)^j (Y)_{a+1,b}}{[i]^j} - \frac{1}{[i]} \sum_{j=0}^{2(n-1)} \frac{\mathrm{ad}(N)^j (Y)_{a,b-1}}{[i]^j} \\ &= \frac{1}{[i]} E(i)_{a+1,b} - \frac{1}{[i]} E(i)_{a,b-1}. \end{aligned}$$

We used $\mathrm{ad}(N)^0 (Y)_{a,b} = \mathrm{ad}(N)^{2(n-1)} (Y)_{a+1,b} = \mathrm{ad}(N)^{2(n-1)} (Y)_{a,b-1} = 0$.

(2). We show item (2) by induction. Note that

$$Y_{a,b} = \begin{cases} 0 & \text{if } a \leq n-1, \\ E(i-1)^{(q)}_{1,b} / [i] & \text{if } a = n. \end{cases}$$

First, we compute $E(i)_{a,1}$. We can write $\mathrm{ad}(N)^j(Y) = N^jY + MN$ for some matrix M. Since $(MN)_{a,1} = 0$, we obtain

$$\mathrm{ad}(N)^j(Y)_{a,1} = (N^jY)_{a,1} = \begin{cases} E(i-1)^{(q)}_{1,1}/[i] & \text{if } j = n-a, \\ 0 & \text{if } j \neq n-a, \end{cases}$$

and hence

$$E(i)_{a,1} = \sum_{j=0}^{\infty} \frac{\mathrm{ad}(N)^j(Y)_{a,1}}{[i]^j} = \frac{E(i-1)^{(q)}_{1,1}/[i]}{[i]^{n-a}} = \left(\frac{1}{[i]}\right)^{n-a+1} E(i-1)^{(q)}_{1,1}.$$

Next, we compute $E(i)_{n,b}$. Similarly, we can write $\mathrm{ad}(N)^j(Y) = NM + (-1)^jYN^j$ for some matrix M. Since $(NM)_{n,b} = 0$, we obtain

$$\mathrm{ad}(N)^j(Y)_{n,b} = (-1)^j(YN^j)_{n,b} = \begin{cases} (-1)^jE(i-1)^{(q)}_{1,b-j}/[i] & \text{if } j \leq b-1, \\ 0 & \text{if } j > b-1, \end{cases}$$

and hence

$$\begin{aligned} E(i)_{n,b} &= \sum_{j=0}^{\infty} \frac{\mathrm{ad}(N)^j(Y)_{n,b}}{[i]^j} = \sum_{j=0}^{b-1} \frac{(-1)^jE(i-1)^{(q)}_{1,b-j}/[i]}{[i]^j} \\ &= \sum_{j=0}^{b-1}(-1)^j \left(\frac{1}{[i]}\right)^{1+j} E(i-1)^{(q)}_{1,b-j}. \end{aligned}$$

Last, we assume that $1 \leq a < n$ and $1 < b \leq n$. By the inductive hypothesis, we have

$$\begin{aligned} E(i)_{a+1,b} &= \sum_{j=0}^{b-1}(-1)^j \binom{n-(a+1)+j}{j} \left(\frac{1}{[i]}\right)^{n-(a+1)+1+j} E(i-1)^{(q)}_{1,b-j}, \\ E(i)_{a,b-1} &= \sum_{j=0}^{(b-1)-1}(-1)^j \binom{n-a+j}{j} \left(\frac{1}{[i]}\right)^{n-a+1+j} E(i-1)^{(q)}_{1,(b-1)-j}. \end{aligned}$$

Using item (1), we obtain

$$\begin{aligned} &E(i)_{a,b} = \frac{1}{[i]}E(i)_{a+1,b} - \frac{1}{[i]}E(i)_{a,b-1} \\ &= \sum_{j=0}^{b-1}(-1)^j \left(\binom{n-a-1+j}{j} + \binom{n-a-1+j}{j-1}\right) \left(\frac{1}{[i]}\right)^{n-a+1+j} E(i-1)^{(q)}_{1,b-j} \\ &= \sum_{j=0}^{b-1}(-1)^j \binom{n-a+j}{j} \left(\frac{1}{[i]}\right)^{n-a+1+j} E(i-1)^{(q)}_{1,b-j}. \end{aligned}$$

Hence, we complete the proof of item (2). $\square$

The main theorem in this subsection is the following.

Theorem 2.6 (Exponential I). *Let $1 \leq l \leq n$ and $1 \leq m \leq n$. Let $(\exp_n)_{l,m}$ denote the (l,m)th entry of the matrix $\exp_n$, and let $E(i)_{l,m}$ denote the coefficient of τ^i of the series $(\exp_n)_{l,m}$:*

$$(\exp_n)_{l,m} = \sum_{i=0}^{\infty} E(i)_{l,m}\tau^i,$$

and

$$\exp_n = \begin{bmatrix} (\exp_n)_{1,1} & (\exp_n)_{1,2} & \cdots & (\exp_n)_{1,m} & \cdots & * \\ \vdots & \vdots & & \vdots & & \vdots \\ * & * & \cdots & (\exp_n)_{l,m} & \cdots & * \\ \vdots & \vdots & & \vdots & & \vdots \\ * & * & \cdots & * & \cdots & * \end{bmatrix}.$$

(1) *For any l and m, the following holds:*

$$(\exp_n)_{l,m} = \sum_{j=0}^{m-1} \binom{l-1}{j} \Delta_0^{(l-1)-j}(\exp_n)_{1,m-j}.$$

In particular, the lth entry of the first column is equal to $(\exp_n)_{l,1} = \Delta_0^{l-1}(\exp_n)_{1,1}$.

(2) *For simplicity, we set $C(i) = C(i)_{\exp} := D_i^n \cdot E(i)$. The entries of the first row are given by*

$$(\exp_n)_{1,1} = \sum_{i=0}^{\infty} \frac{1}{D_i^n}\tau^i \ (= \mathcal{E}_n(\tau)),$$

$$\begin{aligned}(\exp_n)_{1,1+m} &= \sum_{i=1}^{\infty} E(i)_{1,1+m}\tau^i = \sum_{i=1}^{\infty} C(i)_{1,1+m}\frac{1}{D_i^n}\tau^i \\ &= \sum_{i=1}^{\infty}\left(\sum_{s=1}^{m}(-1)\binom{n}{s}\sum_{t=1}^{i}\left(\frac{C(t)_{1,1+m-s}}{[t]^s}\right)^{q^{i-t}}\right)\frac{1}{D_i^n}\tau^i\end{aligned}$$

for any $1 \leq m \leq n-1$. Recall that $E(0)_{1,1+m} = 0$. In particular, for $1 \leq m \leq n-1$, we obtain

$$C(i)_{1,1+m} = \sum_{s=1}^{m}(-1)\binom{n}{s}\sum_{t=1}^{i}\left(\frac{C(t)_{1,1+m-s}}{[t]^s}\right)^{q^{i-t}}.$$

Proof. This can be proven by using the same technique that was used to prove Theorem 2.3.

(1). First, assume that $i = 0$. The coefficient $E(0)_{l,m}$ of τ^0 on the left hand side is

$$E(0)_{l,m} = \begin{cases} 0 & \text{if } l \neq m, \\ 1 & \text{if } l = m. \end{cases}$$

On the other hand, the coefficient of τ^0 on the right hand side is

$$\sum_{j=0}^{m-1} \binom{l-1}{j} [0]^{(l-1)-j} E(0)_{1,m-j} = \binom{l-1}{m-1} [0]^{l-m} E(0)_{1,1} = \begin{cases} 0 & \text{if } l > m, \\ 0 & \text{if } l < m, \\ 1 & \text{if } l = m. \end{cases}$$

Hence, the claim holds for $i = 0$.

Next, suppose that $i \geq 1$. We will show that the equality

$$E(i)_{l,m} = \sum_{j=0}^{m-1} \binom{l-1}{j} [i]^{(l-1)-j} E(i)_{1,m-j}$$

holds, and using induction on l and m, we will prove item (1). By the inductive hypothesis, we have

$$E(i)_{l-1,m} = \sum_{j=0}^{m-1} \binom{l-2}{j} [i]^{(l-2)-j} E(i)_{1,m-j},$$

$$L(i)_{l-1,m-1} = \sum_{j=0}^{m-2} \binom{l-2}{j} [i]^{(l-2)-j} E(i)_{1,(m-1)-j}.$$

Using Lemma 2.5 (1), we obtain

$$\begin{aligned} E(i)_{l,m} &= [i]E(i)_{l-1,m} + E(i)_{l-1,m-1} \\ &= \sum_{j=0}^{m-1} \left(\binom{l-2}{j} + \binom{l-2}{j-1} \right) [i]^{(l-1)-j} E(i)_{1,m-j} \\ &= \sum_{j=0}^{m-1} \binom{l-1}{j} [i]^{(l-1)-j} E(i)_{1,m-j}. \end{aligned}$$

Hence, we finish the proof of item (1).

(2). For an $n \times n$ matrix $M = [M_{a,b}]$, we note that $M^{(q)}_{a,b}$ is equal to $M^q_{a,b}$. Indeed, the former $M^{(q)}_{a,b} = (M^{(q)})_{a,b}$ is the (a, b)th entry of $M^{(q)}$, and the latter $M^q_{a,b} = (M_{a,b})^q$ is the qth power of the (a, b)th entry of M. Recall that $M^{(q)} = [M^q_{a,b}]$. This is useful in the proof.

We prove item (2) using induction on i. First, we compute the diagonal entry $E(i)_{1,1}$. When $i = 0$, we have $E(0)_{1,1} = 1 = 1/D^n_0$. Suppose that $i \geq 1$. By the inductive hypothesis, we have $E(i-1)_{1,1} = 1/D^n_{i-1}$. It follows from Lemma 2.5 (2) that

$$E(i)_{1,1} = \left(\frac{1}{[i]}\right)^n E(i-1)^{(q)}_{1,1} = \left(\frac{1}{[i]}\right)^n E(i-1)^q_{1,1} = \frac{1}{D^n_i}.$$

Recall that $D_i = [i]D^q_{i-1}$. Hence, the first claim has been proven.

Next, we determine the non-diagonal entry $L(i)_{1,1+m}$ for $1 \leq m \leq n-1$. For simplicity, we set $C(i)_{1,1+m} := E(i)_{1,1+m} D_i^n$. When $i = 0$, we have $C(0)_{1,1+m} = 0$. Suppose that $i \geq 1$. We show that

$$C(i)_{1,1+m} = \sum_{s=1}^{m} (-1) \binom{n}{s} \sum_{t=1}^{i} \left(\frac{C(t)_{1,1+m-s}}{[t]^s} \right)^{q^{i-t}} .$$

By the inductive hypothesis, we have

$$C(i-1)_{1,1+m} = \sum_{s=1}^{m} (-1) \binom{n}{s} \sum_{t=1}^{i-1} \left(\frac{C(t)_{1,1+m-s}}{[t]^s} \right)^{q^{i-1-t}} .$$

It follows from Lemma 2.5 (2) and $D_i = [i] D_{i-1}^q$ that

$$\begin{aligned} C(i)_{1,1+m} &= E(i)_{1,1+m} D_i^n \\ &= \left(\sum_{j=0}^{m} (-1)^j \binom{n-1+j}{j} \left(\frac{1}{[i]} \right)^{n+j} E(i-1)_{1,1+m-j}^q \right) D_i^n \\ &= \sum_{j=0}^{m} (-1)^j \binom{n-1+j}{j} \left(\frac{1}{[i]} \right)^{j} E(i-1)_{1,1+m-j}^q D_{i-1}^{qn} \\ &= \sum_{j=0}^{m} (-1)^j \binom{n-1+j}{j} \frac{C(i-1)_{1,1+m-j}^q}{[i]^j}. \end{aligned} \tag{2.2}$$

Using the inductive hypothesis, we obtain

$$\begin{aligned} &\sum_{j=0}^{m} (-1)^j \binom{n-1+j}{j} \frac{C(i-1)_{1,1+m-j}^q}{[i]^j} \\ &= C(i-1)_{1,1+m}^q + \sum_{j=1}^{m} (-1)^j \binom{n-1+j}{j} \frac{C(i-1)_{1,1+m-j}^q}{[i]^j} \\ &= \sum_{s=1}^{m} (-1)^q \binom{n}{s}^q \sum_{t=1}^{i-1} \left(\frac{C(t)_{1,1+m-s}}{[t]^s} \right)^{q^{i-t}} \\ &\qquad + \sum_{j=1}^{m} (-1)^j \binom{n-1+j}{j} \frac{C(i-1)_{1,1+m-j}^q}{[i]^j} \\ &= \sum_{s=1}^{m} (-1) \binom{n}{s} \sum_{t=1}^{i-1} \left(\frac{C(t)_{1,1+m-s}}{[t]^s} \right)^{q^{i-t}} \\ &\qquad + \sum_{j=1}^{m} (-1)^j \binom{n-1+j}{j} \frac{C(i-1)_{1,1+m-j}^q}{[i]^j}. \end{aligned}$$

Recall that $a^q = a$ for any $a \in \mathbb{F}_q$. We transform the last equality as follows:

$$\begin{aligned}
&\sum_{s=1}^{m}(-1)\binom{n}{s}\sum_{t=1}^{i-1}\left(\frac{C(t)_{1,1+m-s}}{[t]^s}\right)^{q^{i-t}} \\
&\qquad+\sum_{j=1}^{m}(-1)^j\binom{n-1+j}{j}\frac{C(i-1)^q_{1,1+m-j}}{[i]^j} \\
&=\sum_{s=1}^{m}(-1)\binom{n}{s}\sum_{t=1}^{i}\left(\frac{C(t)_{1,1+m-s}}{[t]^s}\right)^{q^{i-t}}+\sum_{s=1}^{m}\binom{n}{s}\frac{C(i)_{1,1+m-s}}{[i]^s} \\
&\qquad+\sum_{j=1}^{m}(-1)^j\binom{n-1+j}{j}\frac{C(i-1)^q_{1,1+m-j}}{[i]^j}.
\end{aligned}$$

Next, we check that the sum of the second and third terms equals zero. It follows from equality (2.2) that

$$\begin{aligned}
&\sum_{s=1}^{m}\binom{n}{s}\frac{C(i)_{1,1+m-s}}{[i]^s}+\sum_{j=1}^{m}(-1)^j\binom{n-1+j}{j}\frac{C(i-1)^q_{1,1+m-j}}{[i]^j} \\
&=\sum_{s=1}^{m}\binom{n}{s}\left(\sum_{j=0}^{m-s}(-1)^j\binom{n-1+j}{j}\frac{C(i-1)^q_{1,1+m-s-j}}{[i]^j}\right)\frac{1}{[i]^s} \\
&\qquad+\sum_{j=1}^{m}(-1)^j\binom{n-1+j}{j}\frac{C(i-1)^q_{1,1+m-j}}{[i]^j} \\
&=\sum_{s=1}^{m}\sum_{j=0}^{m-s}\left((-1)^j\binom{n}{s}\binom{n-1+j}{j}\right)\frac{C(i-1)^q_{1,1+m-s-j}}{[i]^{s+j}} \\
&\qquad+\sum_{j=1}^{m}(-1)^j\binom{n-1+j}{j}\frac{C(i-1)^q_{1,1+m-j}}{[i]^j}.
\end{aligned}$$

The next step is key. We transform the last equality as follows:

$$\begin{aligned}
&\sum_{s=1}^{m}\sum_{j=0}^{m-s}\left((-1)^j\binom{n}{s}\binom{n-1+j}{j}\right)\frac{C(i-1)^q_{1,1+m-s-j}}{[i]^{s+j}} \\
&\qquad+\sum_{j=1}^{m}(-1)^j\binom{n-1+j}{j}\frac{C(i-1)^q_{1,1+m-j}}{[i]^j} \\
&=\sum_{j=1}^{m}\left(\sum_{k=0}^{j}(-1)^{j-k}\binom{n}{k}\binom{n-1+j-k}{j-k}\right)\frac{C(i-1)^q_{1,1+m-j}}{[i]^j}.
\end{aligned}$$

By the identity in Lemma 2.2, we see that the last equality is equal to zero.

Hence, the second claim has been proven. ☐

Remarks. We can interpret the entries

$$\Delta_0^{l-m}(\exp_n)_{1,1}, \Delta_0^{l-m+1}(\exp_n)_{1,2}, \ldots, \Delta_0^{l-l}(\exp_n)_{1,m}$$

of the first row as generators of $(\exp_n)_{l,m}$. Comparing Subsection 2.2 with Subsection 2.1, we find that the properties of $\exp_n$ are similar to those of the anti-diagonal transpose of $\log_n$, that is, the entries $(\exp_n)_{l+1,m+1}$ and $(\log_n)_{n-m,n-l}$ are similar.

We have proven the formula for $\exp_n$. We can give a more detailed formula for $\exp_n$. Indeed, all the entries of $\exp_n$ can be expressed by means of $(\exp_n)_{1,1}, (\exp_n)_{1,1+p^r}, \ldots, (\exp_n)_{1,1+(n_0-1)p^r}$.

Corollary 2.7 (Exponential II). *Let $n = n_0 p^r$, $r \in \mathbb{Z}_{\geq 1}$, and $\gcd(n_0, p) = 1$.*

(1) *Let $1 \leq m \leq n-1$. If m is of the form $m = m_0 p^r$, where $m_0 \geq 1$ and $\gcd(m_0, p) = 1$, then we have*

$$(\exp_n)_{1,1+m_0p^r} = \sum_{i=1}^{\infty}\left(\sum_{s_0=1}^{m_0}(-1)\binom{n_0}{s_0}\sum_{t=1}^{i}\left(\frac{C(t)_{1,1+(m_0-s_0)p^r}}{[t]^{s_0p^r}}\right)^{q^{i-t}}\right)\frac{1}{D_i^n}\tau^i,$$

where $C(i) = D_i^n \cdot E(i)$ is the modified coefficient of $\exp_n$.

On the other hand, if $m_0 p^r < m < (m_0+1)p^r$, where $m_0 \geq 0$ and $\gcd(m_0, p) = 1$, then we have $(\exp_n)_{1,1+m} = 0$.

(2) *Let $1 \leq l \leq n$ and $1 \leq m \leq n$. Then we have*

$$(\exp_n)_{l,m} = \sum_{k=0}^{n_0-1}\binom{l-1}{m-(1+kp^r)}\Delta_0^{(l-1)-(m-(1+kp^r))}(\exp_n)_{1,1+kp^r},$$

and so

$$\begin{aligned}\exp_n &= [(\exp_n)_{l,m}] \\ &= \sum_{k=0}^{n_0-1}\left[\binom{l-1}{m-(1+kp^r)}\Delta_0^{(l-1)-(m-(1+kp^r))}(\exp_n)_{1,1+kp^r}\right].\end{aligned}$$

Proof. This can be proven by using the same technique that was used to prove Corollary 2.4.

(1). First, suppose that $0 < m < p^r$. Recall that $\binom{n_0p^r}{s} = 0$ for $1 \leq s \leq m$. Then, it follows from Theorem 2.6 (2) that we have

$$\begin{aligned}(\exp_n)_{1,1+m} &= \sum_{i=1}^{\infty} C(i)_{1,1+m}\frac{1}{D_i^n}\tau^i \\ &= \sum_{i=1}^{\infty}\left(\sum_{s=1}^{m}(-1)\binom{n}{s}\sum_{t=1}^{i}\left(\frac{C(t)_{1,1+m-s}}{[t]^s}\right)^{q^{i-t}}\right)\frac{1}{D_i^n}\tau^i \\ &= 0.\end{aligned}$$

In particular, we obtain $C(i)_{1,1+m} = 0$ for any i when $1 < 1+m < 1+p^r$. Second, assume that $m = p^r$. Recall that $\binom{n_0 p^r}{p^r} = \binom{n_0}{1}$, and $\binom{n_0 p^r}{s} = 0$ if $1 \le s < p^r$. By Theorem 2.3 (2), we have

$$\begin{aligned}
(\exp_n)_{1,1+p^r} &= \sum_{i=1}^{\infty} \left(\sum_{s=1}^{p^r} (-1) \binom{n}{s} \sum_{t=1}^{i} \left(\frac{C(t)_{1,1+p^r-s}}{[t]^s} \right)^{q^{i-t}} \right) \frac{1}{D_i^n} \tau^i \\
&= \sum_{i=1}^{\infty} \left((-1) \binom{n_0}{1} \sum_{t=1}^{i} \left(\frac{C(t)_{1,1}}{[t]^{p^r}} \right)^{q^{i-t}} \right) \frac{1}{D_i^n} \tau^i.
\end{aligned}$$

Third, suppose that $p^r < m < 2p^r$. Note that $C(t)_{1,1+m-p^r} = 0$ for $1 < 1+m-p^r < 1+p^r$. By Theorem 2.6 (2), we have

$$\begin{aligned}
(\exp_n)_{1,1+m} &= \sum_{i=1}^{\infty} C(i)_{1,1+m} \frac{1}{D_i^n} \tau^i \\
&= \sum_{i=1}^{\infty} \left((-1) \binom{n_0}{1} \sum_{t=1}^{i} \left(\frac{C(t)_{1,1+m-p^r}}{[t]^{p^r}} \right)^{q^{i-t}} \right) \frac{1}{D_i^n} \tau^i \\
&= 0.
\end{aligned}$$

In particular, we obtain $C(i)_{1,1+m} = 0$ for any i when $1+p^r < 1+m < 1+2p^r$. Fourth, assume that $l = 2p^r$. By Theorem 2.6 (2), we have

$$\begin{aligned}
(\exp_n)_{1,1+2p^r} = \sum_{i=1}^{\infty} \Bigg(& (-1) \binom{n_0}{1} \sum_{t=1}^{i} \left(\frac{C(t)_{1,1+p^r}}{[t]^{p^r}} \right)^{q^{i-t}} \\
& + (-1) \binom{n_0}{2} \sum_{t=1}^{i} \left(\frac{C(t)_{1,1}}{[t]^{2p^r}} \right)^{q^{i-t}} \Bigg) \frac{1}{D_i^n} \tau^i.
\end{aligned}$$

Repeating the above computations, we can show item (1).

(2). It follows from Theorem 2.6 (2) and item (1). □

Having proven the explicit formula for $\exp_n$, from now on, we present applications. Recall that $A = \mathbb{F}_q[\theta]$. Let $a = \sum_{k=0}^{\deg a} a_k \theta^k \in A$, and let $d[a]_n$ denote the coefficient of τ^0 of $C^{\otimes n}(a)$. Note that $d[\theta]_n = \theta I_n + N$ and $d[1]_n = I_n$. Since $C^{\otimes n}$ is an $\mathbb{F}_q$-algebra homomorphism, we have

$$C^{\otimes n}(a) = \sum_{k=0}^{\deg a} a_k C^{\otimes n}(\theta^k) = \sum_{k=0}^{\deg a} a_k (C^{\otimes n}(\theta))^k,$$

and so

$$d[a]_n = \sum_{k=0}^{\deg a} a_k d[\theta^k]_n = \sum_{k=0}^{\deg a} a_k d[\theta]_n^k.$$

For convenience, we write the (i,j)th entry of $d[a]_n$ (resp. $d[\theta]_n^k$) as $d(a)_{i,j}$ (resp. $c(\theta)_{i,j}^{(k)}$), that is, $d[a]_n = [d(a)_{i,j}]$ (resp. $d[\theta]_n^k = [c(\theta)_{i,j}^{(k)}]$). Let $\partial_\theta^j : A \to A$ denote the jth hyperderivative defined by

$$\partial_\theta^j a = \sum_{k=0}^{\deg a} a_k \binom{k}{j} \theta^{k-j}.$$

It is well known that

$$\begin{aligned}
\partial_\theta^j(ab) &= \sum_{k=0}^{j} \partial_\theta^k a \cdot \partial_\theta^{j-k} b, \\
\partial_\theta^i(\partial_\theta^j a) &= \binom{i+j}{j} \partial_\theta^{i+j} a, \\
\partial_\theta^j a^{p^r} &= \begin{cases} (\partial_\theta^{j_0} a)^{p^r} & \text{if } j = j_0 p^r, \\ 0 & \text{if otherwise.} \end{cases}
\end{aligned}$$

See [21] and [22] for the definition and properties of hyperderivatives.

Using the hyperderivatives, all the entries of $d[a]_n$ can be explicitly determined. The proof is elementary, but we provide it here for completeness.

Lemma 2.8. *Let $a = \sum_{k=0}^{\deg a} a_k \theta^k \in A$. Then the entry $d(a)_{i,j}$ of the matrix $d[a]_n$ equals $\partial_\theta^{j-i} a$, namely, $d(a)_{i,j} = \partial_\theta^{j-i} a$. In particular, for the entry $c(\theta)_{i,j}^{(k)}$ of the matrix $d[\theta^k]_n$, we obtain*

$$c(\theta)_{i,j}^{(k)} = \partial_\theta^{j-i} \theta^k = \binom{k}{j-i} \theta^{k-(j-i)}.$$

Proof. For simplicity, we set $d_{i,j} := d(a)_{i,j}$ and $c_{i,j}^{(k)} := c(\theta)_{i,j}^{(k)}$. Since ∂_θ^j is $\mathbb{F}_q$-linear, we have $\partial_\theta^{j-i} a = \sum_{k=0}^{\deg a} a_k \partial_\theta^{j-i} \theta^k$. Note that $d_{i,j} = \sum_{k=0}^{\deg a} a_k c_{i,j}^{(k)}$. It is sufficient to show $c_{i,j}^{(k)} = \partial_\theta^{j-i} \theta^k$. Recall that $d[\theta^k]_n = (\theta I_n + N)^k$ for all k.

By induction on k, we show that $[c_{i,j}^{(k)}] = (\theta I_n + N)^k$. When $k = 0$, we have $[c_{i,j}^{(0)}] = (\theta I_n + N)^0 = I_n = [\binom{0}{j-i} \theta^{0-(j-i)}]$. Recall that $\binom{0}{0} = 1$, and $\binom{0}{j-i} = 0$ for $j \neq i$. When $k = 1$, we have $[c_{i,j}^{(1)}] = \theta I_n + N = [\binom{1}{j-i} \theta^{1-(j-i)}]$. Recall also that $\binom{1}{j-i} = 1$ for $j = i, i+1$, and $\binom{1}{j-i} = 0$ otherwise. We suppose that $[c_{i,j}^{(k)}] = (\theta I_n + N)^k = [\binom{k}{j-i} \theta^{k-(j-i)}]$ is true, which is an inductive hypothesis. Then we can compute the equality $[c_{i,j}^{(k+1)}] = (\theta I_n + N)^{k+1}$ as follows:

$$\begin{aligned}
[c_{i,j}^{(k+1)}] &= (\theta I_n + N)^{k+1} = (\theta I_n + N)^k(\theta I_n + N) \\
&= \left[\binom{k}{j-i}\theta^{k-(j-i)}\right]\left[\binom{1}{j-i}\theta^{1-(j-i)}\right] \\
&= \left[\sum_{s=1}^{n}\binom{k}{s-i}\theta^{k-(s-i)}\binom{1}{j-s}\theta^{1-(j-s)}\right] \\
&= \left[\left(\binom{k}{j-i}\binom{1}{0} + \binom{k}{j-1-i}\binom{0}{0}\right)\theta^{k-(j-i)}\right] \\
&= \left[\binom{k+1}{j-i}\theta^{k-(j-i)}\right],
\end{aligned}$$

and hence, we finish the proof. □

We now recall the results of Anderson–Thakur [2], Maurischat [21], and Namoijan and Papanikolas [22], respectively. We write the matrix size n as $n = n_0 p^r$, where $r \in \mathbb{Z}_{\geq 0}$ and $\gcd(n_0, p) = 1$. In 1990 Anderson and Thakur determined the period lattice $\Lambda_n := \mathrm{Ker}(\exp_n)$ of $C^{\otimes n}$:

$$\Lambda_n = \{d[a]_n\lambda \mid a \in A\},$$

where $\lambda := {}^t(\lambda_1, \ldots, \lambda_{n-1}, \widetilde{\pi}^n) \in \mathbb{C}_\infty^n$ with the last coordinate $\lambda_n = \widetilde{\pi}^n$ (see [2, Corollary 2.5.8]). In 2018 Maurischat proved that the vector λ is equal to

$$\lambda = {}^t(\overbrace{0, \ldots, 0, \lambda_{p^r}}^{p^r}, \ldots, \overbrace{0, \ldots, 0, \lambda_{(n_0-1)p^r}}^{p^r}, \overbrace{0, \ldots, 0, \widetilde{\pi}^n}^{p^r}) \in \mathbb{C}_\infty^n,$$

and moreover that all the coordinates $\lambda_{p^r}, \ldots, \lambda_{(n_0-1)p^r}$ are nonzero (see [21, Corollary 8.5]). In 2021 Namoijam and Papanikolas [22] stated that for $1 \leq j_0 \leq n_0$, the coordinate $\lambda_{j_0 p^r}$ is given by

$$\lambda_{j_0 p^r} = (-1)^n(\partial_t^{n_0 - j_0}\Omega^{-n_0})^{p^r}|_{t=\theta},$$

where Ω is the Anderson–Thakur power series

$$\Omega := (-\theta)^{-q/(q-1)}\prod_{i=1}^{\infty}\left(1 - \frac{t}{\theta^{q^i}}\right),$$

and $\Omega|_{t=\theta} = -1/\widetilde{\pi}$. Here $(-\theta)^{1/(q-1)}$ is a fixed $(q-1)$th root of $-\theta$ in $\overline{k}_\infty$. Combining the above results with Lemma 2.8, the period $d[a]_n\lambda$ is then given by

$$d[a]_n\lambda = (-1)^n \sum_{j_0=1}^{n_0} \begin{bmatrix} (\partial_\theta^{j_0p^r-1}a)(\partial_t^{n_0-j_0}\Omega^{-n_0})^{p^r}|_{t=\theta} \\ (\partial_\theta^{j_0p^r-2}a)(\partial_t^{n_0-j_0}\Omega^{-n_0})^{p^r}|_{t=\theta} \\ \vdots \\ (\partial_\theta^{j_0p^r-n}a)(\partial_t^{n_0-j_0}\Omega^{-n_0})^{p^r}|_{t=\theta} \end{bmatrix}.$$

In particular, the last coordinate can be computed as

$$\begin{aligned}(-1)^n \sum_{j_0=1}^{n_0} (\partial_\theta^{j_0p^r-n}a)(\partial_t^{n_0-j_0}\Omega^{-n_0})^{p^r}|_{t=\theta} &= (-1)^n(\partial_\theta^0 a)(\partial_t^0\Omega^{-n_0})^{p^r}|_{t=\theta} \\ &= a\widetilde{\pi}^n.\end{aligned}$$

In conclusion, we have the following.

Corollary 2.9. *Let* $e_1(\mathbf{z}), e_2(\mathbf{z}), \dots, e_n(\mathbf{z})$ *be the coordinates of* $\exp_n(\mathbf{z})$*. If* $\mathbf{z} = {}^t(z_1, \dots, z_n) \in \mathbb{C}_\infty^n$ *is a nonzero common zero of* $e_1(\mathbf{z}), e_2(\mathbf{z}), \dots, e_n(\mathbf{z})$*, then we have* $z_n \neq 0$*. Or equivalently,*

$$\{\mathbf{z} = {}^t(z_1, \dots, z_{n-1}, 0) \in \mathbb{C}_\infty^n \mid e_1(\mathbf{z}) = e_2(\mathbf{z}) = \cdots = e_n(\mathbf{z}) = 0\} = \{\mathbf{0}\}.$$

In particular, we have

$$\{z \in \mathbb{C}_\infty \mid \mathcal{E}_n(z) = \Delta_0\mathcal{E}_n(z) = \cdots = \Delta_0^{n-1}\mathcal{E}_n(z) = 0\} = \{0\}.$$

Proof. The second claim follows from the first claim and Corollary 2.7. In fact, we have $e_l(\mathbf{z}) = \Delta_0^{l-1}\mathcal{E}_n(z)$ for all l if we restrict $\mathbf{z}$ to $\mathbf{z} = {}^t(z, 0, \dots, 0)$. Hence, we prove the first claim only.

It is obvious that $\mathbf{z} = \mathbf{0}$ is a common zero. Conversely, suppose that $\mathbf{z} = {}^t(z_1, \dots, z_{n-1}, 0)$ is a common zero. By definition, this element $\mathbf{z}$ is in the period lattice Λ_n, so there exists an element $a \in A$ such that $d[a]_n\lambda = \mathbf{z}$. Since the last coordinate of $d[a]_n\lambda$ equals $a\widetilde{\pi}^n$, we obtain $a\widetilde{\pi}^n = 0$, and so $a = 0$. From $d[0]_n = 0$, we have $\mathbf{z} = d[0]_n\lambda = \mathbf{0}$. □

Remarks. We note that the above corollary also follows from [32, Theorem 2.3].

From now on, we suppose that $n_0 = 1$, namely, $n = p^r$. Then, we can prove the following.

Corollary 2.10. *Suppose that* $n = p^r$*.*

(1) *The matrix* $\exp_{p^r}$ *is lower triangular. More precisely, we have*

$$(\exp_{p^r})_{l,m} = \binom{l-1}{m-1}\Delta_0^{l-m}\mathcal{E}_{p^r},$$

that is,

$$\exp_{p^r} = \begin{bmatrix} \mathcal{E}_{p^r} & 0 & 0 & \cdots & 0 \\ \Delta_0\mathcal{E}_{p^r} & \mathcal{E}_{p^r} & 0 & \cdots & 0 \\ \Delta_0^2\mathcal{E}_{p^r} & \binom{2}{1}\Delta_0\mathcal{E}_{p^r} & \mathcal{E}_{p^r} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \Delta_0^{p^r-1}\mathcal{E}_{p^r} & \binom{p^r-1}{1}\Delta_0^{p^r-2}\mathcal{E}_{p^r} & \binom{p^r-1}{2}\Delta_0^{p^r-3}\mathcal{E}_{p^r} & \cdots & \mathcal{E}_{p^r} \end{bmatrix}.$$

(2) *The element*

$$d[a]_{p^r}\lambda = {}^t((\partial_\theta^{p^r-1}a)\widetilde{\pi}^{p^r}, (\partial_\theta^{p^r-2}a)\widetilde{\pi}^{p^r}, \ldots, (\partial_\theta a)\widetilde{\pi}^{p^r}, a\widetilde{\pi}^{p^r})$$

is a period in Λ_{p^r} for any $a \in A$. That is, for each $1 \le l \le p^r$, we have

$$\sum_{m=1}^{l} \binom{l-1}{m-1} \Delta_0^{l-m}\mathcal{E}_{p^r}((\partial_\theta^{p^r-l}a)\widetilde{\pi}^{p^r}) = 0.$$

In particular, we have $\mathcal{E}_{p^r}((\partial_\theta^{p^r-1}a)\widetilde{\pi}^{p^r}) = 0$ when $l = 1$.

Proof. Recall that $n = p^r$.

(1). First, we consider $(\exp_n)_{1,1+m}$ for $1 \le m \le n-1$. Since $\binom{p^r}{s} = 0$ for $1 \le s \le n-1$, we obtain $(\exp_n)_{1,1+m} = 0$ from Theorem 2.6 (2). Next, suppose that $1 \le l \le n$ and $1 \le m \le n$. It follows from Theorem 2.6 (1) and the first claim that $(\exp_n)_{l,m} = \binom{l-1}{m-1}\Delta_0^{l-m}(\exp_n)_{1,1}$. Then we have $(\exp_n)_{l,m} = 0$ for $l < m$ since $\binom{l-1}{m-1} = 0$ for $l < m$.

(2). From [21, Corollary 8.5], we have $\lambda = {}^t(0, \ldots, 0, \widetilde{\pi}^n)$. By Lemma 2.8, we have

$$d[a]_n\lambda = {}^t((\partial_\theta^{n-1}a)\widetilde{\pi}^n, (\partial_\theta^{n-2}a)\widetilde{\pi}^n, \ldots, (\partial_\theta a)\widetilde{\pi}^n, a\widetilde{\pi}^n).$$

Since $d[a]_n\lambda \in \Lambda_n$ from [2, Corollary 2.5.8], we finish the proof by using item (1). ☐

Remarks.

(1) Fix $a = \sum_{k=0}^{\deg a} a_k\theta^k \in A$. It is known that

$$\mathcal{E}_{p^r}((a\widetilde{\pi})^{p^r}) = \sum_{i=0}^{\infty} \frac{1}{D_i^{p^r}}((a\widetilde{\pi})^{p^r})^{q^i} = (e_C(a\widetilde{\pi}))^{p^r} = C_a(e_C(\widetilde{\pi})) = 0,$$

that is, $(a\widetilde{\pi})^{p^r}$ is a zero of $\mathcal{E}_{p^r}(z)$.

We can prove this using the corollary above, which we will now demonstrate. Fix s such that $1 \le s \le p^r$. First, we know that $\binom{kp^r+p^r-s}{p^r-s} = 1$, which follows from induction on r and $j\binom{i}{j} = i\binom{i-1}{j-1}$. From this, we have $\partial_\theta^{p^r-s}\theta^{kp^r+p^r-s} = \theta^{kp^r}$. Next, for any t with $1 \le t < s$, we see that $\binom{kp^r+p^r-s}{p^r-t} = 0$, which follows from $\binom{i}{j} =$

$\binom{i-1}{j-1}+\binom{i-1}{j}$. Define a new polynomial $\alpha=\sum_{k=0}^{\deg a} a_k^{p^r}\theta^{kp^r+p^r-s}\in A$ by using the coefficients of the above polynomial a. Then, we obtain

$$\partial_\theta^{p^r-s}\alpha=\sum_{k=0}^{\deg a} a_k^{p^r}\partial_\theta^{p^r-s}\theta^{kp^r+p^r-s}=\sum_{k=0}^{\deg a} a_k^{p^r}\theta^{kp^r}=a^{p^r},$$

$$\partial_\theta^{p^r-t}\alpha=\sum_{k=0}^{\deg a} a_k^{p^r}\partial_\theta^{p^r-t}\theta^{kp^r+p^r-s}=0, \qquad \text{(for any } 1\le t<s).$$

Therefore, by the corollary above, we obtain

$$\begin{aligned}d[\alpha]_{p^r}\lambda &= {}^t((\partial_\theta^{p^r-1}\alpha)\widetilde{\pi}^{p^r},\dots,(\partial_\theta^{p^r-(s-1)}\alpha)\widetilde{\pi}^{p^r},(\partial_\theta^{p^r-s}\alpha)\widetilde{\pi}^{p^r},\dots,\alpha\widetilde{\pi}^{p^r})\\ &= {}^t(0,\dots,0,(a\widetilde{\pi})^{p^r},\dots,\alpha\widetilde{\pi}^{p^r})\in\Lambda_{p^r}.\end{aligned}$$

Hence, $\mathcal{E}_{p^r}((a\widetilde{\pi})^{p^r})=0$, and we have finished the proof.

Unfortunately, we cannot get a nontrivial zero of $\Delta_0^{p^r-l}\mathcal{E}_{p^r}$ using this approach.

(2) Recall that $\exp_n(\log_n(\mathbf{z}))=\mathbf{z}$. From Corollaries 2.4 (3) and 2.10 (1), we can obtain many relations between $\mathcal{E}_{p^r}$, $\Delta_0\mathcal{E}_{p^r}$, …, $\Delta_0^{p^r-1}\mathcal{E}_{p^r}$ and $\mathcal{L}_{p^r}$, $\Delta_0\mathcal{L}_{p^r}$, …, $\Delta_0^{p^r-1}\mathcal{L}_{p^r}$. For example,

$$\mathcal{E}_{p^r}(\mathcal{L}_{p^r}(z))=z,$$

$$\sum_{m=1}^{l}\binom{l-1}{m-1}\Delta_0^{(l-1)-(m-1)}\mathcal{E}_{p^r}(\Delta_0^{m-1}\mathcal{L}_{p^r}(z))=0, \quad \text{(for any } 2\le l\le p^r),$$

which is similar to $(X+Y)^{l-1}=\sum_{m=1}^{l}\binom{l-1}{m-1}X^{(l-1)-(m-1)}Y^{m-1}$.

(3) The p^r+1 functions $e_1(\mathbf{z}),\dots,e_{p^r}(\mathbf{z})$ and z_{p^r} are algebraically independent over $\mathbb{C}_\infty$, where

$$e_l(\mathbf{z})=\sum_{m=1}^{l}\binom{l-1}{m-1}\sum_{i=0}^{\infty}\frac{[i]^{l-m}}{D_i^n}z_m^{q^i},$$

that is, there exists no nonzero polynomial $P\in\mathbb{C}_\infty[X_1,\dots,X_{p^r},X_{p^r+1}]$ such that

$$P(e_1(\mathbf{z}),\dots,e_{p^r}(\mathbf{z}),z_{p^r})=0$$

for all $\mathbf{z}={}^t(z_1,\dots,z_{p^r})\in\mathbb{C}_\infty^{p^r}$. This follows from [32, Corollary 1.8].

3. Applications

We begin this section by presenting relations between entries of $\exp_n$ and the Thakur hypergeometric function (or the exp-type hypergeometric function) $F_{\exp}$, which was defined in [25, 26, 27], and as well as relations between entries of $\log_n$ and the log-type hypergeometric function $F_{\log}$, which was defined in [17]. Next, we investigate the convergence of $F_{\log}$. As an application, we study the transcendence of $F_{\log}$.

Let $[0]=0$ and $[i]=\theta^{q^i}-\theta$, $D_0=1$ and $D_i=[i]D_{i-1}^q$, and $L_0=1$ and $L_i=(-[i])L_{i-1}$. Assume that $a_1,a_2,\dots,a_r$ and $b_1,b_2,\dots,b_s$ are integers. First, we present the hypergeometric function introduced by Thakur [25, 26]. The Thakur hypergeometric function F_{exp} is defined as

$$\begin{aligned}F_{\mathrm{exp}}((a);(b);\tau)&=F_{C,\mathrm{exp}}(a_1,a_2,\dots,a_r;b_1,b_2,\dots,b_s;\tau)\\&:=\sum_{i=0}^{\infty}\frac{(a_1)_i^{\mathrm{Thakur}}(a_2)_i^{\mathrm{Thakur}}\cdots(a_r)_i^{\mathrm{Thakur}}}{(b_1)_i^{\mathrm{Thakur}}(b_2)_i^{\mathrm{Thakur}}\cdots(b_s)_i^{\mathrm{Thakur}}}\frac{\tau^i}{D_i},\end{aligned}$$

where the Pochhammer–Thakur symbol $(a)_i^{\mathrm{Thakur}}$ is given by

$$(a)_i=(a)_i^{\mathrm{Thakur}}:=\begin{cases}D_{i+a-1}^{q^{-(a-1)}} & \text{if } a\ge 1,\\ 1/L_{-a-i}^{q^i} & \text{if } a\le -i,\\ 0 & \text{if } -i<a\le 0\end{cases}$$

(see [27, Section 6.5]). Next, we present the other hypergeometric function. The log-type hypergeometric function $F_{\log}$ is defined as

$$\begin{aligned}F_{\log}((a);(b);\tau)&=F_{C,\log}(a_1,a_2,\dots,a_r;b_1,b_2,\dots,b_s;\tau)\\&:=\sum_{i=0}^{\infty}\frac{(a_1)_i^{\mathrm{FF}}(a_2)_i^{\mathrm{FF}}\cdots(a_r)_i^{\mathrm{FF}}}{(b_1)_i^{\mathrm{FF}}(b_2)_i^{\mathrm{FF}}\cdots(b_s)_i^{\mathrm{FF}}}\frac{\tau^i}{L_i},\end{aligned}$$

where the symbol $(a)_i^{\mathrm{FF}}$, a function-field analogue of the Pochhammer symbol, is given by

$$(a)_i=(a)_i^{\mathrm{FF}}:=\begin{cases}1 & \text{if } i=0,\\ [a][a+1][a+2]\cdots[a+i-1] & \text{if } i\ge 1\end{cases}$$

(see [17]). We use the symbol $(a)_i^{\mathrm{Thakur}}$ for the exp-type F_{exp}, and the symbol $(a)_i^{\mathrm{FF}}$ for the log-type $F_{\log}$, so even with the use of the notation $(a)_i$ dropped the markings “Thakur” and “FF”, we do not get confusing.

If $b_j>0$ for all j, then $F_{\mathrm{exp}}((a);(b);\tau)$ and $F_{\log}((a);(b);\tau)$ are well defined. From now on, we assume that $b_j>0$ for all j. If $a_j\le 0$ for some j, then these are polynomial in τ.

3.1. Specializations of the hypergeometric functions. In the subsection, we consider specializations of F_{exp} and $F_{\log}$. Recall that

$$\mathcal{E}_n:=\sum_{i=0}^{\infty}\frac{1}{D_i^n}\tau^i \qquad\text{and}\qquad \mathcal{L}_n:=\sum_{i=0}^{\infty}\frac{1}{L_i^n}\tau^i.$$

In what follows, we show that all of the series $\mathcal{E}_n,\Delta_0\mathcal{E}_n,\dots,\Delta_0^{n-1}\mathcal{E}_n$ are specializations of F_{exp}.

Proposition 3.1. *Let $1 \leq l \leq n-1$. Then the following holds.*

(1) *We obtain $\mathcal{E}_n(z) = F_{\exp}(-; 1^{n-1}; z)$, where $\overbrace{1, 1, \ldots, 1}^{a}$ is denoted by 1^a.*

(2) *We obtain*

$$\Delta_0^l \mathcal{E}_n(z) = F_{\exp}(-; 2^{n-l}, 1^{l-1}; z)^q,$$

where $\overbrace{2, 2, \ldots, 2}^{a}, \overbrace{1, 1, \ldots, 1}^{b}$ is denoted by $2^a, 1^b$.

Proof. In this proof, the symbol $(a)_i^{\mathrm{Thakur}}$ is written as $(a)_i$.

(1). Recall that $(1)_i = D_i$. Item (1) follows from the definition of $F_{\exp}$.

(2). Recall that $(2)_i^q = D_{i+1}$ and $(1)_i^q[i+1] = D_{i+1}$. We can compute $F_{\exp}(-; 2^{n-l}, 1^{l-1}; z)^q$ as follows:

$$\begin{aligned}
F_{\exp}(-; 2^{n-l}, 1^{l-1}; z)^q &= \left(\sum_{i=0}^{\infty} \frac{1}{(2)_i^{n-l}(1)_i^{l-1}} \frac{z^{q^i}}{D_i} \right)^q \\
&= \sum_{i=0}^{\infty} \frac{1}{D_{i+1}^{n-l}(D_{i+1}/[i+1])^{l-1}} \frac{z^{q^{i+1}}}{D_{i+1}/[i+1]} \\
&= \sum_{i=0}^{\infty} \frac{[i+1]^l}{D_{i+1}^n} z^{q^{i+1}} = \sum_{i=1}^{\infty} \frac{[i]^l}{D_i^n} z^{q^i} \\
&= \sum_{i=0}^{\infty} \frac{[i]^l}{D_i^n} z^{q^i} = \Delta_0^l \mathcal{E}_n(z),
\end{aligned}$$

thus completing the proof. □

Next, we prove that all of the series $\mathcal{L}_n, (-\Delta_0)\mathcal{L}_n, \ldots, (-\Delta_0)^{n-1}\mathcal{L}_n$ are specializations of $F_{\log}$.

Proposition 3.2. *Let $1 \leq m \leq n-1$, and let ζ be a fixed $(q-1)$th root of -1 in $\mathbb{F}_{q^2}$, that is, $\zeta^{q-1} = -1$. Then the following holds.*

(1) *We obtain $\mathcal{L}_n(z) = \zeta^{-(n-1)} F_{\log}(-; 1^{n-1}; \zeta^{n-1} z)$.*

(2) *We obtain*

$$(-\Delta_0)^m \mathcal{L}_n(z) = \zeta^{-(n-1)} \frac{(-1)^{m-1}}{[1]^{n-m}} F_{\log}(-; 2^{n-m}, 1^{m-1}; (\zeta^{n-1} z)^q).$$

(3) *Let $\mathbf{z} = {}^t(z_1, \ldots, z_n)$, and let $l_n(\mathbf{z})$ denote the last coordinate of $\log_n(\mathbf{z})$. Then we obtain*

$$l_n(\mathbf{z}) = \zeta^{-(n-1)} \sum_{m=1}^{n-1} \frac{(-1)^{m+1}}{[1]^{n-m}} F_{\log}(-; 2^{n-m}, 1^{m-1}; (\zeta^{n-1} z_{n-m})^q) + \zeta^{-(n-1)} F_{\log}(-; 1^{n-1}; \zeta^{n-1} z_n).$$

Proof. In this proof, the symbol $(a)_i^{\mathrm{FF}}$ is wrtiten as $(a)_i$.

(1). Note that $L_i = (-1)^i(1)_i$ and that

$$\begin{aligned}\zeta^{(n-1)q^i} &= \zeta^{n-1}\cdot\zeta^{(n-1)(q^i-1)} = \zeta^{n-1}\cdot(-1)^{(n-1)(1+q+\cdots+q^{i-1})}\\ &= \zeta^{n-1}\cdot(-1)^{(n-1)i}.\end{aligned}$$

We can compute $F_{\log}(-;1^{n-1};\zeta^{n-1}z)$ as follows:

$$\begin{aligned}F_{\log}(-;1^{n-1};\zeta^{n-1}z) &= \sum_{i=0}^{\infty}\frac{1}{(1)_i^{n-1}}\frac{(\zeta^{n-1}z)^{q^i}}{L_i}\\ &= \sum_{i=0}^{\infty}\frac{(-1)^{(n-1)i}}{L_i^{n-1}}\frac{\zeta^{n-1}(-1)^{(n-1)i}z^{q^i}}{L_i}\\ &= \zeta^{n-1}\mathcal{L}_n(z).\end{aligned}$$

(2). Note that $(1)_i[i+1] = (1)_{i+1}$ and $[1](2)_i = (1)_{i+1}$. Recall that

$$(-\Delta_0)^m\mathcal{L}_n(z) = \sum_{i=0}^{\infty}\frac{(-[i])^m}{L_i^n}z^{q^i} = \sum_{i=1}^{\infty}\frac{(-[i])^m}{L_i^n}z^{q^i}.$$

We can compute $F_{\log}(-;2^{n-m},1^{m-1};(\zeta^{n-1}z)^q)$ as follows:

$$\begin{aligned}&F_{\log}(-;2^{n-m},1^{m-1};(\zeta^{n-1}z)^q) = \sum_{i=0}^{\infty}\frac{1}{(2)_i^{n-m}(1)_i^{m-1}}\frac{((\zeta^{n-1}z)^q)^{q^i}}{L_i}\\ &= \sum_{i=0}^{\infty}\frac{[1]^{n-m}[i+1]^{m-1}}{(1)_{i+1}^{n-m}(1)_{i+1}^{m-1}}\frac{\zeta^{n-1}(-1)^{(n-1)(i+1)}z^{q^{i+1}}}{L_{i+1}/(-[i+1])}\\ &= \zeta^{n-1}[1]^{n-m}(-1)^{m-1}\sum_{i=0}^{\infty}\frac{(-[i+1])^m}{L_{i+1}^n}z^{q^{i+1}}\\ &= \zeta^{n-1}[1]^{n-m}(-1)^{m-1}\sum_{i=1}^{\infty}\frac{(-[i])^m}{L_i^n}z^{q^i}\\ &= \zeta^{n-1}[1]^{n-m}(-1)^{m-1}(-\Delta_0)^m\mathcal{L}_n(z).\end{aligned}$$

(3). It follows from Theorem 2.3 that

$$l_n(\mathbf{z}) = \sum_{m=0}^{n-1}(-\Delta_0)^m\mathcal{L}_n(z_{n-m}).$$

When $m=0$, we have $\mathcal{L}_n(z_n) = \zeta^{-(n-1)}F_{\log}(-;1^{n-1};\zeta^{n-1}z_n)$ by item (1). When $1\le m\le n-1$, we have

$$(-\Delta_0)^m\mathcal{L}_n(z_{n-m}) = \zeta^{-(n-1)}\frac{(-1)^{m-1}}{[1]^{n-m}}F_{\log}(-;2^{n-m},1^{m-1};(\zeta^{n-1}z_{n-m})^q)$$

from item (2), and the proof is finished. □

Remarks. Let ζ be a fixed $(q-1)$th root of -1 in $\mathbb{F}_{q^2}$. Note that $\zeta^{q^r-1} = (-1)^{q^{r-1}+\cdots+q+1} = (-1)^r$.

(1) Kochubei [18, 19, 20] defined the function $\ell_n(z)$ as

$$\ell_n(z) := \sum_{i=1}^{\infty} \frac{z^{q^i}}{[i]^n},$$

which is also a specialization of $F_{\log}$. In fact, we obtain

$$F_{\log}(1^{n+1}; 2^n; \zeta z^q) = \zeta \cdot [1]^n \ell_n(z).$$

(2) Suppose that $n = q^{2r}$. Then we have $\zeta^{n-1} = (-1)^{2r} = 1$, and thus

$$\mathcal{L}_n(z) = F_{\log}(-; 1^{n-1}; z),$$
$$(-\Delta_0)^m \mathcal{L}_n(z) = \frac{(-1)^{m+1}}{[1]^{n-m}} F_{\log}(-; 2^{n-m}, 1^{m-1}; z^q).$$

3.2. Convergences of the hypergeometric functions. In this subsection, we study the convergences of $F_{\exp}$ and $F_{\log}$, respectively. We set $\deg(x) := -v_\infty(x)$ and see that $\deg(a) = \deg_\theta(a)$ for any $a \in A = \mathbb{F}_q[\theta]$.

Thakur proved the following fact.

Fact ([25, Subsection 2.2] and [27, Remarks 6.5.3 (1)])**.** Suppose that $a_1, a_2, \ldots, a_r$ and $b_1, b_2, \ldots, b_s$ are positive integers. Then the convergence of the series $F_{\exp}((a); (b); z)$ is as follows.

(1) If $r > s+1$, then the series converges for $z = 0$ only.
(2) If $r < s+1$, then the series converges for all z.
(3) If $r = s+1$, then the series converges for z with

$$\deg z < \sum_{\beta=1}^{s} (b_\beta - 1) - \sum_{\alpha=1}^{r} (a_\alpha - 1).$$

Next, we show the convergence of the other function $F_{\log}$.

Proposition 3.3. *Suppose that $a_1, a_2, \ldots, a_r$ and $b_1, b_2, \ldots, b_s$ are positive integers. Set*

$$\rho = \rho(F_{\log}) := \frac{q}{q-1} \left(\sum_{\alpha=1}^{r} q^{a_\alpha - 1} - \sum_{\beta=1}^{s} q^{b_\beta - 1} - 1 \right).$$

(1) *If $\rho < 0$, then $F_{\log}((a); (b); z)$ converges for z with* $\deg z < -\rho$*, and diverges for z with* $\deg z > -\rho$.
(2) *If $\rho = 0$, then $F_{\log}((a); (b); z)$ converges for $z = 0$ only. If $\rho > 0$, then $F_{\log}(z)$ does not converge for all z.*

Proof. We prove the claim by using [15, Proposition 2.4]. In this proof, the symbol $(a)_i^{\mathrm{FF}}$ is written by $(a)_i$. Note that $\deg z = -v_\infty(z)$.

Let $a > 0$. It follows from $[a] = \theta^{q^a} - \theta$ and $(a)_i = [a][a+1]\cdots[a+(i-1)]$ that $\deg(a)_i = q^a + q^{a+1} + \cdots + q^{a+(i-1)} = q^a(q^i-1)/(q-1)$. Recall that $(1)_i = (-1)^i L_i$. Therefore, we have

$$\deg \frac{(a_1)_i(a_2)_i\cdots(a_r)_i}{(b_1)_i(b_2)_i\cdots(b_s)_i}\frac{1}{L_i} = (q^i-1)\rho,$$

and thus the order of convergence of $F_{\log}(z)$ is equal to

$$\lim_{i\to\infty} \deg \frac{(a_1)_i(a_2)_i\cdots(a_r)_i}{(b_1)_i(b_2)_i\cdots(b_s)_i}\frac{1}{L_i}\cdot\frac{1}{q^i} = \rho.$$

Hence, $F_{\log}((a);(b);z)$ converges if $\deg z < -\rho$, and diverges if $\deg z > -\rho$. □

Remarks. Let $1 \le m \le n$. If $\deg z < -\rho = (nq/(q-1) - m)q$, then $F_{\log}(-;2^{n-m},1^{m-1};z)$ converges. Note that $\rho = (m - nq/(q-1))q < 0$.

Remarks. Assume that $a_1, a_2, \ldots, a_r$ and $b_1, b_2, \ldots, b_s$ are positive integers.

(1) Suppose that $r < s+1$. It is known that $F_{C,\exp}((a);(b);\tau) = \sum_{i\ge 0}\alpha_i\tau^i$ is an E_q-function in the sense of Yu (see Remarks in [28] for a proof and [31] for the definition of an E_q-function). Note that $\deg\alpha_i = ((r-s-1)i + \sum_{\alpha=1}^{r}(a_\alpha - 1) - \sum_{\beta=1}^{s}(b_\beta-1))q^i = O(iq^i)$.

(2) The other function $F_{C,\log}((a);(b);\tau) = \sum_{i\ge 0}\beta_i\tau^i$ is not an E_q-function because it is not entire. Note that $\deg\beta_i = \rho q^i = O(q^i)$. It has been shown in the classical setting that algebraic functions, the polylogarithm $\mathrm{Li}_n(z)$, and the Gauss hypergeometric function $F_{\text{Classical}}$ with rational parameters are G-functions in the sense of Siegel (see [3] and [12]). In the function-field setting, the notion of G-functions has not been defined yet. It is important to study such a notion, but we leave this task for a future paper.

3.3. Transcendence results for hypergeometric functions. In this subsection, we prove transcendences for $F_{\exp}$ and $F_{\log}$. In the classical setting, a complex number that is not algebraic over $\mathbb{Q}$ is said to be transcendental over $\mathbb{Q}$. Similarly, in the function-field setting, an element in $\mathbb{C}_\infty$ that is not in $\overline{k}$ is called transcendental over k. Recall that $\overline{k}$ is the algebraic closure of $k = \mathbb{F}_q(\theta)$ in $\mathbb{C}_\infty$. Note that $\overline{k}$ equals the algebraic closure of k inside $\overline{k}_\infty$.

In 1882 the transcendence of π was shown by Lindemann. As its function-field analogue, in 1941 Wade [29] proved that $\widetilde{\pi}_C$ is transcendental over k. Recall that $\widetilde{\pi}_C$ is an analogue of $2\pi i$, and that $e^{2\pi i} = 0$ and $e_C(\widetilde{\pi}_C) = 0$. In 1873 the transcendence of e was shown by Hermite. In general, it is known that $\{\gamma \in \overline{\mathbb{Q}} \mid e^\gamma \in \overline{\mathbb{Q}}\} = \{0\}$. Its function-field analogue is

$$\{\gamma \in \overline{k} \mid e_C(\gamma) \in \overline{k}\} = \{0\} \tag{3.1}$$

(see, for example, [31, Theorem 5.1], [27, Theorem 10.2.2], and [28, Theorem 3]). In 1929 Siegel proved that $\{\gamma \in \overline{\mathbb{Q}} \mid J_0(\gamma) \in \overline{\mathbb{Q}}\} = \{0\}$, where $J_0(z) := \sum_{i=0}^{\infty} \frac{(-1)^i}{i!i!} \left(\frac{z}{2}\right)^{2i}$ is the Bessel function (see [24, Hauptsatz]). Its function-field analogue is

$$\{\gamma \in \bar{k} \mid J_m(\gamma) \in \bar{k}\} = \{0\}, \tag{3.2}$$

where $J_m(z) := \sum_{i=0}^{\infty} \frac{z^{q^{m+i}}}{D_{m+i} D_i^{q^m}}$ for $m \in \mathbb{Z}_{\geq 0}$ is the Bessel–Carlitz function (see, for example, [28, Theorem 3]).

In the classical setting, the functions e^z, $\log(z)$, and $J_0(z)$ are transcendental functions. Intuitively, the transcendental functions $f(z)$ can be expected to take transcendental values $f(\gamma)$ at almost all algebraic points γ, but this is incorrect for the Gauss hypergeometric function with rational parameters. Let $F_{\text{Classical}}(a, b; c; z)$ be the Gauss hypergeometric function:

$$F_{\text{Classical}}(a, b; c; z) := \sum_{i=0}^{\infty} \frac{(a)_i (b)_i}{(c)_i} \frac{z^i}{i!}, \qquad (\text{for } |z| < 1),$$

where $(a)_i = (a)(a+1)\dots(a+i-1)$ denotes the Pochhammer symbol. Let $E_{\text{Classical}}(a, b; c)$ be the exceptional set:

$$E_{\text{Classical}}(a, b; c) := \{\gamma \in \overline{\mathbb{Q}} \mid F_{\text{Classical}}(\gamma) \in \overline{\mathbb{Q}}\}.$$

The following is known.

Fact (Wolfart [30], Cohen–Wüstholz [9], and Edixhoven–Yafaev [13])**.** Assume that $a, b, c \in \mathbb{Q}$ and $c \neq 0, -1, -2, \dots$. Then the monodromy group of $F_{\text{Classical}}(a, b; c; z)$ is arithmetic if and only if the set $E_{\text{Classical}}(a, b; c)$ is infinite.

In the function-field setting, it is natural to investigate the exceptional sets for the Thakur hypergeometric function $F_{\exp}((a); (b); z)$ and the log-type hypergeometric function $F_{\log}((a); (b); z)$. We define the exceptional sets $E_{\exp}((a); (b))$ and $E_{\log}((a); (b))$ as

$$E_{\exp}((a); (b)) := \{\gamma \in \bar{k} \mid F_{\exp}((a); (b); \gamma) \in \bar{k}\},$$
$$E_{\log}((a); (b)) := \{\gamma \in \bar{k} \mid F_{\log}((a); (b); \gamma) \in \bar{k}\},$$

respectively. The sets $E_{\exp}((a); (b))$ and $E_{\log}((a); (b))$ are not both empty: $0 \in E_{\exp}((a); (b))$ and $0 \in E_{\log}((a); (b))$. We can rewrite facts (3.1) and (3.2) as

$$E_{\exp}(-; -) = \{0\} \qquad \text{and} \qquad E_{\exp}(-; m+1) = \{0\}.$$

Note that $e_C(z) = F_{\exp}(-; -; z)$ and $J_m(z) = F_{\exp}(-; m+1; z)^{q^m}$.

For the Thakur hypergeometric function $F_{\exp}((a); (b); z)$, the following holds.

Proposition 3.4. *Suppose that $r < s+1$, and that $a_1, \dots, a_r$ and $b_1, \dots, b_s$ are positive integers. Let R denote the set defined by*

$$R := \{\gamma \in \overline{k} \mid \textit{the separable degree } [k(\gamma) : k]_s < q\}.$$

Then, for all $\gamma \in R \setminus \{0\}$, the special value $F_{\exp}((a);(b);\gamma)$ is transcendental over k, that is,

$$R \setminus \{0\} \subseteq \overline{k} \setminus E_{\exp}((a);(b)).$$

In particular, for all $\gamma \in R \setminus \{0\}$, the value $\Delta_0^{l-1}\mathcal{E}_n(\gamma) = e_l({}^t(\gamma, 0, \dots, 0))$, the entry of the first column of $\exp_n$, is transcendental over k.

Proof. The first claim immediately follows from [28, Theorem 2]. In the proof of the second claim, we use Theorem 2.6 and Proposition 3.1:

$$e_l({}^t(z, 0, \dots, 0)) \\ = \begin{cases} \mathcal{E}_n(z) = F_{\exp}(-; 1^{n-1}; z), \\ \Delta_0^{l-1}\mathcal{E}_n(z) = F_{\exp}(-; 2^{n-(l-1)}, 1^{(l-1)-1})^q, & \text{(for any } 2 \le l \le n). \end{cases}$$

Using this and the first claim, we complete the proof. □

For the log-type hypergeometric function $F_{\log}((a);(b);z)$, the following holds.

Theorem 3.5. *Let ζ be a fixed $(q-1)$th root of -1 in $\mathbb{F}_{q^2}$.*

(1) *Let $z_n \in \overline{k}$. Assume that $\deg z_n < nq/(q-1)$. If $F_{\log}(-; 1^{n-1}; z_n)$ is nonzero, then it is transcendental over k, and so $z_n \notin E_{\log}(-; 1^{n-1})$. That is,*

$$E_{\log}(-; 1^{n-1}) = \{\gamma \in \overline{k} \mid F_{\log}(-; 1^{n-1}; \gamma) = 0\}.$$

(2) *Let $m \ge 1$, and let $z_{n-m} \in \overline{k}$. Assume that $\deg z_{n-m} < (nq/(q-1) - m)q$. If $F_{\log}(-; 2^{n-m}, 1^{m-1}; z_{n-m})$ is nonzero, then it is transcendental over k, and so $z_{n-m} \notin E_{\log}(-; 2^{n-m}, 1^{m-1})$. That is,*

$$E_{\log}(-; 2^{n-m}, 1^{m-1}) = \{\gamma \in \overline{k} \mid F_{\log}(-; 2^{n-m}, 1^{m-1}; \gamma) = 0\}.$$

Proof. Before we prove items (1) and (2), we recall several facts. Note that the elements ζ and $[1]$ are algebraic: $\zeta, [1] \in \overline{k}$. From [32, Theorem 2.3], we know that $l_n(\mathbf{z})$ is either zero or transcendental.

(1). Set $\mathbf{z} = {}^t(0, \cdots, 0, \zeta^{-(n-1)} z_n) \in \overline{k}^n$. By Proposition 3.2 (3), we have

$$l_n(\mathbf{z}) = \zeta^{-(n-1)} F_{\log}(-; 1^{n-1}; z_n).$$

Since $F_{\log}(-; 1^{n-1}; z_n)$ is nonzero from the assumption, it is transcendental over k.

(2). Set $\mathbf{z} = {}^t(0, \cdots, 0, \zeta^{-(n-1)} z_{n-m}^{1/q}, 0, \cdots, 0) \in \overline{k}^n$. By Proposition 3.2 (3), we have

$$l_n(\mathbf{z}) = \zeta^{-(n-1)} \frac{(-1)^{m-1}}{[1]^{n-m}} F_{\log}(-; 2^{n-m}, 1^{m-1}; z_{n-m}).$$

Since $F_{\log}(-; 2^{n-m}, 1^{m-1}; z_{n-m})$ is nonzero from the assumption, it is transcendental over k. □

Remarks.

(1) If $a_j \leq 0$ for some j, then $F_{\exp}((a); (b); z)$ and $F_{\log}((a); (b); z)$ are polynomials, namely, algebraic functions. Hence, we obtain

$$\overline{k} = E_{\exp}((a); (b)) \qquad \text{and} \qquad \overline{k} = E_{\log}((a); (b)).$$

(2) In the classical setting, it is well known that

$$\{\gamma \in \overline{\mathbb{Q}} \mid \log \gamma \in \overline{\mathbb{Q}}\} = \{\gamma \in \overline{\mathbb{Q}} \mid \log \gamma = 0\} = \{1\},$$

which is an analogue for item (1) in the theorem above. In fact, item (1) can be rewritten as

$$\{\gamma \in \overline{k} \mid F_{\log}(-; 1^{n-1}; \gamma) \in \overline{k}\} = \{\gamma \in \overline{k} \mid \mathcal{L}_n(\zeta^{-(n-1)}\gamma) = 0\}.$$

In particular, when $n=1$, we have $E_{\log}(-; -) = \{\gamma \in \overline{k} \mid \log_C(\gamma) = 0\}$.

(3) In the theorem above, we studied transcendences for the special parameters

$$(a) = - \qquad \text{and} \qquad (b) = 2^{n-m}, 1^{m-1}.$$

In a future paper, we will investigate the transcendences for the other parameters (a) and (b).

References

[1] G. W. ANDERSON, "t-motives", *Duke Math. J.* **53** (1986), p. 457-502.

[2] G. W. ANDERSON & D. S. THAKUR, "Tensor powers of the Carlitz module and zeta values", *Ann. Math.* **132** (1990), no. 1, p. 159-191.

[3] Y. ANDRÉ, *G-functions and geometry*, Aspects of Mathematics, vol. 13, Vieweg & Sohn, 1989.

[4] L. CARLITZ, "On certain functions connected with polynomials in a Galois field", *Duke Math. J.* **1** (1935), no. 2, p. 137-168.

[5] C.-Y. CHANG, N. GREEN & Y. MISHIBA, "Taylor coefficients of Anderson-Thakur series and explicit formulae", *Math. Ann.* **379** (2021), no. 3-4, p. 1425-1474.

[6] C.-Y. CHANG & Y. MISHIBA, "On multiple polylogarithms in characteristic p: v-adic vanishing versus ∞-adic Eulerianness", *Int. Math. Res. Not.* **2019** (2019), no. 3, p. 923-947.

[7] ———, "On a conjecture of Furusho over function fields", *Invent. Math.* **223** (2021), no. 1, p. 49-102.

[8] C.-Y. CHANG & J. YU, "Determination of algebraic relations among special zeta values in positive characteristic", *Adv. Math.* **216** (2007), no. 1, p. 321-345.

[9] P. B. COHEN & G. WÜSTHOLZ, "Application of the André–Oort conjecture to some questions in transcendence", in *A panorama in number theory or The view from Baker's garden (Zürich, 1999)*, Cambridge University Press, 2002, p. 89-106.

[10] V. G. DRINFELD, "Elliptic modules", *Mat. Sb., N. Ser.* **94** (1974), p. 594-627.

[11] ———, "Elliptic modules. II", *Mat. Sb., N. Ser.* **102** (1977), p. 182-194.
[12] B. Dwork, G. Gerotto & F. J. Sullivan, *An introduction to G-functions*, Annals of Mathematics Studies, vol. 133, Princeton University Press, 1994.
[13] B. Edixhoven & A. Yafaev, "Subvarieties of Shimura varieties", *Ann. Math.* **157** (2003), no. 2, p. 621-645.
[14] A. El-Guindy & M. A. Papanikolas, "Explicit formulas for Drinfeld modules and their periods", *J. Number Theory* **133** (2013), no. 6, p. 1864-1886.
[15] D. Goss, *Basic structures of function field arithmetic*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge, vol. 35, Springer, 1996.
[16] R. Harada, "On the period interpretation for some special values of Thakur hypergeometric functions", `https://arxiv.org/abs/2205.00649`, 2022.
[17] T. Hasegawa, "Logarithmic-type and exponential-type hypergeometric functions for function fields", *J. Number Theory* **233** (2022), p. 87-111.
[18] A. N. Kochubei, "Polylogarithms and a zeta function for finite places of a function field", in *Ultrametric functional analysis (Clermont-Ferrand, 2004)*, Contemporary Mathematics, vol. 384, American Mathematical Society, 2005, p. 157-167.
[19] ———, "Evolution equations and functions of hypergeometric type over fields of positive characteristic", *Bull. Belg. Math. Soc. Simon Stevin* **14** (2007), no. 5, p. 947-959.
[20] ———, "Hypergeometric functions and Carlitz differential equations over function fields", in *Arithmetic and geometry around hypergeometric functions (Istanbul, 2005)*, Progress in Mathematics, vol. 260, Birkhäuser, 2007, p. 163-187.
[21] A. Maurischat, "Prolongations of t-motives and algebraic independence of periods", *Doc. Math.* **23** (2018), p. 815-838.
[22] C. Namoijam & M. A. Papanikolas, "Hyperderivatives of periods and quasi-periods for Anderson t-modules", `https://arxiv.org/abs/2103.05836`, 2021.
[23] M. A. Papanikolas, "Tannakian duality for Anderson-Drinfeld motives and algebraic independence of Carlitz logarithms", *Invent. Math.* **171** (2008), no. 1, p. 123-174.
[24] C. L. Siegel, "Über einige Anwendungen diophantischer Approximationen", in *On some applications of Diophantine approximations. (A translation of Carl Ludwig Siegel's "Über einige Anwendungen diophantischer Approximationen" by Clemens Fuchs)*, Quaderni. Scuola Normale Superiore di Pisa. Monographs, vol. 2, Edizioni della Normale, 2014, p. 81-138.
[25] D. S. Thakur, "Hypergeometric functions for function fields", *Finite Fields Appl.* **1** (1995), no. 2, p. 219-231.
[26] ———, "Hypergeometric functions for function fields. II", *J. Ramanujan Math. Soc.* **15** (2000), no. 1, p. 43-52.
[27] ———, *Function field arithmetic*, World Scientific, 2004.
[28] D. S. Thakur, Z.-Y. Wen, J.-Y. Yao & L. Zhao, "Transcendence in positive characteristic and special values of hypergeometric functions", *J. Reine Angew. Math.* **657** (2011), p. 135-171.
[29] L. I. Wade, "Certain quantities transcendental over $\mathrm{GF}(p^n, x)$", *Duke Math. J.* **8** (1941), p. 701-720.
[30] J. Wolfart, "Werte hypergeometrischer Funktionen", *Invent. Math.* **92** (1988), no. 1, p. 187-216.
[31] J. Yu, "Transcendence and Drinfeld modules", *Invent. Math.* **83** (1986), p. 507-517.
[32] ———, "Transcendence and special zeta values in characteristic p", *Ann. Math.* **134** (1991), no. 1, p. 1-23.

Takehiro Hasegawa
Shiga University, Otsu, Shiga 520-0862
Japan
E-mail: `thasegawa3141592@yahoo.co.jp`