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Résumé. Soit F un corps local sur $\mathbf{Q}_p$ ou $\mathbf{F}_p((t))$ et soit D une algèbre à division centrale simple sur F de degré d. Dans le cas p-adique, on suppose que $p > de+1$ où e est le degré de ramification sur $\mathbf{Q}_p$; sinon on suppose seulement que p et d sont premiers entre eux. Pour le sous-groupe $I_1 = 1 + \varpi_D \mathcal{O}_D$ de $D^\times$, on détermine la structure de $\mathrm{H}^1(I_1, \pi)$ en tant que représentation de $D^\times/I_1$ pour une $\overline{\mathbf{F}}_p$-représentation lisse irréductible quelconque π de $D^\times$. Nous utilisons ceci pour calculer le groupe $\mathrm{Ext}^1_{D^\times}(\pi, \pi')$ pour des représentations lisses irréductibles quelconques π et π' de $D^\times$. Dans le cas p-adique, via la dualité de Poincaré, nous pouvons calculer les groupes de cohomologie supérieurs et les extensions de degré maximal.

Abstract. Let F be a local field over $\mathbf{Q}_p$ or $\mathbf{F}_p((t))$, and let D be a central simple division algebra over F of degree d. In the p-adic case, we assume $p > de + 1$ where e is the ramification degree over $\mathbf{Q}_p$; otherwise, we need only assume p and d are coprime. For the subgroup $I_1 = 1 + \varpi_D \mathcal{O}_D$ of $D^\times$ we determine the structure of $\mathrm{H}^1(I_1, \pi)$ as a representation of $D^\times/I_1$ for an arbitrary smooth irreducible $\overline{\mathbf{F}}_p$-representation π of $D^\times$. We use this to compute the group $\mathrm{Ext}^1_{D^\times}(\pi, \pi')$ for arbitrary smooth irreducible representations π and π' of $D^\times$. In the p-adic case, via Poincaré duality we can compute the top cohomology groups and compute the highest degree extensions.

1. Introduction

Let F be a non-Archimedean local field whose residue field has characteristic p. In the local Langlands program, division algebras play an important role because of the Jacquet–Langlands correspondence. This correspondence states that for a division algebra D of invariant $1/n$ over F

there is an injection

$$\left\{\begin{array}{c}\text{irreducible smooth}\\ \text{representations}\\ \text{of } D^\times \text{ over } \overline{\mathbf{Q}}_\ell\end{array}\right\}/\simeq \;\longhookrightarrow\; \left\{\begin{array}{c}\text{irreducible smooth}\\ \text{representations}\\ \text{of } \mathrm{GL}_n(F) \text{ over } \overline{\mathbf{Q}}_\ell\end{array}\right\}/\simeq$$

here one can describe the image well. Here, we choose the prime ℓ so that $\ell \neq p$. The objects on the right side appear in the Langlands correspondence for GL_n, which is then in natural bijection with certain Weil–Deligne representations over $\overline{\mathbf{Q}}_\ell$.

When we instead take representations over a field of characteristic p, the problem becomes more difficult. There has been some progress towards proving a characteristic p local Langlands correspondence, such as in [1], but in general it is not clear how to formulate the Jacquet–Langlands correspondence given above. One attempt has been made in [10], where for a division algebra of invariant $1/n$ over F, Scholze constructs a functor

$$\left\{\begin{array}{c}\text{smooth admissible}\\ \text{representations}\\ \text{of } \mathrm{GL}_n(F) \text{ over } \mathbf{F}_p\end{array}\right\} \;\xrightarrow{\;\mathcal{F}\;}\; \left\{\begin{array}{c}\text{smooth admissible}\\ \text{representations}\\ \text{of } D^\times \text{ over } \mathbf{F}_p\end{array}\right\}$$

where $\mathcal{F}(\pi)$ also carries an action of $\mathrm{Gal}(\overline{F}/F)$. This gives some evidence for a mod p analogue of the Jacquet–Langlands correspondence. Because of this, understanding the representation theory of $D^\times$ with $\overline{\mathbf{F}}_p$ coefficients can be expected to be of use in the mod p local Langlands program. The irreducible representations of $D^\times$ over $\overline{\mathbf{F}}_p$ are easy to classify, however because $\mathrm{Rep}(D^\times)$ is not semisimple it is important to understand extensions between irreducible representations. These are what we will ultimately characterize.

Let D be a central simple division algebra over a non-Archimedean local field F of degree d, with a choice of uniformizer ϖ_D and ring of integers $\mathcal{O}_D$.

For $a|d$, set $D_a^\times = F^\times \mathcal{O}_D^\times \langle \varpi_D^a \rangle$. Then let $\pi = \mathrm{Ind}_{D_a^\times}^{D^\times} \chi$ and $\pi' = \mathrm{Ind}_{D_{a'}^\times}^{D^\times} \chi'$ be irreducible representations, where χ and χ' are characters of $D_a^\times$ and $D_{a'}^\times$. In Section 5, we prove the following main theorem with some mild constraints on the residue field characteristic of F.

Theorem (Theorem 21). *Assume that $p > de + 1$ if $F/\mathbf{Q}_p$ and that $\gcd(p, d) = 1$ if $F/\mathbf{F}_p((t))$.*

Let χ, χ', π and π' be as defined in the setup above, and define the set

$$S = \{\chi^s : s = \varpi_D^i, 0 \leq i < \gcd(a, a')\},$$

so that the elements s form a set of coset representatives for $D_{a'}^\times \backslash D^\times / D_a^\times$.

There are two types of direct summands in $\mathrm{Ext}^1_{D^\times}(\pi, \pi')$. If $\mathrm{Res}^{D_{a'}^\times}_{D_{\mathrm{lcm}(a,a')}^\times} \chi'$ is equal to some $\chi^s \in S$, we have a nonzero direct summand A_{χ^s} fitting into

an exact sequence

$$0 \longrightarrow \overline{\mathbf{F}}_p \longrightarrow A_{\chi^s} \longrightarrow \mathrm{H}^1(1 + \pi_F \mathcal{O}_F, \overline{\mathbf{F}}_p) \longrightarrow 0.$$

We also get a nonzero direct summand $A_{\chi^s} \simeq \overline{\mathbf{F}}_p$ for each $\chi^s \in S$ for which $\mathrm{Res}_{D_{\mathrm{lcm}(a,a')}^{\times}}^{D_{a'}^{\times}} \chi' \otimes (\chi^s)^*$ is extended trivially from a character $x \mapsto \left(\frac{x}{\sigma(x)} \right)^{p^i}$. Set $A_{\chi^s} = 0$ otherwise. Then $\mathrm{Ext}_{D^{\times}}^1(\pi, \pi') \simeq \bigoplus_{\chi^s \in S} A_{\chi^s}$.

We note that the cohomology ring of the pro-p group $1 + \pi_F \mathcal{O}_F$ with the trivial action is known to be $\bigotimes_{i \in I} \overline{\mathbf{F}}_p[x_i]/x_i^2$ for some index set I when p is large enough so that there is no p-torsion. In this case $1 + \pi_F \mathcal{O}_F$ is a direct product of $|I|$ copies of $\mathbf{Z}_p$ as a topological group. In the p-adic case we have $|I| = [F : \mathbf{Q}_p]$, and in the local function field case we have $I = \mathbf{N}$. Thus, A_{χ^s} of the first type described in the theorem has dimension $[F : \mathbf{Q}_p] + 1$ as an $\overline{\mathbf{F}}_p$-vector space in the p-adic case and countable dimension in the local function field case.

This paper is organized as follows: in Section 2, we go over some basic information about local division algebras and their representations. In Section 3, we translate the problem of computing extensions to one of computing certain cohomology groups. Then, in Section 4, we compute the cohomology group $\mathrm{H}^1(1 + \varpi_D \mathcal{O}_D, \pi)$ for any smooth irreducible representation π. Setting $I_1 = 1 + \varpi_D \mathcal{O}_D$, we do this by computing the Frattini subgroup $[I_1, I_1]I_1^p$. We additionally show that almost all elements of this subgroup are of the form $[x, y]z^p$ for $x, y, z \in I_1$. More explicitly, when $d > 4$ this is true for every element of the Frattini subgroup that does not also lie in the subgroup $I_3 = 1 + \varpi_D^3 \mathcal{O}_D$. In Section 5, we use this to compute $\mathrm{Ext}_{D^{\times}}^1(\pi, \pi')$ for arbitrary smooth irreducible representations π and π' of $D^{\times}$. In the p-adic case, we use Poincaré duality to compute higher extensions and also show how to get partial information about $\mathrm{H}^2(I_1, \overline{\mathbf{F}}_p)$. This is enough to compute all extension groups for a quaternion algebra over $\mathbf{Q}_p$.

1.1. Notation. Throughout, we will fix some notation. We let F be a p-adic field of ramification degree e and residue field degree f over $\mathbf{Q}_p$. We also allow F to be an extension of $\mathbf{F}_p((t))$. We will need $p > de + 1$ in the p-adic case and in the local function field case we need $\gcd(p, d) = 1$.

We denote a choice of uniformizer by π_F, the ring of integers by $\mathcal{O}_F$, and the residue field by $k_F = \mathbf{F}_q$. The choice of discrete valuation is given by ν_F, normalized so that $\nu_F(\pi_F) = 1$.

Over F, we consider a degree $d > 1$ central simple division algebra D, so that $\dim_F D = d^2$. We use ϖ_D for a choice of a uniformizer with respect to $\nu_D := \frac{1}{d}\nu_F \circ \mathrm{Nrd}$, where Nrd is the reduced norm on D. We moreover choose ϖ_D so that $\varpi_D^d = \pi_F$. We denote the ring of integers by $\mathcal{O}_D$, and the

residue field by $k_D = \mathbf{F}_{q^d}$. We fix an algebraic closure $\overline{\mathbf{F}}_p$ and embeddings $k_F \hookrightarrow k_D \hookrightarrow \overline{\mathbf{F}}_p$. The notation $[\,\cdot\,]$ is used for the Teichmüller lift $k_D \to D$. We let σ be a generator of $\mathrm{Gal}(k_D/k_F)$ such that $\varpi_D[x]\varpi_D^{-1} = [\sigma(x)]$ for all $x \in k_D$, and set $I_1 = 1 + \varpi_D \mathcal{O}_D$ to be the unique pro-p Iwahori subgroup of $D^\times$. For $a|d$ we set $D_a^\times := F^\times \mathcal{O}_D^\times \langle \varpi_D^a \rangle$.

Given a smooth representation π of a locally profinite group G, we use $\mathrm{Ext}_G^n(\pi, -)$ to denote the nth derived functor of $\mathrm{Hom}_G(\pi, -)$, where $\mathrm{Hom}_G(\pi, \pi')$ is the space of G-equivariant linear maps from π to another smooth G-representation π'. All representations will be over $\overline{\mathbf{F}}_p$. We use $\mathbf{1}$ to denote the trivial character of G, which is the space $\overline{\mathbf{F}}_p$ with the trivial action. If $\chi : G \to \overline{\mathbf{F}}_p^\times$ is a character of G, then we use χ^* to denote the dual character, so $\chi \otimes \chi^* = \mathbf{1}$. For a normal subgroup $H \leq G$ and a character $\chi : H \to \overline{\mathbf{F}}_p^\times$, define $\chi^s(h)$ to be $\chi(s^{-1}hs)$ for $s \in G$ and $h \in H$ - we use this in the Mackey formula.

Throughout, the cohomology groups $\mathrm{H}^n(G, A)$ are continuous cohomology groups for a topological group G and discrete G-module A. As a special case of this, $\mathrm{Hom}(G, A)$ is the group of continuous homomorphisms from G to A, which we sometimes emphasize by writing it as $\mathrm{Hom}_{\mathrm{cont}}(G, A)$. For a compact open subgroup K of a locally profinite group G, the Hecke algebra $\mathcal{H}_K$ is the algebra $\overline{\mathbf{F}}_p[K \backslash G / K]$ of $\overline{\mathbf{F}}_p$-valued bi-K-invariant continuous functions of compact support under convolution. For any group G and $g, h \in G$, we use the convention $[g, h] = ghg^{-1}h^{-1}$. We use $[G, G]$ and $[G, G]G^p$ to denote the closures of the commutator and Frattini subgroups of an ℓ-group G.

2. Preliminaries

2.1. Structure of division algebras. There is a decomposition

$$D^\times \simeq \varpi_D^{\mathbf{Z}} \ltimes (k_D^\times \ltimes I_1),$$

similar to the analogous decomposition of $F^\times$ except we must use semidirect products due to non-commutativity. The subgroup I_1 has a filtration

$$I_i = 1 + \varpi_D^i \mathcal{O}_D,$$

where $I_i/I_{i+1} \simeq k_D$ as an additive group for $i \geq 1$. We may understand elements of D explicitly via Teichmüller lifts $[\,\cdot\,] : k_D \to D$, which allows a unique representation of every element as $x = \sum_{i \geq n}[x_i]\varpi_D^i$.

Up to isomorphism, the central simple division algebras D over F are classified by the result of the invariant map

$$\mathrm{Br}(F) \xrightarrow{\mathrm{inv}_F} \mathbf{Q}/\mathbf{Z}.$$

This is an isomorphism, and the isomorphism classes $[D]$ of degree d central simple division algebras are obtained as preimages of $\frac{r}{d}$ where $\gcd(r, d) = 1$.

The value r reflects the choice of the generator $\sigma := (x \mapsto x^{q^r})$ of $\mathrm{Gal}(k_D/k_F)$, such that $\varpi_D[x]\varpi_D^{-1} = [\sigma(x)]$.

2.2. Representations of $D^\times$.

We work with smooth representations of $D^\times$ with coefficients in $\overline{\mathbf{F}}_p$, which makes the following well-known lemma crucial:

Lemma 1. *Let G be a pro-p group. Then any nonzero smooth representation V over $\overline{\mathbf{F}}_p$ has $V^G \neq 0$.*

Proof. This is a classical result, which can be easily deduced from Proposition 26 in [11] which deals with finite p-groups. □

To begin, we would like to understand the characters of $D^\times$. There is an exact sequence

$$1 \longrightarrow D_{\mathrm{Nrd}=1}^\times \longrightarrow D^\times \xrightarrow{\mathrm{Nrd}} F^\times \longrightarrow 1,$$

where exactness follows from the fact that Nrd is surjective. This is because the degree d unramified extension E/F is contained in D and one can show $\mathrm{Nrd}|_E = \mathrm{Nm}_{E/F}$, so surjectivity of Nrd follows from surjectivity of the norm map from $\mathcal{O}_E^\times$ to $\mathcal{O}_F^\times$ for unramified extensions and the fact that $\mathrm{Nrd}(\varpi_D)$ is a unit multiple of π_F. It is known that $D_{\mathrm{Nrd}=1}^\times = [D^\times, D^\times]$, see for example [8, §1.4.3]. Then any character $D^\times \to \overline{\mathbf{F}}_p^\times$ arises as

$$\chi : D^\times \xrightarrow{\mathrm{Nrd}} F^\times \xrightarrow{\kappa} \overline{\mathbf{F}}_p^\times$$

because $\overline{\mathbf{F}}_p^\times$ is abelian, so any homomorphism into this group must factor through the abelianization. We can easily classify the characters κ of $F^\times$ via $F^\times \simeq \pi_F^{\mathbf{Z}} \times k_F^\times \times (1 + \pi_F \mathcal{O}_F)$, where the final component is pro-p and hence we only need to compute characters of $\pi_F^{\mathbf{Z}} \times k_F^\times$ by the previous lemma. These are both cyclic groups, so the characters are then determined by where we send the generators of each component.

Now we turn to classifying characters of $D_a^\times$. Recall that this subgroup is defined to be $F^\times \mathcal{O}_D^\times \langle \varpi_D^a \rangle$ where $a|d$. As special cases, $D_d^\times = F^\times \mathcal{O}_D^\times$ and $D_1^\times = D^\times$. We have a decomposition

$$D_a^\times \simeq \varpi_D^{a\mathbf{Z}} \ltimes (k_D^\times \ltimes I_1),$$

and so we can denote elements by (ϖ_D^{an}, x, y) where $x \in k_D^\times$ and $y \in I_1$.

Lemma 2. *The characters of $D_a^\times$ are given by*

$$\chi_{a,\alpha,m} : (\varpi_D^{an}, x, y) \longmapsto \alpha^n \, \mathrm{Nm}_{k_D/\mathbf{F}_{q^a}}(x)^m.$$

Here, $\alpha \in \overline{\mathbf{F}}_p^\times$ and $m \in \mathbf{Z}$.

Proof. As I_1 is a pro-p group, Lemma 1 implies that any character mapping to $\overline{\mathbf{F}}_p^\times$ must only depend on ϖ_D^{an} and x. Because $\varpi_D^{a\mathbf{Z}}$ and $k_D^\times$ are cyclic and $k_D^\times$ embeds into $\overline{\mathbf{F}}_p^\times$, such a character must be of the form $\chi : (\varpi_D^{an}, x, y) \mapsto \alpha^n x^m$. The only thing to work out is exactly which values of m are allowed.

In order for χ to be well-defined on $D_a^\times$, it must be the case that

$$\chi([x]) = \chi(\varpi_D^a[x]\varpi_D^{-a}) = \chi([\sigma^a(x)]).$$

In particular, $x^m = \sigma^a(x^m)$ for all $x \in k_D^\times$. This gives a congruence condition on m. Namely, viewing $m \in \mathbf{Z}/|k_D^\times|\mathbf{Z}$, if σ sends $x \mapsto x^{q^r}$ we have $m \equiv q^{ar}m \pmod{|k_D^\times|}$. This means $\frac{q^d-1}{q^a-1}$ divides m, and so the character restricted to $k_D^\times$ factors through the norm map $\mathrm{Nm}_{k_D/\mathbf{F}_{q^a}}$. From this we know that any character must take the form given in the lemma. $\square$

Corollary 3. *Suppose $a|a'$ are divisors of d. Then*

$$\mathrm{Res}^{D_a^\times}_{D_{a'}^\times} \chi_{a,\alpha,m} = \chi_{a',\alpha^{a'/a},m'}$$

where $m' = \frac{q^{a'}-1}{q^a-1}m$, which implies that $\mathrm{Nm}_{k_D/\mathbf{F}_{q^a}}(x)^m = \mathrm{Nm}_{k_D/\mathbf{F}_{q^{a'}}}(x)^{m'}$ for all $x \in k_D^\times$.

In particular, we see that we can obtain all characters $\chi_{a',\alpha',0}$ from restrictions of characters $\chi_{1,\alpha,0} = \kappa \circ \mathrm{Nrd}$ of $D^\times$ as we may choose α to be an ath root of α' (recall d is assumed to be coprime to p in any case). We will now show that all irreducible representations can be obtained from inductions of characters of $D_a^\times$.

In the following theorem, a character χ of $k_D^\times$ is said to be of order a if a is the minimal integer such that $\chi(x) = \chi(\sigma^a(x))$ for all $x \in k_D^\times$. Such a character can be inflated along the quotient map $\mathcal{O}_D^\times \to k_D^\times$, and then extended trivially to $F^\times \mathcal{O}_D^\times$. By assigning the value of ϖ_D^a to be 1, it is then extended to $D_a^\times$.

Theorem 4 ([6, Prop. 1.3.1])**.** *The smooth irreducible mod p representations V of $D^\times$ are given by*

$$V_{\chi,\kappa} := \mathrm{Ind}^{D^\times}_{D_a^\times}(\chi \otimes \kappa).$$

Here, κ is a character of the type $\chi_{a,\alpha,0}$. The character χ is extended from an order a character of $k_D^\times$.

We have $V_{\chi,\kappa} \simeq V_{\chi',\kappa'}$ if and only if $\kappa = \kappa'$ and $\chi = (\chi')^s$ for some $s \in D^\times$.

Proof. Let V be an irreducible representation of $D^\times$ over $\overline{\mathbf{F}}_p$. Then, again by Lemma 1, we know V^{I_1} is a nonzero subrepresentation because I_1 is

pro-p and normal, so $V^{I_1} = V$. It follows that irreducible representations of $D^\times$ are in bijection with those of

$$D^\times / I_1 \simeq \varpi_D^{\mathbf{Z}} \ltimes k_D^\times.$$

The irreducible representations of this group over $\overline{\mathbf{F}}_p$ are well-known, as it is a semidirect product of $\varpi_D^{\mathbf{Z}} \simeq \mathbf{Z}$ and a finite abelian group with order prime to p. This is shown in [11, §8.2], and the irreducibles match up to those given in the theorem statement. $\qquad\square$

3. Reduction to cohomology

We will begin by explaining how to reduce the computation of the groups $\operatorname{Ext}_{D^\times}^n(\pi, \pi')$ to extensions between characters, after which we can reduce that problem to computation of certain cohomology groups. There are two main facts used in this reduction: first, that π and π' are induced from characters. Secondly, the inductions are from *finite index* subgroups, so Frobenius reciprocity is a two-sided adjunction.

Lemma 5. *For a locally profinite group G, let $H \leq G$ be a closed subgroup such that $[G : H] < \infty$, and let V and W be smooth representations of the groups H and G respectively. Then we have*

$$\operatorname{Ext}_G^n(\operatorname{Ind}_H^G V, W) \simeq \operatorname{Ext}_H^n(V, \operatorname{Res}_H^G W).$$

This also holds in the other direction.

Proof. This is a well-known fact from category theory: Frobenius reciprocity carries on to the derived functors of Hom because we have an adjunction both ways, since for finite index subgroups compact induction agrees with induction. $\qquad\square$

Now we can use this to reduce our problem to computing extensions of characters. Consider irreducible representations $\pi = \operatorname{Ind}_{D_a^\times}^{D^\times} \chi$ and $\pi' = \operatorname{Ind}_{D_{a'}^\times}^{D^\times} \chi'$. Here, we have absorbed the κ component in the notation of Theorem 4 into χ, so that χ and χ' are more general characters of $D_a^\times$ and $D_{a'}^\times$ rather than characters extended from $k_D^\times$. We would like to compute the dimension of $\operatorname{Ext}_{D^\times}^n(\pi, \pi')$.

Theorem 6. *Let χ, χ', π and π' be as given above. Then*

$$\operatorname{Ext}_{D^\times}^n(\pi, \pi') \simeq \bigoplus_{\bar{s} \in D_{a'}^\times \backslash D^\times / D_a^\times} \operatorname{Ext}_{D_{\operatorname{lcm}(a,a')}^\times}^n (\mathbf{1}, (\operatorname{Res}_{D_{\operatorname{lcm}(a,a')}^\times}^{D_{a'}^\times} \chi') \otimes (\chi^s)^*),$$

where s is a coset representative of the double coset $\bar{s}$. The characters χ^s on $D_{\operatorname{lcm}(a,a')}^\times$ are defined as $\chi^s(x) = \chi(s^{-1}xs)$, and so are conjugated restrictions of χ.

Proof. We first apply Frobenius reciprocity, the first time on the right induced representation. By the previous lemma, we may apply it for Ext^n as the subgroups $D_a^\times$ and $D_{a'}^\times$ have finite index in $D^\times$. We have

$$\mathrm{Ext}^n_{D^\times}(\pi, \pi') = \mathrm{Ext}^n_{D^\times}\left(\mathrm{Ind}^{D^\times}_{D_a^\times} \chi, \mathrm{Ind}^{D^\times}_{D_{a'}^\times} \chi'\right) \simeq \mathrm{Ext}^n_{D_{a'}^\times}\left(\mathrm{Res}^{D^\times}_{D_{a'}^\times} \mathrm{Ind}^{D^\times}_{D_a^\times} \chi, \chi'\right).$$

By the Mackey formula, we have

$$\mathrm{Res}^{D^\times}_{D_{a'}^\times} \mathrm{Ind}^{D^\times}_{D_a^\times} \chi \simeq \bigoplus_{\bar{s} \in D_{a'}^\times \backslash D^\times / D_a^\times} \mathrm{Ind}^{D_{a'}^\times}_{sD_a^\times s^{-1} \cap D_{a'}^\times} \chi^s,$$

where s is a representative of the double coset $\bar{s}$. Because $D_a^\times$ is normal, the subgroup we induce from is simply $D_a^\times \cap D_{a'}^\times = D_{\mathrm{lcm}(a,a')}^\times$.

We now pull out the direct sum and apply Frobenius reciprocity on the other side. We then obtain an isomorphism between

$$\mathrm{Ext}^n_{D^\times}(\pi, \pi') \simeq \bigoplus_{\bar{s} \in D_{a'}^\times \backslash D^\times / D_a^\times} \mathrm{Ext}^n_{D_{a'}^\times}\left(\mathrm{Ind}^{D_{a'}^\times}_{D_{\mathrm{lcm}(a,a')}^\times} \chi^s, \chi'\right)$$

and

$$\bigoplus_{\bar{s} \in D_{a'}^\times \backslash D^\times / D_a^\times} \mathrm{Ext}^n_{D_{\mathrm{lcm}(a,a')}^\times}\left(\chi^s, \mathrm{Res}^{D_{a'}^\times}_{D_{\mathrm{lcm}(a,a')}^\times} \chi'\right).$$

The result then follows after tensoring with $(\chi^s)^*$ on each individual extension group. $\qquad\square$

Thus, it suffices to be able to compute $\mathrm{Ext}^n_{D_a^\times}(\mathbf{1}, \chi)$ for any character χ of $D_a^\times$.

We can now employ all the tools of group cohomology to solve our problem. It is tempting to try to use the Hochschild–Serre spectral sequence and identify $\mathrm{Ext}^n_{D_a^\times}(\mathbf{1}, \chi)$ with $\mathrm{H}^n(D_a^\times, \chi)$, but this does not quite work. Since we must work with continuous cohomology and smooth representations, and $D_a^\times$ is not a profinite group, we cannot use the Hochschild–Serre spectral sequence in this case. However, we can recover a similar spectral sequence.

Proposition 7. *There is a first quadrant spectral sequence*

$$E_2^{i,j} = \mathrm{H}^i(D_a^\times / I_1, \mathrm{H}^j(I_1, \chi)) \implies \mathrm{Ext}^{i+j}_{D_a^\times}(\mathbf{1}, \chi).$$

Proof. From equation (32) in [7, §9], letting $\mathcal{H} \simeq \overline{\mathbf{F}}_p[D_a^\times / I_1]$ be the Hecke algebra of I_1, we have a spectral sequence

$$E_2^{i,j} = \mathrm{Ext}^i_{\mathcal{H}}(\mathbf{1}, \mathrm{H}^j(I_1, \chi)) \implies \mathrm{Ext}^{i+j}_{D_a^\times}(\mathbf{1}, \chi).$$

On the left we take Ext in the category of all $\mathcal{H}$-modules, and on the right we take it in the category of smooth representations of $D_a^\times$. As constructed in [7], this spectral sequence is used for GL_2. However a subcase applies in our setting, which we will briefly explain.

We have $\mathrm{End}_{D_a^\times}(c\text{-}\mathrm{Ind}_{I_1}^{D_a^\times} \mathbf{1}) = \mathcal{H} \simeq \overline{\mathbf{F}}_p[D_a^\times/I_1]$. Let $\mathrm{Rep}_{D_a^\times}^{I_1}$ be the full subcategory of the category of smooth $D_a^\times$ representations $\mathrm{Rep}_{D_a^\times}$ consisting of representations generated by their I_1-invariants. There is a functor $\mathcal{I} : \mathrm{Rep}_{D_a^\times} \to \mathrm{Mod}_{\mathcal{H}}$ sending $\pi \mapsto \pi^{I_1} = \mathrm{Hom}_{D_a^\times}(c\text{-}\mathrm{Ind}_{I_1}^{D_a^\times} \mathbf{1}, \pi)$. There is also a functor $\mathcal{J} : M \mapsto M \otimes_{\mathcal{H}} (c\text{-}\mathrm{Ind}_{I_1}^{D_a^\times} \mathbf{1})$ going the other way, from $\mathrm{Mod}_{\mathcal{H}}$ to $\mathrm{Rep}_{D_a^\times}$.

The relevant fact needed for the construction of the spectral sequence is that when viewed as functors between $\mathrm{Rep}_{D_a^\times}^{I_1}$ and $\mathrm{Mod}_{\mathcal{H}}$, $\mathcal{I}$ and $\mathcal{J}$ are quasi-inverse to each other. This can be directly verified in this case because I_1 is normal in $D_a^\times$. Indeed, a representation π being generated by its I_1-invariants means any $v \in \pi$ can be written as a sum of elements of the form $d \cdot w$ for $d \in D_a^\times$ and $w \in \pi^{I_1}$. By normality, I_1 will again fix $d \cdot w$, so in fact arbitrary v is also fixed. It follows that $\mathrm{Rep}_{D_a^\times}^{I_1}$ is equivalent to representations with a trivial action of I_1. Using $\mathrm{Mod}_{\mathcal{H}} \simeq \mathrm{Rep}_{D_a^\times/I_1}$, $\mathcal{I}$ equivalently sends $\pi \mapsto \pi^{I_1}$ as a $D_a^\times/I_1$ representation. The action of $D_a^\times/I_1$ on π^{I_1} recovers the $D_a^\times$ action as π already has a trivial I_1 action, which is precisely what the functor $\mathcal{J}$ does. Thus, in our setting the functors are again quasi-inverse and we have the desired spectral sequence.

Now we return to the spectral sequence $E_2^{i,j} = \mathrm{Ext}_{\mathcal{H}}^i(\mathbf{1}, \mathrm{H}^j(I_1, \chi)) \implies \mathrm{Ext}_{D_a^\times}^{i+j}(\mathbf{1}, \chi)$. As the extension group is computed in the category of all $\mathcal{H}$-modules or equivalently $D_a^\times/I_1$-representations, we have

$$E_2^{i,j} = \mathrm{Ext}_{D_a^\times/I_1,\mathrm{all}}^i(\mathbf{1}, \mathrm{H}^j(I_1, \chi)) \implies \mathrm{Ext}_{D_a^\times}^{i+j}(\mathbf{1}, \chi)$$

where the subscript all means we take the category of all representations. Now use that $D_a^\times/I_1$ is discrete, so we can identify $\mathrm{Ext}_{D_a^\times/I_1,\mathrm{all}}^i(\mathbf{1}, \mathrm{H}^j(I_1, \chi))$ with the continuous group cohomology $\mathrm{H}^i(D_a^\times/I_1, \mathrm{H}^j(I_1, \chi))$. This yields the claimed spectral sequence. $\qquad\square$

As the spectral sequence of Proposition 7 is a first quadrant spectral sequence, we may obtain the following five-term exact sequence of low degree terms:

$$0 \longrightarrow \mathrm{H}^1(D_a^\times/I_1, \chi) \longrightarrow \mathrm{Ext}_{D_a^\times}^1(\mathbf{1}, \chi) \longrightarrow (\mathrm{Hom}(I_1, \overline{\mathbf{F}}_p) \otimes \chi)^{D_a^\times/I_1}$$
$$\longrightarrow \mathrm{H}^2(D_a^\times/I_1, \chi) \longrightarrow \mathrm{Ext}_{D_a^\times}^2(\mathbf{1}, \chi)$$

Here, we have expanded the definition of $\mathrm{H}^1(I_1, \chi)^{D_a^\times/I_1}$. Thus, we obtain an exact sequence similar to the inflation-restriction exact sequence we would obtain via the Hochschild–Serre spectral sequence if $D_a^\times$ were a profinite group.
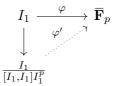
In Corollary 19, we use the same spectral sequence to obtain a description of $\mathrm{Ext}^n_{D_a^\times}(\mathbf{1}, \chi)$ in terms of the cohomology of I_1, so this step in our method works for all degrees of extensions.

4. Cohomology of I_1

Recall that we use I_1 to denote the subgroup $1 + \varpi_D \mathcal{O}_D$ of $D^\times$. A key step in computing all extensions of irreducible representations of $D^\times$ is to understand the space $\mathrm{H}^1(I_1, \pi)$ as a representation of $D_a^\times/I_1$, where π is a smooth irreducible representation of $D_a^\times$. Note that at the end of the previous section we had reduced to the case where π was a character, but the computation is the same if π is an irreducible of any dimension which is what we will assume here.

Since the action of I_1 is trivial on π, we have $\mathrm{H}^1(I_1, \pi) \simeq \mathrm{Hom}(I_1, \overline{\mathbf{F}}_p) \otimes \pi$, where the action of $g \in D_a^\times/I_1$ on $\mathrm{Hom}(I_1, \overline{\mathbf{F}}_p)$ sends a homomorphism φ to $x \mapsto \varphi(g^{-1}xg)$ and we view π as a representation of $D_a^\times/I_1$. Note that we use the normality of I_1 here - in general, this cohomology group is only a module over the Hecke algebra $\mathcal{H}_{I_1}$. Thus, we seek to understand the homomorphism space $\mathrm{Hom}(I_1, \overline{\mathbf{F}}_p)$.

Because the additive group $\overline{\mathbf{F}}_p$ is abelian and every element is p-torsion, any homomorphism $\varphi : I_1 \to \overline{\mathbf{F}}_p$ will factor through the following diagram:

$$
\begin{array}{ccc}
I_1 & \xrightarrow{\;\;\varphi\;\;} & \overline{\mathbf{F}}_p \\
\downarrow & \nearrow{\scriptstyle \varphi'} & \\
\dfrac{I_1}{[I_1, I_1]I_1^p} & &
\end{array}
$$

The homomorphism φ' is unique given φ. The closed subgroup $[I_1, I_1]I_1^p \lhd I_1$ generated by commutators and pth powers is called the Frattini subgroup of I_1.

Hence, we can reduce our problem of computing $\mathrm{Hom}(I_1, \overline{\mathbf{F}}_p)$ to that of computing $\mathrm{Hom}(I_1/[I_1, I_1]I_1^p, \overline{\mathbf{F}}_p)$. We do this by computing the Frattini subgroup itself.

4.1. The Frattini subgroup.

To compute $[I_1, I_1]I_1^p$, we will begin with computing I_1^p. In the p-adic case, this has a simple description.

Proposition 8. *If F is an extension of $\mathbf{Q}_p$, then $I_1^p = 1 + \varpi_D^{de+1}\mathcal{O}_D$ when $p > de + 1$.*

Proof. First we show that $I_1^p \subset 1 + \varpi_D^{de+1}\mathcal{O}_D$. Let $1 + x \in I_1$ for $x \in \varpi_D\mathcal{O}_D$. Then we have $(1 + x)^p = \sum_{0 \leq i \leq p} \binom{p}{i} x^i = 1 + px \sum_{1 \leq i \leq p-1} \left(\binom{p}{i} / p \right) x^{i-1} + x^p$. Because F is a p-adic field, we have $\varpi_D^{de} = \pi_F^e = pu$ for some unit $u \in \mathcal{O}_F^\times$. Thus, because $x \in \varpi_D\mathcal{O}_D$, px is in $\varpi_D^{de+1}\mathcal{O}_D$. Moreover, because $p > de+1$, we have that x^p is in $\varpi_D^{de+1}\mathcal{O}_D$, so $(1+x)^p$ is in $1 + \varpi_D^{de+1}\mathcal{O}_D$ as desired.

Now we show that $I_1^p \supset 1 + \varpi_D^{de+1}\mathcal{O}_D$. Let $1 + y \in 1 + \varpi_D^{de+1}\mathcal{O}_D$ for some $y \in \varpi_D^{de+1}\mathcal{O}_D$. As a formal power series, we know that

$$\left(\sum_{n=0}^{\infty} \binom{1/p}{n} Y^n \right)^p = 1 + Y,$$

so to prove that $1 + y \in I_1^p$ it suffices to show that $\sum_{n=0}^{\infty} \binom{1/p}{n} y^n$ converges in I_1.

Because the $n = 0$ term in this series is 1 and $\varpi_D\mathcal{O}_D$ is closed in D, it suffices to show that $\nu_D \left(\binom{1/p}{n} y^n \right) > 0$ for all $n > 0$ and that $\nu_D \left(\binom{1/p}{n} y^n \right) \to \infty$ as $n \to \infty$. We have the identity

$$\binom{1/p}{n} = \frac{(-1)^n}{n!} \prod_{0 \leq i \leq n-1} \left(\frac{ip-1}{p} \right).$$

Thus, because $\nu_D = \nu_F = e\nu_p$ on $\mathbf{Q}_p$ (where ν_p is the p-adic valuation), we compute

$$\nu_D \left(\binom{1/p}{n} y^n \right) = \nu_D(y^n) - (e\nu_p(n!) + e\nu_p(p^n)) = n\nu_D(y) - e(\nu_p(n!) + n).$$

We also have $\nu_D(\varpi_D) = \frac{1}{d}$, so because $y \in \varpi_D^{de+1}\mathcal{O}_D$, we have $\nu_D(y) \geq \nu_D(\varpi_D^{de+1}) = e\left(1 + \frac{1}{de}\right)$. By Legendre's formula, $\nu_p(n!)$ is bounded above by $\frac{n}{p-1}$. Therefore,

$$\nu_D \left(\binom{1/p}{n} y^n \right) \geq ne\left(1 + \frac{1}{de}\right) - ne\left(\frac{1}{p-1} + 1\right),$$

and because $p - 1 > de$ this will be greater than 0 for $n > 0$ and approach ∞ as $n \to \infty$. $\square$

When F is an extension of $\mathbf{F}_p((t))$, we work in characteristic p, so we have $(1 + x)^p = 1 + x^p$. This implies that

$$I_1^p = 1 + (\varpi_D\mathcal{O}_D)^p.$$

We now turn to computing $[I_1, I_1]$. Consider first the group $1 + \pi_F\mathcal{O}_F$. In the p-adic case, assuming $p > de+1$ ensures this group has no p-torsion, while in the local function field case this fact is a given. This implies that as a topological group we have $1 + \pi_F\mathcal{O}_F$ is isomorphic to $\mathbf{Z}_p^{[F:\mathbf{Q}_p]}$ in the p-adic

case, and $\mathbf{Z}_p^{\mathbf{N}}$ in the local function field case. This leads to the following decomposition, which will be useful here as well as later on.

Lemma 9. *Suppose $p > de + 1$ if $F/\mathbf{Q}_p$ or $\gcd(p, d) = 1$ if $F/\mathbf{F}_p((t))$. We have a decomposition*

$$I_1 = (1 + \pi_F \mathcal{O}_F) \times I_{1, \mathrm{Nrd}=1},$$

where $I_{1,\mathrm{Nrd}=1}$ is the kernel of the reduced norm restricted to I_1.

Proof. Because $1 + \pi_F \mathcal{O}_F$ is a direct product of copies of $\mathbf{Z}_p$ as a multiplicative group, it follows that every element has a unique dth root, which means that $\mathrm{Nrd}\,|_{1+\pi_F\mathcal{O}_F} : x \mapsto x^d$ is surjective. Any element $x \in I_1$ can then be divided by an element of $1+\pi_F\mathcal{O}_F$ to land in $I_{1,\mathrm{Nrd}=1}$, which shows $I_1 = (1 + \pi_F \mathcal{O}_F)I_{1,\mathrm{Nrd}=1}$. These subgroups have trivial intersection, since the only dth root of 1 in $1+\pi_F\mathcal{O}_F$ is 1. Finally, both of these subgroups are normal because $1+\pi_F\mathcal{O}_F$ is central, so we have $I_1 = (1+\pi_F\mathcal{O}_F) \times I_{1,\mathrm{Nrd}=1}$ as desired. $\qquad\square$

As $1 + \pi_F \mathcal{O}_F$ is central in I_1, we then have $[I_1, I_1] = [I_{1,\mathrm{Nrd}=1}, I_{1,\mathrm{Nrd}=1}]$. We compute this using [8, §1.4]. The following lemma is useful when considering the lowest nonzero coefficients of commutators in the subgroup $[I_{1,\mathrm{Nrd}=1}, I_{1,\mathrm{Nrd}=1} \cap I_i]$. This will be used in the theorem that follows.

Lemma 10. *Let $i \geq 0$, and for $y \in k_D$ let $\varphi_{i,y} \in \mathrm{End}_{k_F}(k_D)$ denote the map*

$$\varphi_{i,y} : x \longmapsto \sigma(x)y - x\sigma^i(y).$$

The image of $\varphi_{i,y}$ is the subspace

$$\ker(\mathrm{Tr}_{k_D/k_F}) \cdot \prod_{0 \leq j \leq i} \sigma^j(y),$$

which has codimension one for $y \in k_D^\times$.

Proof. Since σ fixes k_F, the k_F-linearity of this map follows immediately. When $y = 0$, the result is clear, so suppose now that $y \in k_D^\times$.

We first compute the kernel of this map. We have $\varphi_{i,y}(x) = 0$ if and only if $\sigma(x)y = x\sigma^i(y)$. For $x \neq 0$, this is equivalent to $\frac{x}{\sigma(x)} = \frac{y}{\sigma^i(y)}$, which always has a solution x by the multiplicative version of Hilbert's theorem 90 as

$$\mathrm{Nm}_{k_D/k_F}(y/\sigma^i(y)) = 1$$

and σ is a generator of the Galois group. Again since σ is a generator, we have $k_D^{\langle\sigma\rangle} = k_F$, so the kernel is one-dimensional: supposing for nonzero x' we have $\frac{x'}{\sigma(x')} = \frac{x}{\sigma(x)}$, then $\sigma(\frac{x'}{x}) = \frac{x'}{x}$ so the ratio lies in $k_F^\times$. Thus, $\mathrm{im}\,\varphi_{i,y}$ is a codimension one subspace. Noting that

$$\frac{\varphi_{i,y}(x)}{\prod_{0 \leq j \leq i} \sigma^j(y)} = \frac{\sigma(x)}{\prod_{0 < j \leq i} \sigma^j(y)} - \frac{x}{\prod_{0 \leq j < i} \sigma^j(y)},$$

applying σ to the second term on the right hand side yields the first term, so the right hand side is contained in $\ker(\mathrm{Tr}_{k_D/k_F})$ and

$$\mathrm{im}\,\varphi_{i,y} \subset \ker(\mathrm{Tr}_{k_D/k_F}) \cdot \prod_{0 \leq j \leq i} \sigma^j(y).$$

The additive version of Hilbert's theorem 90 says

$$\ker \mathrm{Tr}_{k_D/k_F} = \mathrm{im}(\sigma(x) - x) = \mathrm{im}\,\varphi_{i,1},$$

which was shown to also have codimension one. Hence, we have equality. $\quad\square$

Theorem 11. *We have*

$$[I_1, I_1]I_1^p = I_{1,\mathrm{Nrd}=1}I_1^p \cap I_2,$$

and

$$[I_{1,\mathrm{Nrd}=1}, I_{1,\mathrm{Nrd}=1}]I_{1,\mathrm{Nrd}=1}^p = I_{1,\mathrm{Nrd}=1} \cap I_2 = I_{2,\mathrm{Nrd}=1}.$$

When $F/\mathbf{Q}_p$, we may describe I_1^p as in Proposition 8, and when $F/\mathbf{F}_p\,((t))$ we have $I_1^p = 1 + (\varpi_D \mathcal{O}_D)^p$. Here, we make the same assumptions on p as Lemma 9.

Proof. There is a filtration $I_{i,\mathrm{Nrd}=1} := I_{1,\mathrm{Nrd}=1} \cap I_i$. As previously discussed, by Lemma 9 we have

$$[I_1, I_1] = [I_{1,\mathrm{Nrd}=1}, I_{1,\mathrm{Nrd}=1}].$$

By Theorem 1.9 in [8], for $d > 2$ this commutator subgroup equals $I_{2,\mathrm{Nrd}=1}$. This can be extended to the case of $d = 2$ here because of the restrictions on p. Because $I_1^p \subset I_2$, the first claim of the theorem follows as

$$[I_1, I_1]I_1^p = I_{2,\mathrm{Nrd}=1}I_1^p = I_{1,\mathrm{Nrd}=1}I_1^p \cap I_2.$$

Similarly, the second claim follows by noting that $I_{1,\mathrm{Nrd}=1}^p \subset I_1^p \subset I_2$, hence it is a subgroup of $I_{1,\mathrm{Nrd}=1} \cap I_2 = I_{2,\mathrm{Nrd}=1}$ and so the commutator accounts for the entire Frattini subgroup of $I_{1,\mathrm{Nrd}=1}$.

The argument in [8] is short, so we summarize it here. We show first that under the quotient map $q_i : I_i \to I_i/I_{i+1} \simeq k_D$ we have

$$q_{i+1}([I_{1,\mathrm{Nrd}=1}, I_{i,\mathrm{Nrd}=1}]) = q_{i+1}(I_{i+1,\mathrm{Nrd}=1}).$$

To do this, we first show that the image $q_{i+1}([I_{1,\mathrm{Nrd}=1}, I_{i,\mathrm{Nrd}=1}])$ contains the elements $\varphi_{i,y}(x)$ where $y \in q_1(I_{1,\mathrm{Nrd}=1}) = k_D$ and $x \in q_i(I_{i,\mathrm{Nrd}=1})$, where we have $q_i(I_{i,\mathrm{Nrd}=1}) = k_D$ if $d \nmid i$ and $q_i(I_{i,\mathrm{Nrd}=1}) = \ker(\mathrm{Tr}_{k_D/k_F})$ if $d|i$. Indeed, if $y = 1 + \sum_{n \geq 1}[y_n]\varpi_D^n \in I_{1,\mathrm{Nrd}=1}$ and $x = 1 + \sum_{n \geq i}[x_n]\varpi_D^n \in I_{i,\mathrm{Nrd}=1}$, then we have

$$[y, x] = 1 + ((y - 1)(x - 1) - (x - 1)(y - 1))y^{-1}x^{-1},$$

from which we obtain

$$[y, x] = 1 + ([y_1]\varpi_D[x_i]\varpi_D^i - [x_i]\varpi_D^i[y_1]\varpi_D) + O(\varpi_D^{i+2}),$$

so $q_{i+1}([y, x]) = \varphi_{i,y_1}(x_i)$.

Then Lemma 10 can be used to show that these $\varphi_{i,y}(x)$ generate all of $q_{i+1}(I_{i+1,\mathrm{Nrd}=1})$: when $d|(i + 1)$, we have $\mathrm{im}(\varphi_{i,y}) = \ker(\mathrm{Tr}_{k_D/k_F})$ for any y, so we get all of $q_{i+1}(I_{i+1,\mathrm{Nrd}=1})$; when $d \nmid (i + 1)$, we can show that we generate all of $q_{i+1}(I_{i+1,\mathrm{Nrd}=1}) = k_D$ by first finding $y, y' \in k_D$ so that $\mathrm{im}(\varphi_{i,y}) \neq \mathrm{im}(\varphi_{i,y'})$. Choosing $y = 1$ and y' such that $\prod_{0 \leq j \leq i} \sigma^j(y')$ does not lie in k_F, we can always do this. If, in addition, $d|i$, the values x are restricted to $\ker(\mathrm{Tr}_{k_D/k_F})$ so we need to justify why this still suffices. The image of $\varphi_{i,y}|_{\ker(\mathrm{Tr}_{k_D/k_F})}$ still has codimension one since $k_F = \ker(\varphi_{i,y})$ has trivial intersection with $\ker(\mathrm{Tr}_{k_D/k_F})$ (because p and d are coprime) and so restricted to $\ker(\mathrm{Tr}_{k_D/k_F})$ the map $\varphi_{i,y}$ is an isomorphism. Thus, we have $q_{i+1}([I_{1,\mathrm{Nrd}=1}, I_{i,\mathrm{Nrd}=1}]) = q_{i+1}(I_{i+1,\mathrm{Nrd}=1})$.

It then follows that

$$[I_{1,\mathrm{Nrd}=1}, I_{i,\mathrm{Nrd}=1}]I_{i+2,\mathrm{Nrd}=1} = I_{i+1,\mathrm{Nrd}=1}.$$

Now $[I_{1,\mathrm{Nrd}=1}, I_{1,\mathrm{Nrd}=1}]$ is a non-central normal subgroup of $D_{\mathrm{Nrd}=1}^\times$, and so by Theorem 3.3 in [8] it is open and contains $I_{j,\mathrm{Nrd}=1}$ for some j. Supposing that the minimal such j satisfies $j > 2$, then we have

$$[I_{1,\mathrm{Nrd}=1}, I_{1,\mathrm{Nrd}=1}] \supset [I_{1,\mathrm{Nrd}=1}, I_{j-2,\mathrm{Nrd}=1}]I_{j,\mathrm{Nrd}=1} = I_{j-1,\mathrm{Nrd}=1},$$

which contradicts the fact that j is minimal. Thus, we must have $j \leq 2$. But we also have $[I_{1,\mathrm{Nrd}=1}, I_{1,\mathrm{Nrd}=1}] \subset I_{2,\mathrm{Nrd}=1}$, so we have equality.   □

Remark. In the case of $d = 2$, as shown in [9, §5], the same result about the commutator subgroup of $I_{1,\mathrm{Nrd}=1}$ being $I_{2,\mathrm{Nrd}=1}$ will still hold so long as D is not a dyadic division algebra. This case is ruled out because $\gcd(p, d) = 1$ in any case, so this assumption is important to include.

4.2. A commutator construction. While not needed for computing the group $\mathrm{H}^1(I_1, \pi)$, we can actually show that nearly all elements of $[I_1, I_1]I_1^p$ are products of a single commutator and a pth power, rather than products of many commutators and a pth power as done above. Namely, every element of $[I_1, I_1]I_1^p$ that does not lie in I_3 is a product of a commutator and pth power when $d > 4$.

We first study $q_2([I_1, I_1])$ in detail in Proposition 12 to determine the exact number of ways a given value in k_D can be produced. We use this in the corollary that follows to show there are enough ways to do this that we can inductively choose $x, y \in I_1$ so that $[x, y]$ produces the desired commutator by studying the coefficient of ϖ_D^i in the expansion of $[x, y]$ using Teichmüller representatives.

Proposition 12. *Suppose $d > 4$ and fix $\alpha \in k_D^\times$. There exist $1 + [x]\varpi_D \neq 1$ and $1 + [y]\varpi_D \neq 1$ such that*

$$q_2([1 + [x]\varpi_D, 1 + [y]\varpi_D]) = \alpha,$$

and x/y does not lie in any proper subfield of k_D.

Proof. If we show that there are more than $\frac{q^{d/2+1}-1}{q-1}$ ratios x/y achieve any given value of α, then the claim follows. This is because

$$\frac{q^{d/2+1} - 1}{q - 1} \geq \sum_{a \leq d/2} q^a \geq \left| \bigcup_{a|d, a<d} \mathbf{F}_{q^a} \right|,$$

the number of elements in the union of all proper subfields of k_D. We first show that the number of ratios can be deduced from the number of pairs (x, y).

To have $q_2(1 + [x]\varpi_D, 1 + [y]\varpi_D]) = \alpha$ requires $x\sigma(y) - \sigma(x)y = \alpha$. Viewed as a curve C_α over k_D, this has many symmetries. The $(q + 1)$th roots of unity $\langle \zeta \rangle = \mu_{q+1}(k_D)$ act on C_α. Note that these agree with the $(q^r + 1)$th roots of unity as $(r, d) = 1$, which is why this makes sense.

In particular, given $(x, y) \in C_\alpha(k_D)$ we have $(\zeta x, \zeta y)$ as another solution. If we fix the ratio x/y, then the system

$$x\sigma(y) - \sigma(x)y = \alpha, x/y = \beta$$

gives $x = y\beta$, so we solve $y\beta\sigma(y) - \sigma(y)\sigma(\beta)y = \alpha$. Note that $\beta - \sigma(\beta) \neq 0$ as $\alpha \neq 0$, so if solutions exist there are $|\mu_{q+1}(k_D)|$ of them. Thus, over nonzero values α the number of ratios x/y from $(x, y) \in C_\alpha(k_D)$ is precisely $|C_\alpha(k_D)|/|\mu_{q+1}(k_D)|$, where $|C_\alpha(k_D)|$ denotes the number of points over k_D.

Thus, we now want to show $|C_\alpha(k_D)|$ exceeds $|\mu_{q+1}(k_D)|\frac{q^{d/2+1}-1}{q-1}$. If we are given $x, y \in k_D^\times$, then (x, y) lies on some $C_\alpha(k_D)$ for $\alpha \in k_D$. Via Lemma 10, we may show exactly $|k_D^\times| \cdot |k_F^\times|$ of these yield $\alpha = 0$. Then we conclude

$$\sum_{\alpha \in k_D^\times} |C_\alpha(k_D)| = (|k_D^\times|)^2 - |k_D^\times| \cdot |k_F^\times|.$$

There are two relevant actions on the family of curves C_α:

- For $z \in k_D^\times$, if $(x, y) \in C_\alpha(k_D)$ then $(zx, zy) \in C_{z\sigma(z)\alpha}(k_D)$.
- For $A \in \mathrm{GL}_2(k_F)$, if $(x, y) \in C_\alpha(k_D)$ then $A \cdot (x, y)$ lies in $C_{\det A \cdot \alpha}(k_D)$.

Let d be odd. The first point shows for α the cosets $k_D^\times / k_D^{\times(q+1)}$ we get a common value of $|C_\alpha(k_D)|$ as the image of $z\sigma(z)$ is identical to z^{q+1} (as r is coprime to d). As d is odd, the image of z^{q+1} consists of all the squares. The second item shows that these two common values are the same. Thus, $|C_\alpha(k_D)| = |k_D^\times| - |k_F^\times| = q^d - q$ when $\alpha \in k_D^\times$ because we have produced bijections between $C_\alpha(k_D)$ over all $\alpha \in k_D^\times$.

Now let d be even, and let $\alpha \in k_D$. We claim the curve

$$C_\alpha : x\sigma(y) - \sigma(x)y = \alpha$$

has its k_D points in bijection with

$$X_\alpha : \sigma(x)x + \sigma(y)y = \zeta\alpha.$$

Here, we choose $\zeta \neq 0$ so that $\sigma(\zeta) = -\zeta$.

Explicitly, given $(x, y) \in X_\alpha$ we first choose ω such that $\sigma(\omega)\omega = -1$. Then $(x + \omega y, \omega x + y)$ is a solution to $x\sigma(y) + \sigma(x)y = (\sigma(\omega) + \omega)\zeta\alpha$. We have $(\sigma(\omega) + \omega) \in k_F^\times$. Dividing the x coordinate by ζ, we get a solution to $x\sigma(y) - \sigma(x)y = (\sigma(\omega) + \omega)\alpha$. Using the $\mathrm{GL}_2(k_F)$ action, the solutions are in bijection.

Finally, we observe that points in $X_\alpha(k_D)$ are in bijection with solutions to $x^{q+1} + y^{q+1} = \zeta\alpha$, simply because x^{q+1} and $\sigma(x)x$ take on the same values (as r and d are coprime) and have equal numbers of preimages for each value. The explicit number of solutions to this general type of equation is computed in [15], and will satisfy the desired bound so long as $d > 4$. In particular, the values $|C_\alpha(k_D)|$ take on constant values for each coset of $k_D^\times / k_D^{\times(q+1)}$, with one coset for $\zeta\alpha = -1$ taking the value

$$q^d + 1 + (-q)^{d/2}\frac{q - q^3}{q + 1} - (q + 1)$$

and the others a common value so that the average is $q^d - q$. The coset for $\zeta\alpha = -1$ counts points on $x^{q+1} + y^{q+1} + 1 = 0$, which when projectivized (to add $q + 1$ points) is the Fermat curve

$$x^{q+1} + y^{q+1} + z^{q+1} = 0.$$

The zeta function of this curve is well-known over $\mathbf{F}_{q^2}$ (see [12]). That the remaining cosets take the same value can be seen from the explicit formula of [15]. Thus, we have shown the desired bound on $|C_\alpha(k_D)|$ and the proposition follows. □

The following corollary is analogous to the result at the end of Section 1.4.3 of [8], which states that in $[D^\times, D^\times]$ every element is a product of at most two commutators (this was originally shown in [14]). We will first prove a helpful lemma.

Lemma 13. *Set $x = 1 + \sum_{i \geq 1}[x_i]\varpi_D^i$, and similarly for y.*

Let $de > i > 1$ and assume $x_1, \ldots, x_{i-1}$ and $y_1, \ldots, y_{i-1}$ are fixed in k_D. The ϖ_D^{i+1} coefficient of the commutator $[x, y]$ is a Teichmüller lift of

$$C + \varphi_{i,x_1}(y_i) + \varphi_{i,y_1}(x_i).$$

Here, $C \in k_D$ is a constant only depending on $x_1, \ldots, x_{i-1}$ and $y_1, \ldots, y_{i-1}$.

Proof. To do this calculation, we recall that we have

$$[x] + [y] = [x + y] + O(p).$$

If $F/\mathbf{F}_p\,((t))$, this error is zero, while in the p-adic case note that $p \in \varpi_D^{de}\mathcal{O}_D$ so this error term is quite small.

Write

$$x^{-1} = 1 + \sum_{i \geq 1}(1 - x)^i$$

and the same for y. We will understand $[x, y] = xyx^{-1}y^{-1}$ using these expansions. First, note that the coefficient of ϖ_D^{i+1} in the commutator, via this expansion, only depends on $x_1, \ldots, x_i, x_{i+1}$ and $y_1, \ldots, y_i, y_{i+1}$: this is because the error terms in $[x] + [y] = [x + y] + O(p)$ are small. In particular, if when expanding we have any expression $[x]\varpi_D^i + [y]\varpi_D^i$ then it is $[x+y]\varpi_D^i + O(\varpi_D^i p)$, and hence when calculating $[x, y]$ by expanding when $i > 0$ the error lies in I_{de+1} as $p \in \varpi_D^{de}$, and therefore does not matter (the $i = 0$ case doesn't occur for expansion of the commutator).

Now knowing we can ignore the error terms, we need only consider x^{-1} up to $1 + (1 - x) + (1 - x)^2$ and similarly for y^{-1} as we only want the terms using $[x_i]$ and $[y_i]$. Now observe the contributions from x_{i+1} and y_{i+1} cancel. Further, we can only get a term contributing to the coefficient of ϖ_D^{i+1} depending on x_i or y_i if we combine them with y_1 and x_1 respectively. This gives the claimed formula. $\qquad\square$

Now we are ready to do the commutator construction, using this formula to show we can produce enough commutators.

Corollary 14. *Suppose $d > 4$, and assume $p > de + 1$ in the p-adic case and $\gcd(p, d) = 1$ in the local function field case. Any element $\alpha \in [I_1, I_1]I_1^p$ with $q_2(\alpha) \neq 0$ can be constructed as a product of a commutator and a pth power.*

Proof. We wish to write $\alpha \in [I_1, I_1]I_1^p \backslash I_3$ as $[x, y]z^p$ for $x, y, z \in I_1$. Suppose we have $x = 1 + \sum_{i \geq 1}[x_i]\varpi_D^i$, and similarly for y. We fix x_1 and y_1 to be as in Proposition 12, so $q_2([x, y]) = q_2(\alpha) \in k_D^\times$ and x_1/y_1 does not lie in a proper subfield of k_D. The previous lemma shows that we can fix $x_1, \ldots, x_{i-1}$ and $y_1, \ldots, y_{i-1}$ for any i so that there is $\alpha = [x, y]z^p$ whose coefficient $[\alpha_{i+1}]$ of ϖ_D^{i+1} is

$$C + \varphi_{i, x_1}(y_i) + \varphi_{i, y_1}(x_i).$$

Here, $C \in k_D$ is a constant only depending on $x_1, \ldots, x_{i-1}$ and $y_1, \ldots, y_{i-1}$. We achieve this for all i rather than those in the lemma by absorbing error terms in $I_{de+1} = I_1^p$ into z^p. The idea will be to show that we can modify x_i and y_i in this formula to match coefficients $[\alpha_i]$ in $\alpha = 1 + \sum_{i \geq 1}[\alpha_i]\varpi_D^i$ one by one.

By Lemma 10, we know $\{C + \varphi_{i,x_1}(y_i) + \varphi_{i,y_1}(x_i) : x_i, y_i \in k_D\} \subset k_D$ is a sum of two codimension one subspaces, shifted by C. This is all of k_D if and only if the two subspaces are distinct, which will depend on the choice of x_1 and y_1. By Lemma 10, $\mathrm{im}(\varphi_{i,x_1}) = \mathrm{im}(\varphi_{i,y_1})$ if and only if

$$\prod_{0 \leq j \leq i} \sigma^j(x_1/y_1) \in k_F,$$

since $c \cdot \ker(\mathrm{Tr}_{k_D/k_F}) = \ker(\mathrm{Tr}_{k_D/k_F})$ if and only if $c \in k_F$ due to the non-degeneracy of the trace pairing. A calculation shows that this condition on x_1/y_1 holds if and only if x_1/y_1 is in the degree $\gcd(i+1, d)$ extension of k_F. Because x_1/y_1 does not lie in a proper subfield of k_D, we may obtain any Teichmüller lift of k_D as the coefficient of ϖ_D^{i+1} of $[x,y]z^p$ when $d \nmid (i+1)$. When $d \mid (i+1)$, the values obtained for this coefficient are in the coset

$$C + \mathrm{im}(\varphi_{i,x_1}) = C + \mathrm{im}(\varphi_{i,y_1}),$$

which has codimension one in k_D by Lemma 10.

Thus, we see already that we can obtain many distinct $\alpha \in [I_1, I_1]I_1^p$ as $[x,y]z^p$ by inductively choosing $[x_i]$ and $[y_i]$ to produce the desired coefficient of ϖ_D^{i+1} in the commutator, up to a change in z (given by the error terms in the calculation). This works unless in $\alpha = [x,y]z^p$ the coefficients $[\alpha_{i+1}]$ for $d \mid (i+1)$ lie outside the codimension one subspace we can construct. There is also the constraint that $q_2([x,y]) \in k_D^\times$.

We claim this actually produces all $\alpha \in [I_1, I_1]I_1^p$ with $q_2(\alpha) \neq 0$. We can explicitly count the number of elements in $[I_1, I_1]I_1^p$ modulo I_{de+1} with $q_2(\alpha) \neq 0$. One uses $[I_1, I_1]I_1^p = I_{1,\mathrm{Nrd}=1}I_{de+1} \cap I_2$ to see the result. Modulo I_{de+1}, we can then equivalently count elements in $\alpha \in I_{2,\mathrm{Nrd}=1}$ with $q_2(\alpha) \neq 0$. Using $I_1 = (1 + \pi_F \mathcal{O}_F) \times I_{1,\mathrm{Nrd}=1}$ we see that we get everything in I_2/I_{de+1} with $q_2(\alpha) \neq 0$ with the restriction that coefficients of ϖ_D^j when $d \mid j$ lie in a codimension one subspace (this comes from $1 + \pi_F \mathcal{O}_F$). Our commutator construction then produces exactly the same number of distinct elements in the Frattini subgroup modulo I_{de+1}, so we get all of them.

Since $I_{de+1} = I_1^p$, we have now seen the construction produces representatives for all cosets in $([I_1, I_1]^p I_1^p)/I_1^p$ where q_2 is nonzero. However, because we can vary z^p in $[x,y]z^p$, we actually produce the entire cosets from a representative. We therefore produce all elements of $[I_1, I_1]I_1^p$ where q_2 is nonzero. $\qquad\square$

Following through the same arguments by inspecting the values from Proposition 12 and comparing to $|\mu_{q+1}(k_D)|\frac{q^{d/2+1}-1}{q-1}$, we can get results for $d = 2, 3, 4$ as well. For $d = 2, 3$ we can get the same final result, but for $d = 4$ Proposition 12 does not hold since the exact number of points on $C_\alpha(k_D)$ need not exceed the bound $|\mu_{q+1}(k_D)|\frac{q^{d/2+1}-1}{q-1}$. However, it does hold for $\frac{q}{q+1}|k_D^\times|$ values in $k_D^\times$.

4.3. Computing cohomology. With our understanding of the Frattini subgroup, we can now compute the structure of $\mathrm{H}^1(I_1, \pi)$ as a representation of $D_a^\times / I_1$. Assume $p > de + 1$ in the p-adic case and $\gcd(p, d) = 1$ in the local function field case throughout this section, as we will be applying Theorem 11.

For most of this section, we focus on computing the $D^\times / I_1$-representation structure where π is a smooth irreducible representation of $D^\times$ because, as shown after the proof of Proposition 17, we can obtain the $D_a^\times / I_1$-representation structure from this simply by restricting.

First we review why we have a $D^\times / I_1$ representation structure on the group $\mathrm{H}^1(I_1, \pi)$ for an irreducible smooth representation π. Being the pro-p Iwahori subgroup of $D^\times$, there is the structure of a module over the Hecke algebra $\mathcal{H}_{I_1} \simeq \overline{\mathbf{F}}_p[I_1 \backslash D^\times / I_1]$ on π^{I_1}. Since I_1 is normal, this has the structure of a representation of $D^\times / I_1$. The derived functors of this are the continuous cohomology groups $\mathrm{H}^i(I_1, \pi)$, hence there is a structure of a representation of $D^\times / I_1$ on $\mathrm{H}^1(I_1, \pi)$. As $\mathrm{H}^1(I_1, \pi) \simeq \mathrm{H}^1(I_1, \overline{\mathbf{F}}_p) \otimes \pi$, we will only need to compute this structure for the trivial representation. For $\mathrm{H}^1(I_1, \overline{\mathbf{F}}_p)$, the structure as a representation can be described very explicitly as the conjugation action of $D^\times / I_1$ on homomorphisms. Namely, $g \in D^\times / I_1$ sends a homomorphism $\varphi(x)$ to $\varphi(g^{-1} x g)$.

Theorem 15. *We have a decomposition*

$$\mathrm{H}^1(I_1, \overline{\mathbf{F}}_p) \simeq \mathrm{H}^1(I_{1,\mathrm{Nrd}=1}, \overline{\mathbf{F}}_p) \oplus \mathrm{H}^1(1 + \pi_F \mathcal{O}_F, \overline{\mathbf{F}}_p),$$

where $\mathrm{H}^1(I_{1,\mathrm{Nrd}=1}, \overline{\mathbf{F}}_p) \simeq \mathrm{Hom}(k_D, \overline{\mathbf{F}}_p) \simeq \overline{\mathbf{F}}_p^{df}$ *and* $\mathrm{H}^1(1 + \pi_F \mathcal{O}_F, \overline{\mathbf{F}}_p)$ *is isomorphic to* $\overline{\mathbf{F}}_p^{ef}$ *in the p-adic case and* $\bigoplus_{i \in \mathbf{N}} \overline{\mathbf{F}}_p$ *in the local function field case.*

Proof. Due to Lemma 9, the decomposition into these cohomology groups follows immediately. This is for the first cohomology group so we do not need more advanced methods, but it is worth noting this is a subcase of the Künneth theorem for profinite groups.

By Theorem 11, the Frattini subgroup of $I_{1,\mathrm{Nrd}=1}$ is $I_{1,\mathrm{Nrd}=1} \cap I_2 = I_{2,\mathrm{Nrd}=1}$. We then have

$$\mathrm{H}^1(I_{1,\mathrm{Nrd}=1}, \overline{\mathbf{F}}_p) \simeq \mathrm{Hom}(I_{1,\mathrm{Nrd}=1}/I_{2,\mathrm{Nrd}=1}, \overline{\mathbf{F}}_p).$$

The quotient $I_{1,\mathrm{Nrd}=1}/I_{2,\mathrm{Nrd}=1}$ is isomorphic to k_D: we know it is a finite dimensional $\mathbf{F}_p$-vector space since it is finite, abelian, and p-torsion. In the decomposition of Lemma 9, upon taking the elements $1 + [x]\varpi_D$ over $x \in k_D$, the $I_{1,\mathrm{Nrd}=1}$ components yield coset representatives of $I_{1,\mathrm{Nrd}=1}/I_{2,\mathrm{Nrd}=1} \subset k_D$ for any element of k_D. This is because $d \geq 2$, so division by an element of $1 + \pi_F \mathcal{O}_F$ cannot affect the coefficient of ϖ_D. Thus, we have

$$\mathrm{H}^1(I_{1,\mathrm{Nrd}=1}, \overline{\mathbf{F}}_p) \simeq \mathrm{Hom}(k_D, \overline{\mathbf{F}}_p),$$

where this last space is isomorphic to $\overline{\mathbf{F}}_p^{df}$ because the additive group of k_D is $\mathbf{F}_p^{df}$. Now we turn to $1 + \pi_F \mathcal{O}_F$, which is isomorphic to $\mathbf{Z}_p^{[F:\mathbf{Q}_p]}$ in the p-adic case, and $\mathbf{Z}_p^{\mathbf{N}}$ in the local function field case. As $\mathrm{Hom}_{\mathrm{cont}}(\mathbf{Z}_p, \overline{\mathbf{F}}_p) \simeq \overline{\mathbf{F}}_p$, in the p-adic case the claim follows. In the local function field case

$$\mathrm{Hom}_{\mathrm{cont}}(1 + \pi_F \mathcal{O}_F, \overline{\mathbf{F}}_p) \simeq \mathrm{Hom}_{\mathrm{cont}}(\mathbf{Z}_p^{\mathbf{N}}, \overline{\mathbf{F}}_p) \simeq \bigoplus_{i \in \mathbf{N}} \mathrm{Hom}_{\mathrm{cont}}(\mathbf{Z}_p, \overline{\mathbf{F}}_p),$$

which is what we wanted.   □

For $0 \le i < df$ and $\eta_i := (x \mapsto x^{p^i}) \in \mathrm{Aut}_{\mathbf{F}_p}(k_D)$, define the homomorphisms $\phi^{\eta_i} : I_1 \to \overline{\mathbf{F}}_p$ by the compositions

$$I_1 \longrightarrow I_1/I_2 \xrightarrow{\simeq} k_D \xrightarrow{\eta_i} k_D \longrightarrow \overline{\mathbf{F}}_p$$

where the last map is the inclusion that we have fixed.

For homomorphisms $\eta : 1 + \pi_F \mathcal{O}_F \to \overline{\mathbf{F}}_p$, define homomorphisms $\psi^\eta : I_1 \to \overline{\mathbf{F}}_p$ by

$$I_1 \xrightarrow{\mathrm{Nrd}} 1 + \pi_F \mathcal{O}_F \xrightarrow{\eta} \overline{\mathbf{F}}_p.$$

Corollary 16. *The maps ϕ^{η_i} form a basis of the $\overline{\mathbf{F}}_p$-vector space*

$$\mathrm{H}^1(I_{1,\mathrm{Nrd}=1}, \overline{\mathbf{F}}_p) \subset \mathrm{H}^1(I_1, \overline{\mathbf{F}}_p).$$

Taking a basis $\{\eta_j\}$ of $\mathrm{Hom}_{\mathrm{cont}}(1 + \pi_F \mathcal{O}_F, \overline{\mathbf{F}}_p)$, the maps $\{\psi^{\eta_j}\}$ form a basis of $\mathrm{H}^1(1 + \pi_F \mathcal{O}_F, \overline{\mathbf{F}}_p) \subset \mathrm{H}^1(I_1, \overline{\mathbf{F}}_p)$.

Proof. We note that $\phi^{\eta_i}|_{1+\pi_F \mathcal{O}_F} = 0$ and that $\psi^{\eta_j}|_{I_{1,\mathrm{Nrd}=1}} = 0$, so the proposed basis elements lie in the correct components of the decomposition in Theorem 15, and we would like to show that they indeed form bases of these components.

We do this for the ϕ^{η_i} first. We already computed the dimension of $\mathrm{H}^1(I_{1,\mathrm{Nrd}=1}, \overline{\mathbf{F}}_p) \subset \mathrm{H}^1(I_1, \overline{\mathbf{F}}_p)$, so it suffices to show that the ϕ^{η_i} are linearly independent. By surjectivity of the quotient map $I_1 \to I_1/I_2$, it suffices to show that the $\eta_i \in \mathrm{Hom}(k_D, \overline{\mathbf{F}}_p)$ are linearly independent. Viewing these as polynomials $x \mapsto x^{p^i} \in \overline{\mathbf{F}}_p[x]$, any linear combination that is equal to zero corresponds to a degree $< p^{df}$ polynomial having p^{df} roots, which means the coefficients must all be zero. It follows that these homomorphisms are linearly independent.

Now consider the ψ^{η_j}. The reduced norm Nrd restricts to $x \mapsto x^d$ on F, so as in Lemma 9 we see $\mathrm{Nrd} : I_1 \to 1 + \pi_F \mathcal{O}_F$ is a surjection since $1 + \pi_F \mathcal{O}_F \subset I_1$. The maps η_j are a basis of $\mathrm{Hom}_{\mathrm{cont}}(1 + \pi_F \mathcal{O}_F, \overline{\mathbf{F}}_p) = \mathrm{H}^1(1 + \pi_F \mathcal{O}_F, \overline{\mathbf{F}}_p)$, and so by surjectivity of the reduced norm the maps ψ^{η_j} form a basis of $\mathrm{H}^1(1 + \pi_F \mathcal{O}_F, \overline{\mathbf{F}}_p) \subset \mathrm{H}^1(I_1, \overline{\mathbf{F}}_p)$.   □

We may now compute the structure of $\mathrm{H}^1(I_1, \overline{\mathbf{F}}_p)$ as a representation of $D^\times/I_1$.

Proposition 17. *As a representation of $D^\times/I_1$ via the conjugation action, we have*

$$\mathrm{H}^1(I_1, \overline{\mathbf{F}}_p) \simeq \mathrm{H}^1(1 + \pi_F \mathcal{O}_F, \overline{\mathbf{F}}_p) \oplus \bigoplus_{i \in \mathbf{Z}/f\mathbf{Z}} \mathrm{Ind}_{D_d^\times/I_1}^{D^\times/I_1} \chi_{\eta_i}$$

where the action on $\mathrm{H}^1(1 + \pi_F \mathcal{O}_F, \overline{\mathbf{F}}_p)$ is trivial and χ_{η_i} is extended trivially from the order d character $k_D^\times \to \overline{\mathbf{F}}_p^\times$ given by $x \mapsto \left(\frac{\sigma(x)}{x}\right)^{p^i}$. Up to isomorphism, the choice of a coset representative of i does not matter.

Proof. By Corollary 16, we see that every map contained in the component $\mathrm{H}^1(1 + \pi_F \mathcal{O}_F, \overline{\mathbf{F}}_p) \subset \mathrm{H}^1(I_1, \overline{\mathbf{F}}_p)$ factors through Nrd. As a result, the conjugation action of $D^\times/I_1$ is trivial on this component and it is a direct sum of copies of the trivial representation.

We now turn to the action on $\mathrm{H}^1(I_{1,\mathrm{Nrd}=1}, \overline{\mathbf{F}}_p)$. To do this, we will use the basis ϕ^{η_i} provided by Corollary 16. As $D^\times/I_1 \simeq \varpi_D^{\mathbf{Z}} \ltimes k_D^\times$, it suffices to study the $k_D^\times$ and $\varpi_D^{\mathbf{Z}}$ actions. For any $y = 1 + \sum_{n>0}[y_n]\varpi_D^n \in I_1$, as we are given that $\varpi_D[y_n]\varpi_D^{-1} = [\sigma(y_n)]$ and σ sends $y_n \mapsto y_n^{q^r}$, we can compute that

$$\varpi_D \cdot \phi^{\eta_i}(y) = \phi^{\eta_i}\left(1 + \sum_{n>0} \varpi_D^{-1}[y_n]\varpi_D^{n+1}\right)$$

$$= \eta_i(\sigma^{-1}(y_1)) = y_1^{p^{i-rf}} = \phi^{\eta_{i-rf}}(y),$$

hence the conjugation action of ϖ_D is by

$$\varpi_D \cdot \phi^{\eta_i} = \phi^{\eta_{i-rf}}.$$

Similarly, for any y as above and any $x \in k_D^\times$, we have

$$[x] \cdot \phi^{\eta_i}(y) = \phi^{\eta_i}\left(1 + \sum_{n>0} [x^{-1}y_n]\varpi_D^n[x]\right)$$

$$= \eta_i(x^{-1}y_1\sigma(x)) = \left(\frac{\sigma(x)}{x}\right)^{p^i} \phi^{\eta_i}(y),$$

from which we obtain $[x] \cdot \phi^{\eta_i} = \left(\frac{\sigma(x)}{x}\right)^{p^i} \phi^{\eta_i}$. Defining V_i to be the span of the $\phi^{\eta_{i'}}$ where $i' \equiv i \pmod{f}$, each is a subrepresentation so we obtain

$$\mathrm{H}^1(I_{1,\mathrm{Nrd}=1}, \overline{\mathbf{F}}_p) = \bigoplus_{i \in \mathbf{Z}/f\mathbf{Z}} V_i.$$

Each V_i is a dimension d representation. We have

$$\mathrm{Res}_{D_d^\times/I_1}^{D^\times/I_1} V_i \simeq \bigoplus_{i' \equiv i \,(\mathrm{mod}f)} \chi_{\eta_{i'}}$$

where we define $\chi_{\eta_{i'}}$ for $i' \in \mathbf{Z}/(df)\mathbf{Z}$ as sending $x \in k_D^{\times}$ to $\left(\frac{\sigma(x)}{x}\right)^{p^{i'}}$ and extend trivially to $D_d^{\times}/I_1$. It follows that

$$\mathrm{Hom}_{D_d^{\times}/I_1}\left(\chi_{\eta_i}, \mathrm{Res}_{D_d^{\times}/I_1}^{D^{\times}/I_1} V_i\right) \neq 0,$$

which by Frobenius reciprocity means that $\mathrm{Hom}_{D^{\times}/I_1}\left(\mathrm{Ind}_{D_d^{\times}/I_1}^{D^{\times}/I_1} \chi_{\eta_i}, V_i\right) \neq 0$. However, by the classification of irreducible representations, the induced representation is irreducible of dimension d. As $\dim V_i = d$, it follows that V_i is isomorphic to $\mathrm{Ind}_{D_d^{\times}/I_1}^{D^{\times}/I_1} \chi_{\eta_i}$. $\qquad\square$

For any smooth irreducible representation π of $D^{\times}$, this immediately gives us

$$\mathrm{H}^1(I_1, \pi) \simeq (\mathrm{H}^1(1 + \pi_F \mathcal{O}_F, \overline{\mathbf{F}}_p) \otimes \pi) \oplus \bigoplus_{i \in \mathbf{Z}/f\mathbf{Z}} \mathrm{Ind}_{D_d^{\times}/I_1}^{D^{\times}/I_1}\left(\chi_{\eta_i} \otimes \mathrm{Res}_{D_d^{\times}/I_1}^{D^{\times}/I_1} \pi\right).$$

We have already determined how to compute the restriction of a character of $D^{\times}$ in general in Corollary 3, so when π is a character it is easy to write this down explicitly.

We can now derive the $D_a^{\times}$-representation structure for other values of $a|d$. For the trivial representation, this is

$$\mathrm{Res}_{D_a^{\times}/I_1}^{D^{\times}/I_1} \mathrm{H}^1(I_1, \overline{\mathbf{F}}_p) \simeq \mathrm{H}^1(1 + \pi_F \mathcal{O}_F, \overline{\mathbf{F}}_p) \oplus \bigoplus_{i \in \mathbf{Z}/f\mathbf{Z}} \mathrm{Res}_{D_a^{\times}/I_1}^{D^{\times}/I_1} \mathrm{Ind}_{D_d^{\times}/I_1}^{D^{\times}/I_1} \chi_{\eta_i}.$$

The Mackey formula decomposes each term in the direct sum as

$$\mathrm{Res}_{D_a^{\times}/I_1}^{D^{\times}/I_1} \mathrm{Ind}_{D_d^{\times}/I_1}^{D^{\times}/I_1} \chi_{\eta_i} \simeq \bigoplus_{\bar{s} \in D_a^{\times} \backslash D^{\times}/D_d^{\times}} \mathrm{Ind}_{D_d^{\times}/I_1}^{D_a^{\times}/I_1} \chi_{\eta_i}^s.$$

We can similarly compute the tensor product with a smooth irreducible representation π of $D_a^{\times}$.

Remark. In [3], for a split reductive p-adic group G and for a pro-p Iwahori subgroup I_1 of G, the group $\mathrm{H}^1(I_1, \overline{\mathbf{F}}_p)$ is computed as a module over the Hecke algebra $\mathcal{H}_{I_1}$. Our result is somewhat analogous to Theorem 6.4 in that paper in how $\mathrm{H}_1(I_1, \overline{\mathbf{F}}_p)$ splits into two components, but the fact that $D^{\times}$ is anisotropic modulo its center makes its structure far more interesting than that of $\mathrm{GL}_1(F)$. Notably, our decomposition as a module over $\mathcal{H}_{I_1}$ similarly does not involve supersingular modules, but computing the same cohomology group for $\mathrm{GL}_2(D)$ using Theorem 11 does actually produce supersingular modules if we assume $d > 1$, as would be expected from the main theorem of [3].

5. Extensions of irreducibles

Having computed the $D_a^\times/I_1$-representation structure of $\mathrm{H}^1(I_1, \pi)$, we are now ready to use this to compute extensions.

5.1. $\mathrm{Ext}^1_{D^\times}$ for all irreducible representations. Let χ be any smooth character of $D_a^\times$. To use the exact sequence of low degree terms at the end of Section 3 to compute $\mathrm{Ext}^1_{D_a^\times}(\mathbf{1}, \chi)$, and ultimately to compute $\mathrm{Ext}^1_{D^\times}(\pi, \pi')$ for any π and π', we will also need to compute the structure of $\mathrm{H}^i(D_a^\times/I_1, \chi)$.

Proposition 18. *We have* $\mathrm{H}^i(D_a^\times/I_1, \chi) = 0$ *unless the action by* $\chi : D_a^\times \to \overline{\mathbf{F}}_p^\times$ *is trivial, in which case* $\mathrm{H}^1(D_a^\times/I_1, \chi) \simeq \overline{\mathbf{F}}_p$ *and* $\mathrm{H}^i(D_a^\times/I_1, \chi) = 0$ *for* $i > 1$.

Proof. Because we have a semidirect product $D_a^\times/I_1 \simeq \varpi_D^{a\mathbf{Z}} \ltimes k_D^\times$, we have a normal subgroup $k_D^\times$ and a quotient $\varpi_D^{a\mathbf{Z}}$ making an exact sequence

$$0 \longrightarrow k_D^\times \longrightarrow D_a^\times/I_1 \longrightarrow \varpi_D^{a\mathbf{Z}} \longrightarrow 0.$$

The group in question is discrete, and hence without requiring cochains to be continuous the cohomology is the same as taking all cochains. We may therefore use the Hochschild–Serre spectral sequence in ordinary cohomology.

We first show $\mathrm{H}^i(k_D^\times, \chi) = 0$ for any χ and $i > 0$. Observe that the composition

$$\mathrm{H}^i(k_D^\times, \chi) \xrightarrow{\mathrm{res}} \mathrm{H}^i(1, \chi) \xrightarrow{\mathrm{cores}} \mathrm{H}^i(k_D^\times, \chi)$$

is multiplication by $|k_D^\times|$, and hence an isomorphism as we work in characteristic p. But the middle group is 0, so this is also the zero map. It follows the cohomology group $\mathrm{H}^i(k_D^\times, \chi)$ is trivial. One can also see this by observing that $\widehat{\mathrm{H}}^0(k_D^\times, \chi)$ and $\widehat{\mathrm{H}}_0(k_D^\times, \chi)$ are both 0 since $|k_D^\times|$ is prime to p, and then using Tate periodicity.

Because $\mathrm{H}^i(k_D^\times, \chi) = 0$ is trivial for $i > 0$, we have $\mathrm{H}^i(\varpi_D^{a\mathbf{Z}}, \chi^{k_D^\times}) \simeq \mathrm{H}^i(\varpi_D^{a\mathbf{Z}} \ltimes k_D^\times, \chi)$ via the inflation map in the higher inflation-restriction exact sequences arising from the Hochschild–Serre spectral sequence. The cohomological dimension of $\varpi_D^{a\mathbf{Z}} \simeq \mathbf{Z}$ is 1, so for $i > 1$ we have now shown the claim. For $i = 1$, $\mathrm{H}^1(\varpi_D^{a\mathbf{Z}}, \chi^{k_D^\times}) = 0$ unless $\chi^{k_D^\times}$ is trivial, in which case we get $\overline{\mathbf{F}}_p$. We only get $\chi^{k_D^\times}$ as the trivial $\varpi_D^{a\mathbf{Z}}$-character when χ itself is trivial. $\qquad\square$

With this, we can now determine $\mathrm{Ext}^1_{D_a^\times}(\mathbf{1}, \chi)$ via the spectral sequence Proposition 7. We also obtain the following corollary:

Corollary 19. *There is an exact sequence*

$$0 \longrightarrow \mathrm{H}^1(D_a^\times/I_1, \mathrm{H}^{i-1}(I_1, \chi)) \longrightarrow \mathrm{Ext}^i_{D_a^\times}(\mathbf{1}, \chi) \longrightarrow \mathrm{H}^i(I_1, \chi)^{D_a^\times/I_1} \longrightarrow 0.$$

Proof. Proposition 18 shows that $\mathrm{H}^i(D_a^\times/I_1, V) = 0$ for $i \geq 2$ for any representation V of $D_a^\times/I_1$. Indeed, the previous argument can be adapted directly for $i \geq 2$ because $\mathbf{Z}$ still has cohomological dimension 1 and $\mathrm{Rep}(k_D^\times)$ is semisimple as $|k_D^\times|$ is prime to p, and so we can decompose $V = \bigoplus_j \chi_j$ as a representation of $k_D^\times$ to get $\mathrm{H}^i(k_D^\times, V) = \bigoplus_j \mathrm{H}^i(k_D^\times, \chi_j) = 0$. We have a spectral sequence

$$E_2^{i,j} = \mathrm{H}^i(D_a^\times/I_1, \mathrm{H}^j(I_1, \chi)) \implies \mathrm{Ext}_{D_a^\times}^{i+j}(\mathbf{1}, \chi)$$

via Proposition 7 which we now know consists of two columns on the E_2 page, from which we get the above exact sequence as the E_2 page equals the E_∞ page. □

We now compute $\mathrm{Ext}_{D_a^\times}^1(\mathbf{1}, \chi)$.

Theorem 20. *Let D be a degree d division algebra over F where $p > de+1$ in the p-adic case, and $\gcd(p, d) = 1$ in the local function field case. Let χ be a character of $D_a^\times$ where $a|d$. There are two cases where the extensions $\mathrm{Ext}_{D_a^\times}^1(\mathbf{1}, \chi)$ can be nontrivial:*

- *When χ is trivial, there is an exact sequence*

$$0 \longrightarrow \overline{\mathbf{F}}_p \longrightarrow \mathrm{Ext}_{D_a^\times}^1(\mathbf{1}, \chi) \longrightarrow \mathrm{H}^1(1 + \pi_F\mathcal{O}_F, \overline{\mathbf{F}}_p) \longrightarrow 0$$

  where $\mathrm{H}^1(1 + \pi_F\mathcal{O}_F, \overline{\mathbf{F}}_p) \simeq \mathrm{Hom}(I_1, \overline{\mathbf{F}}_p)^{D_a^\times/I_1}$ is as in Theorem 15.
- *When $a = d$, and χ is extended trivially from a character $x \mapsto \left(\frac{x}{\sigma(x)}\right)^{p^i}$ of $k_D^\times$ for some i, we have $\mathrm{Ext}_{D_a^\times}^1(\mathbf{1}, \chi) \simeq \overline{\mathbf{F}}_p$.*

Otherwise, $\mathrm{Ext}_{D_a^\times}^1(\mathbf{1}, \chi) = 0$.

Proof. Due to Proposition 18, in the exact sequence of low degree terms arising from Proposition 7, we obtain

$$0 \longrightarrow \mathrm{H}^1(D_a^\times/I_1, \chi) \longrightarrow \mathrm{Ext}_{D_a^\times}^1(\mathbf{1}, \chi) \longrightarrow (\mathrm{Hom}(I_1, \overline{\mathbf{F}}_p) \otimes \chi)^{D_a^\times/I_1} \longrightarrow 0.$$

When $\chi = \mathbf{1}$, by Proposition 17 we see that

$$\mathrm{Hom}(I_1, \overline{\mathbf{F}}_p)^{D_a^\times/I_1} \simeq \mathrm{H}^1(1 + \pi_F\mathcal{O}_F, \overline{\mathbf{F}}_p)$$

since this is the trivial component of the representation. Additionally, Proposition 18 tells us that $\mathrm{H}^1(D_a^\times/I_1, \overline{\mathbf{F}}_p) \simeq \overline{\mathbf{F}}_p$. We then recover the first case of the theorem statement.

When χ is nontrivial, $\mathrm{H}^1(D_a^\times/I_1, \chi) = 0$ so we have $\mathrm{Ext}_{D_a^\times}^1(\mathbf{1}, \chi) \simeq (\mathrm{Hom}(I_1, \overline{\mathbf{F}}_p) \otimes \chi)^{D_a^\times/I_1}$ via the restriction map. We know $\mathrm{Hom}(I_1, \overline{\mathbf{F}}_p) \otimes \chi \simeq \mathrm{H}^1(I_1, \chi)$ as a $D_a^\times/I_1$-representation, so we just need to compute the trivial component.

Recall that as a $D_a^\times/I_1$-representation, we have already shown

$$\mathrm{H}^1(I_1, \chi) \simeq (\mathrm{H}^1(1 + \pi_F \mathcal{O}_F, \overline{\mathbf{F}}_p) \otimes \chi) \oplus \bigoplus_{i \in \mathbf{Z}/f\mathbf{Z}} \left(\mathrm{Res}_{D_a^\times}^{D^\times} \mathrm{Ind}_{D_d^\times}^{D^\times} \chi_{\eta_i} \right) \otimes \chi.$$

As $\mathrm{H}^1(1 + \pi_F \mathcal{O}_F, \overline{\mathbf{F}}_p)$ has a trivial action and χ is nontrivial, we know the $(\mathrm{H}^1(1 + \pi_F \mathcal{O}_F, \overline{\mathbf{F}}_p) \otimes \chi)$ component is nontrivial. By the Mackey formula, the remaining component before tensoring with χ is

$$\bigoplus_{i \in \mathbf{Z}/f\mathbf{Z}} \mathrm{Res}_{D_a^\times}^{D^\times} \mathrm{Ind}_{D_d^\times}^{D^\times} \chi_{\eta_i} \simeq \bigoplus_{i \in \mathbf{Z}/f\mathbf{Z}} \bigoplus_{\bar{s} \in D_a^\times \backslash D^\times / D_d^\times} \mathrm{Ind}_{D_d^\times}^{D_a^\times} \chi_{\eta_i}^s.$$

By the Mackey irreducibility criterion, $\mathrm{Ind}_{D_d^\times}^{D_a^\times} \chi_{\eta_i}^s$ is irreducible as it breaks down as a direct sum of distinct characters upon restriction to $D_d^\times$.

When we tensor with χ, we can further pull χ into the induction via the push-pull formula to obtain a direct sum of inductions of the form

$$\mathrm{Ind}_{D_d^\times}^{D_a^\times}(\chi_{\eta_i}^s \otimes \mathrm{Res}_{D_d^\times}^{D_a^\times} \chi) =: \mathrm{Ind}_{D_d^\times}^{D_a^\times} \chi'.$$

This remains irreducible since we have tensored with a character. Now the trivial representation is irreducible as well, so if $\mathrm{Ind}_{D_d^\times}^{D_a^\times} \chi'$ contains a copy of the trivial representation then it must itself be trivial. It follows that we must have $a = d$ in this case, and also that $\chi_{\eta_i}^s \otimes \mathrm{Res}_{D_d^\times}^{D_a^\times} \chi \simeq \mathbf{1}$. As $a = d$, this just says that χ must be dual to some $\chi_{\eta_i}^s$, which can only be the case for at most one i and $\bar{s}$.

We are now left with determining what exactly the characters $\chi_{\eta_i}^s$ are when $a = d$. We see they are trivially extended from $x \mapsto \sigma^{-n} \left(\frac{\sigma(x)}{x} \right)^{p^i}$ over $i \in \mathbf{Z}/f\mathbf{Z}$. Here, we take coset representatives $s = \varpi_D^n$ for $n = 0$ to $d - 1$ for $D_d^\times \backslash D^\times / D_d^\times$. In particular, the resulting characters are precisely $x \mapsto \left(\frac{\sigma(x)}{x} \right)^{p^i}$ for $i \in \mathbf{Z}/(df)\mathbf{Z}$, which is what was claimed. $\qquad \square$

This gives a way to compute any particular extension group.

Theorem 21. *Assume $p > de+1$ when F is a p-adic field, and $\gcd(p, d) = 1$ in the local function field case. Let χ, χ', π and π' be as in Theorem 6, and define the set $S = \{\chi^s : s = \varpi_D^i, 0 \le i < \gcd(a, a')\}$, so that the elements s form a set of coset representatives for $D_{a'}^\times \backslash D^\times / D_a^\times$.*

There are two types of direct summands in $\mathrm{Ext}_{D^\times}^1(\pi, \pi')$. If $\mathrm{Res}_{D_{\mathrm{lcm}(a,a')}^\times}^{D_{a'}^\times} \chi'$ is equal to some $\chi^s \in S$, we have a nonzero direct summand A_{χ^s} fitting into an exact sequence

$$0 \longrightarrow \overline{\mathbf{F}}_p \longrightarrow A_{\chi^s} \longrightarrow \mathrm{H}^1(1 + \pi_F \mathcal{O}_F, \overline{\mathbf{F}}_p) \longrightarrow 0.$$

We also get a nonzero direct summand $A_{\chi^s} \simeq \overline{\mathbf{F}}_p$ for each $\chi^s \in S$ for which $\mathrm{Res}_{D^\times_{\mathrm{lcm}(a,a')}}^{D^\times_{a'}} \chi' \otimes (\chi^s)^*$ *is extended trivially from a character $x \mapsto \left(\frac{x}{\sigma(x)}\right)^{p^i}$.*
Set $A_{\chi^s} = 0$ otherwise. Then

$$\mathrm{Ext}^1_{D^\times}(\pi, \pi') \simeq \bigoplus_{\chi^s \in S} A_{\chi^s}.$$

Proof. We get this result simply by applying the result of Theorem 20 in Theorem 6. ∎

If $\mathrm{lcm}(a, a') < d$, only the first type of direct summand occurs a single time; it is only in the case that $\mathrm{lcm}(a, a') = d$ that the situation can be more complicated.

We illustrate the example of $d = 2$ below.

Example 22. Let D be a quaternion algebra over a p-adic field F. We will consider the case where $\pi = \mathrm{Ind}_{D_2^\times}^{D^\times}(\chi \otimes \kappa)$ and $\pi' = \mathrm{Ind}_{D_2^\times}^{D^\times}(\chi' \otimes \kappa')$ (in the notation of Theorem 4), since this is the case where the most interesting types of extensions occur.

Explicitly, these characters of $D_2^\times$ are of the form $\chi_{2,\alpha,m}$, where $q + 1 \nmid m$ so that the characters have order 2. There are four different possibilities for $\dim \mathrm{Ext}^1_{D^\times}(\pi, \pi') : 0, 1, ef + 1$, and $ef + 2$.

We obtain dimension $ef + 1$ when $\kappa = \kappa'$ and $\chi \otimes (\chi')^*$ is trivial, or $\kappa = \kappa'$ and $\chi \otimes (\chi'^{\varpi_D})^*$ is trivial. The character χ'^{ϖ_D}, when we set $\chi' = \chi_{2,1,m'}$, is $\chi_{2,1,qm'}$. These conditions cannot both occur at the same time, which is why dimension $2ef + 2$ is not possible.

Dimension $ef + 2$ can occur when one of the above conditions holds, for example $\kappa = \kappa'$ and $\chi \otimes (\chi')^*$ is trivial. We then additionally require $\chi \otimes (\chi'^{\varpi_D})^*$ to be some character $x \mapsto (x^{1-q})^{p^i}$ when restricted to $k_D^\times$. This situation can occur, as well as the opposite situation where $\kappa = \kappa'$ and $\chi \otimes (\chi'^{\varpi_D})^*$ is trivial and $\chi \otimes (\chi')^*$ is some character $x \mapsto (x^{1-q})^{p^i}$ on $k_D^\times$.

We can obtain dimension 1 when $\chi \otimes (\chi')^*$ is $x \mapsto (x^{1-q})^{p^i}$ on $k_D^\times$, but $\chi \otimes (\chi'^{\varpi_D})^*$ is nontrivial and vice versa. Finally, we get dimension 0 in all other cases.

5.2. Higher degree extensions in the p-adic case. In [5], many fundamental results about p-adic analytic pro-p groups are shown which will be helpful in understanding the structure of the higher cohomology groups in the case where F is a p-adic field. Here, by a p-adic analytic group we mean a topological group with the structure of a p-adic analytic manifold over $\mathbf{Q}_p$ such that the group multiplication and inversion operations are analytic.

Throughout this section, we will need to assume $p > de + 1$ to make sure that I_1 has no p-torsion to apply the following result.

Theorem ([5, §2.5.8])**.** *Let G be a p-adic analytic pro-p group and with no p-torsion, and let $r = \dim_{\mathbf{Q}_p} G$. Then G is a Poincaré duality group of dimension r over $\mathbf{F}_p$. That is,*

- *$\mathrm{H}^n(G, \mathbf{F}_p)$ is finite dimensional for all $n \in \mathbf{N}$.*
- *$\dim_{\mathbf{F}_p} \mathrm{H}^r(G, \mathbf{F}_p) = 1$.*
- *The cup product*

$$\mathrm{H}^n(G, \mathbf{F}_p) \times \mathrm{H}^{r-n}(G, \mathbf{F}_p) \longrightarrow \mathrm{H}^r(G, \mathbf{F}_p)$$

  is a non-degenerate bilinear form.

Here, as always, all cohomology taken is continuous.

We will be interested in the case of $I_1 = G$, and we will also want to understand how the cup product interacts with the $D_a^\times/I_1$-representation structure on the cohomology group. We note that I_1 is a p-adic analytic group as it is an open subgroup of $D^\times$, and since $p > de + 1$ it has no p-torsion. Thus, I_1 is a Poincaré duality group.

The most difficult part of the following result is that the action of $D_a^\times/I_1$ on the top cohomology is trivial. This is the main result of [4] for general connected reductive groups. Using the method of Theorem 7.2 in [3], we can alternatively show this by finding the uniform pro-p subgroup I_{de+1} and computing explicitly the action on its first cohomology, which determines the action on its top cohomology as the cohomology ring is an exterior algebra. We can then show that this action is trivial, and that it must agree with the action on the top cohomology of I_1 using the corestriction map.

Proposition 23 ([4])**.** *Let $r = d^2ef$. As $D_a^\times/I_1$-representations, we have that $\mathrm{H}^n(I_1, \overline{\mathbf{F}}_p)^* \simeq \mathrm{H}^{r-n}(I_1, \overline{\mathbf{F}}_p)$. Here, V^* denotes the dual representation.*

Proof. As representations of $D^\times/I_1$ are equivalent to modules over $\mathcal{H}_{I_1}$ and $\dim_{\mathbf{Q}_p} I_1 = d^2ef$, the structure on $\mathrm{H}^r(I_1, \mathbf{F}_p)$ as a $D^\times/I_1$-representation, and hence as a $D_a^\times/I_1$-representation, is trivial by [4]. The cup product

$$\mathrm{H}^n(I_1, \mathbf{F}_p) \times \mathrm{H}^{r-n}(I_1, \mathbf{F}_p) \longrightarrow \mathrm{H}^r(I_1, \mathbf{F}_p) \simeq \mathbf{F}_p$$

is a non-degenerate bilinear form, which behaves well with respect to the $D_a^\times/I_1$-action. Let d be an element of $D_a^\times/I_1$. Then we have for $\alpha \in \mathrm{H}^n(I_1, \mathbf{F}_p)$ and $\beta \in \mathrm{H}^{r-n}(I_1, \mathbf{F}_p)$ that

$$d \cdot (\alpha \smile \beta) = (d \cdot \alpha) \smile (d \cdot \beta).$$

It follows from the triviality of the top cohomology that Poincaré duality yields an isomorphism $\mathrm{H}^n(I_1, \mathbf{F}_p)^* \simeq \mathrm{H}^{r-n}(I_1, \mathbf{F}_p)$. Tensoring with $\overline{\mathbf{F}}_p$ gives the desired isomorphism. $\qquad\square$

We will now again use the spectral sequence

$$E_2^{i,j} = \mathrm{H}^i(D_a^\times/I_1, \mathrm{H}^j(I_1, \chi)) \implies \mathrm{Ext}^{i+j}_{D_a^\times}(\mathbf{1}, \chi)$$

from Proposition 7. The work has already been done in Corollary 19 and the previous proposition.

Lemma 24. *Let $r = d^2 ef$ and let χ be a smooth character of $D_a^\times$. There is an exact sequence*

$$0 \longrightarrow \mathrm{H}^1(D_a^\times / I_1, \mathrm{H}^{r-i+1}(I_1, \chi)^*) \longrightarrow \mathrm{Ext}_{D_a^\times}^i(\mathbf{1}, \chi^*)$$
$$\longrightarrow (\mathrm{H}^{r-i}(I_1, \chi)^*)^{D_a^\times / I_1} \longrightarrow 0.$$

Proof. This follows from Corollary 19 combined with the previous proposition. ∎

For $i = r + 1$, we get $\mathrm{Ext}_{D_a^\times}^{r+1}(\mathbf{1}, \chi^*) \simeq \mathrm{H}^1(D_a^\times / I_1, \chi^*)$, which is trivial unless $\chi = \mathbf{1}$ in which case we get $\overline{\mathbf{F}}_p$. After $r + 1$, the extensions are all trivial.

We can make use of this lemma when $i = r$ as well, since we have results about the surrounding terms in the exact sequence. We have

$$0 \longrightarrow \mathrm{H}^1(D_a^\times / I_1, \mathrm{H}^1(I_1, \chi)^*) \longrightarrow \mathrm{Ext}_{D_a^\times}^r(\mathbf{1}, \chi^*) \longrightarrow (\chi^*)^{D_a^\times / I_1} \longrightarrow 0.$$

We may determine the first group via Proposition 18, the dimension is the multiplicity of the trivial representation. This is precisely what was determined in Theorem 20.

In the case of $d = 2$ and $F = \mathbf{Q}_p$, this is enough to determine all extension groups because $r = d^2 ef = 4$. For a smooth character χ of $D^\times$, we can compute all $\mathrm{H}^i(I_1, \chi)$ to do this. It is worth noting that this cohomology computation is known and of independent interest in stable homotopy theory, for example see [2, Prop. 7]. The group I_1 appears as the Morava stabilizer group attached to a formal group law, and $\mathrm{H}^\bullet(I_1, \mathbf{F}_p)$ controls certain localization functors $L_{K(n)}$ in Morava K-theory. Here, we provide an alternative proof using our results.

We have $\mathrm{H}^0(I_1, \chi) \simeq \chi$, so $\mathrm{H}^4(I_1, \chi) \simeq \chi$. As for H^1, it becomes

$$\mathrm{H}^1(I_1, \chi) \simeq \left(\overline{\mathbf{F}}_p \oplus \mathrm{Ind}_{D_2^\times}^{D^\times} \chi_\eta \right) \otimes \chi,$$

where χ_η is extended from the character of $\mathbf{F}_{p^2}^\times$ sending $x \mapsto x^{p-1}$. The representation $\mathrm{H}^3(I_1, \chi)$ is the dual of $\mathrm{H}^1(I_1, \overline{\mathbf{F}}_p)$ tensored with χ, so it is given by

$$\mathrm{H}^3(I_1, \chi) \simeq \left(\overline{\mathbf{F}}_p \oplus \mathrm{Ind}_{D_2^\times}^{D^\times} \chi_\eta^* \right) \otimes \chi.$$

What remains is to compute $\mathrm{H}^2(I_1, \overline{\mathbf{F}}_p)$. In this case, the Euler characteristic of I_1 will be 0 using the main result of [13], so we can deduce that

$\dim \mathrm{H}^2(I_1, \overline{\mathbf{F}}_p) = 4$. Via Lemma 9 and the Künneth formula for profinite groups,

$$\mathrm{H}^2(I_1, \overline{\mathbf{F}}_p) \simeq \bigoplus_{i+j=2} \mathrm{H}^i(1 + p\mathbf{Z}_p, \overline{\mathbf{F}}_p) \otimes \mathrm{H}^j(I_{1,\mathrm{Nrd}=1}, \overline{\mathbf{F}}_p).$$

The isomorphism is via the cross product, which makes this an isomorphism on the level of representations as well because $1 + p\mathbf{Z}_p$ and $I_{1,\mathrm{Nrd}=1}$ are normal. As $1 + p\mathbf{Z}_p \simeq \mathbf{Z}_p$, its second cohomology is trivial, so there are only two nontrivial terms in the Künneth formula. The first term is $\mathrm{H}^1(1 + p\mathbf{Z}_p, \overline{\mathbf{F}}_p) \otimes \mathrm{H}^1(I_{1,\mathrm{Nrd}=1}, \overline{\mathbf{F}}_p)$, which we already know is $\mathrm{Ind}_{D_2^\times}^{D^\times} \chi_\eta$ via the decomposition of $\mathrm{H}^1(I_1, \overline{\mathbf{F}}_p)$ in Theorem 15. The other nontrivial component is $\mathrm{H}^0(1 + p\mathbf{Z}_p, \overline{\mathbf{F}}_p) \otimes \mathrm{H}^2(I_{1,\mathrm{Nrd}=1}, \overline{\mathbf{F}}_p)$. We can in general produce some of this as the image of the Bockstein

$$\beta_p : \mathrm{H}^1(I_{1,\mathrm{Nrd}=1}, \mathbf{F}_p) \longrightarrow \mathrm{H}^2(I_{1,\mathrm{Nrd}=1}, \mathbf{F}_p),$$

after noting that the elements in $\mathrm{H}^1(I_{1,\mathrm{Nrd}=1}, \mathbf{Z}/p^2\mathbf{Z}) \simeq \mathrm{Hom}(k_D, \mathbf{Z}/p^2\mathbf{Z})$ are sent to 0 so that the map becomes an injection. Note that the isomorphism

$$\mathrm{H}^1(I_{1,\mathrm{Nrd}=1}, \mathbf{Z}/p^2\mathbf{Z}) \simeq \mathrm{Hom}(k_D, \mathbf{Z}/p^2\mathbf{Z})$$

still follows from the commutator calculation: we have $[I_{1,\mathrm{Nrd}=1}, I_{1,\mathrm{Nrd}=1}] = I_{2,\mathrm{Nrd}=1}$. Tensoring with $\overline{\mathbf{F}}_p$, we see that

$$\mathrm{H}^2(I_1, \chi) \supseteq \left(\mathrm{Ind}_{D_2^\times}^{D^\times} \chi_\eta\right)^{\oplus 2} \otimes \chi.$$

This has dimension four, so it must be the entire cohomology group. Via Corollary 19 we can determine all extension groups as well. Note that the fact that we have accounted for the entire cohomology group in the above calculation implies that the injective Bockstein map β_p is actually an isomorphism in this case.

Remark. When F is a more general p-adic field or we consider a general division algebra, this method only provides partial information about $\mathrm{H}^2(I_1, \chi)$. The Künneth decomposition can provide a subrepresentation of $\mathrm{H}^2(I_1, \overline{\mathbf{F}}_p)$ by considering the image of the Bockstein in $\mathrm{H}^2(I_{1,\mathrm{Nrd}=1}, \overline{\mathbf{F}}_p)$ isomorphic to $\mathrm{H}^1(I_{1,\mathrm{Nrd}=1}, \overline{\mathbf{F}}_p)$. The other two components are completely determined, as $\mathrm{H}^1(I_1, \overline{\mathbf{F}}_p)$ is known and all cohomology groups of $1 + \pi_F \mathcal{O}_F$ are easy to find since it is isomorphic to a direct product of ef copies of $\mathbf{Z}_p$ as a topological group. We can thus use another Künneth decomposition on it to show that $\mathrm{H}^i(1 + \pi_F\mathcal{O}_F, \overline{\mathbf{F}}_p) \simeq \overline{\mathbf{F}}_p^{\binom{ef}{i}}$. In summary, we have

$$\mathrm{H}^1(I_{1,\mathrm{Nrd}=1}, \overline{\mathbf{F}}_p)^{\oplus 2} \oplus \overline{\mathbf{F}}_p^{\binom{ef}{2}} \subset \mathrm{H}^2(I_1, \overline{\mathbf{F}}_p),$$

where each summand is also as representations of $D^\times/I_1$, with $\overline{\mathbf{F}}_p^{\binom{ef}{2}}$ having the trivial action.

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