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Reverse engineered Diophantine equations over \mathbb{Q}

par KATERINA SANTICOLA

RÉSUMÉ. Soit $\mathcal{P}_{\mathbb{Q}} = \{\alpha^n : \alpha \in \mathbb{Q}, n \geq 2\}$ l'ensemble des puissances parfaites rationnelles, et soit S un sous-ensemble fini de $\mathcal{P}_{\mathbb{Q}}$. Nous prouvons l'existence d'un polynôme $f_S \in \mathbb{Z}[X]$ tel que $f(\mathbb{Q}) \cap \mathcal{P}_{\mathbb{Q}} = S$. Ceci généralise un théorème récent de Gajović qui a démontré un résultat similaire pour les sous-ensembles finis de puissances parfaites entières. Notre approche fait appel à la résolution de l'équation de Fermat généralisée de signature $(2, 4, n)$ dans [2, 4, 7], ainsi qu'à la finitude des puissances parfaites dans les suites récurrentes binaires non dégénérées, prouvée par Pethő et par Shorey et Stewart.

ABSTRACT. Let $\mathcal{P}_{\mathbb{Q}} = \{\alpha^n : \alpha \in \mathbb{Q}, n \geq 2\}$ be the set of rational perfect powers, and let S be a finite subset of $\mathcal{P}_{\mathbb{Q}}$. We prove the existence of a polynomial $f_S \in \mathbb{Z}[X]$ such that $f(\mathbb{Q}) \cap \mathcal{P}_{\mathbb{Q}} = S$. This generalizes a recent theorem of Gajović who proved a similar result for finite subsets of integer perfect powers. Our approach makes use of the resolution of the generalized Fermat equation of signature $(2, 4, n)$ in [2, 4, 7], as well as the finiteness of perfect powers in non-degenerate binary recurrence sequences, proved by Pethő and by Shorey and Stewart.

1. Introduction

By a *Schinzel–Tijdeman equation* [14] we mean an equation of the form

$$(1.1) \quad f(X) = Y^n$$

where f is a polynomial with integer or rational coefficients, and the unknowns X, Y are taken to be either integral or rational. The exponent n is also an unknown, usually taken to run through the integers $n \geq 2$. A famous theorem of Schinzel and Tijdeman [12] concerning such equations asserts that if $f \in \mathbb{Z}[X]$ has at least two distinct roots, then there are finitely many solutions (X, Y, n) to (1.1) with $X, Y, n \in \mathbb{Z}, n \geq 3$ and $|Y| > 1$. One of the earliest examples of a Schinzel–Tijdeman equation is

$$(1.2) \quad X^2 + 1 = Y^n, \quad X, Y, n \in \mathbb{Z}, \quad n \geq 2;$$

In 1850 Victor Lebesgue [9] proved that the only solutions are $(X, Y) = (0, 1)$ if n is odd, and $(X, Y) = (0, \pm 1)$ if n is even. Lebesgue's argument

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is elementary, and some version of this argument, with the help of the Primitive Divisor Theorem of Bilu, Hanrot and Voutier [3] is capable of resolving many equations of the form $X^2 + C = Y^n$ (e.g. [6]). In contrast, the equation $X^2 + 7 = Y^n$ (which generalizes the famous Ramanujan–Nagell equation) apparently cannot be tackled by elementary arguments, and was solved by Bugeaud, Mignotte and Siksek [5] using a combination of tools from Diophantine approximation and Galois representations of elliptic curves. We can think of the Catalan equation

$$X^m - Y^n = 1, \quad X, Y, m, n \in \mathbb{Z}, \quad m, n \geq 2$$

as an infinite family of Schinzel–Tijdeman equations of the form (1.1) by rewriting it as $X^m + 1 = Y^n$, giving such an equation for each value of m . The famous Catalan conjecture, proved by Mihăilescu [10] in 2004, states the only solution with $XY \neq 0$ is $3^2 - 2^3 = 1$.

We can view equation (1.1) as the question of which perfect powers belong to $f(\mathbb{Z})$ (if the unknowns X, Y belong to \mathbb{Z}) or to $f(\mathbb{Q})$ (if the unknowns X, Y belong to \mathbb{Q}). It is therefore natural to ask whether there is a restriction on the set of perfect powers that can belong to $f(\mathbb{Z})$ or $f(\mathbb{Q})$. Indeed, at the recent “Rational Points” conference (Schney, April 2022), Siksek posed the following two questions.

Question 1.1. *Let*

$$\mathcal{P}_{\mathbb{Z}} = \{a^n : a \in \mathbb{Z}, n \geq 2\}$$

be the set of perfect powers in \mathbb{Z} . Let S be a finite subset of $\mathcal{P}_{\mathbb{Z}}$. Is there a polynomial $f_S \in \mathbb{Z}[X]$ such that $f_S(\mathbb{Z}) \cap \mathcal{P}_{\mathbb{Z}} = S$?

Question 1.2. *Let*

$$(1.3) \quad \mathcal{P}_{\mathbb{Q}} = \{\alpha^n : \alpha \in \mathbb{Q}, n \geq 2\}$$

be the set of perfect powers in \mathbb{Q} . Let S be a finite subset of $\mathcal{P}_{\mathbb{Q}}$. Is there a polynomial $f_S \in \mathbb{Q}[X]$ such that $f_S(\mathbb{Q}) \cap \mathcal{P}_{\mathbb{Q}} = S$?

Question 1.1 was answered affirmatively by Gajović [8, Theorem 3.1], and we briefly recall (and slightly simplify) his elegant argument which yields an explicit polynomial f_S in terms of S . Let $S = \{b_1, b_2, \dots, b_r\} \subseteq \mathcal{P}_{\mathbb{Z}}$. Let

$$g(X) = \prod_{i=1}^r (X - b_i)^2 + 1, \quad h(X) = (X - 1) \cdot g(X) + 1, \quad f_S(X) = g(X) \cdot h(X).$$

It is clear that $f_S(b_i) = b_i$, so $S \subseteq f_S(\mathbb{Z}) \cap \mathcal{P}_{\mathbb{Z}}$. To prove the reverse inclusion let $x \in \mathbb{Z}$ such that $f_S(x) = y^n$ for some $y \in \mathbb{Z}$ and $n \geq 2$; it is enough to prove that $x = b_i$ for some i . Write $c = \prod_{i=1}^r (x - b_i)$. We consider two cases:

- $y = 0$. Then $h(x) = 0$, so $x - 1 = -1/(c^2 + 1)$. Since $x \in \mathbb{Z}$, we have $c = 0$, and so $x = b_i$ for some i .

- $y \neq 0$. Note that $g(x) > 0$ and $h(x)$ are coprime integers whose product is y^n . Thus $g(x) = z^n$ for some integer z , and so $c^2 + 1 = z^n$. This implies $c = 0$, by the aforementioned theorem of Lebesgue concerning (1.2). Hence $x = b_i$ for some i , completing the proof.

We point out that for the last step Gajović invokes Mihăilescu's theorem [10] (the Catalan conjecture), although it is enough to invoke the special (and elementary) case due to Lebesgue.

The purpose of this paper is to give an affirmative answer to Question 1.2. Concisely, our theorem can be stated as follows.

Theorem 1.3. *Let S be a finite subset of $\mathcal{P}_{\mathbb{Q}}$. Then there is a polynomial $f_S \in \mathbb{Z}[X]$ such that $f_S(\mathbb{Q}) \cap \mathcal{P}_{\mathbb{Q}} = S$.*

We give a recipe for f_S in terms of a certain effectively computable (though currently inexplicit) constant. Instead of Lebesgue's theorem, or Mihăilescu's theorem, our proof that $f_S(\mathbb{Q}) \cap \mathcal{P}_{\mathbb{Q}} = S$ crucially relies on the following theorem due to Ellenberg, to Bennett, Ellenberg and Ng, and to Bruin.

Theorem 1.4 (Ellenberg et al.). *Let $n \geq 4$. Then the equation $a^2 + b^4 = c^n$ has no solutions in coprime non-zero integers.*

The theorem is due to Bruin [4] for $n = 6$, who treats this case using an elliptic Chabauty argument. It is due to Ellenberg [7] for prime $n \geq 211$, and to Bennett, Ellenberg and Ng [2] for all other $n \geq 4$. These two papers make use of deep results in the theory of Galois representations of \mathbb{Q} -curves as well as a careful study of critical values of Hecke L -functions of modular forms.

We shall also need a famous theorem on perfect powers in non-degenerate binary recurrence sequences, due to Pethő [11] and independently to Shorey and Stewart [13]. We shall not need the theorem in its full generality, so we only state a special case.

Theorem 1.5 (Pethő, Shorey and Stewart). *Let a, b, α, β be non-zero integers with $\alpha \neq \pm\beta$. Let*

$$u_t = a\alpha^t + b\beta^t, \quad t \in \mathbb{Z}, \quad t \geq 0.$$

Then there is an effectively computable constant $C(a, b, \alpha, \beta)$ such that $u_t \notin \mathcal{P}_{\mathbb{Z}}$ for $t > C(a, b, \alpha, \beta)$.

The theorem is proved using Baker's theory of lower bounds for linear forms in logarithms of algebraic numbers.

2. Preliminary results

We shall need the following consequence of Theorem 1.5.

Lemma 2.1. *Let $\gamma \in \mathbb{Q}$, $\gamma \neq 0$. Then there is an effectively computable constant $D(\gamma)$ such for $t > D(\gamma)$,*

$$\gamma - 2^t \notin \mathcal{P}_{\mathbb{Q}}.$$

Proof. Write $\gamma = u/v$ where $u, v \in \mathbb{Z}$, $u \neq 0$, $v \geq 1$ and $\gcd(u, v) = 1$. We let

$$D(\gamma) = \max\{\log_2 |\gamma|, C(u, -v, 1, 2)\}$$

where $C(a, b, \alpha, \beta)$ is as in Theorem 1.5. Note that since $C(a, b, \alpha, \beta)$ is effectively computable, so is $D(\gamma)$. Let $t > D(\gamma)$. We suppose $\gamma - 2^t \in \mathcal{P}_{\mathbb{Q}}$ and derive a contradiction.

Since $t > \log_2 |\gamma|$, we have $\gamma - 2^t \neq 0$. Observe that $\gamma - 2^t = (u - 2^t v)/v$ where the numerator $u - 2^t v$ and the denominator v are coprime. As $\gamma - 2^t \in \mathcal{P}_{\mathbb{Q}} \setminus \{0\}$ we conclude that $u - 2^t v \in \mathcal{P}_{\mathbb{Z}}$. We apply Theorem 1.5 with $a = u$, $b = -v$, $\alpha = 1$, $\beta = 2$. Since $t > C(u, -v, 1, 2)$, the theorem tells us that $u - 2^t v \notin \mathcal{P}_{\mathbb{Z}}$ giving a contradiction. \square

We also need the following two corollaries to Theorem 1.4.

Corollary 2.2. *Let $n \geq 2$. Then the only solutions to the equation $A^4 + B^4 = 2C^n$ with $A, B, C \in \mathbb{Z}$ and $\gcd(A, B) = 1$ satisfy $A = \pm 1$, $B = \pm 1$.*

Proof. Note that A, B are both odd. Write

$$U = AB, \quad V = \frac{A^4 - B^4}{2}.$$

Then U, V are coprime integers and satisfy

$$U^4 + V^2 = \left(\frac{A^4 + B^4}{2} \right)^2 = C^{2n}.$$

By Theorem 1.4 we have $UVC = 0$. However, U is odd so $U \neq 0$. Thus $C \neq 0$. Hence $V = 0$, so $A^4 = B^4$. Since A, B are coprime, $A = \pm 1$, $B = \pm 1$ as required. \square

Corollary 2.3. *Let $n \geq 2$. Then the only solutions to the equation $A^4 + B^4 = C^n$ with $A, B, C \in \mathbb{Z}$ and $\gcd(A, B) = 1$ satisfy $A = 0$, $B = \pm 1$, or $B = 0$, $A = \pm 1$.*

Proof. For $n = 2$ this is a famous result of Fermat, proved by infinite descent. For $n = 3$ it is in fact a result of Lucas [1, Section 5]. Suppose $n \geq 4$. Then we can rewrite the equation as $(A^2)^2 + B^4 = C^n$ and conclude from Theorem 1.4. \square

3. Proof of Theorem 1.3

Let S be a finite subset of $\mathcal{P}_{\mathbb{Q}}$. We would like to give a polynomial $f_S \in \mathbb{Z}[X]$ such that $f_S(\mathbb{Q}) \cap \mathcal{P}_{\mathbb{Q}} = S$. For $S = \emptyset$ we may take any constant polynomial whose value is not a perfect power. Thus we may suppose $S \neq \emptyset$. We write

$$S = \{\beta_1, \dots, \beta_r\}$$

where $r \geq 1$, and $\beta_i \in \mathcal{P}_{\mathbb{Q}}$. As the β_i are rational numbers, we can write

$$\beta_i = \frac{a_i}{c_i}, \quad a_i, c_i \in \mathbb{Z}, \quad c_i \geq 1, \quad \gcd(a_i, c_i) = 1.$$

Let p_1, p_2, \dots, p_t be the distinct prime divisors of $\prod_{i=1}^r c_i$, and let

$$(3.1) \quad k = \text{lcm}(4, p_1 - 1, p_2 - 1, \dots, p_t - 1).$$

Let

$$(3.2) \quad F(X) = \prod_{i=1}^r (c_i X - a_i)^2 - 1.$$

Since $c_i \geq 1$ we note that F has degree $2r$, and hence at most $2r$ roots. We are only interested in the non-zero rational roots of F , and we let these be $\delta_1, \delta_2, \dots, \delta_m$. We let s be an integer satisfying the following:

$$(3.3) \quad s = 2^\kappa - 1 \text{ for some } \kappa \geq 1,$$

$$(3.4) \quad s \geq D(4\delta_j), \quad j = 1, \dots, m.$$

Here $D(\cdot)$ is as in Lemma 2.1. It is clear that such an s is effectively computable. Let

$$(3.5) \quad g(X) = \prod_{i=1}^r (c_i X - a_i)^k + 1, \quad h(X) = (X - 2^s)g(X) + 2^s,$$

$$(3.6) \quad f_S(X) = g(X) \cdot h(X) \in \mathbb{Z}[X].$$

To prove Theorem 1.3 we need to show that $f_S(\mathbb{Q}) \cap \mathcal{P}_{\mathbb{Q}} = S$. Observe that $g(\beta_i) = 1$, $h(\beta_i) = \beta_i$ and so $f_S(\beta_i) = \beta_i$. Thus $\beta_i \in f_S(\mathbb{Q}) \cap \mathcal{P}_{\mathbb{Q}}$, and therefore $S \subseteq f_S(\mathbb{Q}) \cap \mathcal{P}_{\mathbb{Q}}$. We would like to prove the reverse inclusion. Thus let $x \in \mathbb{Q}$ and suppose that $f_S(x) \in \mathcal{P}_{\mathbb{Q}}$. To prove Theorem 1.3 it will be enough to show that $x = \beta_i$ for some $1 \leq i \leq r$.

Write

$$(3.7) \quad x = \frac{u}{v}, \quad u, v \in \mathbb{Z}, \quad v \geq 1, \quad \gcd(u, v) = 1.$$

Let

$$(3.8) \quad A = \prod_{i=1}^r (c_i u - a_i v), \quad B = \frac{A}{\gcd(A, v^r)}, \quad w = \frac{v^r}{\gcd(A, v^r)},$$

and note that

$$(3.9) \quad \gcd(B, w) = 1, \quad w \geq 1.$$

Recall that we would like to show that $x = \beta_i$ (or equivalently $u/v = a_i/c_i$) for some $1 \leq i \leq r$. This is equivalent to $A = 0$, which is equivalent to $B = 0$. Thus our proof will be complete on showing that $B = 0$.

We note that

$$(3.10) \quad \begin{aligned} g(x) &= \frac{A^k + v^{kr}}{v^{kr}} = \frac{B^k + w^k}{w^k}, \\ h(x) &= \frac{(u - 2^s v) \cdot (B^k + w^k) + 2^s v w^k}{v w^k}. \end{aligned}$$

Lemma 3.1. $\gcd(B^k + w^k, v)$ is a power of 2.

Proof. Let p be an odd prime dividing both $B^k + w^k$ and v . From (3.8) we see that $B^k + w^k$ divides $A^k + v^{kr}$. Thus $p \mid (A^k + v^{kr})$ and so $p \mid A$. From the product definition of A in (3.8), and since $\gcd(u, v) = 1$ from (3.7), we see that $p \mid c_i$ for some $1 \leq i \leq r$. It follows from the definition of k in (3.1) that $(p-1) \mid k$. Since $\gcd(B, w) = 1$ (3.9), we have that $p \nmid w$ and $p \nmid B$, so $B, w \in \mathbb{Z}_p^*$. Then Fermat's little theorem gives $B^k + w^k \equiv 2 \pmod{p}$. This contradicts $p \mid (B^k + w^k)$, and our claim follows. \square

Lemma 3.2. $B^k + w^k \neq 2$.

Proof. Recall that k is an even integer by (3.1), and that $w \geq 1$ from (3.9). Suppose $B^k + w^k = 2$. Then $|B| = w = 1$. In particular, from (3.10) we have $g(x) = 2$. Thus

$$(3.11) \quad f_S(x) = g(x) \cdot h(x) = 4x - 2^{s+1}$$

from (3.5) and (3.6). First we show that $x \neq 0$. Recall our assumption that $f_S(x) \in \mathcal{P}_{\mathbb{Q}}$, and so $f_S(x) = y^n$ for some rational y and some integer $n \geq 2$. If $x = 0$, then $y^n = -2^{s+1} = -2^{2^\kappa}$ by (3.3). Thus $n \mid 2^\kappa$ and so n is even. Hence $y^n > 0$ giving a contradiction. We conclude that $x \neq 0$.

Recall, from (3.7), that $v \geq 1$ and $x = u/v$. From (3.8), as $|B| = w = 1$, we see that

$$|A| = \gcd(A, v^r) = v^r.$$

Thus $A = \pm v^r$. Dividing both sides of $A = \pm v^r$ by v^r gives

$$\prod_{i=1}^r (c_i x - a_i) = \frac{A}{v^r} = \pm 1,$$

from the product expression for A in (3.8). Thus x is a non-zero root of the polynomial $F(X)$ given in (3.2). We have previously labelled the roots of F by $\delta_1, \delta_2, \dots, \delta_m$. Thus $x = \delta_j$ for some $1 \leq j \leq m$. By (3.4) we have $s+1 > D(4\delta_j)$. Hence, by Lemma 2.1 and (3.11) we have $f_S(x) = 4x - 2^{s+1} = 4\delta_j - 2^{s+1} \notin \mathcal{P}_{\mathbb{Q}}$, giving a contradiction. \square

Lemma 3.3. If $f_S(x) = 0$ then $B = 0$.

Proof. Suppose $f_S(x) = 0$ and recall that $f_S(x) = g(x)h(x)$. Note that k is even by (3.1), and also $w \geq 1$ by (3.9). Thus $B^k + w^k$ is a positive integer and hence, from (3.10) we have $g(x) \neq 0$. Hence $h(x) = 0$, whence from (3.10),

$$(3.12) \quad (u - 2^s v) \cdot (B^k + w^k) = -2^s v w^k.$$

We claim that $B^k + w^k$ is a power of 2. To prove this claim let $p \mid (B^k + w^k)$ be an odd prime. As $\gcd(B, w) = 1$ by (3.9), we have $p \nmid w$. From Lemma 3.1 we note that $p \nmid v$. However, from (3.12), we have $p \mid 2^s v w^k$ giving a contradiction. Thus $B^k + w^k$ is indeed a power of 2. Since k is even and $\gcd(B, w) = 1$ we see that $4 \nmid (B^k + w^k)$ and so $B^k + w^k = 1$ or 2 . However, $B^k + w^k \neq 2$ from Lemma 3.2. We conclude that $B^k + w^k = 1$. As $w \geq 1$, we obtain $w = 1$ and $B = 0$ as required. \square

Recall that we have supposed that $f_S(x) \in \mathcal{P}_{\mathbb{Q}}$ and to complete the proof of Theorem 1.3, it is enough to show that $B = 0$. Lemma 3.3 establishes this if $f_S(x) = 0$. Thus we may suppose $f_S(x) \neq 0$. Hence $f_S(x) = y^n$ where y is a non-zero rational and $n \geq 2$. We claim that $B^k + w^k = z^n$ or $2z^n$ for some odd positive integer z . To prove this let p be an odd prime dividing $B^k + w^k$. Then, as before $p \nmid w$ since $\gcd(B, w) = 1$, and $p \nmid v$ from Lemma 3.1. From the expression for $h(x)$ in (3.10) we see that p divides neither the numerator nor the denominator of $h(x)$, and hence $\text{ord}_p(h(x)) = 0$. However $g(x)h(x) = f_S(x) = y^n$, so

$$\text{ord}_p(B^k + w^k) = \text{ord}_p(g(x)) = \text{ord}_p(y^n) \equiv 0 \pmod{n}.$$

As this is true for every odd prime dividing $B^k + w^k$ we have $B^k + w^k = 2^e z^n$ for some $e \geq 0$ and some odd integer z . Now as before $4 \nmid (B^k + w^k)$, since k is even and $\gcd(B, w) = 1$. Thus $B^k + w^k = z^n$ or $B^k + w^k = 2z^n$ where z is odd.

Suppose first that $B^k + w^k = 2z^n$. Recall that $4 \mid k$ by (3.1). By Corollary 2.2 we have $B = \pm 1$, $w = 1$, contradicting Lemma 3.2.

Thus $B^k + w^k = z^n$. We apply Corollary 2.3 to conclude that $B = 0$ or $w = 0$. However, $w \geq 1$, so $B = 0$ completing the proof.

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