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Résumé. Soit $\mathbb{Q}_p$ le corps des nombres p-adiques et $\mathbb{G}$ un schéma en groupes réductif, connexe et déployé sur $\mathbb{Z}_p$. Nous introduirons un faisceau d'opérateurs différentiels arithmétiques tordus sur la variété des drapeaux formelle de $\mathbb{G}$, associée à un caractère général. En particulier, nous généraliserons les résultats de [21], concernant la $\mathcal{D}^\dagger$-affinité de la variété des drapeaux formelle lisse de $\mathbb{G}$, de certains gerbes d'opérateurs différentiels arithmétiques tordus p-adiquement complets, associés à un caractère algébrique, et les résultats de [24] concernant le calcul des sections globales.

Abstract. Let $\mathbb{Q}_p$ be the field of p-adic numbers and $\mathbb{G}$ a split connected reductive group scheme over $\mathbb{Z}_p$. In this work we will introduce a sheaf of twisted arithmetic differential operators on the formal flag variety of $\mathbb{G}$, associated to a general character. In particular, we will generalize the results of [21], concerning the $\mathcal{D}^\dagger$-affinity of the smooth formal flag variety of $\mathbb{G}$, of certain sheaves of p-adically complete twisted arithmetic differential operators associated to an algebraic character, and the results of [24] concerning the calculation of the global sections.

1. Introduction

An important theorem in group theory is the so-called *Beilinson–Bernstein localization theorem* [2]. Let us briefly recall its statement. Let G be a semi-simple complex algebraic group with Lie algebra $\mathfrak{g}_\mathbb{C} := \mathrm{Lie}(G)$. Let $\mathfrak{t}_\mathbb{C} \subset \mathfrak{g}_\mathbb{C}$ be a Cartan subalgebra and $Z(\mathcal{U}(\mathfrak{g}_\mathbb{C})) \subset \mathcal{U}(\mathfrak{g}_\mathbb{C})$ the center of the universal enveloping algebra $\mathcal{U}(\mathfrak{g}_\mathbb{C})$ of $\mathfrak{g}_\mathbb{C}$. For every character $\lambda \in \mathfrak{t}_\mathbb{C}^*$ we denote by $\mathfrak{m}_\lambda \subset Z(\mathcal{U}(\mathfrak{g}_\mathbb{C}))$ the corresponding maximal ideal determined by the *Harish–Chandra isomorphism*. We put $\mathcal{U}_\lambda := \mathcal{U}(\mathfrak{g}_\mathbb{C})/\mathfrak{m}_\lambda$. The theorem states that if X is the flag variety associated to G and $\mathcal{D}_{X,\lambda}$ is the sheaf of λ-twisted differential operators on X, then $\mathrm{Mod}_{\mathrm{qc}}(\mathcal{D}_{X,\lambda}) \cong \mathrm{Mod}(\mathcal{U}_\lambda)$, provided that λ is a dominant and regular character. Here $\mathrm{Mod}_{\mathrm{qc}}(\mathcal{D}_{X,\lambda})$

denotes the category of $\mathcal{O}_X$-quasi-coherent $\mathcal{D}_{X,\lambda}$-modules. Moreover, under this equivalence of categories, coherent $\mathcal{D}_{X,\lambda}$-modules correspond to finitely generated $\mathcal{U}_\lambda$-modules [2, Théorème Principal].

The localization theorem has been proved independently by A. Beilinson and J. Bernstein in [2], and by J.L. Brylinski and M. Kashiwara in [12]. This result is an essential tool in the proof of Kazhdan–Lusztig's multiplicity conjecture [27]. In mixed characteristic, an important progress has been accomplished by C. Huyghe in [20, 21] and Huyghe–Schmidt in [24]. They use Berthelot's arithmetic differential operators, [5], to prove an arithmetic version of the Beilinson–Bernstein localization theorem for the smooth formal flag variety over a discrete valuation ring. They achieve this result by working with algebraic characters. In this setting, the global sections of these operators equal a crystalline version of the classical distribution algebra $\mathrm{Dist}(G)$ of G.

Let $\mathbb{Q}_p$ be the field of p-adic numbers. Throughout this paper $\mathbb{G}$ will denote a split connected reductive group scheme over $\mathbb{Z}_p$, $\mathbb{B} \subset \mathbb{G}$ a Borel subgroup and $\mathbb{T} \subset \mathbb{B}$ a split maximal torus of $\mathbb{G}$. We will also denote by $X = \mathbb{G}/\mathbb{B}$ the smooth flag $\mathbb{Z}_p$-scheme associated to $\mathbb{G}$. In this work, we introduce sheaves of *twisted differential operators* on the formal flag scheme $\mathfrak{X}$ and we show an arithmetic Beilinson–Bernstein correspondence for general characters of $\mathrm{Lie}(\mathbb{T})$. Here the twist is respect to a morphism of $\mathbb{Z}_p$-algebras $\lambda : \mathrm{Dist}(\mathbb{T}) \to \mathbb{Z}_p$, where $\mathrm{Dist}(\mathbb{T})$ denotes the distribution algebra in the sense of [13]. Those sheaves are denoted by $\mathcal{D}^\dagger_{\mathfrak{X},\lambda}$. In particular, there exists a basis $\mathcal{S}$ of $\mathfrak{X}$ consisting of affine open subsets, such that for every $\mathfrak{U} \in \mathcal{S}$ we have

$$\mathcal{D}^\dagger_{\mathfrak{X},\lambda}|_{\mathfrak{U}} \cong \mathcal{D}^\dagger_{\mathfrak{U}}.$$

In other words, locally we recover the sheaf of Berthelot's differential operators (of course, this clarifies why they are called twisted arithmetic differential operators).

To calculate their global sections, we remark for the reader that we actually dispose of a description of the distribution algebra $\mathrm{Dist}(\mathbb{G})$ as an inductive limit of filtered noetherian $\mathbb{Z}_p$-algebras $\mathrm{Dist}(\mathbb{G}) = \varinjlim_{m \in \mathbb{N}} D^{(m)}(\mathbb{G})$, such that for every $m \in \mathbb{N}$ we have

$$D^{(m)}(\mathbb{G}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = \mathcal{U}(\mathrm{Lie}(\mathbb{G}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)$$

the universal enveloping algebra of $\mathrm{Lie}(\mathbb{G}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. In particular, every homomorphism of $\mathbb{Z}_p$-algebras $\lambda : \mathrm{Dist}(\mathbb{T}) \to \mathbb{Z}_p$ induces, taking tensor product with $\mathbb{Q}_p$, an infinitesimal central character χ_λ. This last property allows us to define $\widehat{D}^{(m)}(\mathbb{G})_\lambda$ as the p-adic completion of the central reduction $D^{(m)}(\mathbb{G})/(D^{(m)}(\mathbb{G}) \cap \ker(\chi_{\lambda+\rho}))$, where ρ is the *Weyl character*, and we

denote by $D^\dagger(\mathbb{G})_\lambda$ the colimit of the system

$$\widehat{D}^{(m)}(\mathbb{G})_\lambda \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \longrightarrow \widehat{D}^{(m')}(\mathbb{G})_\lambda \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.$$

In this work we show the following result.

Theorem. *Let us suppose that $\lambda : \mathrm{Dist}(\mathbb{T}) \to \mathbb{Z}_p$ is a homomorphism of $\mathbb{Z}_p$-algebras, such that the character*

$$(\lambda \otimes_{\mathbb{Z}_p} 1_{\mathbb{Q}_p})|_{\mathrm{Lie}(\mathbb{T}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p} + \rho \in \mathrm{Hom}_{\mathbb{Q}_p\text{-mod}}(\mathrm{Lie}(\mathbb{T}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p, \mathbb{Q}_p)$$

is dominant and regular.[1] *Then the global sections functor induces an equivalence between the categories of coherent $\mathcal{D}^\dagger_{\mathfrak{X},\lambda}$-modules and finitely presented $D^\dagger(\mathbb{G})_\lambda$-modules.*

This theorem is based on a refined version for the sheaves of *level m twisted arithmetic differential operators* $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X},\lambda,\mathbb{Q}}$ (Definition 3.30). As in the classical case, the inverse functor is determined by the *localization functor*

$$\mathcal{L}oc^\dagger_{\mathfrak{X},\lambda}(\bullet) := \mathcal{D}^\dagger_{\mathfrak{X},\lambda} \otimes_{D^\dagger(\mathbb{G})_\lambda} (\bullet),$$

with a completely analogous definition for every $m \in \mathbb{N}$.

Let us explain the structure of this text. The first section is devoted to fix some important arithmetic definitions. They are necessary to define an arithmetic analogue of the usual sheaf of twisted differential operators on the smooth flag variety $X_{\mathbb{Q}} := X \times_{\mathrm{Spec}(\mathbb{Z}_p)} \mathrm{Spec}(\mathbb{Q}_p)$, associated to the split connected reductive algebraic group $\mathbb{G}_{\mathbb{Q}} := \mathbb{G} \times_{\mathrm{Spec}(\mathbb{Z}_p)} \mathrm{Spec}(\mathbb{Q}_p)$. One of the most important is the *algebra of distributions of level m*, which is denoted by $D^{(m)}(\mathbb{G})$ and it is introduced in Section 2.4. This is a filtered noetherian $\mathbb{Z}_p$-algebra which plays a fundamental roll in this text.

Inspired by the works [1, 8, 11], in the first part of the third section we construct our *level m twisted arithmetic differential operators* on the formal flag scheme $\mathfrak{X}$. To do this, we denote by $\mathfrak{t}$ the commutative $\mathbb{Z}_p$-Lie algebra of the maximal torus $\mathbb{T}$ and by $\mathfrak{t}_{\mathbb{Q}} := \mathfrak{t} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. They are Cartan subalgebras of $\mathfrak{g} := \mathrm{Lie}(\mathbb{G})$ and $\mathfrak{g}_{\mathbb{Q}} := \mathrm{Lie}(\mathbb{G}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$, respectively. Let $\boldsymbol{N}$ be the unipotent radical subgroup of the Borel subgroup $\mathbb{B}$ and let us define

$$\widetilde{X} := \mathbb{G}/\boldsymbol{N} \quad \text{and} \quad X := \mathbb{G}/\mathbb{B},$$

the *basic affine space* and the *flag scheme* of $\mathbb{G}$. These are smooth and separated schemes over $\mathbb{Z}_p$, and $\widetilde{X}$ is endow with commuting $(\mathbb{G}, \mathbb{T})$-actions making the canonical projection $\xi : \widetilde{X} \to X$ a $\mathbb{T}$-torsor for the Zariski topology on X (Definition 3.1, Notation 3.2 and Remark 3.3). Following [11], the

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[1] We have used the canonical isomorphism $\mathrm{Dist}(\mathbb{T}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \cong \mathcal{U}(\mathrm{Lie}(\mathbb{T}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)$, [26, Part I, 7.10 (1)].

right $\mathbb{T}$-action on $\widetilde{X}$ allows to define the *level m relative enveloping algebra* of the torsor ξ as the sheaf of $\mathbb{T}$-invariants of $\xi_* \mathcal{D}_{\widetilde{X}}^{(m)}$:

$$(1.1) \qquad\qquad \widetilde{\mathcal{D}^{(m)}} := \left(\xi_* \mathcal{D}_{\widetilde{X}}^{(m)} \right)^{\mathbb{T}}.$$

As we will explain later, this is a sheaf of $D^{(m)}(\mathbb{T})$-modules and we will show that over an affine open subset $U \subseteq X$ that trivialises the torsor, the sheaf (1.1) may be described as the tensor product $\mathcal{D}_X^{(m)}|_U \otimes_{\mathbb{Z}_p} D^{(m)}(\mathbb{T})$ (this is the arithmetic analogue of [11, p. 180]).

Two fundamental properties of the distribution algebra $\mathrm{Dist}(\mathbb{T})$ are: the $\mathbb{Q}_p$-algebra of algebraic distributions $\mathrm{Dist}(\mathbb{T}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is a noetherian algebra which is canonically isomorphic to the universal enveloping algebra $\mathcal{U}(\mathfrak{t}_{\mathbb{Q}})$, and $\mathrm{Dist}(\mathbb{T}) = \varinjlim_m D^{(m)}(\mathbb{T})$. These properties allow us to introduce the *central reduction* of the sheaves (1.1) as follows. First of all, we say that a morphism of $\mathbb{Z}_p$-algebras $\lambda : \mathrm{Dist}(\mathbb{T}) \to \mathbb{Z}_p$ is a *character* of $\mathrm{Dist}(\mathbb{T})$ (3.21). By the properties just stated, it induces a character [2] of the Cartan subalgebra $\mathfrak{t}_{\mathbb{Q}}$

$$(\lambda =)\ (\lambda \otimes_{\mathbb{Z}_p} 1_{\mathbb{Q}_p})|_{\mathfrak{t}_{\mathbb{Q}}} : \mathfrak{t}_{\mathbb{Q}} \longhookrightarrow \mathcal{U}(\mathfrak{t}_{\mathbb{Q}}) \cong \mathrm{Dist}(\mathbb{T}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \longrightarrow \mathbb{Q}_p.$$

Now, via $\lambda^{(m)} : D^{(m)}(\mathbb{T}) \to \mathrm{Dist}(\mathbb{T}) \xrightarrow{\lambda} \mathbb{Z}_p$ we may consider the ring $\mathbb{Z}_p$ as a $D^{(m)}(\mathbb{T})$-module, and define the sheaf of *level m twisted arithmetic differential operators* on the flag scheme X by

$$(1.2) \qquad\qquad \mathcal{D}_{X,\lambda}^{(m)} := \widetilde{\mathcal{D}^{(m)}} \otimes_{D^{(m)}(\mathbb{T}),\lambda^{(m)}} \mathbb{Z}_p.$$

This is a sheaf of $\mathbb{Z}_p$-algebras satisfying $(\mathcal{D}_{X,\lambda}^{(m)} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)|_{X_{\mathbb{Q}_p}} = \mathcal{D}_{X_{\mathbb{Q}},\lambda}$ (the process of taking tensor product with $\mathbb{Q}_p$ and restricting to the generic fiber $X_{\mathbb{Q}} \hookrightarrow X$ is equal to the usual sheaf of twisted differential operators $\mathcal{D}_{X_{\mathbb{Q}},\lambda}$ in [11, p. 170]).

The second part of the third section is dedicated to explore some finiteness properties of the cohomology of coherent $\mathcal{D}_{X,\lambda}^{(m)}$-modules. Notably important is the case when the character $\lambda + \rho \in \mathfrak{t}_{\mathbb{Q}}^*$ is dominant and regular. Under this assumption, the cohomology groups of every coherent $\mathcal{D}_{X,\lambda}^{(m)}$-module have the nice property of being of finite p-torsion. This is a central result in this work.

In Section 4, we will consider the p-adic completion of (1.2). It will be denoted by $\widehat{\mathcal{D}}_{\mathfrak{X},\lambda}^{(m)}$ and $\widehat{\mathcal{D}}_{\mathfrak{X},\lambda,\mathbb{Q}}^{(m)} := \widehat{\mathcal{D}}_{\mathfrak{X},\lambda}^{(m)} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ will be our sheaf of *level m twisted arithmetic differential operators* on the formal flag scheme $\mathfrak{X}$.

Let $Z(\mathfrak{g}_{\mathbb{Q}})$ be the center of the universal enveloping algebra $\mathcal{U}(\mathfrak{g}_{\mathbb{Q}})$. From now on, we will always assume that $\lambda \in \mathrm{Hom}_{\mathbb{Z}_p\text{-alg}}(\mathrm{Dist}(\mathbb{T}), \mathbb{Z}_p)$ is such that

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[2] In order to soft the notation through this work we will always denote by λ the character $(\lambda \otimes_{\mathbb{Z}_p} 1_{\mathbb{Q}_p})|_{\mathfrak{t}_{\mathbb{Q}_p}}$ of $\mathfrak{t}_{\mathbb{Q}}$. This should not cause any confusion to the reader.

$\lambda + \rho \in \mathfrak{t}^*_{\mathbb{Q}}$ is dominant and regular, and denote by $\chi_\lambda : Z(\mathfrak{g}_{\mathbb{Q}}) \to \mathbb{Q}_p$ the central character induced by λ via the classical Harish–Chandra isomorphism. In Section 4.3 we will show that if

$$\mathrm{Ker}(\chi_{\lambda+\rho})_{\mathbb{Z}_p} := D^{(m)}(\mathbb{G}) \cap \mathrm{Ker}(\chi_{\lambda+\rho})$$

and $\widehat{D}^{(m)}(\mathbb{G})_\lambda$ denotes the formal p-adic completion of the central reduction $D^{(m)}(\mathbb{G})_\lambda := D^{(m)}(\mathbb{G})/D^{(m)}(\mathbb{G}) \mathrm{Ker}(\chi_{\lambda+\rho})_{\mathbb{Z}_p}$, then we have a canonical isomorphism of $\mathbb{Q}_p$-algebras

$$\widehat{D}^{(m)}(\mathbb{G})_\lambda \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \xrightarrow{\cong} H^0\left(\mathfrak{X}, \widehat{\mathcal{D}}^{(m)}_{\mathfrak{X},\lambda,\mathbb{Q}}\right).$$

Still in Section 4.3, we will introduce the localization functor $\mathcal{L}oc^{(m)}_{\mathfrak{X},\lambda}$ from the category of finitely generated $\widehat{D}^{(m)}(\mathbb{G})_\lambda \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$-modules to the category of coherent $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X},\lambda,\mathbb{Q}}$-modules, as the sheaf associated to the presheaf defined by

$$\mathcal{U} \subseteq \mathfrak{X} \longmapsto \widehat{\mathcal{D}}^{(m)}_{\mathfrak{X},\lambda,\mathbb{Q}}(\mathcal{U}) \otimes_{\widehat{D}^{(m)}(\mathbb{G})_\lambda \otimes_{\mathbb{Z}_p} \mathbb{Q}_p} E,$$

where E is a finitely generated $\widehat{D}^{(m)}(\mathbb{G})_\lambda \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$-module. We will show

Theorem. *Let us suppose that $\lambda : \mathrm{Dist}(\mathbb{T}) \to \mathbb{Z}_p$ is a character of $\mathrm{Dist}(\mathbb{T})$ such that $\lambda + \rho \in \mathfrak{t}^*_{\mathbb{Q}}$ is a dominant and regular character of $\mathfrak{t}_{\mathbb{Q}}$.*

  (i) *The functors $\mathcal{L}oc^{(m)}_{\mathfrak{X},\lambda}$ and $H^0(\mathfrak{X}, \bullet)$ induce quasi-inverse equivalence of categories between the abelian categories of finitely generated (left) $\widehat{D}^{(m)}(\mathbb{G})_\lambda \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$-modules and coherent $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X},\lambda,\mathbb{Q}}$-modules.*

  (ii) *The functor $\mathcal{L}oc^{(m)}_{\mathfrak{X},\lambda}$ is an exact functor.*

Finally, Section 5 of this work is devoted to treat the problem of passing to the inductive limit. In fact, if $m \leq m'$ the sheaves $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X},\lambda,\mathbb{Q}}$ and $\widehat{\mathcal{D}}^{(m')}_{\mathfrak{X},\lambda,\mathbb{Q}}$ are related via a natural map $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X},\lambda,\mathbb{Q}} \to \widehat{\mathcal{D}}^{(m')}_{\mathfrak{X},\lambda,\mathbb{Q}}$ and we may define $\mathcal{D}^\dagger_{\mathfrak{X},\lambda}$ to be the colimit. Similarly, we will denote by $D^\dagger(\mathbb{G})_\lambda$ the colimit of the system $\widehat{D}^{(m)}(\mathbb{G})_\lambda \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to \widehat{D}^{(m')}(\mathbb{G})_\lambda \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. Introducing the localization functor $\mathcal{L}oc^\dagger_{\mathfrak{X},\lambda}$ exactly as we have made before, we get an analogue of the previous theorem for the sheaves $\mathcal{D}^\dagger_{\mathfrak{X},\lambda}$.

The work developed by Huyghe in [20] and by D. Patel, T. Schmidt and M. Strauch in [22, 33, 34], shows that the Beilinson–Bernstein theorem is an important tool in the following localization theorem [22, Theorem 5.3.8]: if $\mathfrak{X}$ denotes the formal flag scheme of a split connected reductive group $\mathbb{G}$, then the theorem provides an equivalence of categories between the category of *admissible locally analytic $\mathbb{G}(\mathbb{Q}_p)$-representations* (with trivial character!) [36] and a category of *coadmissible equivariant arithmetic $\mathcal{D}$-modules* (on the family of formal models of the rigid analytic flag variety

of $\mathbb{G}$). Our motivation is to study this localization in the twisted case. In a recent work, [35], we have proved the affinity of an admissible formal blow-up of $\mathfrak{X}$ for the sheaf of twisted arithmetic differential operators with a congruence level k. This is a fundamental result to achieve the previous localization theorem in the twisted setting, [22].

The case of a finite extension L of $\mathbb{Q}_p$ seems to present several technical problems (the reader can take a look to Proposition 3.21 to see one of this obstacles). We will treat this case in a future work.

Notation. Throughout this work p will denote a prime number and $\mathbb{Z}_p$ the ring of p-adic integers. If X is an arbitrary noetherian scheme over $\mathbb{Z}_p$ and $j \in \mathbb{N}$, then we will denote by $X_j \coloneqq X \times_{\mathrm{Spec}(\mathbb{Z}_p)} \mathrm{Spec}(\mathbb{Z}_p/p^{j+1})$ the reduction modulo p^{j+1}, and by

$$\mathfrak{X} = \varinjlim_{j} X_j$$

the formal completion of X along the special fiber. Moreover, if $\mathcal{E}$ is a sheaf of $\mathbb{Z}_p$-modules on X then its p-adic completion $\mathcal{E} \coloneqq \varprojlim_{j} \mathcal{E}/p^{j+1}\mathcal{E}$ will be considered as a sheaf on $\mathfrak{X}$. Finally, the base change of a sheaf of $\mathbb{Z}_p$-modules on X (resp. on $\mathfrak{X}$) to $\mathbb{Q}_p$ will always be denoted by the subscript $\mathbb{Q}$. For instance, $\mathcal{E}_{\mathbb{Q}} \coloneqq \mathcal{E} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ (resp. $\mathcal{E}_{\mathbb{Q}} \coloneqq \mathcal{E} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$).

2. Arithmetic Definitions

In this section we will describe the arithmetic objects on which the definitions and constructions of our work are based. We will give their functorial constructions and we will enunciate their most remarkable properties. For a more detailed approach, the reader is invited to take a look to the references [5, 7, 20, 23].

2.1. Partial Divided Power Structures of Level m. Let $p \in \mathbb{Z}$ be a prime number. In this subsection $\mathbb{Z}_{(p)}$ denotes the localization of $\mathbb{Z}$ with respect to the prime ideal (p).

We start by recalling the following definition [7, Definition 3.1].

Definition 2.1. Let A be a commutative ring and $I \subset A$ an ideal. By a structure of divided powers on I we mean a collection of maps $\gamma_i : I \to A$ for all integers $i \geq 0$, such that

(i) For all $x \in I$, $\gamma_0(x) = 1$, $\gamma_1(x) = x$ and $\gamma_i(x) \in I$ if $i \geq 2$.
(ii) For $x, y \in I$ and $k \geq 1$ we have $\gamma_k(x + y) = \sum_{i+j=k} \gamma_i(x)\gamma_j(y)$.
(iii) For $a \in A$ and $x \in I$ we have $\gamma_k(ax) = a^k \gamma_k(x)$.
(iv) For $x \in I$ we have $\gamma_i(x)\gamma_j(x) = (i + j)!(i!)^{-1}(j!)^{-1}\gamma_{i+j}(x)$.
(v) We have $\gamma_p(\gamma_q(x)) = C_{p,q}\gamma_{pq}(x)$, where $C_{p,q} := (pq)!(p!)^{-1}(q!)^{-p}$.

Throughout this paper we will use the terminology: (I, γ) is a *PD-ideal*, (A, I, γ) is a *PD-ring* and γ is a *PD-structure* on I. Moreover, we say that $\phi : (A, I, \gamma) \to (B, J, \delta)$ is a *PD-homomorphism* if $\phi : A \to B$ is a homorphism of rings such that $\phi(I) \subset J$ and $\delta_k \circ \phi|_I = \phi \circ \gamma_k$, for every $k \geq 0$.

Example 2.2 ([7, §3, Example 3.2 (3)]). Let $L|\mathbb{Q}_p$ be a finite extension of $\mathbb{Q}_p$, such that $\mathfrak{o}$ is its valuation ring and ϖ is a uniformizing parameter. Let us write $p = u\varpi^e$, with u a unit and e a positive integer (called the absolute ramification index of $\mathfrak{o}$). Then $\gamma_k(x) := x^k/k!$ defines a PD-structure on (ϖ) if and only if $e \leq p - 1$. In particular, we dispose of a PD-structure on $(p) \subset \mathbb{Z}_p$. We let $x^{[k]} := \gamma_k(x)$ and we denote by $((p), [\,])$ this PD-ideal.

Let us fix a positive integer $m \in \mathbb{Z}$. For the next terminology we will always suppose that (A, I, γ) is a $\mathbb{Z}_{(p)}$-PD-algebra whose PD-structure is *compatible* (in the sense of [5, §1.2]) with the PD-structure induced by $((p), [\,])$ (we recall for the reader that the PD-structure $((p), [\,])$ always *extends* to a PD-structure on any $\mathbb{Z}_{(p)}$-algebra [7, Proposition 3.15]). We will also denote by $I^{(p^m)}$ the ideal generated by x^{p^m}, with $x \in I$.

Definition 2.3. Let m be a positive integer. Let A be a $\mathbb{Z}_{(p)}$-algebra and $I \subset A$ an ideal. We call an m-PD-structure on I a PD-ideal $(J, \gamma) \subset A$ such that $I^{(p^m)} + pI \subset J$.

We will say that (I, J, γ) is an *m-PD-ideal* of A. Moreover, we say that $\phi : (A, I, J, \gamma) \to (A', I', J', \gamma')$ is an *m-PD-morphism* if $\phi : A \to A'$ is a ring morphism such that $\phi(I) \subset I'$, and such that $\phi : (A, J, \gamma) \to (A', J', \gamma')$ is a PD-morphism.

For every $k \in \mathbb{N}$ we denote by $k = p^m q + r$ the Euclidean division of k by p^m, and for every $x \in I$ we define $x^{\{k\}(m)} := x^r(\gamma_q(x^{p^m}))$. We remark for the reader that the relation $q!\gamma_q(x) = x^q$ (which is an easy consequence of (i) and (iv) of Definition 2.1) implies that $q!x^{\{k\}(m)} = x^k$.

On the other hand, the m-PD-structure (I, J, γ) allows us to define an increasing filtration $(I^{\{n\}})_{n \in \mathbb{N}}$ on the ring A which is finer that the I-adic filtration and called the *m-PD-filtration*. It is characterized by the following conditions [6, 1.3]:

(i) $I^{\{0\}} = A$, $I^{\{1\}} = I$.

(ii) For every $n \geq 1$, $x \in I^{\{n\}}$ and $k \geq 0$ we have $x^{\{k\}} \in I^{\{kn\}}$.

(iii) For every $n \geq 0$, $(J + pA) \cap I^{\{n\}}$ is a PD-subideal of $(J + pA)$.

Proposition 2.4 ([7, Proposition 1.4.1])**.** *Let R be a $\mathbb{Z}_{(p)}$-algebra endowed with an m-PD-structure $(\mathfrak{a}, \mathfrak{b}, \alpha)$. Let A be an R-algebra and $I \subset A$ an ideal. There exists an R-algebra $P_{(m)}(I)$, an ideal $\bar{I} \subset P_{(m)}(I)$ endowed with an m-PD-structure $(\widetilde{I}, [\,])$ compatible with $(\mathfrak{b}, \alpha)$, and a homomorphism of rings $\phi : A \to P_{(m)}(I)$ such that $\phi(I) \subset \bar{I}$. Moreover, the objects $(P_{(m)}(I), \bar{I}, \widetilde{I}, [\,], \phi)$ satisfy the following universal property: for every R-homomorphism $f : A \to A'$ sending I to an ideal I' which is endowed with an m-PD-structure (J', γ') compatible with $(\mathfrak{b}, \alpha)$, there exists a unique m-PD-morphism $g : (P_{(m)}(I), \bar{I}, \widetilde{I}, [\,]) \to (A', I', J', \gamma')$ such that $g \circ \phi = f$.*

Definition 2.5. Under the hypothesis of the preceding proposition, we call the R-algebra $P_{(m)}(I)$, endowed with the m-PD-ideal $(\bar{I}, \widetilde{I}, [\,])$, the m-PD-envelope of (A, I).

Finally, if we endow $P_{(m)}^n(I) := P_{(m)}(I)/\bar{I}^{\{n+1\}}$ with the quotient m-PD-structure [5, 1.3.4] we have tthe following result.

Corollary 2.6. [5, Corollary 1.4.2] *Under the hypothesis of Proposition 2.4, there exists an R-algebra $P_{(m)}^n(I)$ endowed with an m-PD-structure $(\bar{I}, \widetilde{I}, [\,])$ compatible with $(\mathfrak{b}, \alpha)$ and such that $\bar{I}^{\{n+1\}} = 0$. Moreover, there exists an R-homomorphism $\phi_n : A \to P_{(m)}^n(I)$ such that $\phi(I) \subset \bar{I}$, and universal for the R-homomorphisms $A \to (A', I', J', \gamma')$ sending I into an m-PD-ideal I' compatible with $(\mathfrak{b}, \alpha)$ and such that $I'^{\{n+1\}} = 0$.*

2.2. Arithmetic Differential Operators. Let us suppose that $\mathbb{Z}_p$ is endowed with the m-PD-structure defined in Example 2.2 (cf. [5, §1.3, Example (i)]). Let X be a smooth $\mathbb{Z}_p$-scheme, and $\mathcal{I} \subset \mathcal{O}_X$ a quasi-coherent ideal. The presheaves

$$U \subseteq X \longmapsto P_{(m)}(\Gamma(U, \mathcal{I})) \quad \text{and} \quad U \subseteq X \longmapsto P_{(m)}^n(\Gamma(U, \mathcal{I}))$$

are sheaves of quasi-coherent $\mathcal{O}_X$-modules which we denote by $\mathcal{P}_{(m)}(\mathcal{I})$ and $\mathcal{P}_{(m)}^n(\mathcal{I})$, respectively. In a completely analogous way, we can define a canonical ideal $\bar{\mathcal{I}}$ of $\mathcal{P}_{(m)}(\mathcal{I})$, a sub-PD-ideal $(\widetilde{\mathcal{I}}, [\,]) \subset \bar{\mathcal{I}}$, and the sequence of ideals $(\bar{\mathcal{I}}^{\{n\}})_{n \in \mathbb{N}}$ defining the m-PD-filtration. Those are also quasi-coherent sheaves [5, §1.4].

Now, let us consider the diagonal embedding $\Delta : X \hookrightarrow X \times_{\mathbb{Z}_p} X$ and let $W \subset X \times_{\mathbb{Z}_p} X$ be an open subset such that $X \subset W$ is a closed subset, defined by a quasi-coherent sheaf $\mathcal{I} \subset \mathcal{O}_W$. For every $n \in \mathbb{N}$, the algebra $\mathcal{P}^n_{X,(m)} := \mathcal{P}^n_{(m)}(\mathcal{I})$ is quasi-coherent and its support is contained in X. In particular, it is independent of the open subset W [5, 2.1]. Moreover, by Proposition 2.4 the projections $p_1, p_2 : X \times_{\mathbb{Z}_p} X \to X$ induce two morphisms $d_1, d_2 : \mathcal{O}_X \to \mathcal{P}^n_{X,(m)}$ endowing $\mathcal{P}^n_{X,(m)}$ of a *left* and a *right* structure of $\mathcal{O}_X$-algebra, respectively.

Definition 2.7. Let m, n be positive integers. The sheaf of differential operators of level m and order less or equal to n on X is defined by

$$\text{(2.1)} \qquad \mathcal{D}^{(m)}_{X,n} := \mathcal{H}om_{\mathcal{O}_X}(\mathcal{P}^n_{X,(m)}, \mathcal{O}_X).$$

If $n \leq n'$ corollary 2.6 gives us a canonical surjection

$$\mathcal{P}^{n'}_{X,(m)} \longrightarrow \mathcal{P}^n_{X,(m)}$$

inducing the injection $\mathcal{D}^{(m)}_{X,n} \hookrightarrow \mathcal{D}^{(m)}_{X,n'}$. The sheaf of *differential operators of level m* is defined by

$$\text{(2.2)} \qquad \mathcal{D}^{(m)}_X := \bigcup_{n \in \mathbb{N}} \mathcal{D}^{(m)}_{X,n}.$$

We remark for the reader that by definition $\mathcal{D}^{(m)}_X$ is endowed with a natural filtration called the *order filtration*, and like the sheaves $\mathcal{P}^n_{X,(m)}$, the sheaves $\mathcal{D}^{(m)}_{X,n}$ are endowed with two natural structures of $\mathcal{O}_X$-modules. Moreover, the sheaf $\mathcal{D}^{(m)}_X$ acts on $\mathcal{O}_X$: if $P \in \mathcal{D}^{(m)}_{X,n}$, then this action is given by the composition $\mathcal{O}_X \xrightarrow{d_1} \mathcal{P}^n_{X,(m)} \xrightarrow{P} \mathcal{O}_X$.

Finally, let us give a local description of $\mathcal{D}^{(m)}_{X,n}$. Let U be a smooth open affine subset of X endowed with a family of local coordinates $x_1, \ldots, x_N$. Let $dx_1, \ldots, dx_N$ be a basis of $\Omega_X(U)$ and $\partial_{x_1}, \ldots, \partial_{x_N}$ the dual basis of $\mathcal{T}_X(U)$ (as usual, $\mathcal{T}_X$ and Ω_X denote the tangent and cotangent sheaf on X, respectively). Let $\underline{k} \in \mathbb{N}^N$. Let us denote by $|\underline{k}| = \sum_{i=1}^N k_i$ and $\partial_i^{[k_i]} = \partial_{x_i}/k_i!$ for every $1 \leq i \leq N$. Then, using multi-index notation, we have $\underline{\partial}^{[\underline{k}]} = \prod_{i=1}^N \partial_i^{[k_i]}$ and $\underline{\partial}^{\langle \underline{k} \rangle} = q_{\underline{k}}! \underline{\partial}^{[\underline{k}]}$. In this case, the sheaf $\mathcal{D}^{(m)}_{X,n}$ has the following description on U

$$\text{(2.3)} \qquad \mathcal{D}^{(m)}_{X,n}(U) = \left\{ \sum_{|\underline{k}| \leq n} a_{\underline{k}} \underline{\partial}^{\langle \underline{k} \rangle} \,\middle|\, a_{\underline{k}} \in \mathcal{O}_X(U) \text{ and } \underline{k} \in \mathbb{N}^N \right\}.$$

2.3. Symmetric Algebra of Finite Level. In this subsection we will focus on introducing the constructions in [20]. Let X be a $\mathbb{Z}_p$-scheme, $\mathcal{L}$

a locally free $\mathcal{O}_X$-module of finite rank, $\boldsymbol{S}_X(\mathcal{L})$ the symmetric algebra associated to $\mathcal{L}$ and $\mathcal{I}$ the ideal of homogeneous elements of degree 1. Using the notation of Section 2.1 we define

$$\Gamma_{X,(m)}(\mathcal{L}) := \mathcal{P}_{\boldsymbol{S}_X(\mathcal{L}),(m)}(\mathcal{I}) \quad \text{and} \quad \Gamma_{X,(m)}^n(\mathcal{L}) := \Gamma_{X,(m)}(\mathcal{L})/\overline{\mathcal{I}}^{\{n+1\}}.$$

Those algebras are graded [20, Proposition 1.3.3], and if $\eta_1, \ldots, \eta_N$ is a local basis of $\mathcal{L}$, we have

$$\Gamma_{X,(m)}^n(\mathcal{L}) = \bigoplus_{|\underline{l}| \leq n} \mathcal{O}_X \underline{\eta}^{\{\underline{l}\}}.$$

As before $\underline{\eta}^{\{\underline{l}\}} = \prod_{i=1}^N \eta_i^{\{l_i\}}$ and $q_i! \eta_i^{\{l_i\}} = \eta_i^{l_i}$. We define by duality

$$\mathrm{Sym}^{(m)}(\mathcal{L}) := \bigcup_{k \in \mathbb{N}} \mathcal{H}om_{\mathcal{O}_X}\left(\Gamma_{X,(m)}^k(\mathcal{L}^\vee), \mathcal{O}_X\right).$$

By [20, Propositions 1.3.1, 1.3.3 and 1.3.6] we may conclude that

$$\mathrm{Sym}^{(m)}(\mathcal{L}) = \bigoplus_{n \in \mathbb{N}} \mathrm{Sym}_n^{(m)}(\mathcal{L})$$

is a commutative graded algebra with noetherian sections over any open affine subset. Moreover, locally over a basis $\eta_1, \ldots, \eta_N$, we have

$$\mathrm{Sym}_n^{(m)}(\mathcal{L}) = \bigoplus_{|\underline{l}|=n} \mathcal{O}_X \underline{\eta}^{\langle \underline{l} \rangle}, \quad \text{where} \quad \frac{l_i!}{q_i!}\eta_i^{\langle l_i \rangle} = \eta_i^{l_i}.$$

Remark 2.8. By [7, A.10] we have that $\mathrm{Sym}^{(0)}(\mathcal{L})$ equals the symmetric algebra of $\mathcal{L}$. This justifies the terminology.

Finally, let $\mathcal{I}$ be the kernel of the comorphism $\Delta^\sharp$ associated to the diagonal embedding $\Delta : X \to X \times_{\mathrm{Spec}(\mathbb{Z}_p)} X$. In [20, Proposition 1.3.7.3] Huyghe shows that the graded algebra associated to the m-PD-adic filtration of $\mathcal{P}_{X,(m)}$ it is identified with the graded m-PD-algebra

$$\Gamma_{X,(m)}(\mathcal{I}/\mathcal{I}^2) = \Gamma_{X,(m)}(\Omega_X^1).$$

More exactly, we have canonical isomorphisms

$$\Gamma_{X,(m)}^n := \Gamma_{X,(m)}^n(\Omega_X^1) \xrightarrow{\cong} \mathrm{gr}_\bullet(\mathcal{P}_{X,(m)}^n)$$

which, by definition, induce a graded isomorphism of algebras

$$(2.4) \qquad\qquad \mathrm{Sym}^{(m)}(\mathcal{T}_X) \xrightarrow{\cong} \mathrm{gr}_\bullet \mathcal{D}_X^{(m)}.$$

2.4. Arithmetic Distribution Algebra of Finite Level. As in the introduction, let us consider $\mathbb{G}$ a split connected reductive group scheme over $\mathbb{Z}_p$ and $m \in \mathbb{N}$ fixed. We propose to give a description of the algebra of distributions of level m introduced in [23]. Let I denote the kernel of the surjective morphism of $\mathbb{Z}_p$-algebras $\epsilon_{\mathbb{G}} : \mathbb{Z}_p[\mathbb{G}] \to \mathbb{Z}_p$, given by the identity element of $\mathbb{G}$. We know that I/I^2 is a free $\mathbb{Z}_p = \mathbb{Z}_p[\mathbb{G}]/I$-module of finite rank. Let $t_1, \ldots, t_l \in I$ such that modulo I^2, these elements form a basis of I/I^2. The m-divided power enveloping of $(\mathbb{Z}_p[\mathbb{G}], I)$ (Proposition 2.4) denoted by $P_{(m)}(\mathbb{G})$, is a free $\mathbb{Z}_p$-module with basis the elements $\underline{t}^{\{\underline{k}\}} = t_1^{\{k_1\}} \ldots t_l^{\{k_l\}}$, where $q_i! t_i^{\{k_i\}} = t_i^{k_i}$, for every $k_i = p^m q_i + r_i$ and $0 \leq r_i < p^m$. These algebras are endowed with a decreasing filtration by ideals $\bar{I}^{\{n\}}$ (Section 2.1), such that $\bar{I}^{\{n\}} = \bigoplus_{|\underline{k}| \geq n} \mathbb{Z}_p \, \underline{t}^{\{\underline{k}\}}$. The quotients $P_{(m)}^n(\mathbb{G}) := P_{(m)}(\mathbb{G})/\bar{I}^{\{n+1\}}$ are therefore $\mathbb{Z}_p$-modules generated by the elements $\underline{t}^{\{\underline{k}\}}$ with $|\underline{k}| \leq n$ [5, Proposition 1.5.3 (ii)]. Moreover, there exists an isomorphism of $\mathbb{Z}_p$-modules

$$P_{(m)}^n(\mathbb{G}) \cong \bigoplus_{|\underline{k}| \leq n} \mathbb{Z}_p \, \underline{t}^{\{\underline{k}\}}.$$

Corollary 2.6 gives us for any two integers n, n' such that $n \leq n'$ a canonical surjection $\pi^{n',n} : P_{(m)}^{n'}(\mathbb{G}) \to P_{(m)}^n(\mathbb{G})$. Furthermore, for every $m' \geq m$, the universal property of the divided powers gives us a unique morphism of filtered $\mathbb{Z}_p$-algebras $\psi_{m,m'} : P_{(m')}(\mathbb{G}) \to P_{(m)}(\mathbb{G})$ which induces a homomorphism of $\mathbb{Z}_p$-algebras $\psi_{m,m'}^n : P_{(m')}^n(\mathbb{G}) \to P_{(m)}^n(\mathbb{G})$. The module of *distributions of level m and order n* is $D_n^{(m)}(\mathbb{G}) := \operatorname{Hom}(P_{(m)}^n(\mathbb{G}), \mathbb{Z}_p)$. *The algebra of distributions of level m is*

$$D^{(m)}(\mathbb{G}) := \varinjlim_n D_n^{(m)}(\mathbb{G}),$$

where the limit is formed respect to the morphisms $\operatorname{Hom}_{\mathbb{Z}_p}(\pi^{n',n}, \mathbb{Z}_p)$. The multiplication is defined as follows. By universal property (Corollary 2.6) there exists a canonical map $\delta^{n,n'} : P_{(m)}^{n+n'}(\mathbb{G}) \to P_{(m)}^n(\mathbb{G}) \otimes_{\mathbb{Z}_p} P_{(m)}^{n'}(\mathbb{G})$. If $(u,v) \in D_n^{(m)}(\mathbb{G}) \times D_{n'}^{(m)}(\mathbb{G})$, we define $u.v$ as the composition

$$u.v : P_{(m)}^{n+n'}(\mathbb{G}) \xrightarrow{\delta^{n,n'}} P_{(m)}^n(\mathbb{G}) \otimes_{\mathbb{Z}_p} P_{(m)}^{n'}(\mathbb{G}) \xrightarrow{u \otimes v} \mathbb{Z}_p.$$

Let us denote by $\mathfrak{g} := \operatorname{Hom}_{\mathbb{Z}_p}(I/I^2, \mathbb{Z}_p)$ the Lie algebra of $\mathbb{G}$. This is a free $\mathbb{Z}_p$-module with basis $\xi_1, \ldots, \xi_l$ defined as the dual basis of the elements $t_1, \ldots, t_l$. Moreover, if for every multi-index $\underline{k} \in \mathbb{N}^l$, $|\underline{k}| \leq n$, we denote by $\underline{\xi}^{\langle \underline{k} \rangle}$ the dual of the element $\underline{t}^{\{\underline{k}\}} \in P_{(m)}^n(\mathbb{G})$, then $D_n^{(m)}(\mathbb{G})$ is a free $\mathbb{Z}_p$-module of finite rank with a basis given by the elements $\underline{\xi}^{\langle \underline{k} \rangle}$ with $|\underline{k}| \leq n$ [23, Proposition 4.1.6].

Remark 2.9. Let us exemplify the local situation when $X = \mathrm{Spec}(A)$ with A a $\mathbb{Z}_p$-algebra [21, Section 1.3.1]. Let E be a free A-module of finite rank with base $(x_1, \ldots, x_N)$ and let $(y_1, \ldots, y_N)$ be the dual base of $E^\vee :=$ $\mathrm{Hom}_A(E, A)$. As in the preceding subsection, let $\boldsymbol{S}(E^\vee)$ be the symmetric algebra and $\boldsymbol{I}(E^\vee)$ the augmentation ideal. Let $\Gamma_{(m)}(E^\vee)$ be the m-PD-envelope of $(\boldsymbol{S}(E^\vee),\ \boldsymbol{I}(E^\vee))$. We put $\Gamma_{(m)}^n(E^\vee) := \Gamma_{(m)}(E^\vee)/\bar{I}^{\{n+1\}}$. These are free A-modules with base $y_1^{\{k_1\}} \ldots y_N^{\{k_N\}}$ with $\sum k_i \leq n$ [20, 1.1.2]. Let $\{\underline{x}^{\langle \underline{k} \rangle}\}_{|\underline{k}| \leq n}$ be the dual base of $\mathrm{Hom}_A(\Gamma_{(m)}^n(E^\vee), A)$. We have

$$\mathrm{Sym}^{(m)}(E) = \bigcup_{n \in \mathbb{N}} \mathrm{Hom}_A\Big(\Gamma_{(m)}^n(E^\vee), A\Big).$$

This is a free A-module with a base given by all the $\underline{x}^{\langle \underline{k} \rangle}$. The inclusion $\mathrm{Sym}^{(m)}(E) \subseteq \mathrm{Sym}^{(m)}(E) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ gives the relation

$$(2.5) \qquad\qquad\qquad x_i^{\langle k_i \rangle} = \frac{k_i!}{q_i!} x^{k_i}.$$

Moreover, it also has a structure of algebra defined as follows. By [20, Proposition 1.3.1] there exists a canonical map

$$\Delta_{n,n'} : \Gamma_{(m)}^{n+n'}(E^\vee) \longrightarrow \Gamma_{(m)}^n(E^\vee) \otimes_A \Gamma_{(m)}^{n'}(E^\vee),$$

which allows us to define the product of $u \in \mathrm{Hom}_A(\Gamma_{(m)}^n(E^\vee), A)$ and $v \in \mathrm{Hom}_A(\Gamma_{(m)}^{n'}(E^\vee), A)$ by the composition

$$u.v : \Gamma_{(m)}^{n+n'}(E^\vee) \xrightarrow{\Delta_{n,n'}} \Gamma_{(m)}^n(E^\vee) \otimes_A \Gamma_{(m)}^{n'}(E^\vee) \xrightarrow{u \otimes v} A.$$

This morphism endows $\mathrm{Sym}^{(m)}(E)$ of a structure of a graded noetherian $\mathbb{Z}_p$-algebra [20, Propositions 1.3.1, 1.3.3 and 1.3.6].

We have the following important properties [23, Proposition 4.1.15].

Proposition 2.10.
  (i) *We have a canonical isomorphism of graded $\mathbb{Z}_p$-algebras* $\mathrm{gr}_\bullet(D^{(m)}(\mathbb{G})) \cong \mathrm{Sym}^{(m)}(\mathfrak{g})$.
  (ii) *The $\mathbb{Z}_p$-algebras $\mathrm{gr}_\bullet(D^{(m)}(\mathbb{G}))$ and $D^{(m)}(\mathbb{G})$ are noetherian.*

2.5. The Infinitesimal Action. Throughout this paper $\mathfrak{g}$ denotes the Lie algebra of a connected reductive group scheme $\mathbb{G}$ and $\mathcal{U}(\mathfrak{g})$ its universal enveloping algebra. If $\mathfrak{g}_{\mathbb{Q}}$ denotes the Lie algebra of the algebraic group $\mathbb{G}_{\mathbb{Q}} = \mathbb{G} \times_{\mathrm{Spec}(\mathbb{Z}_p)} \mathrm{Spec}(\mathbb{Q}_p)$ and $\mathcal{U}(\mathfrak{g}_{\mathbb{Q}})$ its universal enveloping algebra, then it is known that $\mathcal{U}(\mathfrak{g}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = \mathcal{U}(\mathfrak{g}_{\mathbb{Q}})$. Moreover, the algebra of distributions of level m, introduced in the preceding subsection, it also satisfies this property. In other words, $D^{(m)}(\mathbb{G}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \cong \mathcal{U}(\mathfrak{g}_{\mathbb{Q}})$.

In the following discussion we will assume that X is a smooth $\mathbb{Z}_p$-scheme endowed with a right $\mathbb{G}$-action.

Proposition 2.11. *The right $\mathbb{G}$-action induces a canonical homomorphism of filtered $\mathbb{Z}_p$-algebras*

$$\Phi^{(m)} : D^{(m)}(\mathbb{G}) \longrightarrow H^0(X, \mathcal{D}_X^{(m)}).$$

Proof. The reader can find the proof of this proposition in [23, Proposition 4.4.1 (ii)], we will briefly discuss the construction of $\Phi^{(m)}$. The central idea in the construction is that if $\rho : X \times_{\mathbb{Z}_p} \mathbb{G} \to X$ denotes the action, then the comorphism $\rho^\natural : \mathcal{O}_X \to \mathcal{O}_X \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[\mathbb{G}]$ induces a morphism

$$\rho_m^{(n)} : \mathcal{P}_{X,(m)}^n \longrightarrow \mathcal{O}_X \otimes_{\mathbb{Z}_p} P_{(m)}^n(\mathbb{G})$$

for every $n \in \mathbb{N}$. Those maps are compatible when varying n. Taking $u \in D_n^{(m)}(\mathbb{G})$ we define $\Phi^{(m)}(u)$ by

$$\Phi^{(m)}(u) : \mathcal{P}_{X,(m)}^n \xrightarrow{\rho_m^{(n)}} \mathcal{O}_X \otimes_{\mathbb{Z}_p} P_{(m)}^n(\mathbb{G}) \xrightarrow{id \otimes u} \mathcal{O}_X.$$

Again, those maps are compatible when varying n and we get the morphism of the proposition. $\qquad\square$

Remark 2.12.

(i) If X is endowed with a left $\mathbb{G}$-action, then it turns out that $\Phi^{(m)}$ is an anti-homomorphism.

(ii) In [23, Theorem 4.4.8.3] Huyghe and Schmidt have shown that if $X = \mathbb{G}$ and we consider the right (resp. left) regular action, then the morphism of the preceding proposition is in fact a canonical filtered isomorphism (resp. an anti-isomorphism) between $D^{(m)}(\mathbb{G})$ and $H^0(\mathbb{G}, \mathcal{D}_{\mathbb{G}}^{(m)})^{\mathbb{G}}$, the $\mathbb{Z}_p$-submodule of (left) $\mathbb{G}$-invariant global sections (cf. Definition 3.5). This isomorphism induces a bijection between $D_n^{(m)}(\mathbb{G})$ and $H^0(\mathbb{G}, \mathcal{D}_{\mathbb{G},n}^{(m)})^{\mathbb{G}}$, and it is compatible when varying m.

Let us define $\mathcal{A}_X^{(m)} := \mathcal{O}_X \otimes_{\mathbb{Z}_p} D^{(m)}(\mathbb{G})$, which is a priori considered as a subsheaf of $\mathbb{Z}_p$-modules of the sheaf of $\mathbb{Q}_p$-vector spaces

$$\mathcal{U}^\circ := \mathcal{O}_{X_{\mathbb{Q}}} \otimes_{\mathbb{Q}_p} \mathcal{U}(\mathfrak{g}_{\mathbb{Q}})$$

under the identification $(\mathcal{A}_X^{(m)} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)|_{X_{\mathbb{Q}}} = \mathcal{U}^\circ$. We remark for the reader that we can endow $\mathcal{U}^\circ$ with the skew ring multiplication (see (2.7) below) coming from the action of $\mathcal{U}(\mathfrak{g}_{\mathbb{Q}})$ on $\mathcal{O}_{X_{\mathbb{Q}}}$ via a morphism of $\mathbb{Q}_p$-Lie algebras $\tau : \mathfrak{g}_{\mathbb{Q}} \to H^0(X_{\mathbb{Q}}, \mathcal{T}_{X_{\mathbb{Q}}})$ (see (2.6) below). This multiplication defines over $\mathcal{U}^\circ$ a structure of a sheaf of associative $\mathbb{Q}_p$-algebras, which preserves the submodule $\mathcal{A}_X^{(m)}$ and endows $\mathcal{A}_X^{(m)}$ with an associative ring structure.

In order to formalize this discussion, let us recall how the multiplicative structure of the sheaf $\mathcal{U}^\circ$ is defined (cf. [30, §2] or [33, §5.1]).

Differentiating the right action of $\mathbb{G}_\mathbb{Q}$ on $X_\mathbb{Q}$ we get a morphism of $\mathbb{Q}_p$-Lie algebras

$$(2.6) \qquad\qquad \tau : \; \mathfrak{g}_\mathbb{Q} \longrightarrow H^0(X_\mathbb{Q}, \mathcal{T}_{X_\mathbb{Q}}).$$

This implies that $\mathfrak{g}_\mathbb{Q}$ acts on $\mathcal{O}_{X_\mathbb{Q}}$ by derivations and we can endow $\mathcal{U}^\circ$ with the skew ring multiplication

$$(2.7) \qquad\qquad (f \otimes \eta)(g \otimes \zeta) = f\tau(\eta)g \otimes \zeta + fg \otimes \eta\zeta$$

for $\eta \in \mathfrak{g}_L$, $\zeta \in \mathcal{U}(\mathfrak{g}_\mathbb{Q})$ and $f, g \in \mathcal{O}_{X_\mathbb{Q}}$. With this product, the sheaf $\mathcal{U}^\circ$ becomes a sheaf of associative algebras and this multiplicative structure preserves the submodule $\mathcal{A}_X^{(m)}$. We have the following result from [23, Corollary 4.4.6].

Proposition 2.13.
  (i) *The sheaf $\mathcal{A}_X^{(m)}$ is a locally free $\mathcal{O}_X$-module.*
  (ii) *There exist a unique structure over $\mathcal{A}_X^{(m)}$ of filtered $\mathcal{O}_X$-rings and an isomorphism of graded $\mathcal{O}_X$-algebras $gr(\mathcal{A}_X^{(m)}) = \mathcal{O}_X \otimes_{\mathbb{Z}_p} \mathrm{Sym}^{(m)}(\mathfrak{g})$.*
  (iii) *The sheaf $\mathcal{A}_X^{(m)}$ (resp. $gr(\mathcal{A}_X^{(m)})$) is a coherent sheaf of $\mathcal{O}_X$-rings (resp. a coherent sheaf of $\mathcal{O}_X$-algebras), with noetherian sections over open affine subsets of X.*

Remark 2.14. If we take the tensor product with $\mathbb{Q}_p$ in Proposition 2.11, then by construction (proof of Proposition 2.11) we get the following commutative diagram

$$
\begin{array}{ccc}
D^{(m)}(\mathbb{G}) & \xrightarrow{\;\Phi^{(m)}\;} & H^0(X, \mathcal{D}_X^{(m)}) \\
\downarrow & & \downarrow \\
\mathcal{U}(\mathfrak{g}_\mathbb{Q}) & \xrightarrow{\;\Psi_{X_\mathbb{Q}}\;} & H^0(X_\mathbb{Q}, \mathcal{D}_{X_\mathbb{Q}})
\end{array}
$$

where $\Psi_{X_\mathbb{Q}}$ is the morphism induced by (2.6) and it is called the *operator-representation* [10].

3. Twisted Arithmetic Differential Operators

3.1. Torsors. Let us suppose that $\mathbb{T}$ is a smooth affine algebraic group over $\mathbb{Z}_p$ with Lie algebra denoted by $\mathfrak{t}$, and that $\widetilde{X}$ and X are smooth separated schemes over $\mathbb{Z}_p$, such that $\widetilde{X}$ is endowed with a right $\mathbb{T}$-action $\sigma : \widetilde{X} \times_{\mathrm{Spec}(\mathbb{Z}_p)} \mathbb{T} \to \widetilde{X}$. We will also assume that $\mathbb{T}$ acts trivially on X (for example if $\mathbb{T} \subset \mathbb{B}$ is a split maximal torus contained in the Borel group $\mathbb{B}$ and $X = \mathbb{G}/\mathbb{B}$ is the flag $\mathbb{Z}_p$-scheme), and that there exists a faithfully

flat and locally of finite type morphism $\xi : \widetilde{X} \to X$. We have the following definition (cf. [31, III, §4]).

Definition 3.1. We say that $\widetilde{X}$ is locally trivial $\mathbb{T}$-torsor for the Zariski topology, if X can be covered by a family $\{U_i\}_{i \in I}$ of affine open subschemes, such that for every $i \in I$ there exists a $\mathbb{T}$-equivariant isomorphism

$$(\xi^{-1}(U_i) \simeq) \ U_i \times_{X, \xi} \widetilde{X} \xrightarrow{\simeq} U_i \times_{\mathrm{Spec}(\mathbb{Z}_p)} \mathbb{T},$$

with $\mathbb{T}$ acting on $U_i \times_{\mathrm{Spec}(\mathbb{Z}_p)} \mathbb{T}$ by right translations on the second factor. The covering $\{U_i\}_{i \in I}$ is called a trvialisation.

Remark 3.2. In the next sections we will abuse of the notations and simple say that $\xi : \widetilde{X} \to X$ is a locally trivial $\mathbb{T}$-torsor.

Remark 3.3. In [11] the authors built the *relative enveloping algebra* for any fibration $X \to Y$ and any algebraic group $\mathbb{H}$. Under this generality, they needed to consider the more general notion of *torsor* for the *étale topology* or even for the *fpqc topology*. Although the constructions of Sections 3.2 and 3.3 are valid in this generality, we will only consider torsors which admit trivialisations for the *Zariski topology*; this because in this work we will always consider fibrations such as $\mathbb{G} \to \mathbb{G}/\mathbb{B}$, for the Borel subgroup $\mathbb{B} \subseteq \mathbb{G}$, or $\mathbb{G}/\boldsymbol{N} \to \mathbb{G}/\mathbb{B}$, for the split maximal torus $\mathbb{T} \subseteq \mathbb{B}$. Here $\boldsymbol{N}$ denotes the unipotent radical of $\mathbb{B}$, [26, (1), 1.8. Chapter II]. It is known that these fibrations admit trivializations for the Zariski topology (essentially, open immersions of open cells; to see the argument right after Remark 3.17).

3.2. $\mathbb{T}$-Equivariant Sheaves and Sheaves of $\mathbb{T}$-Invariant Sections.
Let us suppose for a moment that Y is a separated $\mathbb{Z}_p$-scheme endowed with a right $\mathbb{T}$-action $\sigma : Y \times_{\mathbb{Z}_p} \mathbb{T} \to Y$. Let us denote by $\mathrm{mult} : \mathbb{T} \times_{\mathrm{Spec}(\mathbb{Z}_p)} \mathbb{T} \to \mathbb{T}$ the group law of $\mathbb{T}$ and by

$$p_1 : Y \times_{\mathrm{Spec}(\mathbb{Z}_p)} \mathbb{T} \longrightarrow Y, \quad p_{1,2} : Y \times_{\mathrm{Spec}(\mathbb{Z}_p)} \mathbb{T} \times_{\mathrm{Spec}(\mathbb{Z}_p)} \mathbb{T} \longrightarrow Y \times_{\mathrm{Spec}(\mathbb{Z}_p)} \mathbb{T}$$

the respective projections. We will also denote by

$$f_1, \ f_2, \ f_3 : Y \times_{\mathrm{Spec}(\mathbb{Z}_p)} \mathbb{T} \times_{\mathrm{Spec}(\mathbb{Z}_p)} \mathbb{T} \longrightarrow Y$$

the morphisms defined by

$$f_1(x, t_1, t_2) = x, \quad f_2(x, t_1, t_2) = xt_1 \quad \text{and} \quad f_3(x, t_1, t_2) = xt_1t_2.$$

Following [32, Chapter 0, §3], we say that a couple $(\mathcal{M}, \Psi)$, consisting of an $\mathcal{O}_Y$-modules and an isomorphism

$$(3.1) \qquad\qquad\qquad \sigma^* \mathcal{M} \xrightarrow{\Psi} p_1^* \mathcal{M},$$

is $\mathbb{T}$-*equivariant*, if

$$(3.2) \qquad\qquad p_{1,2}^* \Psi \ \circ \ (\sigma \times \mathrm{id}_{\mathbb{T}}) \Psi = (\mathrm{id}_Y \times \mathrm{mult})^* \Psi.$$

Cocycle condition [19, (9.10.10)]. We will need the following lemmas.

Lemma 3.4. *Let* $(\mathcal{L}, \Psi)$ *be a* $\mathbb{T}$-*equivariant locally free* $\mathcal{O}_Y$-*module of finite rank. Then* $(\mathcal{L}^\vee, (\Psi^{-1})^\vee)$ *is also* $\mathbb{T}$-*equivariant.*

Proof. This is proved in [23, Proposition 3.3.2.1]. $\square$

Let $(\mathcal{M}, \Psi)$ be a $\mathbb{T}$-equivariant quasi-coherent $\mathcal{O}_Y$-module. In light of the Künneth formula [17, Theorem 6.7.8] we have a canonical isomorphism

$$H^0(Y \times_{\mathbb{Z}_p} \mathbb{T}, p_1^* \mathcal{M}) \cong H^0(Y, \mathcal{M}) \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[\mathbb{T}].$$

By composing the previous isomorphism with the map

$$H^0(Y, \mathcal{M}) \longrightarrow H^0(Y \times_{\mathbb{Z}_p} \mathbb{T}, \sigma^* \mathcal{M}) \overset{H^0(\Psi)}{\longrightarrow} H^0(Y \times_{\mathbb{Z}_p} \mathbb{T}, p_1^* \mathcal{M})$$

(the first one being induced via the canonical morphism $\mathcal{M} \to \sigma_* \sigma^* \mathcal{M}$) we obtain a morphism

$$\Delta : H^0(Y, \mathcal{M}) \longrightarrow H^0(Y, \mathcal{M}) \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[\mathbb{T}],$$

defining a structure of $\mathbb{T}$-comodule on $H^0(Y, \mathcal{M})$ [32, Chapter 1, §3, p. 32].

Definition 3.5. The $\mathbb{T}$-invariant elements of $H^0(Y, \mathcal{M})$ are the elements $P \in H^0(Y, \mathcal{M})$ such that $\Delta(P) = P \otimes 1$. This subspace will be denoted by $H^0(Y, \mathcal{M})^{\mathbb{T}}$.

1. Sheaf of $\mathbb{T}$-Invariant Sections. Let us suppose now that Y is a separated $\mathbb{Z}_p$-scheme endowed with a right $\mathbb{T}$-action $\sigma : Y \times_{\mathbb{Z}_p} \mathbb{T} \to Y$, and whose Zariski topology admits a basis $\mathcal{S}_Y$ consisting of affine open subsets which are invariant under the $\mathbb{T}$-action. Let $(\mathcal{M}, \Psi)$ be a $\mathbb{T}$-equivariant $\mathcal{O}_Y$-module. For every $U \in \mathcal{S}_Y$ the morphism σ gives rise to a right $\mathbb{T}$-action $\sigma_U : U \times_{\mathbb{Z}_p} \mathbb{T} \to U$ on $U \subseteq Y$. By pulling back Ψ under the inclusion $U \times_{\mathbb{Z}_p} \mathbb{T} \hookrightarrow Y \times_{\mathbb{Z}_p} \mathbb{T}$ we get an isomorphism $\Psi_U : \sigma_U^* \mathcal{M}|_U \to p_1^* \mathcal{M}|_U$ which satisfies the respective cocycle condition (3.2), and, as before, we obtain a comodule morphism

$$\Delta_U : \Gamma(U, \mathcal{M}) \longrightarrow \Gamma(U, \mathcal{M}) \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[\mathbb{T}].$$

As in Definition 3.5, we can define the $\mathbb{Z}_p$-submodule of $\mathbb{T}$-*invariant sections* on U by

$$(3.3) \qquad \Gamma(U, \mathcal{M})^{\mathbb{T}} := \{ P \in \Gamma(U, \mathcal{M}) \mid \Delta_U(P) = P \otimes 1 \}.$$

Now, let us take $V, U \in \mathcal{S}_Y$ satisfying $V \subseteq U$. By functoriality, we have $\text{rest}_V^U \otimes \text{id}_{\mathbb{Z}_p[T]} \circ \Delta_U = \Delta_V \circ \text{rest}_V^U$, and therefore the restriction map $\text{rest}_V^U : \Gamma(U, \mathcal{M})^{\mathbb{T}} \to \Gamma(V, \mathcal{M})^{\mathbb{T}}$ is well-defined. In order to prove that the previous arguments define an $\mathcal{S}_Y$-sheaf we need to verify the glueing condition, but this is clear because $\mathcal{M}$ is already a sheaf and the restriction maps are $\mathbb{T}$-equivariant. This construction induces a sheaf $(\mathcal{M})^{\mathbb{T}}$ over Y.

As an application, let us point out that if $\xi : \widetilde{X} \to X$ is a $\mathbb{T}$-torsor, then we dispose of a subsheaf of $\mathbb{T}$-invariant sections of the direct image

sheaf $\xi_*\mathcal{M}$, with $\mathcal{M}$ a $\mathbb{T}$-equivariant $\mathcal{O}_{\widetilde{X}}$-module. In fact, if $\mathcal{S}$ denotes the collection of all affine open subsets that trivialises the torsor ξ, then for every $U \in \mathcal{S}$ we know that $\xi^{-1}(U)$ is stable under the right $\mathbb{T}$-action and, as in (3.3), we can define

$$\Gamma(U, \xi_*\mathcal{M})^{\mathbb{T}} := \{P \in \Gamma(U, \xi_*\mathcal{M}) \mid \Delta_U(P) = P \otimes 1\}.$$

As before, this process defines an $\mathcal{S}$-sheaf and therefore we get a subsheaf of $\mathbb{T}$-*invariant sections*

(3.4) $$(\xi_*\mathcal{M})^{\mathbb{T}} \subseteq \xi_*\mathcal{M}.$$

Definition 3.6. Let Y be a smooth separated $\mathbb{Z}_p$-scheme endowed with a right $\mathbb{T}$-action, which admits a basis $\mathcal{S}_Y$ for the Zariski topology consisting of $\mathbb{T}$-stable affine subschemes. For every $\mathbb{T}$-equivariant $\mathcal{O}_Y$-module $\mathcal{M}$, the subsheaf $(\mathcal{M})^{\mathbb{T}}$ is called the subsheaf of $\mathbb{T}$-invariant sections of $\mathcal{M}$.

For the rest of this subsection we will always suppose that $\xi : \widetilde{X} \to X$ is a $\mathbb{T}$-torsor.

Lemma 3.7. *If $\xi : \widetilde{X} \to X$ is a $\mathbb{T}$-torsor, then ξ induces an isomorphism* $\xi^{\natural} : \mathcal{O}_X \to \left(\xi_*\mathcal{O}_{\widetilde{X}}\right)^{\mathbb{T}}$.

Proof. This is exactly as in [1, Lemma 4.3]. $\qquad\square$

Lemma 3.8. *Let $\xi : \widetilde{X} \to X$ be a $\mathbb{T}$-torsor and $\mathcal{M}$ be a $\mathbb{T}$-equivariant quasi-coherent $\mathcal{O}_{\widetilde{X}}$-module, then $(\xi_*\mathcal{M})^{\mathbb{T}}$ is a quasi-coherent $\mathcal{O}_X$-module.*

Proof. By definition and the previous lemma, it is enough to show that the $\mathbb{T}$-action respects the $\xi_*\mathcal{O}_{\widetilde{X}}$-structure of $\xi_*\mathcal{M}$. This can be proved locally over an affine open subset $U \in \mathcal{S}$. In this case, we know that $\xi^{-1}(U)$ is endowed with a $\mathbb{T}$-action and therefore $\Gamma(U, \xi_*\mathcal{M})^{\mathbb{T}}$ is a $\Gamma(U, \xi_*\mathcal{O}_{\widetilde{X}})$-module by [37, 1.4]. $\qquad\square$

3.3. Relative Enveloping Algebras of Finite Level. Let $m \in \mathbb{Z}_{>0}$ be a positive integer. We start this section by giving a description of the sheaf of level m arithmetic differential operators on a fiber product. We will use these arguments to endow the sheaf $\mathcal{D}^{(m)}_{\widetilde{X}}$ with a $\mathbb{T}$-equivariant structure, which will allow us to consider the subsheaf

$$(\xi_*\mathcal{D}^{(m)}_{\widetilde{X}})^{\mathbb{T}} \subseteq \xi_*\mathcal{D}^{(m)}_{\widetilde{X}}$$

of $\mathbb{T}$-invariant sections. We will also use the following tools to give a local description of this subsheaf (Proposition 3.14). The reader will find more details of 1, here below, in [23, §2.2.2].

1. Tensor Product Filtration. Let $\mathcal{A}$ be a filtered sheaf of commutative rings on a topological space Y [9, A: III. 2]. Let $\mathcal{M}$ and $\mathcal{N}$ be filtered $\mathcal{A}$-modules [9, A: III. 2.5]. The sheaf of $\mathcal{A}$-modules $\mathcal{M} \otimes_{\mathcal{A}} \mathcal{N}$ carries a natural

filtration called *the tensor product filtration* and it is defined as follows. Let $n \in \mathbb{N}$ fix. For every $U \subset Y$ we let $F_n(\mathcal{M}(U) \otimes_{\mathcal{A}(U)} \mathcal{N}(U))$ be the abelian subgroup of $\mathcal{M}(U) \otimes_{\mathcal{A}(U)} \mathcal{N}(U)$ generated by elements of type $x \otimes y$ with $x \in \mathcal{M}_l(U)$, $y \in \mathcal{N}_s(U)$, and such that $l + s \leq n$. This process defines a presheaf on Y and we let $F_n(\mathcal{M} \otimes_{\mathcal{A}} \mathcal{N})$ be its sheafification. The sheaf $\mathcal{M} \otimes_{\mathcal{A}} \mathcal{N}$ becomes therefore a filtered sheaf of $\mathcal{A}$-modules

$$F_0(\mathcal{M} \otimes_{\mathcal{A}} \mathcal{N}) \subseteq \cdots \subseteq F_n(\mathcal{M} \otimes_{\mathcal{A}} \mathcal{N}) \subseteq \cdots \subseteq \mathcal{M} \otimes_{\mathcal{A}} \mathcal{N}.$$

Furthermore, for every open subset $U \subset Y$ we have a canonical map

$$\mathrm{gr}_\bullet(\mathcal{M}(U)) \otimes_{\mathrm{gr}_\bullet(\mathcal{A}(U))} \mathrm{gr}_\bullet(\mathcal{N}(U)) \longrightarrow \mathrm{gr}_\bullet(\mathcal{M}(U) \otimes_{\mathcal{A}(U)} \mathcal{N}(U))$$

defined by $x_{(l)} \otimes y_{(s)} \to (x \otimes y)_{l+s}$, where $x \in F_l \mathcal{M}(U) \setminus F_{l-1} \mathcal{M}(U)$, $y \in F_s \mathcal{N}(U) \setminus F_{s-1} \mathcal{N}(U)$, $x_{(l)} := x + F_{l-1} \mathcal{M}(U)$ and $y_{(s)} := y + F_{s-1} \mathcal{N}(U)$. These morphisms are compatible under restrictions and we get a morphism of graded sheaves

$$(3.5) \qquad \mathrm{gr}_\bullet(\mathcal{M}) \otimes_{\mathrm{gr}_\bullet(\mathcal{A})} \mathrm{gr}_\bullet(\mathcal{N}) \longrightarrow \mathrm{gr}_\bullet (\mathcal{M} \otimes_{\mathcal{A}} \mathcal{N}),$$

which is surjective by [28, §I, 6.13].

1. Let $n \in \mathbb{N}$ fix. Let Y_1 and Y_2 be smooth $\mathbb{Z}_p$ schemes and $Z = Y_1 \times_{\mathbb{Z}_p} Y_2$. Let p_1 and p_2 be the projections. By [5, (2.1.4.3)] we have two canonical $\mathcal{O}_Z$-linear morphisms

$$d^n p_1 : p_1^* \mathcal{P}_{Y_1,(m)}^n \longrightarrow \mathcal{P}_{Z,(m)}^n \quad \text{and} \quad d^n p_2 : p_2^* \mathcal{P}_{Y_2,(m)}^n \longrightarrow \mathcal{P}_{Z,(m)}^n.$$

For $i \in \{1, 2\}$, let $\mathcal{J}_i$ be the m-PD-ideal of the m-PD-algebra $\mathcal{P}_{Y_i,(m)}^n$. We have canonical m-PD-morphisms

$$s_i : \mathcal{P}_{Z,(m)}^n \longrightarrow \mathcal{P}_{Z,(m)}^n / p_i^*(\mathcal{J}_i).$$

Let us give a local description of the sheaves involved in the previous morphisms. Let $t_1^{(1)}, \ldots, t_{N_1}^{(1)}$ and $t_1^{(2)}, \ldots, t_{N_2}^{(2)}$ be local coordinates on Y_1 and Y_2, respectively. Considering

$$p_1^*(t_1^{(1)}), \ldots, p_1^*(t_{N_1}^{(1)}), \ p_2^*(t_1^{(2)}), \ldots, p_2^*(t_{N_2}^{(2)})$$

we get a coordinated system on Z.

On the other hando, if we denote by $\tau_i^{(1)} := 1 \otimes t_i^{(1)} - t_i^{(1)} \otimes 1$ for $1 \leq i \leq N_1$, and $\tau_i^{(2)} := 1 \otimes t_i^{(2)} - t_i^{(2)} \otimes 1$ for $1 \leq i \leq N_2$, we can locally identify

$$\mathcal{P}_{Y_i,(m)}^n = \bigoplus_{|\underline{\nu_i}| \leq n} \mathcal{O}_{Y_i}(\underline{\tau^{(i)}})^{\{\underline{\nu_i}\}} \quad \text{with } i \in \{1, 2\}, \text{ and}$$

$$(3.6)$$

$$\mathcal{P}_{Z,(m)}^n = \bigoplus_{|\underline{\nu_1}| + |\underline{\nu_2}| \leq n} \mathcal{O}_Z p_1^*((\underline{\tau^{(1)}})^{\{\underline{\nu_1}\}}) p_2^*((\underline{\tau^{(2)}})^{\{\underline{\nu_2}\}}).$$

Under the previous identification we have that $p_i^*(\mathcal{J}_i)$ can be identified with the $\mathcal{O}_Z$-submodule generated by the $p_1^*((\tau^{(1)})^{\{\underline{\nu_1}\}})p_2^*((\tau^{(2)})^{\{\underline{\nu_2}\}})$ with $|\underline{\nu_i}| \geq 1$, which shows that

$$\mathcal{P}^n_{Z,(m)}/p_i^*(\mathcal{J}_i) = \bigoplus_{|\underline{\nu_{\sigma(i)}}| \leq n} \mathcal{O}_Z p_{\sigma(i)}^*((\tau^{(\sigma(i))})^{\{\underline{\nu_{\sigma(i)}}\}}),$$

where $\sigma(1) = 2$ and $\sigma(2) = 1$. Therefore $\rho_i := s_{\sigma(i)} \circ d^n p_i$ is an m-PD-isomorphism, and we have a canonical section of $d^n p_i$ defined by

$$(3.7) \qquad q_i^n := \rho_i^{-1} \circ s_{\sigma(i)} : \mathcal{P}^n_{Z,(m)} \longrightarrow p_i^* \mathcal{P}^n_{Y_i,(m)}.$$

Now, for every $n \in \mathbb{N}$, the morphisms (3.7) induce two maps

$$\mathcal{H}om_{\mathcal{O}_Z}(p_i^*\mathcal{P}^n_{Y_i,(m)}, \mathcal{O}_Z) \longrightarrow \mathcal{H}om_{\mathcal{O}_Z}(\mathcal{P}^n_{Z,(m)}, \mathcal{O}_Z),$$

and (3.6) allows us to identify $\mathcal{H}om_{\mathcal{O}_Z}(p_i^*\mathcal{P}^n_{Y_i,(m)}, \mathcal{O}_Z) = p_i^*\mathcal{D}^{(m)}_{Y_i,n}$. Taking co-limits we get two morphisms of $\mathbb{Z}_p$-algebras $p_i^*\mathcal{D}^{(m)}_{Y_i} \to \mathcal{D}^{(m)}_Z$, which induce a canonical morphism of graded $\mathbb{Z}_p$-algebras

$$(3.8) \qquad p_1^*\mathcal{D}^{(m)}_{Y_1} \otimes_{\mathcal{O}_Z} p_2^*\mathcal{D}^{(m)}_{Y_2} \longrightarrow \mathcal{D}^{(m)}_Z.$$

The filtration on the left-hand side is the tensor product filtration (1). Passing to local coordinates, as before, we see that (3.8) is in fact a filtered isomorphism.

Let us go back to the situation in which we are interested where $\widetilde{X}$ and X denote smooth separated $\mathbb{Z}_p$-schemes, $\mathbb{T}$ a smooth affine commutative group $\mathbb{Z}_p$-scheme, and $\xi : \widetilde{X} \to X$ is a $\mathbb{T}$-torsor. Let $p_1 : \widetilde{X} \times_{\mathrm{Spec}(\mathbb{Z}_p)} \mathbb{T} \to \widetilde{X}$ be the projection. By [5, (2.4.3.1)] we have, for every $n \in \mathbb{N}$, an $\mathcal{O}_{\widetilde{X} \times_{\mathbb{Z}_p} \mathbb{T}}$-linear morphism

$$d^n \sigma : \sigma^* \mathcal{P}^n_{\widetilde{X},(m)} \longrightarrow \mathcal{P}^n_{\widetilde{X} \times_{\mathbb{Z}_p} \mathbb{T},(m)}.$$

The $\mathbb{T}$-equivariant structure of $\mathcal{P}^n_{\widetilde{X}}$ is then defined by $\Phi^n_{(m)} := q_1^n \circ d^n \sigma$ [23, Proposition 3.4.1]. Definition 2.7 and Lemma 3.4 allow us to conclude that for every $n \in \mathbb{N}$ the sheaf $\mathcal{D}^{(m)}_{\widetilde{X},n}$ is $\mathbb{T}$-equivariant and the inclusions

$$\mathcal{D}^{(m)}_{\widetilde{X},n} \longhookrightarrow \mathcal{D}^{(m)}_{\widetilde{X},n+1}$$

are $\mathbb{T}$-equivariant morphisms. In particular, by (2.2), the sheaf of level m differential operators is $\mathbb{T}$-equivariant.

Remark 3.9 (Notation as at the end of Section 2.3)**.** Following the preceding lines of reasoning we can also show that, for every $n \in \mathbb{N}$, there exists an m-PD-morphism

$$q_1'^n : \Gamma^n_{\widetilde{X} \times \mathbb{T},(m)} \longrightarrow p_1^* \Gamma^n_{\widetilde{X},(m)}.$$

which is a section of the m-PD-morphism induced by p_1 [23, §2.2.2]. Let

$$\Phi'^n := \sigma^* \Gamma^n_{\widetilde{X},(m)} \xrightarrow{\Gamma^n(\sigma)} \Gamma^n_{\widetilde{X} \times \mathbb{T},(m)} \xrightarrow{q_1'^n} \Gamma^n_{\widetilde{X} \times \mathbb{T},(m)}$$

where $\Gamma^n(\sigma)$ is the canonical m-PD-morphism induced by σ. Then Φ'^n is a $\mathbb{T}$-equivariant structure for $\Gamma^n_{\widetilde{X},(m)}$. As before, this implies that $\mathrm{Sym}^{(m)}(\mathcal{T}_{\widetilde{X}})$ is $\mathbb{T}$-equivariant.

Remark 3.10. Although it is well-known that the tangent sheaf $\mathcal{T}_{\widetilde{X}}$ carries a $\mathbb{T}$-equivariant structure, we point out to the reader that since the category of $\mathbb{T}$-equivariant quasi-coherent sheaves is an abelian category [29, Lemma 29.4], and also

$$\mathcal{P}^0_{\widetilde{X},(m)} = \mathcal{O}_{\widetilde{X}} \quad \text{and} \quad \mathcal{P}^1_{\widetilde{X},(m)} = \mathcal{O}_{\widetilde{X}} \oplus \Omega^1_{\widetilde{X}}$$

then Lemma 3.4 gives us the $\mathbb{T}$-equivariance of $\mathcal{T}_{\widetilde{X}}$. In particular, we dispose of the sheaves $(\mathcal{T}_{\mathbb{T}})^{\mathbb{T}}$ and $(\xi_* \mathcal{T}_{\widetilde{X}})^{\mathbb{T}}$.

Let us recall the following discussion from [1, §4.4]. Let us suppose $U \in \mathcal{S}$ and $\tau \in \mathcal{T}_{\widetilde{X}}(\xi^{-1}(U))^{\mathbb{T}}$. This assumption in particular implies that τ is a $\mathbb{T}$-invariant vector field on $\xi^{-1}(U)$ and therefore a $\mathbb{T}$-invariant endomorphism of $\mathcal{O}_{\widetilde{X}}(\xi^{-1}(U))$. Hence it preserves $\mathcal{O}_X(\xi^{-1}(U))^{\mathbb{T}}$ and by Lemma 3.7 it induces a vector field $\nu(\tau) \in \mathcal{T}_X(U)$. We get then a morphism of $\mathcal{O}_X$-modules

$$\nu : \left(\xi_* \mathcal{T}_{\widetilde{X}} \right)^{\mathbb{T}} \longrightarrow \mathcal{T}_X.$$

On the other hand, differentiating the right $\mathbb{T}$-action on $\widetilde{X}$ we obtain a $\mathbb{Z}_p$-linear Lie homomorphism $\mathfrak{t} \to \mathcal{T}_{\widetilde{X}}$, which induces a morphism of $\mathcal{O}_X$-modules

$$\mathfrak{t} \otimes_{\mathbb{Z}_p} \mathcal{O}_X \longrightarrow \left(\xi_* \mathcal{T}_{\widetilde{X}} \right)^{\mathbb{T}}.$$

Both morphisms fit into a sequence of $\mathcal{O}_X$-modules

$$\mathfrak{t} \otimes_{\mathbb{Z}_p} \mathcal{O}_X \longrightarrow \left(\xi_* \mathcal{T}_{\widetilde{X}} \right)^{\mathbb{T}} \xrightarrow{\nu} \mathcal{T}_X$$

which is functorial in $\widetilde{X}$ ([1, §4.4] or [11, p. 187]). The reader can find the proof of the following lemma in [1, Lemma 4.4].

Lemma 3.11. *If $\xi : \widetilde{X} \to X$ is a $\mathbb{T}$-torsor, then the restriction of the previous sequence to any $U \in \mathcal{S}$ is split exact.*

Remark 3.12. Lemma 3.11 shows that $(\xi_* \mathcal{T}_{\widetilde{X}})^{\mathbb{T}}$ is a locally free $\mathcal{O}_X$-module of finite rank. In particular $\mathrm{Sym}^{(m)}((\xi_* \mathcal{T}_{\widetilde{X}})^{\mathbb{T}})$ is well-defined.

Definition 3.13. Let $\xi : \widetilde{X} \to X$ be a $\mathbb{T}$-torsor. We define the level m relative enveloping algebra of the torsor to be the sheaf of $\mathbb{T}$-invariants sections of $\xi_* \mathcal{D}_{\widetilde{X}}^{(m)}$:

$$\widetilde{\mathcal{D}^{(m)}} := \left(\xi_* \mathcal{D}_{\widetilde{X}}^{(m)} \right)^{\mathbb{T}}.$$

The preceding sheaf is endowed with a canonical filtration

$$(3.9) \qquad \mathrm{Fil}_d \left(\widetilde{\mathcal{D}^{(m)}} \right) = \left(\xi_* \mathcal{D}_{\widetilde{X}, d} \right)^{\mathbb{T}}.$$

Proposition 3.14. *For any $U \in \mathcal{S}$ there exists an isomorphism of sheaves of filtered $\mathbb{Z}_p$-algebras*

$$\widetilde{\mathcal{D}^{(m)}}|_U \xrightarrow{\cong} \mathcal{D}_X^{(m)}|_U \otimes_{\mathbb{Z}_p} D^{(m)}(\mathbb{T}).$$

Proof. Let $U \in \mathcal{S}$ and let $\xi^{-1}(U) \cong U \times_{\mathrm{Spec}(\mathbb{Z}_p)} \mathbb{T}$ be a trivialization of ξ over U. We obtain the following isomorphisms of filtered $\mathbb{Z}_p$-algebras

$$\left(\xi_* \mathcal{D}_{\widetilde{X}}^{(m)} \right)^{\mathbb{T}} (U) = \mathcal{D}_{\widetilde{X}}^{(m)}(\xi^{-1}(U))^{\mathbb{T}} \cong \mathcal{D}_{U \times \mathbb{T}}^{(m)}(U \times \mathbb{T})^{\mathbb{T}}$$
$$\cong \mathcal{D}_X^{(m)}(U) \otimes_{\mathbb{Z}_p} H^0(\mathbb{T}, \mathcal{D}_{\mathbb{T}}^{(m)})^{\mathbb{T}} \cong \mathcal{D}_X^{(m)}(U) \otimes_{\mathbb{Z}_p} D^{(m)}(\mathbb{T})$$

where the first isomorphism follows from the fact that U trivializes the $\mathbb{T}$-torsor ξ, the second isomorphism comes from (3.8) and the Kunneth formula [17, Theorem 6.7.8]), and the third isomorphism is given by (ii) in Remark 2.12. Since the previous isomorphisms are compatible with restrictions to open affine subsets contained in U, we obtain the desired isomorphism of sheaves of filtered $\mathbb{Z}_p$-algebras. □

Lemma 3.15. *For every $m \in \mathbb{N}$, there exists a canonical isomorphism of sheaves of graded $\mathcal{O}_X$-algebras*

$$\mathrm{Sym}^{(m)} \left(\left(\xi_* \mathcal{T}_{\widetilde{X}} \right)^{\mathbb{T}} \right) \cong \left(\xi_* \left(\mathrm{Sym}^{(m)} \left(\mathcal{T}_{\widetilde{X}} \right) \right) \right)^{\mathbb{T}}.$$

Proof. We start the proof by remarking for the reader that, by universal property of the symmetric algebra, the canonical map of $\mathcal{O}_X$-modules $(\xi_* \mathcal{T}_{\widetilde{X}})^{\mathbb{T}} \to (\xi_* \boldsymbol{S}(\mathcal{T}_{\widetilde{X}}))^{\mathbb{T}}$ induces a canonical global morphism of graded $\mathcal{O}_X$-algebras

$$(3.10) \qquad \varphi : \boldsymbol{S} \left(\left(\xi_* \mathcal{T}_{\widetilde{X}} \right)^{\mathbb{T}} \right) \longrightarrow \left(\xi_* \boldsymbol{S} \left(\mathcal{T}_{\widetilde{X}} \right) \right)^{\mathbb{T}}.$$

We want to see that (3.10) induces the isomorphism stated in the lemma for every $m \in \mathbb{N}$. The idea will be to locally construct the isomorphism and then to globalise using (3.10). Let us take $m \in \mathbb{N}$ arbitrary and $U \in \mathcal{S}$. We

have a commutative diagram

$$\widetilde{U} := \xi^{-1}(U) \xrightarrow{\quad \cong \quad} U \times_{\mathbb{Z}_p} \mathbb{T}$$

$$\xi \searrow \quad U \quad \swarrow p_1$$

which tells us that (cf. [18, §II, Exercise 8.3])

$$(3.11) \qquad \mathcal{T}_{\widetilde{U}} = \xi^* \mathcal{T}_U \oplus p_2^* \mathcal{T}_{\mathbb{T}} = \xi^* \mathcal{T}_U \oplus \left(\mathcal{O}_{\widetilde{U}} \otimes_{\mathbb{Z}_p} \mathfrak{t} \right).$$

By Lemma 3.11, we have

$$(3.12) \qquad \operatorname{Sym}^{(m)} \left(\left(\xi_* \mathcal{T}_{\widetilde{X}} \right)^{\mathbb{T}} \right)(U) = \operatorname{Sym}^{(m)} \left(\left(\xi_* \mathcal{T}_{\widetilde{X}}(U) \right)^{\mathbb{T}} \right)$$

$$= \operatorname{Sym}^{(m)} \left(\mathcal{T}_U(U) \oplus \left(\mathcal{O}_U(U) \otimes_{\mathbb{Z}_p} \mathfrak{t} \right) \right)$$

On the other hand, by (3.11) and [20, Proposition 1.3.5] we have the relation

$$(3.13) \qquad \operatorname{Sym}^{(m)}(\mathcal{T}_{\widetilde{U}}) = \operatorname{Sym}^{(m)} (\xi^* \mathcal{T}_U) \otimes_{\mathcal{O}_{\widetilde{U}}} \operatorname{Sym}^{(m)} \left(\mathcal{O}_{\widetilde{U}} \otimes_{\mathbb{Z}_p} \mathfrak{t} \right)$$

$$= \xi^* \operatorname{Sym}^{(m)} (\mathcal{T}_U) \otimes_{\mathcal{O}_{\widetilde{U}}} \operatorname{Sym}^{(m)} \left(\mathcal{O}_{\widetilde{U}} \otimes_{\mathbb{Z}_p} \mathfrak{t} \right)$$

which implies, by the projection formula [18, Ch. II, §5, Exercise 5.1 (d)], that

$$(3.14) \qquad \xi_* \operatorname{Sym}^{(m)}(\mathcal{T}_{\widetilde{U}}) = \operatorname{Sym}^{(m)} (\mathcal{T}_U) \otimes_{\mathcal{O}_U} \xi_* \operatorname{Sym}^{(m)} \left(\mathcal{O}_{\widetilde{U}} \otimes_{\mathbb{Z}_p} \mathfrak{t} \right).$$

Taking $\mathbb{T}$-invariants and sections on U we get

$$(3.15) \quad \left(\xi_* \operatorname{Sym}^{(m)}(\mathcal{T}_{\widetilde{U}}) \right)^{\mathbb{T}}(U)$$

$$= \operatorname{Sym}^{(m)} (\mathcal{T}_U(U)) \otimes_{\mathcal{O}_U(U)} \operatorname{Sym}^{(m)}(\mathcal{O}_U(U) \otimes_{\mathbb{Z}_p} \mathfrak{t}).$$

By [20, Proposition 1.3.5], we have that (3.12) and (3.15) are canonically isomorphic, so in order to globalize this map, which we denote by $\varphi_U^{(m)}$, we need to check that the following diagram is commutative

$$\boldsymbol{S}^{(m)}(\mathcal{T}_U(U) \oplus (\mathcal{O}_U(U) \otimes_{\mathbb{Z}_p} \mathfrak{t})) \hookrightarrow \boldsymbol{S} (\mathcal{T}_U(U) \oplus (\mathcal{O}_U(U) \otimes_{\mathbb{Z}_p} \mathfrak{t})) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$$

$$\wr \downarrow \varphi_U^{(m)} \qquad\qquad\qquad \wr \downarrow \varphi_U \otimes_{\mathbb{Z}_p} 1_{\mathbb{Q}_p}$$

$$\boldsymbol{S}^{(m)} (\mathcal{T}_U(U)) \otimes_{\mathcal{O}_U} \boldsymbol{S}^{(m)} (\mathcal{O}_U(U) \otimes_{\mathbb{Z}_p} \mathfrak{t}) \longrightarrow \boldsymbol{S} (\mathcal{T}_U(U)) \otimes_{\mathcal{O}_U} \boldsymbol{S} (\mathcal{O}_U(U) \otimes_{\mathbb{Z}_p} \mathfrak{t}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.$$

Here we have used the notation $\operatorname{Sym}^{(m)} = \boldsymbol{S}^{(m)}$. Shrinking U if necessary, we can suppose that U is endowed with a set of local coordinates $x_1, \ldots, x_N$, in such a way that $\mathcal{T}_U(U)$ is generated as $\mathcal{O}_U(U)$-module by the derivations $\partial_{x_1}, \ldots, \partial_{x_N}$. Furthermore, if $\zeta_1, \ldots, \zeta_l$ denotes a $\mathbb{Z}_p$-basis of $\mathfrak{t}$, then $\operatorname{Sym}^{(m)} \left(\mathcal{T}_U(U) \oplus \left(\mathcal{O}_U(U) \otimes_{\mathbb{Z}_p} \mathfrak{t} \right) \right)$ is generated (as $\mathcal{O}_U(U)$-module) by all

the elements of the form $\underline{\partial}^{\langle \underline{k} \rangle} \cdot \underline{\zeta}^{\langle \underline{v} \rangle}$ (here we use the multi-index notation introduced in sections 2.2 and 2.4.). In particular

$$\varphi_U^{(m)} \left(\underline{\partial}^{\langle \underline{k} \rangle} \cdot \underline{\zeta}^{\langle \underline{v} \rangle} \right) = \frac{\underline{k}!}{q_{\underline{k}}!} \frac{\underline{v}!}{q_{\underline{v}}!} \, \underline{\partial}^{\underline{k}} \otimes_{\mathbb{Q}_p} \underline{\zeta}^{\underline{v}} = \varphi_U \otimes_{\mathbb{Z}_p} 1_{\mathbb{Q}_p} \left(\underline{\partial}^{\langle \underline{k} \rangle} \cdot \underline{\zeta}^{\langle \underline{v} \rangle} \right).$$

This shows that the previous diagram is commutative and ends the proof of the lemma. $\qquad\square$

Proposition 3.16. *If ξ is a locally trivial $\mathbb{T}$-torsor, then there exists a canonical and graded isomorphism*

$$\mathrm{Sym}^{(m)} \left(\left(\xi_* \mathcal{T}_{\widetilde{X}} \right)^{\mathbb{T}} \right) \xrightarrow{\cong} \mathrm{gr}_\bullet \left(\widetilde{\mathcal{D}^{(m)}} \right).$$

Proof. Applying ξ_* to (2.4) and then taking $\mathbb{T}$-invariant sections we get a canonical isomorphism of graded $\mathcal{O}_X$-algebras

$$\left(\xi_* \mathrm{Sym}^{(m)} \left(\mathcal{T}_{\widetilde{X}} \right) \right)^{\mathbb{T}} \xrightarrow{\cong} \mathrm{gr}_\bullet \left(\widetilde{\mathcal{D}^{(m)}} \right).$$

The proof follows from the previous lemma. $\qquad\square$

3.4. Affine Algebraic Groups and Homogeneous Spaces. Let us suppose that $\mathbb{G}$ is a split connected reductive group scheme over $\mathbb{Z}_p$, $\mathbb{B} \subseteq \mathbb{G}$ is a Borel subgroup and $\mathbb{T} \subseteq \mathbb{B}$ is a split maximal torus in $\mathbb{G}$, contained in $\mathbb{B}$. Let $\boldsymbol{N}$ be the *unipotent radical* of $\mathbb{B}$, [26, (1), 1.8. Chapter II]. We put

$$\widetilde{X} := \mathbb{G}/\boldsymbol{N} \ \ \text{and} \ \ X := \mathbb{G}/\mathbb{B}$$

for the corresponding quotients (the *basic affine space* and *the flag scheme* of $\mathbb{G}$ [1, §4.7]). Since $\mathbb{Z}_p$ is in particular a Dedekind domain, these are smooth and separated schemes over $\mathbb{Z}_p$ [1, Lemma 4.7 (a)].

Remark 3.17. For technical reasons (cf. Proposition 2.11) in this work we will suppose that the group $\mathbb{G}$, and the schemes $\widetilde{X}$ and X are endowed with the right regular $\mathbb{G}$-action. This means that for any $\mathbb{Z}_p$-algebra A and $g_0, g \in \mathbb{G}(A)$ we have

$$g_0 \bullet g = g^{-1} g_0, \ g_0 \, \boldsymbol{N}(A) \bullet g = g^{-1} g_0 \, \boldsymbol{N}(A) \ \text{and} \ g_0 \, \mathbb{B}(A) \bullet g = g^{-1} g_0 \, \mathbb{B}(A).$$

Under these actions, the canonical quotient maps $\mathbb{G} \to \widetilde{X}$ and $\mathbb{G} \to X$ are clearly $\mathbb{G}$-equivariant.

Now, as $\mathbb{T}$ normalises $\boldsymbol{N}$ we have

$$(g\boldsymbol{N}(A)).t \subset g\mathbb{T}(A)\boldsymbol{N}(A),$$

for any $\mathbb{Z}_p$-algebra A, $g \in \mathbb{G}(A)$ and $t \in \mathbb{T}(A)$. This defines a right $\mathbb{T}$-action on $\widetilde{X}$ which clearly commutes with the right regular $\mathbb{G}$-action. Moreover, this right $\mathbb{T}$-action makes the canonical map $\xi : \widetilde{X} \to X$ a $\mathbb{T}$-torsor. To see this we recall first that the *abstract Cartan group* $\boldsymbol{H} := \mathbb{B}/\boldsymbol{N}$ is canonically

isomorphic to $\mathbb{T}$. Let us consider the covering of X given by the open subschemes U_w, $w \in W := N_{\mathbb{G}}(\mathbb{T})/\mathbb{T}$ (the Weyl group) where

$$U_w := \text{image of } w\boldsymbol{N}^-\mathbb{B}$$

under the quotient map $\mathbb{G} \to X$, [26, Part II, Chapter 1, 1.9 (7)], and $\boldsymbol{N}^{-1}$ is the *opposite unipotent radical* of $\mathbb{B}$, [26, (1), 1.8. Chapter II]. For every $w \in W$ we can find a morphism $\pi_w : U_w \to \mathbb{G}$ splitting the canonical map $\mathbb{G} \to X$ [26, Part II, Chapter 1, 1.10 (1) and (2)]. This map gives $\bar{\pi}_w : U_w \to \widetilde{X}$ such that $\xi \circ \bar{\pi}_w = \mathrm{id}_{U_w}$. The map $(u, b\boldsymbol{N}) \mapsto \pi_w(u)b\boldsymbol{N}$ is the required $\mathbb{T}$-invariant isomorphism $U_w \times \mathbb{T} \xrightarrow{\cong} U_w \times \boldsymbol{H} \xrightarrow{\cong} \xi^{-1}(U_w)$.

3.5. Relative Enveloping Algebras of Finite Level on Homogeneous Spaces. In this subsection we adopt the notation of the preceding subsection. In particular, we recall for the reader that the set $\mathcal{S}$, of all affine open subsets of X that trivialise the torsor ξ forms a base for the Zariski topology of X. We also recall that gothic letters denote Lie algebras which always correspond to the respective uppercase letter. For instance, $\mathfrak{g} := \mathrm{Lie}(\mathbb{G})$ and $\mathfrak{t} := \mathrm{Lie}(\mathbb{T})$. Let us recall that by Proposition 2.11 and Remark 3.17 the right regular $\mathbb{G}$-action on $\widetilde{X}$ (introduced in Remark 3.17) induces a homomorphism

$$\Phi^{(m)} : D^{(m)}(\mathbb{G}) \longrightarrow H^0\Big(\widetilde{X}, \mathcal{D}_{\widetilde{X}}^{(m)}\Big)$$

which equals the operator-representation if we tensor with $\mathbb{Q}_p$ (notation at the end of Section 2.5)

$$\Psi_{\widetilde{X}_{\mathbb{Q}}} : \mathcal{U}(\mathfrak{g}_{\mathbb{Q}}) \longrightarrow H^0\Big(\widetilde{X}_{\mathbb{Q}}, \mathcal{D}_{\widetilde{X}_{\mathbb{Q}}}\Big).$$

Here $\mathcal{D}_{\widetilde{X}_{\mathbb{Q}}}$ denotes the usual sheaf of differential operators on $\widetilde{X}_{\mathbb{Q}}$. Let us consider the base change $\mathbb{T}_{\mathbb{Q}} := \mathbb{T} \times_{\mathrm{Spec}(\mathbb{Z}_p)} \mathrm{Spec}(\mathbb{Q}_p)$. We know by [26, Part I, 2.10 (3)] that

$$(3.16) \qquad H^0\Big(\widetilde{X}_{\mathbb{Q}}, \mathcal{D}_{\widetilde{X}_{\mathbb{Q}}}\Big)^{\mathbb{T}_{\mathbb{Q}}} = H^0\Big(\widetilde{X}, \mathcal{D}_{\widetilde{X}}^{(m)}\Big)^{\mathbb{T}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.$$

Given that the right regular $\mathbb{G}$-action on $\widetilde{X}$ commutes with the right action of the torus $\mathbb{T}$, the vector fields by which $\mathfrak{g}_{\mathbb{Q}}$ acts on $\widetilde{X}_{\mathbb{Q}}$ must be invariant under the $\mathbb{T}_{\mathbb{Q}}$-action [10, Lemma 4.5]. This means that the *operator-representation* $\Psi_{\widetilde{X}_{\mathbb{Q}}}$ satisfies

$$(3.17) \qquad \Psi_{\widetilde{X}_{\mathbb{Q}}}(\mathfrak{g}_{\mathbb{Q}}) \subset H^0\Big(\widetilde{X}_{\mathbb{Q}}, \mathcal{D}_{\widetilde{X}_{\mathbb{Q}}}\Big)^{\mathbb{T}_{\mathbb{Q}}} = H^0\Big(\widetilde{X}, \mathcal{D}_{\widetilde{X}_{\mathbb{Q}}}\Big)^{\mathbb{T}}.$$

On the other hand, since the inclusion $\mathcal{D}_{\widetilde{X}}^{(m)} \hookrightarrow \mathcal{D}_{\widetilde{X},\mathbb{Q}}$ is compatible with the $\mathbb{T}$-action, we have the relation

$$H^0\left(\widetilde{X}, \mathcal{D}_{\widetilde{X}_{\mathbb{Q}}}\right)^{\mathbb{T}} \cap H^0\left(\widetilde{X}, \mathcal{D}_{\widetilde{X}}^{(m)}\right) = H^0\left(\widetilde{X}, \mathcal{D}_{\widetilde{X}}^{(m)}\right)^{\mathbb{T}}.$$

This property together with (3.17) tell us that $\Phi^{(m)}$ induces the filtered morphism

$$(3.18) \qquad \Phi^{(m)} : D^{(m)}(\mathbb{G}) \longrightarrow H^0\left(\widetilde{X}, \mathcal{D}_{\widetilde{X}}^{(m)}\right)^{\mathbb{T}} = H^0\left(X, \xi_* \mathcal{D}_{\widetilde{X}}^{(m)}\right)^{\mathbb{T}}.$$

The reasoning given in Section 2.5 applies to define an $\mathcal{O}_X$-morphism of sheaves of filtered $\mathbb{Z}_p$-algebras

$$(3.19) \qquad \Phi_X^{(m)} : \mathcal{A}_X^{(m)} \longrightarrow \widetilde{\mathcal{D}^{(m)}}.$$

The sheaf $\mathcal{A}_X^{(m)} := \mathcal{O}_X \otimes_{\mathbb{Z}_p} D^{(m)}(\mathbb{G})$ of associative $\mathbb{Z}_p$-algebras has been introduced in the Section 2.5. We remark that $(\mathcal{A}_X^{(m)} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)|_{X_{\mathbb{Q}}} = \mathcal{U}^\circ$, where $\mathcal{U}^\circ := \mathcal{O}_{X_{\mathbb{Q}}} \otimes_{\mathbb{Q}_p} \mathcal{U}(\mathfrak{g}_{\mathbb{Q}})$.

To consider the central redaction of the sheaves $\widetilde{\mathcal{D}^{(m)}}$ introduced in the Section 3.3 we need to take into account the classical distribution algebra as in [13, Chapter II, 4.6.1]. To define it, we suppose that $\varepsilon : \mathrm{Spec}(\mathbb{Z}_p) \to \mathbb{T}$ is the identity of $\mathbb{T}$ and we take $J := \{f \in \mathbb{Z}_p[\mathbb{T}] \mid f(\varepsilon) = 0\}$. Then $\mathbb{Z}_p[\mathbb{T}] = \mathbb{Z}_p \oplus J$. We put

$$\mathrm{Dist}_n(\mathbb{T}) := \left(Z_p[\mathbb{T}]/J^{n+1}\right)^* = \mathrm{Hom}_{\mathbb{Z}_p\text{-mod}}(\mathbb{Z}_p[\mathbb{T}]/J^{n+1}, \mathbb{Z}_p) \subset \mathbb{Z}_p[\mathbb{T}]^*$$

the space of *distributions of order n*, and then $\mathrm{Dist}(\mathbb{T}) := \varinjlim_{n \in \mathbb{N}} \mathrm{Dist}_n(\mathbb{T})$. Moreover, if $\Delta_{\mathbb{T}} : \mathbb{Z}_p[\mathbb{T}] \to \mathbb{Z}_p[\mathbb{T}] \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[\mathbb{T}]$ denotes the coproduct of $\mathbb{T}$, then the product

$$uv : \mathbb{Z}_p[\mathbb{T}] \xrightarrow{\Delta_{\mathbb{T}}} \mathbb{Z}_p[\mathbb{T}] \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[\mathbb{T}] \xrightarrow{u \otimes v} \mathbb{Z}_p \qquad u, v \in \mathbb{Z}_p[\mathbb{T}]^*$$

defines a structure of algebra on $\mathbb{Z}_p[\mathbb{T}]^*$ and $\mathrm{Dist}(\mathbb{T})$ is a subalgebra with $\mathrm{Dist}_n(\mathbb{T}) . \mathrm{Dist}_m(\mathbb{T}) \subset \mathrm{Dist}_{m+n}(\mathbb{T})$ [26, Part I, 7.7]. Moreover, given that $\mathrm{Dist}_1(\mathbb{T}) = \mathrm{Lie}(\mathbb{T})$, the canonical map $\mathrm{Dist}_1(\mathbb{T}) \to \mathrm{Dist}(\mathbb{T})$ induces, by universal property, a morphism of $\mathbb{Z}_p$-algebras $\mathcal{U}(\mathrm{Lie}(\mathbb{T})) \to \mathrm{Dist}(\mathbb{T})$ sending the $\mathbb{Z}_p$-submodule $\mathcal{U}_n(\mathrm{Lie}(\mathbb{T}))$, in the canonical filtration $\{\mathcal{U}_k(\mathrm{Lie}(\mathbb{T}))\}_{k \in \mathbb{N}}$ of $\mathcal{U}(\mathrm{Lie}(\mathbb{T}))$, to $\mathrm{Dist}_n(\mathbb{T})$.

Proposition 3.18.

(i) [23, §4.1] *The morphisms*

$$\mathrm{Hom}_{\mathbb{Z}_p}(\psi_{m,m'}, \mathbb{Z}_p) : D^{(m)}(\mathbb{T}) \longrightarrow D^{(m')}(\mathbb{T}),$$

with $\psi_{m,m'}$ as in Section 2.4, induce an isomorphism of filtered $\mathbb{Z}_p$-algebras $\varinjlim_{m \in \mathbb{N}} D^{(m)}(\mathbb{T}) \xrightarrow{\cong} Dist(\mathbb{T})$.

(ii) [26, Part I, 7.10 (1)] *The $\mathbb{Q}_p$-algebra of distributions $\mathrm{Dist}(\mathbb{T}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is canonically isomorphic to the universal enveloping algebra $\mathcal{U}(\mathfrak{t}_{\mathbb{Q}})$.*

Remark 3.19. The canonical isomorphism stated in the second part of the previous proposition comes after tensoring with $\mathbb{Q}_p$ the canonical morphism of $\mathbb{Z}_p$-algebras $\mathcal{U}(\mathfrak{t}) \to \mathrm{Dist}(\mathbb{T})$. If $\lambda \in \mathfrak{t}^*$, we get, by universal property, a morphism of $\mathbb{Z}_p$-algebras $\mathcal{U}(\lambda) : \mathcal{U}(\mathfrak{t}) \to \mathbb{Z}_p$. We will abuse of the notation and we will denote by $\mathcal{U}(\lambda) \otimes_{Z_p} 1_{\mathbb{Q}_p}$ the morphism of $\mathbb{Q}_p$-algebras resulting from the composition

$$\mathrm{Dist}(\mathbb{T}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \xrightarrow{\cong} \mathcal{U}(\mathfrak{t}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \longrightarrow \mathbb{Q}_p.$$

Example 3.20. Let us suppose that $\mathbb{T} = \mathbb{G}_m = \mathrm{Spec}(\mathbb{Z}_p[T, T^{-1}])$. In this case J is generated by $T-1$, and the residue classes of 1, $T-1, \ldots, (T-1)^n$ form a basis of $\mathbb{Z}_p[\mathbb{T}]/J^{n+1}$. Let $\delta_n \in \mathrm{Dist}(\mathbb{T})$ such that $\delta_n((T-1)^i) = \delta_{n,i}$ (the Kronecker delta). By [26, Part I, 7.8] all the δ_n with $n \in \mathbb{N}$ form a basis of $\mathrm{Dist}(\mathbb{T})$ and they satisfy the relation

$$(3.20) \qquad\qquad n!\delta_n = \delta_1(\delta_1 - 1) \ldots (\delta_1 - n + 1).$$

Therefore $\mathrm{Dist}(\mathbb{T}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = \mathbb{Q}_p[\delta_1]$. Since $\mathfrak{t} = (J/J^2)^*$, we may conclude that $\mathrm{Dist}(\mathbb{T}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = \mathcal{U}(\mathfrak{t}_{\mathbb{Q}})$.

Proposition 3.21. *Let $\mathbb{T}$ be a split maximal torus and $\mathfrak{t}$ its $\mathbb{Z}_p$-Lie algebra. There exists a canonical bijection between the characters of $\mathfrak{t}$ and morphisms of $\mathbb{Z}_p$-algebras $\mathrm{Dist}(\mathbb{T}) \to \mathbb{Z}_p$:*

$$(3.21) \qquad \begin{array}{c} \mathrm{Hom}_{\mathbb{Z}_p\text{-mods}}(\mathfrak{t}, \mathbb{Z}_p) \xrightarrow{\cong} \mathrm{Hom}_{\mathbb{Z}_p\text{-alg}}(\mathrm{Dist}(\mathbb{T}), \mathbb{Z}_p) \\ \lambda \longmapsto \left. \left(\mathcal{U}(\lambda) \otimes_{\mathbb{Z}_p} 1_{\mathbb{Q}_p} \right) \right|_{\mathrm{Dist}(\mathbb{T})}. \end{array}$$

Proof. We have remarked that there exists a canonical morphism of $\mathbb{Z}_p$-algebras $\alpha : \mathcal{U}(\mathfrak{t}) \to \mathrm{Dist}(\mathbb{T})$ which becomes an isomorphism after tensorization with $\mathbb{Q}_p$. This in particular implies that every morphism of $\mathbb{Z}_p$-algebras $\beta \in \mathrm{Hom}_{\mathbb{Z}_p\text{-alg}}(\mathrm{Dist}(\mathbb{T}), \mathbb{Z}_p)$ induces a character $(\beta \circ \alpha)|_{\mathfrak{t}}$ of the Cartan subalgebra $\mathfrak{t} \subset \mathfrak{g}$. This correspondence is clearly injective. Let us proof that this is surjective when the base ring is $\mathbb{Z}_p$.

Let us first assume that $\mathbb{T} = \mathbb{G}_m = \mathrm{Spec}(\mathbb{Z}_p[T, T^{-1}])$ is the multiplicative group. From the previous example we know that the set of distributions $\{\delta_n\}_{n \in \mathbb{N}}$, where $\delta_n((T-1)^i) = 0$ if $i < n$ and $\delta_n((T-1)^n) = 1$, is a basis for $\mathrm{Dist}(\mathbb{T})$. Moreover, $\mathrm{Dist}(\mathbb{T}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = \mathbb{Q}_p[\delta_1]$. Now, let us take $\lambda \in \mathfrak{t}^*$. By universal property, this character induces a morphism of $\mathbb{Z}_p$-algebras $\mathcal{U}(\lambda) : \mathcal{U}(\mathfrak{t}) \to \mathbb{Z}_p$. Taking tensor product with $\mathbb{Q}_p$ and using the canonical isomorphism $\mathrm{Dist}(\mathbb{T})_{\mathbb{Q}} \cong \mathcal{U}(\mathfrak{t}_{\mathbb{Q}})$ we obtain a morphism of $\mathbb{Z}_p$-algebras $\left. (\mathcal{U}(\lambda) \otimes_{\mathbb{Z}_p} 1_{\mathbb{Q}_p}) \right|_{\mathrm{Dist}(\mathbb{T})} : \mathrm{Dist}(\mathbb{T}) \to \mathbb{Q}_p$ (we use the notation introduced in Remark 3.19). We want to see that its image is contained in

$\mathbb{Z}_p$. To do that we only need to check that $(\mathcal{U}(\lambda) \otimes_{\mathbb{Z}_p} 1_{\mathbb{Q}_p})|_{\mathrm{Dist}(\mathbb{T})}(\delta_n) \in \mathbb{Z}_p$. By (3.20) we have that

$$(\mathcal{U}(\lambda) \otimes_{\mathbb{Z}_p} 1_{\mathbb{Q}_p})|_{\mathrm{Dist}(\mathbb{T})}(\delta_n) = (\mathcal{U}(\lambda) \otimes_{\mathbb{Z}_p} 1_{\mathbb{Q}_p})|_{\mathrm{Dist}(\mathbb{T})}\left(\binom{\delta_1}{n}\right) = \binom{\lambda(\delta_1)}{n}$$

lies in $\mathbb{Z}_p$ because the binomial coefficients extend to continuous functions from $\mathbb{Z}_p$ to $\mathbb{Z}_p$, and $\delta_1 \in \mathfrak{t}$.

If $\mathbb{T}$ denotes a split maximal torus $\mathbb{T} = \mathbb{G}_m \times_{\mathrm{Spec}(\mathbb{Z}_p)} \cdots \times_{\mathrm{Spec}(\mathbb{Z}_p)} \mathbb{G}_m$ (n-times), the reader may follow the same reasoning using the canonical isomorphism $\mathrm{Dist}(\mathbb{T}) \cong \mathrm{Dist}(\mathbb{G}_m) \otimes_{\mathbb{Z}_p} \cdots \otimes_{\mathbb{Z}_p} \mathrm{Dist}(\mathbb{G}_m)$ (n-times) [26, Part I, 7.9 (3)]. $\qquad\square$

Remark 3.22. Let $\lambda \in \mathfrak{t} := \mathrm{Hom}_{\mathbb{Z}_p\text{-mods}}(\mathfrak{t}, \mathbb{Z}_p)$. From now on, we will abuse of the notations and we will denote by λ both the homomorphism of $\mathbb{Q}_p$-algebras $\mathcal{U}(\lambda) \otimes_{\mathbb{Z}_p} 1_{\mathbb{Q}_p} : \mathcal{U}(\mathfrak{t}_{\mathbb{Q}}) \to \mathbb{Q}_p$, where $\mathcal{U}(\lambda) : \mathcal{U}(\mathfrak{t}) \to \mathbb{Z}_p$ denotes the morphism of $\mathbb{Z}_p$-algebras induced by universal property, and the homomorphism of $\mathbb{Z}_p$-algebras $(\mathcal{U}(\lambda) \otimes_{\mathbb{Z}_p} 1_{\mathbb{Q}_p})|_{\mathrm{Dist}(\mathbb{T})} : \mathrm{Dist}(\mathbb{T}) \to \mathbb{Z}_p$. Furthermore, if $\lambda \in \mathrm{Hom}_{\mathbb{Z}_p\text{-alg}}(\mathrm{Dist}(\mathbb{T}), \mathbb{Z}_p)$, then $\lambda^{(m)} : D^{(m)}(\mathbb{T}) \to \mathbb{Z}_p$ will denote the morphism of $\mathbb{Z}_p$-algebras coming from Proposition 3.18 (ii), and again by λ the morphisms of $\mathbb{Q}_p$-vector spaces

$$(\lambda^{(m)} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)|_{\mathfrak{t}_{\mathbb{Q}}} = (\lambda \otimes_{\mathbb{Z}_p} 1_{\mathbb{Q}_p})|_{\mathfrak{t}_{\mathbb{Q}}} : \mathfrak{t}_{\mathbb{Q}} \longrightarrow \mathbb{Q}_p.$$

Remark 3.23. Let us consider the positive system $\Lambda^+ \subset \Lambda \subset X(\mathbb{T})$ ($X(\mathbb{T})$ the group of *algebraic characters*) associated to the Borel subgroup scheme $\mathbb{B} \subset \mathbb{G}$. The Weyl subgroup $W := N_{\mathbb{G}}(\mathbb{T})/\mathbb{T}$ acts naturally on the space $\mathfrak{t}_{\mathbb{Q}}^* := \mathrm{Hom}_{\mathbb{Q}_p\text{-mod}}(\mathfrak{t} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p, \mathbb{Q}_p)$, and via differentiation $d : X(\mathbb{T}) \hookrightarrow \mathfrak{t}^*$ we may view $X(\mathbb{T})$ as a subgroup of $\mathfrak{t}^*$ in such a way that $X^*(\mathbb{T}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = \mathfrak{t}_{\mathbb{Q}}^*$. Let $\check{\alpha}$ be a *coroot* of $\alpha \in \Lambda$ viewed as an element of $\mathfrak{t}_{\mathbb{Q}}$. An arbitrary weight $\lambda \in \mathfrak{t}_{\mathbb{Q}}^*$ is called *dominant* if $\lambda(\check{\alpha}) \geq 0$ for all $\alpha \in \Lambda^+$. The weight λ is called *regular* if its stabilizer under the W-action is trivial.

We will always denote by $\rho := \frac{1}{2} \sum_{\alpha \in \Lambda^+} \alpha$ the so-called *Weyl character*.

Definition 3.24. We say that a morphism of $\mathbb{Z}_p$-algebras $\lambda : \mathrm{Dist}(\mathbb{T}) \to \mathbb{Z}_p$ (resp. the induced morphism $\lambda^{(m)} : D^{(m)}(\mathbb{T}) \to \mathbb{Z}_p$) is a character of the distribution algebra $\mathrm{Dist}(\mathbb{T})$ (resp. a character of the level m distribution algebra $D^{(m)}(\mathbb{T})$). Moreover, taking into account the notation 3.22, we say that a character $\lambda : \mathrm{Dist}(\mathbb{T}) \to \mathbb{Z}_p$ (resp. a character $\lambda^{(m)} : D^{(m)}(\mathbb{T}) \to \mathbb{Z}_p$) is a dominant and regular character, if the $\mathbb{Q}_p$-linear map $\lambda : \mathfrak{t}_{\mathbb{Q}} \to \mathbb{Q}_p$, induced by the correspondence (3.21) and tensorization with $\mathbb{Q}_p$, is a dominant and regular character of $\mathfrak{t}_{\mathbb{Q}}$.

The reader can easily verify the following elementary lemma.

Lemma 3.25. *Let A be a $\mathbb{Q}_p$-algebra and $A_0 \subset A$ be a $\mathbb{Z}_p$-subalgebra such that $A_0 \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = A$. If $Z(A)$ denotes the center of A (resp. $Z(A_0)$ denotes the center of A_0), then $Z(A) = Z(A_0) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$.*

Let us consider $\mathcal{D}_{\widetilde{X}_{\mathbb{Q}}}$ the usual sheaf of differential operators [7, 16] on $\widetilde{X}_{\mathbb{Q}} := \widetilde{X} \times_{\mathrm{Spec}(\mathbb{Z}_p)} \mathrm{Spec}(\mathbb{Q}_p)$. By [26, Part I, 2.10 (3)] we have

$$(3.22) \quad H^0\Big(X_{\mathbb{Q}}, (\xi \times_{\mathbb{Z}_p} \mathrm{id}_{\mathbb{Q}_p})_* \mathcal{D}_{\tilde{X}_{\mathbb{Q}}}\Big)^{\mathbb{T}_{\mathbb{Q}}} = H^0(\widetilde{X}_{\mathbb{Q}}, \mathcal{D}_{\widetilde{X}_{\mathbb{Q}}})^{\mathbb{T}_{\mathbb{Q}}}$$
$$= H^0\Big(\widetilde{X}, \mathcal{D}_{\widetilde{X}}^{(m)}\Big)^{\mathbb{T}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.$$

On the other hand, we know by 2.11 that the right $\mathbb{T}$-action on $\widetilde{X}$ induces a canonical morphism of filtered $\mathbb{Z}_p$-algebras

$$\Phi_{\mathbb{T}}^{(m)} : D^{(m)}(\mathbb{T}) \longrightarrow H^0\Big(\widetilde{X}, \mathcal{D}_{\widetilde{X}}^{(m)}\Big).$$

Additionally, by [4, p. 7] we have that the morphism $\Phi_{\mathbb{T}}^{(m)} \otimes_{\mathbb{Z}_p} 1_{\mathbb{Q}_p}$ factors through the center of $H^0(X_{\mathbb{Q}}, (\xi \times_{\mathbb{Z}_p} \mathrm{id}_{\mathbb{Q}_p})_* \mathcal{D}_{\widetilde{X}_{\mathbb{Q}}})^{\mathbb{T}_{\mathbb{Q}}}$. By (3.22) and the preceding lemma we get the following morphism

$$D^{(m)}(\mathbb{T}) \hookrightarrow \mathcal{U}(\mathfrak{t}_{\mathbb{Q}}) \xrightarrow{\Phi_{\mathbb{T}}^{(m)} \otimes_{\mathbb{Z}_p} 1_{\mathbb{Q}_p}} Z\Big(H^0\Big(\widetilde{X}, \mathcal{D}_{\widetilde{X}}^{(m)}\Big)^{\mathbb{T}}\Big) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.$$

Following the same lines of reasoning given in page 25 (more specifically (3.16) and (3.17)), and taking into account the previous lemma, we can conclude that $\Phi_{\mathbb{T}}^{(m)}$ induces a morphism of filtered $\mathbb{Z}_p$-algebras

$$\Phi_{\mathbb{T}}^{(m)} : D^{(m)}(\mathbb{T}) \longrightarrow H^0\Big(X, Z(\widetilde{\mathcal{D}^{(m)}})\Big).$$

Here $Z(\widetilde{\mathcal{D}^{(m)}})$ is the center of $\widetilde{\mathcal{D}^{(m)}}$ and its filtration is the induced filtration coming from (3.9).

Finally, let $\lambda : \mathrm{Dist}(\mathbb{T}) \to \mathbb{Z}_p$ be a character of the distribution algebra of $\mathbb{T}$. We recall for the reader that we may consider the ring $\mathbb{Z}_p$ as a $D^{(m)}(\mathbb{T})$-module via $\lambda^{(m)} : D^{(m)}(\mathbb{T}) \to \mathbb{Z}_p$.

The previous morphisms give rise to the following definition.

Definition 3.26. Let $\lambda : \mathrm{Dist}(\mathbb{T}) \to \mathbb{Z}_p$ be a character. We define the sheaf of level m twisted arithmetic differential operators $\mathcal{D}_{X,\lambda}^{(m)}$ on the flag scheme X by

$$\mathcal{D}_{X,\lambda}^{(m)} := \widetilde{\mathcal{D}^{(m)}} \otimes_{\Phi_{\mathbb{T}}^{(m)}, D^{(m)}(\mathbb{T}), \lambda^{(m)}} \mathbb{Z}_p.$$

If we endow $\mathbb{Z}_p$ with the *trivial filtration* as a $D^{(m)}(\mathbb{T})$-module, this is $0 =: \mathrm{F}_{-1}\mathbb{Z}_p$ and $\mathrm{F}_i\mathbb{Z}_p := \mathbb{Z}_p$ for all $i \geq 0$, then using (3.9) we may view $\mathcal{D}_{X,\lambda}^{(m)}$ as a sheaf of filtered $\mathbb{Z}_p$-algebras equipped with the tensor product filtration.

Proposition 3.27. *Let $U \in \mathcal{S}$. Then $\mathcal{D}_{X,\lambda}^{(m)}|_U$ is isomorphic to $\mathcal{D}_X^{(m)}|_U$ as a sheaf of filtered $\mathbb{Z}_p$-algebras.*

Proof. Let us recall that by Proposition 3.14 for every $U \in \mathcal{S}$ we have an isomorphism of filtered $\mathbb{Z}_p$-algebras

$$\widetilde{\mathcal{D}^{(m)}}|_U \xrightarrow{\cong} \mathcal{D}_X^{(m)}|_U \otimes_{\mathbb{Z}_p} D^{(m)}(\mathbb{T})$$

which induces an isomorphism $\mathcal{D}_{X,\lambda}^{(m)}|_U \xrightarrow{\cong} \mathcal{D}_X^{(m)}|_U$ of filtered $\mathbb{Z}_p$-algebras. $\square$

Remark 3.28. This proposition justifies the name of *twisted arithmetic differential operators*.

Let us recall that, as X is a smooth $\mathbb{Z}_p$-scheme, the sheaf $\mathcal{D}_X^{(m)}$ is a sheaf of $\mathcal{O}_X$-rings with noetherian sections over all open affine subsets of X [5, Corollary 2.2.5]. The preceding proposition and the same reasoning given in [25, Proposition 2.2.2 (iii)] imply the following meaningful result.

Proposition 3.29. *The sheaf $\mathcal{D}_{X,\lambda}^{(m)}$ is a sheaf of $\mathcal{O}_X$-rings with noetherian sections over all open affine subsets of X.*

Definition 3.30. Let $\lambda : \mathrm{Dist}(\mathbb{T}) \to \mathbb{Z}_p$ be a character. We will denote by

$$(3.23) \qquad \widehat{\mathcal{D}}_{\mathfrak{X},\lambda}^{(m)} := \varprojlim_j \mathcal{D}_{X,\lambda}^{(m)}/p^{j+1}\mathcal{D}_{X,\lambda}^{(m)}$$

the p-adic completion of $\mathcal{D}_{X,\lambda}^{(m)}$ and we will consider it as a sheaf on $\mathfrak{X}$.

Following the notation given at the beginning of this work, the sheaf $\widehat{\mathcal{D}}_{\mathfrak{X},\lambda,\mathbb{Q}}^{(m)}$ will denote our sheaf of level m twisted differential operators on the formal flag scheme $\mathfrak{X}$.

Proposition 3.31.

(i) *There exists a basis $\mathcal{B}$ of the topology of $\mathfrak{X}$, consisting of open affine subsets, such that for every $\mathfrak{U} \in \mathcal{B}$ the ring $\widehat{\mathcal{D}}_{\mathfrak{X},\lambda}^{(m)}(\mathfrak{U})$ is two-sided noetherian.*

(ii) *The sheaf of rings $\widehat{\mathcal{D}}_{\mathfrak{X},\lambda,\mathbb{Q}}^{(m)}$ is coherent.*

Proof. To show (i) we can take an open affine subset $U \in \mathcal{S}$ and to consider $\mathfrak{U}$ its formal completion along the special fiber. We have

$$H^0(\mathfrak{U}, \widehat{\mathcal{D}}_{\mathfrak{X},\lambda}^{(m)}) \cong H^0(\widehat{U, \mathcal{D}_{X,\lambda}^{(m)}}) \cong H^0(\widehat{U, \mathcal{D}_X^{(m)}}) \cong H^0(\mathfrak{U}, \widehat{\mathcal{D}}_{\mathfrak{X}}^{(m)}).$$

The 1st and 3rd isomorphism follow from [15, $(0_I, 3.2.6)$]. The 2nd one arises from the preceding proposition. By [5, 3.2.3 (iv)] the ring $H^0(\mathfrak{U}, \widehat{\mathcal{D}}_{\mathfrak{X}}^{(m)})$ is two-sided noetherian. Therefore, we can take $\mathcal{B}$ as the set of affine open subsets of $\mathfrak{X}$ contained in the p-adic completion of an affine open subset

$U \in \mathcal{S}$. This proves (i). By [5, Proposition 3.3.4] we can conclude that (ii) is an immediately consequence of (i) because

$$H^0(\mathfrak{U}, \widehat{\mathcal{D}}^{(m)}_{\mathfrak{X},\lambda,\mathbb{Q}}) = H^0(\mathfrak{U}, \widehat{\mathcal{D}}^{(m)}_{\mathfrak{X},\lambda}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$$

([5, (3.4.0.1)]). $\square$

Using the morphism $\Phi^{(m)}_X$ defined in (3.19) and the canonical projection from $\widetilde{\mathcal{D}^{(m)}}$ onto $\mathcal{D}^{(m)}_{X,\lambda}$ we may define a canonical map

(3.24) $$\Phi^{(m)}_{X,\lambda} : \mathcal{A}^{(m)}_X \longrightarrow \mathcal{D}^{(m)}_{X,\lambda}.$$

Proposition 3.32.

   (i) *There exists a canonical isomorphism*

$$\mathrm{Sym}^{(m)}(\mathcal{T}_X) \cong \mathrm{gr}_\bullet(\mathcal{D}^{(m)}_{X,\lambda}).$$

  (ii) *The canonical morphism $\Phi^{(m)}_{X,\lambda}$ is surjective.*

 (iii) *The sheaf $\mathcal{D}^{(m)}_{X,\lambda}$ is a coherent $\mathcal{A}^{(m)}_X$-module.*

Proof. By (3.5) we have a canonical map

$$\mathrm{gr}_\bullet\left(\widetilde{\mathcal{D}^{(m)}}\right) \otimes_{\mathrm{gr}_\bullet(D^{(m)}(\mathbb{T}))} \mathrm{gr}_\bullet(\mathbb{Z}_p) \longrightarrow \mathrm{gr}_\bullet\left(\mathcal{D}^{(m)}_{X,\lambda}\right).$$

By Proposition 3.16 we know that $\mathrm{gr}_\bullet(\widetilde{\mathcal{D}^{(m)}}) \cong \mathrm{Sym}^{(m)}((\xi_* \mathcal{T}_{\widetilde{X}})^{\mathbb{T}})$. Moreover, by definition, we know that $\mathrm{gr}_\bullet(\mathbb{Z}_p) = \mathbb{Z}_p$ as a $\mathrm{gr}_\bullet(D^{(m)}(\mathbb{T}))$-module. We obtain a morphism of sheaves of graded $\mathbb{Z}_p$-algebras

$$\mathrm{Sym}^{(m)}\left((\xi_* \mathcal{T}_{\widetilde{X}})^{\mathbb{T}}\right) \otimes_{\mathrm{Sym}^{(m)}(\mathfrak{t})} \mathbb{Z}_p \longrightarrow \mathrm{gr}_\bullet\left(\mathcal{D}^{(m)}_{X,\lambda}\right)$$

(the structure of $\mathrm{Sym}^{(m)}(\mathfrak{t})$-module is guaranteed by (3.12)). Using the short exact sequence of Lemma 3.11 we see that

$$\mathrm{Sym}^{(m)}\left((\xi_* \mathcal{T}_{\widetilde{X}})^{\mathbb{T}}\right) \otimes_{\mathrm{Sym}^{(m)}(\mathfrak{t})} \mathbb{Z}_p \xrightarrow{\mathrm{Sym}^{(m)}(\nu) \otimes 1} \mathrm{Sym}^{(m)}(\mathcal{T}_X)$$

is an isomorphism and we get a canonical morphism of $\mathbb{Z}_p$-algebras

$$\varphi : \mathrm{Sym}^{(m)}(\mathcal{T}_X) \longrightarrow \mathrm{gr}_\bullet\left(\mathcal{D}^{(m)}_{X,\lambda}\right).$$

By Proposition 3.27, we have a commutative diagram for any $U \in \mathcal{S}$

here the left diagonal arrow is the isomorphism given by (2.4). As $\mathcal{S}$ is a basis for the Zariski topology of X we can conclude that φ is an isomorphism.

For the second claim we can calculate $\mathrm{gr}_{\bullet}(\Phi_{X,\lambda}^{(m)})$. By the first part of the proof and Proposition 2.13 this morphism is identified with

$$\mathcal{O}_X \otimes_{\mathbb{Z}_p} \mathrm{Sym}^{(m)}(\mathfrak{g}) \longrightarrow \mathrm{Sym}^{(m)}(\mathcal{T}_X)$$

which is surjective by [21, Proposition 1.6.1]. Finally, item (iii) follows from (ii) and Proposition 2.13 (i). $\qquad\square$

Remark 3.33.

   (a) By construction $(\mathcal{D}_{X,\lambda}^{(m)} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)|_{X_{\mathbb{Q}}} = \mathcal{D}_{X_{\mathbb{Q}},\lambda}$ is the sheaf of usual λ twisted differential operators on the flag variety $X_{\mathbb{Q}}$ [11, p. 170].

   (b) The regular right action of $\mathbb{G}$ on $\widetilde{X}$ (Remark 3.17) induces a natural map $\Phi_\lambda : \mathcal{U}(\mathfrak{g}_{\mathbb{Q}}) \to H^0(X_{\mathbb{Q}}, \mathcal{D}_{X_{\mathbb{Q}},\lambda})$. This implies that if $\Phi_\lambda^{(m)}$ denotes the canonical map induced by $\Phi_{X,\lambda}^{(m)}$ by taking global sections, then $\Phi_\lambda^{(m)} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = \Phi_\lambda$, [11, pp. 170 and 186].

The relation given in (3.21) tells us that, as in the classical case (cf. [3, 10]), the sheaf $\widetilde{\mathcal{D}^{(m)}} := \left(\xi_* \mathcal{D}_{\widetilde{X}}^{(m)} \right)^{\mathbb{T}}$ can be regarded as a family of twisted differential operators on X parametrized by $\mathfrak{t}^* := \mathrm{Hom}_{\mathbb{Z}_p\text{-mods}}(\mathfrak{t}, \mathbb{Z}_p)$.

Remark 3.34. Before investigating the finiteness properties of the sheaves introduced in Definition 3.30, let us study the particular case when $\lambda \in \mathfrak{t}^*$ is an *algebraic character*. This means, λ is obtained by differentiating a character of $\mathbb{T}$. As is shown in [26, Part I, Chapter 5, (5.8)], the character $\lambda \in \mathrm{Hom}(\mathbb{T}, \mathbb{G}_m)$ induces an invertible sheaf $\mathcal{L}(\lambda)$ on X and we may consider the *sheaf of arithmetic differential operators acting on* $\mathcal{L}(\lambda)$

$$\mathcal{D}_X^{(m)}(\lambda) := \mathcal{L}(\lambda) \otimes_{\mathcal{O}_X} \mathcal{D}_X^{(m)} \otimes_{\mathcal{O}_X} \mathcal{L}(\lambda)^{\vee}.$$

The (left) action of $\mathcal{D}_X^{(m)}(\lambda)$ on $\mathcal{L}(\lambda)$ is defined by

$$(t \otimes P \otimes t^{\vee}) \bullet s := (P \bullet \langle t^{\vee}, s \rangle)\, t \qquad (s, t \in \mathcal{L}(\lambda), \text{ and } t^{\vee} \in \mathcal{L}(\lambda)^{\vee}).$$

Let us denote by $\widehat{\mathcal{D}}_{\mathfrak{X}}^{(m)}(\lambda)$ its p-adic completion (considered as a sheaf on $\mathfrak{X}$) and by $\mathcal{D}_{\mathfrak{X}}^{\dagger}(\lambda)$ its inductive limit tensored with $\mathbb{Q}_p$. These sheaves have been studied in [21, 24]. As before, we put $\widehat{\mathcal{D}}_{\mathfrak{X},\mathbb{Q}}^{(m)}(\lambda) := \widehat{\mathcal{D}}_{\mathfrak{X}}^{(m)}(\lambda) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$.

For the next result, we will abuse of the notation and we will suppose that $\lambda \in \mathrm{Hom}(\mathbb{T}, \mathbb{G}_m)$ is an algebraic character of $\mathfrak{t}$ and that $\lambda' : \mathrm{Dist}(\mathbb{T}) \to \mathbb{Z}_p$ denotes the character of $\mathrm{Dist}(\mathbb{T})$ induced by (3.21).

Proposition 3.35. *The sheaves $\widehat{\mathcal{D}}_{\mathfrak{X},\mathbb{Q}}^{(m)}(\lambda)$ and $\widehat{\mathcal{D}}_{\mathfrak{X},\lambda',\mathbb{Q}}^{(m)}$ are canonically isomorphic.*

Before starting the proof let us recall the following facts (these are detailed in [21]). First of all, the order filtration of $\mathcal{D}_X^{(m)}$ (Definition 2.7) induces a filtration of $\mathcal{D}_X^{(m)}(\lambda)$ by

$$\mathcal{D}_{X,d}^{(m)}(\lambda) := \mathcal{L}(\lambda) \otimes_{\mathcal{O}_X} \mathcal{D}_{X,d}^{(m)} \otimes_{\mathcal{O}_X} \mathcal{L}(\lambda)^{\vee}, \ \ d \in \mathbb{N},$$

such that

$$\mathrm{gr}_{\bullet}\left(\mathcal{D}_X^{(m)}(\lambda)\right) \cong \mathrm{Sym}^{(m)}\left(\mathcal{T}_X\right).$$

It is easy to see that in order to achieve the preceding isomorphism we only need to use (2.4) and the commutativity of the symmetric algebra. On the other hand, by the work developed by Huyghe–Schmidt in [24] we have a canonical morphism of filtered $\mathbb{Z}_p$-algebras

$$\Phi^{(m)}(\lambda) : \mathcal{A}_X^{(m)} := \mathcal{O}_X \otimes_{\mathbb{Z}_p} D^{(m)}(\mathbb{G}) \longrightarrow \mathcal{D}_X^{(m)}(\lambda)$$

which is surjective thanks to the previous isomorphism and the fact that X is an homogeneous space (cf. [21, §1.6]). Moreover, by [18, Chapter II, Excercise 1.19] we have $\mathcal{D}_{X,\lambda}^{(m)} \otimes \mathbb{Q}_p = \mathcal{D}_X^{(m)}(\lambda) \otimes \mathbb{Q}_p$, because their restriction to $X_{\mathbb{Q}}$ are canonically isomorphic by [2, Second example].

Proof of Proposition 3.35. By the preceding discussion and Proposition 3.32 we have two canonical surjective morphisms of filtered $\mathbb{Z}_p$-algebras

$$\Phi^{(m)}(\lambda) : \mathcal{A}_X^{(m)} \longrightarrow \mathcal{D}_X^{(m)}(\lambda) \quad \text{and} \quad \Phi_{X,\lambda'}^{(m)} : \mathcal{A}_X^{(m)} \longrightarrow \mathcal{D}_{X,\lambda'}^{(m)}.$$

Given that by Proposition 3.27 and [35, Proposition 3.3.6] both $\mathcal{D}_{X,\lambda}^{(m)}$ and $\mathcal{D}_X^{(m)}(\lambda)$ are torsion free, the results follows by a diagram chase of the following commutative diagram:

3.6. Finiteness Properties. Let $\lambda : \mathrm{Dist}(\mathbb{T}) \to \mathbb{Z}_p$ be a character. In this section we start the study of the cohomological properties of coherent $\mathcal{D}_{X,\lambda}^{(m)}$-modules. We follow the arguments in [21] to show a technical important finiteness property about the p-torsion of the cohomology groups of coherent $\mathcal{D}_{X,\lambda}^{(m)}$-modules, when the character $\lambda + \rho \in \mathfrak{t}_{\mathbb{Q}}^*$ is dominant and regular (Proposition 3.39 below). To start with, let us recall the twist by the sheaf $\mathcal{O}(1)$. As X is a projective $\mathbb{Z}_p$-scheme, we may fix a very ample

invertible sheaf $\mathcal{O}(1)$ on X [18, Chapter II, Remark 5.16.1]. Therefore, for any arbitrary $\mathcal{O}_X$-module $\mathcal{E}$ we may consider the twist

$$\mathcal{E}(r) := \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{O}(r),$$

where $r \in \mathbb{Z}$ and $\mathcal{O}(r)$ means the r-th tensor product of $\mathcal{O}(1)$ with itself. We recall for the reader that there exists $r_0 \in \mathbb{Z}$, depending of $\mathcal{O}(1)$, such that for every $k \in \mathbb{Z}_{>0}$ and for every $s \geq r_0$, $H^k(X, \mathcal{O}(s)) = 0$ [18, Chapter II, Theorem 5.2 (b)].

We start the results of this section with the following proposition which states three important properties of coherent $\mathcal{A}_X^{(m)}$-modules. The reader can find the proof of the following proposition in [23, Proposition A.2.6.1].

Proposition 3.36. *Let $\mathcal{E}$ be a coherent $\mathcal{A}_X^{(m)}$-module.*

(i) $H^0(X, \mathcal{A}_X^{(m)}) = D^{(m)}(\mathbb{G})$ *is a noetherian $\mathbb{Z}_p$-algebra.*

(ii) *There exists a surjection of $\mathcal{A}_X^{(m)}$-modules $(\mathcal{A}_X^{(m)}(-r))^{\oplus a} \to \mathcal{E} \to 0$ for suitable $r \in \mathbb{Z}$ and $a \in \mathbb{N}$.*

(iii) *For any $k \geq 0$ the group $H^k(X, \mathcal{E})$ is a finitely generated $D^{(m)}(\mathbb{G})$-module.*

The next two results will play an important role when considering formal completions.

Lemma 3.37. *For every coherent $\mathcal{A}_X^{(m)}$-module $\mathcal{E}$, there exists $r = r(\mathcal{E}) \in \mathbb{Z}$ such that $H^k(X, \mathcal{E}(s)) = 0$ for every $s \geq r$ and $k \in \mathbb{Z}_{>0}$.*

Proof. Let us fix $r_0 \in \mathbb{Z}$ such that $H^k(X, \mathcal{O}(s)) = 0$ for every $k > 0$ and $s \geq r_0$. We have

$$H^k(X, \mathcal{A}_X^{(m)}(s)) = H^k(X, \mathcal{O}(s)) \otimes_{\mathbb{Z}_p} D^{(m)}(\mathbb{G}) = 0,$$

where the first equality follows from the fact that $D^{(m)}(\mathbb{G})$ is, by definition, a direct limit of free $\mathbb{Z}_p$-modules of finite rank. By the second part of Proposition 3.36 there exist $a_0 \in \mathbb{N}$ and $s_0 \in \mathbb{Z}$ together with an epimorphism of $\mathcal{A}_X^{(m)}$-modules

$$\mathcal{E}_0 := \left(\mathcal{A}_X^{(m)}(s_0) \right)^{\oplus a_0} \longrightarrow \mathcal{E} \longrightarrow 0.$$

If $r \geq r_0 - s_0$ we see that $H^k(X, \mathcal{E}_0(r)) = 0$. Now, we way use this relation and the preceding proposition to follow word by word the reasoning given in [21, Proposition 2.2.1] to conclude the lemma. $\qquad\square$

Lemma 3.38. *For every coherent $\mathcal{D}_{X,\lambda}^{(m)}$-module $\mathcal{E}$, there exist $r \in \mathbb{Z}$, a natural number $a \in \mathbb{N}$ and an epimorphism of $\mathcal{D}_{X,\lambda}^{(m)}$-modules*

$$\left(\mathcal{D}_{X,\lambda}^{(m)}(-r) \right)^{\oplus a} \longrightarrow \mathcal{E} \longrightarrow 0.$$

Proof. Using the epimorphism in Proposition 3.32 we can suppose that $\mathcal{E}$ is also a coherent $\mathcal{A}_X^{(m)}$-module. In this case, by the second part of Proposition 3.36, there exist $r = r(\mathcal{E}) \in \mathbb{Z}$, a natural number $a \in \mathbb{N}$ and an epimorphism of $\mathcal{A}_X^{(m)}$-modules

$$\left(\mathcal{A}_X^{(m)}(-r)\right)^{\oplus a} \longrightarrow \mathcal{E} \longrightarrow 0.$$

Taking the tensor product with $\mathcal{D}_{X,\lambda}^{(m)}$ we get the desired epimorphism of $\mathcal{D}_{X,\lambda}^{(m)}$-modules

$$\left(\mathcal{D}_{X,\lambda}^{(m)}(-r)\right)^{\oplus a} \cong \mathcal{D}_{X,\lambda}^{(m)} \otimes_{\mathcal{A}_X^{(m)}} \left(\mathcal{A}_X^{(m)}(-r)\right)^{\oplus a} \longrightarrow \mathcal{D}_{X,\lambda}^{(m)} \otimes_{\mathcal{A}_X^{(m)}} \mathcal{E} \cong \mathcal{E} \longrightarrow 0. \ \square$$

We recall for the reader that the distribution algebra of level m, which has been denoted by $D^{(m)}(\mathbb{G})$ in Section 2.4, is a noetherian $\mathbb{Z}_p$-algebra. This finiteness property is essential in the following proposition.

Proposition 3.39. *Let* $\lambda : \mathrm{Dist}(\mathbb{T}) \to \mathbb{Z}_p$ *be a character and let us suppose that* $\lambda + \rho \in \mathfrak{t}_{\mathbb{Q}}^*$ *is a dominant and regular character (Definition 3.24).*

(i) *Let us fix* $r \in \mathbb{Z}$*. There exists a positive integer* $n(r) \in \mathbb{Z}_{>0}$*, such that* $p^{n(r)} H^k(X, \mathcal{D}_{X,\lambda}^{(m)}(r)) = 0$ *for all* $k \in \mathbb{Z}_{>0}$*. In other words, the cohomology groups* $H^k(X, \mathcal{D}_{X,\lambda}^{(m)}(r))$ *have bounded p-torsion for every* $k \in \mathbb{Z}_{>0}$*.*

(ii) *For every coherent* $\mathcal{D}_{X,\lambda}^{(m)}$*-module* $\mathcal{E}$*, the cohomology group* $H^k(X, \mathcal{E})$ *has bounded p-torsion for all* $k > 0$*.*

Proof. We recall that $(\mathcal{D}_{X,\lambda}^{(m)} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)|_{X_{\mathbb{Q}}} = \mathcal{D}_{X_{\mathbb{Q}},\lambda}$ is the usual sheaf of twisted differential operators on the flag variety $X_{\mathbb{Q}}$ (Remark 3.33). From this point and using the Proposition 3.32, the reader can follow word by word the argument given in [22, Proposition 4.1.9 (i)] to prove (i). The same inductive arguments exhibited in [21, Corollary 2.2.2], or in [22, Proposition 4.1.19 (ii)], apply in our case and show (ii). $\square$

4. Passing to Formal Completions

Let us start by recalling the formal completion of the sheaves introduced in the Section 3.5. Let $\lambda \in \mathrm{Hom}_{\mathbb{Z}_p\text{-alg}}(\mathrm{Dist}(\mathbb{T}), \mathbb{Z}_p)$ be a character of the distribution algebra of $\mathbb{T}$. We recall for the reader that in this work we are abusing of the notation and we are denoting by λ the character $(\lambda \otimes_{\mathbb{Z}_p} 1_{\mathbb{Q}_p})|_{\mathfrak{t}_{\mathbb{Q}}}$ of the Cartan subalgebra $\mathfrak{t}_{\mathbb{Q}} \subset \mathfrak{g}_{\mathbb{Q}}$ (Notation 3.22).

We have introduced the following sheaves of p-adically complete $\mathbb{Z}_p$-algebras on the formal p-adic completion $\mathfrak{X}$ of X

$$\widehat{\mathcal{D}}_{\mathfrak{X},\lambda}^{(m)} := \varprojlim_j \mathcal{D}_{X,\lambda}^{(m)}/p^{j+1}\mathcal{D}_{X,\lambda}^{(m)}.$$

The sheaf $\widehat{\mathcal{D}}_{\mathfrak{X},\lambda,\mathbb{Q}}^{(m)}$ is our sheaf of *level m twisted differential operators* on the formal flag scheme $\mathfrak{X}$.

From now on, we will always assume that $\lambda \in \mathrm{Hom}_{\mathbb{Z}_p\text{-alg}}(\mathrm{Dist}(\mathbb{T}), \mathbb{Z}_p)$ is a character of the distribution algebra of $\mathbb{T}$, such that $\lambda + \rho \in \mathfrak{t}_{\mathbb{Q}}^*$ is a dominant and regular character of $\mathfrak{t}_{\mathbb{Q}}$ (3.23).

4.1. Cohomological Properties. Our objective in this subsection is to prove an analogue of Proposition 3.39 for coherent $\widehat{\mathcal{D}}_{\mathfrak{X},\lambda}^{(m)}$-modules and to conclude that $H^0(\mathfrak{X}, \bullet)$ is an exact functor over the category of coherent $\widehat{\mathcal{D}}_{\mathfrak{X},\lambda,\mathbb{Q}}^{(m)}$-modules.

Proposition 4.1. *Let $\mathcal{E}$ be a coherent $\mathcal{D}_{X,\lambda}^{(m)}$-module and $\widehat{\mathcal{E}} := \varprojlim_j \mathcal{E}/p^{j+1}\mathcal{E}$ be its p-adic completion, which we consider as a sheaf on $\mathfrak{X}$.*

   (i) *For all $k \geq 0$ one has $H^k(\mathfrak{X}, \widehat{\mathcal{E}}) = \varprojlim_j H^k(X_j, \mathcal{E}/p^{j+1}\mathcal{E})$.*

   (ii) *For all $k > 0$ one has $H^k(\mathfrak{X}, \widehat{\mathcal{E}}) = H^k(X, \mathcal{E})$.*

   (iii) *The global section functor $H^0(\mathfrak{X}, \bullet)$ satisfies*

$$H^0(\mathfrak{X}, \widehat{\mathcal{E}}) = \varprojlim_j H^0(X, \mathcal{E})/p^{j+1}H^0(X, \mathcal{E}).$$

Proof. Taking into account Propositions 3.29 and 3.39, the proof follows word by word the cohomological arguments given in [20, Proposition 3.2]. □

Proposition 4.2. *Let $\mathcal{E}$ be a coherent $\widehat{\mathcal{D}}_{\mathfrak{X},\lambda}^{(m)}$-module.*

   (i) *There exists $r_2 = r_2(\mathcal{E}) \in \mathbb{Z}$ such that, for all $r \geq r_2$ there is $a \in \mathbb{Z}$ and an epimorphism of $\widehat{\mathcal{D}}_{\mathfrak{X},\lambda}^{(m)}$-modules*

$$\left(\widehat{\mathcal{D}}_{\mathfrak{X},\lambda}^{(m)}(-r)\right)^{\oplus a} \longrightarrow \mathcal{E} \longrightarrow 0.$$

   (ii) *There exists $r_3 = r_3(\mathcal{E}) \in \mathbb{Z}$ such that, for all $r \geq r_3$ we have $H^i(\mathfrak{X}, \mathcal{E}(r)) = 0$, for all $i \in \mathbb{Z}_{>0}$.*

Proof. Taking lemmas 3.37 and 3.38 into account, the proof follows word by word the arguments given in [22, Proposition 4.2.2]. □

The same inductive argument cited in the second part of Proposition 3.39 shows

Corollary 4.3. *Let $\mathcal{E}$ be a coherent $\widehat{\mathcal{D}}_{\mathfrak{X},\lambda}^{(m)}$-module. There exists $c(\mathcal{E}) \in \mathbb{N}$ such that for all $k \in \mathbb{Z}_{>0}$ the cohomology group $H^k(\mathfrak{X}, \mathcal{E})$ is annihilated by $p^{c(\mathcal{E})}$.*

Now, we want to extend part (i) of Proposition 4.2 to the sheaves $\widehat{\mathcal{D}}_{\mathfrak{X},\lambda,\mathbb{Q}}^{(m)}$. We will need the following discussion.

Let $\mathrm{Coh}(\widehat{\mathcal{D}}_{\mathfrak{X},\lambda}^{(m)})$ be the category of coherent $\widehat{\mathcal{D}}_{\mathfrak{X},\lambda}^{(m)}$-modules and let us consider $\mathrm{Coh}(\widehat{\mathcal{D}}_{\mathfrak{X},\lambda}^{(m)})_{\mathbb{Q}}$ the category of coherent $\widehat{\mathcal{D}}_{\mathfrak{X},\lambda}^{(m)}$-modules up to isogeny. This means that $\mathrm{Coh}(\widehat{\mathcal{D}}_{\mathfrak{X},\lambda}^{(m)})_{\mathbb{Q}}$ has the same class of objects as $\mathrm{Coh}(\widehat{\mathcal{D}}_{\mathfrak{X},\lambda}^{(m)})$ and, for any two objects $\mathcal{M}$ and $\mathcal{N}$ in $\mathrm{Coh}(\widehat{\mathcal{D}}_{\mathfrak{X},\lambda}^{(m)})_{\mathbb{Q}}$ one has

$$\mathrm{Hom}_{\mathrm{Coh}(\widehat{\mathcal{D}}_{\mathfrak{X},\lambda}^{(m)})_{\mathbb{Q}}}(\mathcal{M},\mathcal{N}) = \mathrm{Hom}_{\mathrm{Coh}(\widehat{\mathcal{D}}_{\mathfrak{X},\lambda}^{(m)})}(\mathcal{M},\mathcal{N}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.$$

Proposition 4.4. *The functor $\mathcal{M} \mapsto \mathcal{M} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ induces an equivalence of categories between $\mathrm{Coh}(\widehat{\mathcal{D}}_{\mathfrak{X},\lambda}^{(m)})_{\mathbb{Q}}$ and $\mathrm{Coh}(\widehat{\mathcal{D}}_{\mathfrak{X},\lambda,\mathbb{Q}}^{(m)})$.*

Proof. By definition, the sheaf $\widehat{\mathcal{D}}_{\mathfrak{X},\lambda,\mathbb{Q}}^{(m)}$ satisfies [5, Conditions 3.4.1] and therefore [5, Proposition 3.4.5] allows to conclude the proposition. $\square$

The proof of the next theorem follows exactly the same lines of reasoning exhibited in [22, Theorem 4.2.8].

Theorem 4.5. *Let $\mathcal{E}$ be a coherent $\widehat{\mathcal{D}}_{\mathfrak{X},\lambda,\mathbb{Q}}^{(m)}$-module.*

(i) *There is $r(\mathcal{E}) \in \mathbb{Z}$ such that, for every $r \geq r(\mathcal{E})$ there exists $a \in \mathbb{N}$ and an epimorphism of $\widehat{\mathcal{D}}_{\mathfrak{X},\lambda,\mathbb{Q}}^{(m)}$-modules*

$$\left(\widehat{\mathcal{D}}_{\mathfrak{X},\lambda,\mathbb{Q}}^{(m)}(-r)\right)^{\oplus a} \longrightarrow \mathcal{E} \longrightarrow 0.$$

(ii) *For all $i \in \mathbb{Z}_{>0}$ one has $H^i(\mathfrak{X}, \mathcal{E}) = 0$.*

Proof. We remark for the reader that $\mathfrak{X}$ is a noetherian topological space. The previous proposition gives us a coherent $\widehat{\mathcal{D}}_{\mathfrak{X},\lambda}^{(m)}$-module $\mathcal{F}$ such that $\mathcal{F}_{\mathbb{Q}} = \mathcal{E}$. The first part of the theorem follows from Proposition 4.2. The second part follows from [5, (3.4.0.1)] and corollary 4.3. $\square$

4.2. Calculation of Global Sections.
In this subsection we propose to calculate the global sections of the sheaf $\widehat{\mathcal{D}}_{\mathfrak{X},\lambda,\mathbb{Q}}^{(m)}$ inspired in the arguments exhibited in [24].

Let us identify the universal enveloping algebra $\mathcal{U}(\mathfrak{t}_{\mathbb{Q}})$ of the Cartan subalgebra $\mathfrak{t}_{\mathbb{Q}}$ with the symmetric algebra $\boldsymbol{S}(\mathfrak{t}_{\mathbb{Q}})$, and let $Z(\mathfrak{g}_{\mathbb{Q}})$ denote the center of the universal enveloping algebra $\mathcal{U}(\mathfrak{g}_{\mathbb{Q}})$ of $\mathfrak{g}_{\mathbb{Q}}$. The classical Harish–Chandra isomorphism $Z(\mathfrak{g}_{\mathbb{Q}}) \cong \boldsymbol{S}(\mathfrak{t}_{\mathbb{Q}})^W$ (the subalgebra of Weyl invariants) [14, Theorem 7.4.5] allows us to define for every linear form $\lambda \in \mathfrak{t}_{\mathbb{Q}}^*$ a central character ([14, 7.4.6]) $\chi_{\lambda+\rho} : Z(\mathfrak{g}_{\mathbb{Q}}) \to \mathbb{Q}_p$ which induces the central reduction $\mathcal{U}(\mathfrak{g}_{\mathbb{Q}})_\lambda := \mathcal{U}(\mathfrak{g}_{\mathbb{Q}}) \otimes_{Z(\mathfrak{g}_{\mathbb{Q}}),\chi_{\lambda+\rho}} \mathbb{Q}_p$.

If $\mathrm{Ker}(\chi_{\lambda+\rho})_{\mathbb{Z}_p} := D^{(m)}(\mathbb{G}) \cap \mathrm{Ker}(\chi_{\lambda+\rho})$, we may consider the central redaction

$$D^{(m)}(\mathbb{G})_\lambda := D^{(m)}(\mathbb{G})/D^{(m)}(\mathbb{G})\,\mathrm{Ker}(\chi_{\lambda+\rho})_{\mathbb{Z}_p}$$

and its p-adic completion $\widehat{D}^{(m)}(\mathbb{G})_\lambda$. It is clear that

$$D^{(m)}(\mathbb{G})_\lambda \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = \mathcal{U}(\mathfrak{g}_{\mathbb{Q}})_\lambda.$$

Theorem 4.6. *The homomorphism of $\mathbb{Z}_p$-algebras*

$$\Phi_\lambda^{(m)} : D^{(m)}(\mathbb{G}) \longrightarrow H^0(X, \mathcal{D}_{X,\lambda}^{(m)}),$$

defined by taking global sections in (3.24), induces an isomorphism of $\mathbb{Q}_p$-algebras

$$\widehat{D}^{(m)}(\mathbb{G})_\lambda \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \xrightarrow{\cong} H^0\left(\mathfrak{X}, \widehat{\mathcal{D}}_{\mathfrak{X},\lambda,\mathbb{Q}}^{(m)}\right).$$

Proof. The key point in the proof of the theorem is the following commutative diagram, which is an immediate consequence of Remark 3.33

$$
\begin{array}{ccc}
D^{(m)}(\mathbb{G}) & \xrightarrow{\Phi_\lambda^{(m)}} & H^0(X, \mathcal{D}_{X,\lambda}^{(m)}) \\
\downarrow & & \downarrow \\
\mathcal{U}(\mathfrak{g}_{\mathbb{Q}}) & \xrightarrow{\Phi_\lambda} & H^0(X_{\mathbb{Q}}, \mathcal{D}_{X_{\mathbb{Q}},\lambda}).
\end{array}
$$

By the classical Beilinson–Bernstein theorem [2] and the preceding commutative diagram, we have that $\Phi_\lambda^{(m)}$ factors through the morphism

$$\overline{\Phi^{(m)}}_\lambda : D^{(m)}(\mathbb{G})_\lambda \longrightarrow H^0(X, \mathcal{D}_{X,\lambda}^{(m)})$$

which becomes an isomorphism after tensoring with $\mathbb{Q}_p$. Lemma 3.3 of [24] and Proposition 4.1 imply that

$$\widehat{\overline{\Phi^{(m)}}}_\lambda \otimes_{\mathbb{Z}_p} 1_{\mathbb{Q}_p} : \widehat{D}^{(m)}(\mathbb{G})_\lambda \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \xrightarrow{\cong} H^0\left(\mathfrak{X}, \widehat{\mathcal{D}}_{\mathfrak{X},\lambda,\mathbb{Q}}^{(m)}\right)$$

is the desired isomorphism. $\square$

4.3. The Arithmetic Beilinson–Bernstein Theorem. In this section we will introduce the localization functor and we will prove the main result of this paper. Let E be a finitely generated $\widehat{D}^{(m)}(\mathbb{G})_{\lambda,\mathbb{Q}}$-module. We define $\mathcal{L}oc_{\mathfrak{X},\lambda}^{(m)}(E)$ as the associated sheaf to the presheaf on $\mathfrak{X}$ defined by

$$\mathfrak{U} \subseteq \mathfrak{X} \longmapsto \widehat{\mathcal{D}}_{\mathfrak{X},\lambda,\mathbb{Q}}^{(m)}(\mathfrak{U}) \otimes_{\widehat{D}^{(m)}(\mathbb{G})_{\lambda,\mathbb{Q}}} E.$$

By Proposition 3.31, it is clear that $\mathcal{L}oc_{\mathfrak{X},\lambda}^{(m)}$ is a functor from the category of finitely generated $\widehat{D}^{(m)}(\mathbb{G})_{\lambda,\mathbb{Q}}$-modules to the category of coherent $\widehat{\mathcal{D}}_{\mathfrak{X},\lambda,\mathbb{Q}}^{(m)}$-modules.

Proposition 4.7. *If $\mathcal{E}$ is a coherent $\widehat{\mathcal{D}}_{\mathfrak{X},\lambda,\mathbb{Q}}^{(m)}$-module, then $\mathcal{E}$ is generated by its global sections as $\widehat{\mathcal{D}}_{\mathfrak{X},\lambda,\mathbb{Q}}^{(m)}$-module. Furthermore, every coherent $\widehat{\mathcal{D}}_{\mathfrak{X},\lambda,\mathbb{Q}}^{(m)}$-module admits a resolution by finite free $\widehat{\mathcal{D}}_{\mathfrak{X},\lambda,\mathbb{Q}}^{(m)}$-modules.*

Proof. Theorem 4.5, Proposition 3.36 and Theorem 4.6 allow us to follow the same reasoning given in [22, Proposition 4.3.1]. $\qquad\square$

Theorem 4.8 (Principal Theorem). *Let* $\lambda : \mathrm{Dist}(\mathbb{T}) \to \mathbb{Z}_p$ *be a character of* $\mathrm{Dist}(\mathbb{T})$ *such that* $\lambda + \rho \in \mathfrak{t}_{\mathbb{Q}}^*$ *is a dominant and regular character of* $\mathfrak{t}_{\mathbb{Q}}$.

  (i) *The functors* $\mathcal{L}oc_{\mathfrak{X},\lambda}^{(m)}$ *and* $H^0(\mathfrak{X}, \bullet)$ *are quasi-inverse equivalence of categories between the abelian categories of finitely generated (left)* $\widehat{D}^{(m)}(\mathbb{G})_\lambda \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$-*modules and coherent* $\widehat{\mathcal{D}}_{\mathfrak{X},\lambda,\mathbb{Q}}^{(m)}$-*modules.*
  (ii) *The functor* $\mathcal{L}oc_{\mathfrak{X},\lambda}^{(m)}$ *is an exact functor.*

Proof. Taking into account Proposition 4.7, Theorem 4.5 (ii) and Theorem 4.6, we may follow word by word the reasoning given in [20, Proposition 5.2.1] in order to obtain (i). The second assertion follows from the fact that any equivalence between abelian categories is exact. $\qquad\square$

5. Sheaves of Twisted Differential Operators of Infinite Order

Throughout this section we will continue adopting the notation fixed in Section 3.5. For instance, $\lambda : \mathrm{Dist}(\mathbb{T}) \to \mathbb{Z}_p$ will always denote a character of $\mathrm{Dist}(\mathbb{T})$, such that $\lambda + \rho \in \mathfrak{t}_{\mathbb{Q}}^*$ is a dominant and regular character of $\mathfrak{t}_{\mathbb{Q}}$. We recall for the reader that in this text we are abusing of the notation and denoting again by λ the character $(\lambda \otimes_{\mathbb{Z}_p} 1_{\mathbb{Q}_p})|_{\mathfrak{t}_{\mathbb{Q}}}$ of the Cartan subalgebra $\mathfrak{t}_{\mathbb{Q}} \subset \mathfrak{g}_{\mathbb{Q}}$ (Notation 3.22).

In this section we will study the problem of passing to the inductive limit when m varies.

5.1. The Sheaves $\mathcal{D}_{\mathfrak{X},\lambda}^\dagger$. Let us recall that $\xi : \widetilde{X} := \mathbb{G}/\boldsymbol{N} \to X := \mathbb{G}/\mathbb{B}$ is a $\mathbb{T}$-torsor. For every couple of positive integers $m \le m'$ there exists a canonical homomorphism of sheaves of filtered rings [5, (2.2.1.5)]

$$(5.1) \qquad\qquad \rho_{m',m} : \mathcal{D}_{\widetilde{X}}^{(m)} \longrightarrow \mathcal{D}_{\widetilde{X}}^{(m')}.$$

Let us fix a character $\lambda : \mathrm{Dist}(\mathbb{T}) \to \mathbb{Z}_p$. By Proposition 3.18 we have a commutative diagram

$$(5.2) \qquad\qquad D^{(m)}(\mathbb{T}) \overset{\lambda^{(m)}}{\underset{}{\xhookrightarrow{}}} D^{(m')}(\mathbb{T}) \xrightarrow{\lambda^{(m')}} \mathbb{Z}_p$$

Moreover, by [5, (1.4.7.1)] we have a canonical morphism $\mathcal{P}_{\widetilde{X},(m')}^n \to \mathcal{P}_{\widetilde{X},(m)}^n$.

In Section 3.3 we have defined a $\mathbb{T}$-equivariant structure

$$\Phi_{(m)}^n : \sigma^* \mathcal{P}_{\widetilde{X},(m)}^n \longrightarrow p_1^* \mathcal{P}_{\widetilde{X},n}^n$$

on $\mathcal{P}^n_{\widetilde{X},(m)}$ (we recall for the reader that σ denotes the right action of $\mathbb{T}$ on $\widetilde{X}$ and p_1 the first projection). By universal property of $\mathcal{P}^n_{\widetilde{X},(m)}$ the preceding $\mathbb{T}$-equivariant structures fit into a commutative diagram

$$
\begin{array}{ccc}
\sigma^* \mathcal{P}^n_{\widetilde{X},(m')} & \xrightarrow{\Phi^n_{(m')}} & p_1^* \mathcal{P}^n_{\widetilde{X},(m')} \\
\downarrow & & \downarrow \\
\sigma^* \mathcal{P}^n_{\widetilde{X},(m)} & \xrightarrow{\Phi^n_{(m)}} & p_1^* \mathcal{P}^{(m)}_{\widetilde{X},(m)}.
\end{array}
$$

This implies that the morphisms $\mathcal{P}^n_{\widetilde{X},(m')} \to \mathcal{P}^n_{\widetilde{X},(m)}$ are $\mathbb{T}$-equivariant and therefore by Lemma 3.4, we can conclude that the canonical maps in (5.1) are $\mathbb{T}$-equivariant. In this way, we have morphisms $\widetilde{\mathcal{D}^{(m)}} \to \widetilde{\mathcal{D}^{(m')}}$. The diagram (5.2) implies that we also have maps $\mathcal{D}^{(m)}_{X,\lambda} \to \mathcal{D}^{(m')}_{X,\lambda}$ and therefore an inductive system

$$
(5.3) \qquad \xi_* \widehat{(\rho_{m',m})}^{\mathbb{T}} : \widehat{\mathcal{D}}^{(m)}_{\mathfrak{X},\lambda} \longrightarrow \widehat{\mathcal{D}}^{(m')}_{\mathfrak{X},\lambda}.
$$

Definition 5.1. We will denote by $\mathcal{D}^{\dagger}_{\mathfrak{X},\lambda}$ the limit of the inductive system (5.3) tensored with $\mathbb{Q}_p$

$$
\mathcal{D}^{\dagger}_{\mathfrak{X},\lambda} := \left(\varinjlim_m \widehat{\mathcal{D}}^{(m)}_{\mathfrak{X},\lambda} \right) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.
$$

Remark 5.2. Let us suppose that $\lambda \in \mathrm{Hom}(\mathbb{T}, \mathbb{G}_m)$ is an algebraic character and let us denote by $\lambda' : \mathrm{Dist}(\mathbb{T}) \to \mathbb{Z}_p$ the character of $\mathrm{Dist}(\mathbb{T})$ induced by the correspondence (3.21). Let us denote by $\mathcal{D}^{\dagger}_{\mathfrak{X}}(\lambda)$ the inductive limit of the sheaves defined in 3.34. By Proposition 3.35 we have $\mathcal{D}^{\dagger}_{\mathfrak{X}}(\lambda) = \mathcal{D}^{\dagger}_{\mathfrak{X},\lambda'}$.

5.2. The Infinite Order Localization Functor and the Arithmetic Beilinson–Bernstein Theorem. In this subsection we will extend the Beilinson–Bernstein theorem for the sheaf of rings $\mathcal{D}^{\dagger}_{\mathfrak{X},\lambda}$. We start by computing the global section of the sheaf $\mathcal{D}^{\dagger}_{\mathfrak{X},\lambda}$ and defining the *infinite order localization functor*.

Let us recall that in the Section 4.2 we have shown that there exists a canonical isomorphism of $\mathbb{Q}_p$-algebras

$$
\widehat{D}^{(m)}(\mathbb{G})_{\lambda,\mathbb{Q}} \xrightarrow{\cong} H^0 \left(\mathfrak{X}, \widehat{\mathcal{D}}^{(m)}_{\mathfrak{X},\lambda,\mathbb{Q}} \right).
$$

Taking inductive limits we may conclude that if $D^{\dagger}(\mathbb{G})_{\lambda} := \varinjlim_m \widehat{D}^{(m)}(\mathbb{G})_{\lambda,\mathbb{Q}}$, then we also have a canonical isomorphism of $\mathbb{Q}_p$-algebras

$$
(5.4) \qquad D^{\dagger}(\mathbb{G})_{\lambda} \xrightarrow{\cong} H^0(\mathfrak{X}, \mathcal{D}^{\dagger}_{\mathfrak{X},\lambda}).
$$

On the other hand, in a completely analogous way as we have done in the Section 4.3, we may define the localization functor $\mathcal{L}oc^{\dagger}_{\mathfrak{X},\lambda}$ from the category of $D^{\dagger}(\mathbb{G})_{\lambda}$-modules to the category of $\mathcal{D}^{\dagger}_{\mathfrak{X},\lambda}$-modules. This is, if E denotes a finitely presented $D^{\dagger}(\mathbb{G})_{\lambda}$-module, then $\mathcal{L}oc^{\dagger}_{\mathfrak{X},\lambda}(E)$ denotes the associated sheaf to the presheaf on $\mathfrak{X}$ defined by

$$\mathfrak{U} \subseteq \mathfrak{X} \longmapsto \mathcal{D}^{\dagger}_{\mathfrak{X},\lambda}(\mathfrak{U}) \otimes_{D^{\dagger}(\mathbb{G})_{\lambda}} E.$$

By Proposition 5.5 below, we see that $\mathcal{L}oc^{\dagger}_{\mathfrak{X},\lambda}$ induces a functor from the category of finitely presented $D^{\dagger}(\mathbb{G})_{\lambda}$-modules to the category of coherent $\mathcal{D}^{\dagger}_{\mathfrak{X},\lambda}$-modules.

Theorem 5.3. *Let us suppose that $\lambda : \mathrm{Dist}(\mathbb{T}) \to \mathbb{Z}_p$ is a character of $\mathrm{Dist}(\mathbb{T})$ such that $\lambda + \rho \in \mathfrak{t}^*_{\mathbb{Q}}$ is a dominant and regular character of $\mathfrak{t}_{\mathbb{Q}}$.*

   (i) *The functors $\mathcal{L}oc^{\dagger}_{\mathfrak{X},\lambda}$ and $H^0(\mathfrak{X}, \bullet)$ are quasi-inverse equivalence of categories between the abelian categories of finitely presented (left) $D^{\dagger}(\mathbb{G})_{\lambda}$-modules and coherent $\mathcal{D}^{\dagger}_{\mathfrak{X},\lambda}$-modules.*

  (ii) *The functor $\mathcal{L}oc^{\dagger}_{\mathfrak{X},\lambda}$ is an exact functor.*

In order to prove this theorem we will adapt our situation to the one studied in [21]. We will need the following.

Remark 5.4.

   (i) Let us recall that in Remark 2.12 we have stated that $D^{(m)}(\mathbb{T})$ is isomorphic to the subspace of $\mathbb{T}$-invariants $H^0(\mathbb{T}, \mathcal{D}^{(m)}_{\mathbb{T}})^{\mathbb{T}}$. The isomorphism is in fact induced by the action of $\mathbb{T}$ on itself by right translations, [23, Theorem 4.4.8.3], and it is compatible with m variable. This means that if Q_m and Q'_m denote those isomorphisms for $m \le m'$, then we have a commutative diagram

$$
\begin{array}{ccc}
D^{(m)}(\mathbb{T}) & \xrightarrow{\ Q_m\ } & H^0(\mathbb{T}, \mathcal{D}^{(m)}_{\mathbb{T}})^{\mathbb{T}} \\
\ \ \downarrow{\scriptstyle \phi_{m',m}} & & \ \ \downarrow{\scriptstyle (\gamma_{m',m})^{\mathbb{T}}} \\
D^{(m')}(\mathbb{T}) & \xrightarrow{\ Q_{m'}\ } & H^0(\mathbb{T}, \mathcal{D}^{(m')}_{\mathbb{T}})^{\mathbb{T}},
\end{array}
$$

       where the morphisms $\phi_{m',m}$ are obtained by dualizing the canonical morphisms $\psi_{m',m}$ of Section 2.4 and the morphisms $\gamma_{m',m}$ are defined in (5.1).

  (ii) Again by Remark 2.12 the isomorphism of Proposition 3.27 are compatible for varying m.

Proposition 5.5. *The sheaf of rings $\mathcal{D}^{\dagger}_{\mathfrak{X},\lambda}$ is coherent.*

Proof. By Proposition 3.31 and [5, Proposition 3.6.1] we only need to show that the morphisms $\xi_*\widehat{(\rho_{m',m})}^{\mathbb{T}}{}_{\mathbb{Q}}$ are flat. Given that this is a local property we may take $U \in \mathcal{S}$ and to verify this property over its formal completion $\mathfrak{U}$. In this case, Remark 5.4 and the argument used in the proof of the first part of Proposition 3.31 give us, by functoriality, the following commutative diagram

$$
\begin{array}{ccc}
\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X},\lambda,\mathbb{Q}}(\mathfrak{U}) & \xrightarrow{\;\xi_*\widehat{(\rho_{m',m})}^{\mathbb{T}}{}_{\mathbb{Q}}(\mathfrak{U})\;} & \widehat{\mathcal{D}}^{(m')}_{\mathfrak{X},\lambda,\mathbb{Q}}(\mathfrak{U}) \\[2mm]
\Big\downarrow{\wr|} & & \Big\downarrow{\wr|} \\[2mm]
\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X},\mathbb{Q}}(\mathfrak{U}) & \xrightarrow{\;\widehat{\rho}_{m',m,\mathbb{Q}}(\mathfrak{U})\;} & \widehat{\mathcal{D}}^{(m')}_{\mathfrak{X},\mathbb{Q}}(\mathfrak{U}).
\end{array}
$$

Theorem [5, Theorem 3.5.3] states that the lower morphism is flat and so is the morphism on the top. $\qquad\square$

Lemma 5.6. *For every coherent $\mathcal{D}^{\dagger}_{\mathfrak{X},\lambda}$-module $\mathcal{E}$ there exists $m \geq 0$, a coherent $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X},\lambda,\mathbb{Q}}$-module $\mathcal{E}_m$ and an isomorphism of $\mathcal{D}^{\dagger}_{\mathfrak{X},\lambda}$-modules*

$$
\tau : \mathcal{D}^{\dagger}_{\mathfrak{X},\lambda} \otimes_{\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X},\lambda,\mathbb{Q}}} \mathcal{E}_m \xrightarrow{\;\cong\;} \mathcal{E}.
$$

Moreover, if $(m', \mathcal{E}_{m'}, \tau')$ is another such triple, then there exists $l \in \mathbb{N}$ and an isomorphism of $\widehat{\mathcal{D}}^{(l)}_{\mathfrak{X},\lambda,\mathbb{Q}}$-modules

$$
\tau_l : \widehat{\mathcal{D}}^{(l)}_{\mathfrak{X},\lambda,\mathbb{Q}} \otimes_{\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X},\lambda,\mathbb{Q}}} \mathcal{E}_m \xrightarrow{\;\cong\;} \widehat{\mathcal{D}}^{(l)}_{\mathfrak{X},\lambda,\mathbb{Q}} \otimes_{\widehat{\mathcal{D}}^{(m')}_{\mathfrak{X},\lambda,\mathbb{Q}}} \mathcal{E}_{m'}
$$

such that $\tau' \circ \left(\mathrm{id}_{\mathcal{D}^{\dagger}_{\mathfrak{X},\lambda}} \otimes \tau_l \right) = \tau.$

Proof. This is [5, Proposition 3.6.2 (ii)]. We remark that $\mathfrak{X}$ is quasi-compact and separated, and the sheaf $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X},\lambda,\mathbb{Q}}$ satisfies the conditions in [5, 3.4.1]. $\quad\square$

Proposition 5.7. [3] *Let $\mathcal{E}$ be a coherent $\mathcal{D}^{\dagger}_{\mathfrak{X},\lambda}$-module.*

    (i) *There exists an integer $r(\mathcal{E})$ such that, for all $r \geq r(\mathcal{E})$ there is $a \in \mathbb{N}$ and an epimorphism of $\mathcal{D}^{\dagger}_{\mathfrak{X},\lambda}$-modules*

$$
\left(\mathcal{D}^{\dagger}_{\mathfrak{X},\lambda}(-r) \right)^{\oplus a} \longrightarrow \mathcal{E} \longrightarrow 0.
$$

    (ii) *For all $i > 0$ one has $H^i(\mathfrak{X}, \mathcal{E}) = 0$.*

Proof. Taking into account the previous lemma, the results follow from the respective analogues for the sheaves $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X},\lambda,\mathbb{Q}}$ (Theorem 4.5), and the fact that on a noetherian space the cohomology commutes with direct limits. $\quad\square$

---

[3]This is exactly as in [22, Theorem 4.2.8]

Proposition 5.8. [4] *Let $\mathcal{E}$ be a coherent $\mathcal{D}_{\mathfrak{X},\lambda}^{\dagger}$-module. Then $\mathcal{E}$ is generated by its global sections as $\mathcal{D}_{\mathfrak{X},\lambda}^{\dagger}$-module. Moreover, $\mathcal{E}$ has a resolution by finite free $\mathcal{D}_{\mathfrak{X},\lambda}^{\dagger}$-modules and $H^0(\mathfrak{X},\mathcal{E})$ is a $D^{\dagger}(\mathbb{G})_{\lambda}$-module of finite presentation.*

Proof. The proposition is an easy consequence of Lemma 5.6, Propositions 4.7 and 5.7(ii), and (5.4). □

Proof of Theorem 5.3. All in all, we may follow the same arguments of [21, Corollary 2.3.7]. The relation with the category of finitely presented $D^{\dagger}(\mathbb{G})_{\lambda}$-modules follows from (5.4). □

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