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Mahler measure of P_d polynomials

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RÉSUMÉ. Nous étudions la mesure de Mahler d'une famille de polynômes à deux variables, désignée par P_d avec $d \geq 1$, non bornée en degré et en genre. En utilisant une formule de forme close de la mesure de Mahler [13], nous sommes capables de calculer $m(P_d)$, avec d arbitraire, comme une somme des valeurs du dilogarithme en certaines racines de l'unité. Nous prouvons que $m(P_d)$ converge et que la limite est un multiple de $\zeta(3)$, où ζ est la fonction zêta de Riemann. La preuve que nous donnons est computationnelle et est basée sur l'estimation de l'erreur des sommes de Riemann d'une fonction bivariée. Nous exposons une deuxième preuve possible et plus courte basée sur une généralisation conjecturée du théorème de Boyd–Lawton et sur un résultat de D'Andrea et Lalín [11].

ABSTRACT. This article investigates the Mahler measure of a family of 2-variate polynomials, denoted by P_d, for $d \geq 1$, unbounded in both degree and genus. By using a closed formula for the Mahler measure [13], we are able to compute $m(P_d)$, for arbitrary d, as a sum of the values of dilogarithm at special roots of unity. We prove that $m(P_d)$ converges, and the limit is proportional to $\zeta(3)$, where ζ is the Riemann zeta function. The proof we give is computational and based on the estimation of the error of Riemann sums of a bivariate function. We describe a second possible shorter proof based on a conjectural generalization of the theorem of Boyd–Lawton and a result of D'Andrea and Lalín [11].

1. Introduction

Mahler measure is an interesting notion, used in number theory, analysis, special functions, random walks, etc. To delve deeper into this fascinating notion, explore the latest developments in Mahler measure theory, and grasp the essential prerequisite material for this paper, we recommend the book [10] to the reader. The (logarithmic) Mahler measure of a multi-variate polynomial, $P(x_1, \ldots, x_n) \in \mathbb{C}[x_1, \ldots, x_n]$, denoted by $m(P)$, is defined by

the following formula :

$$m(P) = \frac{1}{(2\pi)^n} \int_0^{2\pi} \int_0^{2\pi} \cdots \int_0^{2\pi} \log|P(e^{i\theta_1}, e^{i\theta_2}, \ldots, e^{i\theta_n})| \, \mathrm{d}\theta_1 \, \mathrm{d}\theta_2 \cdots \mathrm{d}\theta_n.$$

It is possible to prove that this integral is not singular, and $m(P)$ always exists [16], but there is no general closed formula to compute it. Moreover, it is not easy to approximate it with arbitrary precision. Guilloux and Marché [13] found a closed formula for a specific class of 2-variate polynomials, called regular exact polynomials, which expresses their Mahler measure as a finite sum. Boyd and Rodriguez-Villegas [7] found a similar closed formula for the Mahler measure of a specific family of exact 2 variable polynomials with a different language. Boyd [6], Bertin, and Zudilin [3, 4] investigated families of curves of genus 2. Furthermore, Bertin [1] computed the Mahler measure of a family of 3-variate polynomials $Q_k(x, y, z)$, defining $K3$ surfaces. Lalín [14] developed a new method for expressing Mahler measures of some families of polynomials in terms of polylogarithms. Also [7] and [8] give many information about the relation between Mahler measures of exact polynomials and the values of Dilogarithm function at certain algebraic numbers.

In this article we study a specific family of 2-variate exact polynomials. We compute their Mahler measures. Furthermore, we compute the limit of the Mahler measures of this family. In Section 3, we introduce the family $P_d(x, y) := \sum_{0 \leq i+j \leq d} x^i y^j$, presented to us by François Brunault. He noted that P_d is exact. To apply the formula in [13], we need to determine its terms. To do so, we compute the toric points, a volume function, and a kind of sign function. Section 4 is devoted to the computation of $m(P_d)$. In Section 5, to achieve the objective, finding a new explicit formula to compute $m(P_d)$ in terms of the values of the Dilogarithm at roots of unity, we introduce $\mathrm{vol}(\theta, \alpha) := D(e^{i\theta}) - D(e^{i(\theta+\alpha)}) + D(e^{i\alpha})$. Using this function we prove the following theorem;

Theorem 1.1. *The Mahler measure $m(P_d)$ is expressed in terms of* vol *as follows:*

$$2\pi m(P_d) = \frac{2}{d+1} \sum_{0 < k < k' \leq d+1} \mathrm{vol}\left(\frac{2k\pi}{d+2}, \frac{2(k'-k)\pi}{d+2}\right)$$

$$- \frac{2}{d+2} \sum_{0 < k < k' \leq d} \mathrm{vol}\left(\frac{2k\pi}{d+1}, \frac{2(k'-k)\pi}{d+1}\right).$$

The above theorem asserts that the Mahler measure of P_d for arbitrary $d \geq 1$ can be expressed in terms of the finite sum over the Dilogarithm function at certain roots of unity. This theorem is the key point for connecting the values of $m(P_d)$ to special values of L-functions, which is a reminiscence of the work of Smyth [17] and Boyd [5, 6].

Moreover, in the above formula, when d goes to infinity, each summation is proportional to a Riemann sum of vol. Hence, $\lim_{d\to\infty} m(P_d)$ is written as a subtraction of two expressions; each of them is proportional to a Riemann sum of vol and both go to infinity. In order to find the limit, we use the Riemann sum technics and studying the errors. We prove the following theorem:

Theorem 1.2. *The $\lim_{d\to\infty} m(P_d)$ exists and we have:*

$$(1.1) \qquad \lim_{d\to\infty} m(P_d) = \frac{9}{2\pi^2}\zeta(3).$$

After having determined the value of the limit of $m(P_d)$, we noted that there exists a polynomial in 4 variables defined by D'Andrea and Lalín [11], called $P_\infty := (x-1)(y-1) - (z-1)(w-1)$ for which they proved that $m(P_\infty) = \frac{9}{2\pi^2}\zeta(3)$. The fact that $\lim_{d\to\infty} m(P_d) = m(P_\infty)$, then reminds us of the theorem of Boyd–Lawton. However, we are not allowed to apply this theorem to P_d for concluding Theorem 1.2, since it is a family of bivariate polynomials. We will prove a generalized version of this theorem in a forthcoming article of Brunault, Guilloux, Mehrabdollahei and Pengo. This generalization gives another proof of Theorem 1.2. Since this conjectural proof is more conceptual and shorter, we state a special case of this generalization and explain the alternative proof of Theorem 1.2 in the following section.

There also exists a family of 3-variable polynomials whose limit of its Mahler measure is $m(P_\infty)$, studied by Gu and Lalin [12]. The family has two parameters, a and b and is defined by $P_{a,b}(x,y,z) := x^{a+b} + 1 + (x^a+1)y + (x^b-1)z$. In contrast to P_d Gu and Lalin applied the actual form of the theorem of Boyd–Lawton and proved that $\lim_{\substack{a\to\infty \\ b\to\infty}} P_{a,b} = m(P_\infty)$.

2. Conjectural proof of convergence of $(m(P_d))_{d\in\mathbb{N}}$

In this section we explain a conjectural method to prove Theorem 1.2. Before explaining the short proof, let us explain to you some clues that guide us toward this method. As we have mentioned in the introduction, D'Andrea and Lalín [11] defined a 4 variable polynomial $P_\infty := (x-1)(y-1) - (z-1)(w-1)$ and proved that $m(P_\infty) = \frac{9}{2\pi^2}\zeta(3)$. Therefore, according to the computations done in the previous section $\lim_{d\to\infty} m(P_d) = m(P_\infty)$. This circumstance remind us of Boyd–Lawton's theorem:

Theorem ([5, 15]). *For $P \in \mathbb{C}[x_1,\ldots,x_n]$, we have:*

$$\lim_{k_2\to\infty} \cdots \lim_{k_n\to\infty} m(P(x, x^{k_2},\ldots, x^{k_n})) = m(P(x_1,\ldots,x_n)).$$

After noticing this coincidence, one may link P_d and P_∞ as follows:

$$P_d(x,y) = \frac{P_\infty(x^{d+2}, y, x, y^{d+2})}{(1-x)(1-y)(x-y)},$$

and since the Mahler measures of the denominator is zero we have:

(2.1) $$m(P_d(x,y)) = m(P_\infty(x^{d+2}, y, x, y^{d+2})).$$

However, we are not allowed to apply the theorem of Boyd–Lawton to P_d since it is a sequence of two variable polynomials. In fact, we need to have a generalization of Boyd–Lawton which explains how to compute the Mahler measure of a polynomial in n variable, by using certain sequence of polynomials in m variables with $m, n \in \mathbb{N}$ and $m < n$. To explain the short proof for Theorem 1.2, we only announce a special case of this generalization, for $m = 2$ and $n = 4$. The general case and its proof is explained in the preprint [9]. Since, the proof of the following theorem is not yet officially published it is called conjectural variant of Boyd–Lawton.

Conjecture 2.1 (Conjectural variant of Boyd–Lawton, [9]). *Let P be a four variable polynomial, then the Mahler measure of P can be computed by the following limit:*

$$\lim_{d \to \infty} m(P(x^d, y, x, y^d)) = m(P(x, y, z, w)).$$

From the above explanation and Conjecture 2.1 another proof for Theorem 1.2 is concluded:

Conjectural proof of Theorem 1.2. Thanks to (2.1) and Conjecture 2.1 we have $\lim_{d \to \infty} m(P_d) = m(P_\infty)$. Then by using the result of D'Andrea and Lalín [11] which is $m(P_\infty) = \frac{9}{2\pi^2}\zeta(3)$, the proof is complete. □

This proof is short and without any complicated computation, but it has some other types of difficulties. For instance, for applying Conjecture 2.1 first, we need to guess a 4 variable polynomial which will be the limit polynomial for $P_d(x,y)$. Here, we did a kind of reverse engineering, moreover we were lucky that D'Andrea and Lalín had done the computation for $m(P_\infty)$. In fact just by looking at P_d without knowing $\lim_{d \to \infty} m(P_d) = \frac{9}{2\pi^2}\zeta(3) = m(P_\infty)$, finding a suitable 4 variables polynomial seems impossible. Therefore, the computational method, though long, still seems necessary. In the remaining sections of this article we explain this computational method to obtain explicitly the limit of $m(P_d)$.

3. A family of exact polynomials

In this section, the class of exact polynomial is introduced. The relevance of exact polynomial is the existence of a closed formula to compute the Mahler measure. We introduce the family of polynomials, called P_d, which

are examples of exact polynomials. In the future sections, using the closed formula of Mahler's measure of exact polynomials, we compute $m(P_d)$.

Definition 3.1. *The real differential 1-form η on $\mathbb{C}^{*2}$ is defined by $\eta = \log|y|\, d\arg(x) - \log|x|\, d\arg(y)$.*

Remark 3.2. *Let $P \in \mathbb{C}[x, y]$ and C be the algebraic curve defined by*

$$C = \left\{ (x, y) \in \mathbb{C}^{*2} \,\middle|\, P(x, y) = 0, dP(x, y) \neq 0 \right\};$$

Then the form η restricted to C is closed.

After the previous remark one may ask about the exactness of $\eta|_C$. In general, the answer is that $\eta|C$ is not always exact, but this question leads to the definition of exact polynomials.

Definition 3.3. *A polynomial $P \in \mathbb{C}[X, Y]$ is called* exact *if the form η restricted to the algebraic curve C is exact. In this case, any primitive for η is called a* Volume function *associated with the exact polynomial P.*

To see a simple example of exact polynomials, we need the following definition.

Definition 3.4. *The* Bloch–Wigner *Dilogarithm function $D(z)$ is defined by:*

$$D(z) = \operatorname{Im}(Li_2(z)) + \arg(1 - z)\log|z|,$$

where $\arg$ *denotes the branch of the argument, lying between $-\pi$ and π, and $Li_2(z)$ is the following function:*

$$Li_2(z) = -\int_0^z \log(1 - u)\frac{\mathrm{d}u}{u} \quad \text{for } z \in \mathbb{C} \setminus [1, \infty).$$

The function $D(z)$ is real analytic on $\mathbb{C}$ except at the two points 0 and 1, where it is continuous but not differentiable. The properties of this function which we will use in this article are: $D(\bar{z}) = -D(z)$; If $|z| = 1$, then $D(z) = D(e^{i\theta}) = \sum_{n=1}^{\infty} \frac{\sin(n\theta)}{n^2}$, in particular we have $D(e^{k\pi i}) = 0$. For more information see [18]. The link between the differential of D and η is a well known fact and $-D(z)$ is a primitive for η restricted to $\{(z, 1 - z) \in \mathbb{C}^{*2}\}$. In other words we have $-dD(z) = \eta(z, 1 - z)$. For more information and a proof you can see [2] or [10, Theorem 7.2];

Example 3.5. *The polynomial $P_1(x, y) = x + y + 1$, is exact and a volume function is $-D(-x)$;*

See [10, Chapter 7] and [14] for the proof and further information.

3.1. P_d polynomials. We generalize the first example P_1 to a family of polynomials, called $P_d(x, y)$, with $d \geq 1$;

Notation 3.6. *For every $d \in \mathbb{N}$ let:*

$$P_d(x, y) := \sum_{0 \leq i+j \leq d} x^i y^j.$$

We now prove that P_d is exact, for all $d \in \mathbb{N}$. The best way to prove the exactness of η restricted to P_d is by an abstract algebraization of η. Consider the multiplicative group K_d^* of the field $K_d = \text{Frac} \frac{\overline{\mathbb{Q}}[X,Y]}{<P_d>}$, as a $\mathbb{Z}$-module. The second exterior product of K_d^* is $K_d^* \wedge K_d^*$. Note that the associated group operation in K_d^* and $K_d^* \wedge K_d^*$ are respectively multiplication and addition. Consider the alternating bi-linear map $\imath : K_d^* \times K_d^* \to \Omega_C^1$ defined by:

$$\imath \colon K_d^* \times K_d^* \longrightarrow \Omega_C^1$$
$$(f, g) \longmapsto \log|g| d \arg f - \log|f| d \arg g.$$

Where, $d \arg f = \text{Im}(d \log f) = \text{Im}(df/f)$, C is the curve of P_d, minus the set of zeros and poles of f and g. Moreover, Ω_C^1 is the $\mathbb{C}$-vector space of smooth differential one-forms on C. According to the universal property of the exterior product, there is a unique morphism of $\mathbb{Z}$-modules, $\bar{\imath} : K_d^* \wedge K_d^* \to \Omega_C^1$, such that the following diagram commutes.

$$\begin{array}{ccc} K_d^* \wedge K_d^* & & \\ \wedge \uparrow & \searrow^{\bar{\imath}} & \\ K_d^* \times K_d^* & \xrightarrow{\imath} & \Omega_C^1 \end{array}$$

where $\bigwedge$ is defined by:

$$\bigwedge \colon K_d^* \times K_d^* \longrightarrow K_d^* \wedge K_d^*$$
$$(f, g) \longmapsto f \wedge g$$

Note that according to the definitions of $\imath(f, g)$ and η we have $\eta(f, g) = \imath(f, g)$.

The following proposition give us an algorithm to compute a volume function associated to P_d, and to conclude the exactness.

Proposition 3.7. *If $x, y \in K_d^*$ and $x \wedge y = \sum_{i=1}^n z_i \wedge (1 - z_i)$ modulo some torsion elements in $K_d^* \wedge K_d^*$, then $(-\sum_{i=1}^n D(z_i))$ is a primitive form for η restricted to smooth zeroes of $P_d(x, y)$.*

Proof. The above proposition is formula 13 in [8, p. 6]. $\square$

Remark 3.8. *We notice that $\wedge$ computation for finding a volume function does not depend on the torsion elements, so in the sequel of this section*

we use the notation $\doteq$ which refers to equality up to torsion elements; For example, for all f, g we have $(-f) \wedge (-g) \doteq f \wedge (-g) \doteq (-f) \wedge g \doteq f \wedge g$.

We remind you that P_1 is exact and the proof is based on Proposition 3.7. Smyth [17] proved that $m(x + y + 1) = \frac{3\sqrt{3}}{4\pi} L(\chi_{-3}, 2)$ which illustrates an important application of Mahler measure in Number theory and special values. In the following we prove that the whole family is exact;

Theorem 3.9. *For all $d \geq 2$, P_d is an exact polynomial and a volume function, denoted by V, is defined as follows:*

$$V(x, y) = \frac{1}{(d+1)(d+2)}[D(y^{d+1}) - D(x^{d+1}) - D((y/x)^{d+1})]$$

$$+ \frac{1}{(d+2)}[D(x) - D(y) - D(x/y)].$$

Proof. For $d \geq 2$ we have the following equations:

$$P_d(x, y) = P_{d-1}(x, y) + y^d \left(\frac{1 - (x/y)^{d+1}}{1 - (x/y)} \right)$$

$$\text{and} \quad P_d(x, y) = y P_{d-1}(x, y) + \left(\frac{1 - x^{d+1}}{1 - x} \right).$$

Therefore, at smooth zeros of P_d, we have $P_{d-1}(x, y) = -y^d (\frac{1-(x/y)^{d+1}}{1-(x/y)}) = -1/y(\frac{1-x^{d+1}}{1-x})$. Hence $y^{d+1} = \frac{1-x^{d+1}}{1-x} \frac{1-(x/y)}{1-(x/y)^{d+1}}$, which by replacing it in $\frac{1}{d+1}x \wedge y^{d+1} = x \wedge y$ we have :

$$(3.1) \quad x \wedge y = \frac{1}{d+1}\Big(x \wedge (1 - x^{d+1}) - x \wedge (1 - x)$$

$$+ x \wedge (1 - x/y) - x \wedge (1 - (x/y)^{d+1})\Big).$$

Since P_d for $d \geq 1$ is a symmetric polynomial, we can switch x and y; Similarly, we have:

$$(3.2) \quad y \wedge x = \frac{1}{d+1}\Big(y \wedge (1 - y^{d+1}) - y \wedge (1 - y)$$

$$+ y \wedge (1 - y/x) - y \wedge (1 - (y/x)^{d+1})\Big).$$

By subtracting (3.2) from (3.1), and using the following easily verify equations:

$$(3.3) \qquad x \wedge (1 - x/y) - y \wedge (1 - y/x) \doteq x/y \wedge (1 - x/y) - x \wedge y,$$

$$(3.4) \quad y \wedge (1 - (y/x)^{d+1}) - x \wedge (1 - (x/y)^{d+1})$$

$$\doteq (y/x) \wedge (1 - (y/x)^{d+1})) + (d+1)x \wedge y,$$

we have:

$$(x \wedge y) \doteq \frac{1}{(d+1)(d+2)} \left(x^{d+1} \wedge (1 - x^{d+1}) - y^{d+1} \wedge (1 - y^{d+1}) \right.$$

$$\left. + (y/x)^{d+1} \wedge (1 - (y/x)^{d+1}) \right)$$

$$+ \frac{1}{d+2} \left(y \wedge (1 - y) - x \wedge (1 - x) + x/y \wedge (1 - x/y) \right).$$

Then applying Proposition 3.7 completes the proof. □

4. Computing the Mahler measure of P_d

As already mentioned, there is a closed formula in [13] to compute the Mahler measure of regular exact polynomials (see Definition 4.3) as follows:

$$(4.1) \qquad\qquad m(P) = \frac{1}{2\pi} \sum \epsilon(x, y) V(x, y).$$

The summation will be on the set of toric points of P (see Definition 4.1); $\epsilon(x, y)$ is the opposite of the sign of the imaginary part of $\frac{x\partial_x P}{y\partial_y P}$ at toric point (x, y) and V is a volume function. Since P_d is exact, we use the formula to compute $m(P_d)$. To apply the formula we first compute the toric points of P_d and then the sign of the imaginary part of $\frac{x\partial_x P_d}{y\partial_y P_d}$ at toric points.

4.1. Toric points of P_d. Let us first, introduce the set of the toric point of a polynomial, and then we find this set for P_d polynomials.

Definition 4.1. *The set of* toric points *of $P \in \mathbb{C}[X, Y]$ is defined by:*

$$\{(x, y) \in \mathbb{C}^{*2} | P(x, y) = 0, |x| = |y| = 1\}.$$

In fact the necessary condition on the exact polynomial P to apply (4.1) is that, the value of $\frac{x\partial_x P}{y\partial_y P}$ at each toric point of P is not real. This property leads to the definition of regular polynomials. Here, we briefly explain about regularity, but for more information see [13].

Definition 4.2. *The* logarithmic Gauss map $\gamma : C \to \mathbb{P}^1(\mathbb{C})$ *is defined by* $\gamma(x, y) = [x\partial_x P, y\partial_y P]$.

Using the logarithmic Gauss map we can define regular polynomials:

Definition 4.3. *An exact polynomial $P(x, y)$ is called* regular *if for each toric point, (x, y), we have $\gamma(x, y) \notin \mathbb{P}^1(\mathbb{R})$.*

From the previous definition, $\gamma(x, y)$ is a point in projective plane. If P is a regular polynomial, then in particular $y\partial_y P|_{(x,y)} \neq 0$ and $x\partial_x P|_{(x,y)} \neq 0$, and consequently, $[x\partial_x P, y\partial_y P] = [\frac{x\partial_x P}{y\partial_y P}, 1] \in \mathbb{P}^1(\mathbb{C}) \setminus \mathbb{P}^1(\mathbb{R})$. Therefore, for the regular polynomial P, the value of $\frac{x\partial_x P}{y\partial_y P}$ at a toric point (x, y), is a non

real number, so we can use the mentioned formula to compute $m(P)$. We use the two point of views in this article: $\frac{x\partial_x P}{y\partial_y P}$ and $\gamma(x,y)$.

The goal of this section is to prove the following proposition:

Proposition 4.4. *The set of toric pints of $P_d(x,y)$ is as follows:*

$$\left\{(x,y) \in \mathbb{C}^{*2} \,\middle|\, x^{d+1} = y^{d+1} = 1, x \neq 1, y \neq 1, x \neq y\right\}$$

$$\cup$$

$$\left\{(x,y) \in \mathbb{C}^{*2} \,\middle|\, x^{d+2} = y^{d+2} = 1, x \neq 1, y \neq 1, x \neq y\right\}.$$

For convenience, the first set in Proposition 4.4 is denoted by U_{d+1}, and the second one by U_{d+2}.

Remark 4.5. *If $P(x,y) \in \mathbb{R}[X,Y]$, then the set of toric points of $P(x,y)$ and $P^*(x,y)$ are equal, where $P^*(x,y) = P(1/x, 1/y)$, with x, y not equal to zero.*

Let (x,y) be a toric point of P_d, using Remark 4.5 we have:

$$(4.2) \qquad\qquad P_d(x,y) = P_d^*(x,y) = 0.$$

Therefore we have $P_d(x,y) + x^{d+1}y^d P_d^*(x,y) = 0$. A simple computation implies the following equation:

$$(4.3) \qquad P_d(x,y) + x^{d+1}y^d P_d^*(x,y) = \frac{y^{d+2} - 1}{y - 1} \frac{x^{d+1} - 1}{x - 1}.$$

Using the previous remark we can prove the following lemma;

Lemma 4.6. *The toric points of $P_d(x,y)$ are contained in:*

$$\left\{(x,y) \in \mathbb{C}^{*2} \,\middle|\, x^{d+1} = y^{d+1} = 1, x \neq 1, y \neq 1\right\}$$

$$\cup$$

$$\left\{(x,y) \in \mathbb{C}^{*2} \,\middle|\, x^{d+2} = y^{d+2} = 1, x \neq 1, y \neq 1\right\}.$$

Proof. If (x,y) is a toric point of P_d, then (4.2) and (4.3) hold, so we have $\frac{x^{d+2}-1}{x-1} = 0$ or $\frac{y^{d+1}-1}{y-1} = 0$. Moreover, $P_d(x,y)$ is a symmetric polynomial, so $P_d(x,y) = P_d(y,x)$. Thus, we switch x and y, and (y,x) is a toric point. Hence, $P_d(y,x) + y^{d+1}x^d P_d^*(y,x) = \frac{y^{d+2}-1}{y-1} \frac{x^{d+1}-1}{x-1} = 0$, which implies $\frac{y^{d+2}-1}{y-1} = 0$ or $\frac{x^{d+1}-1}{x-1} = 0$. Since $x \neq 1$ and $y \neq 1$ the lemma is proved. $\square$

Lemma 4.7. *If (x,y) is a toric point of $P_d(x,y)$, then $x \neq y$.*

Proof. Let x is a $(d+1)$ or $(d+2)$ root of unity. We prove by contradiction that $P_d(x,x)$ is not equal to zero.

$$0 = P_d(x,x) = \sum_{0 \leq i+j \leq d} x^{i+j} = \sum_{0 \leq k \leq d} (k+1)x^k = \left(\frac{\mathrm{d}}{\mathrm{d}x} \sum_{k=0}^{d} x^{k+1}\right).$$

Therefore, x is a root of $\frac{\mathrm{d}}{\mathrm{d}x}\left(\sum_{k=0}^{d} x^{k+1}\right)$. The Gauss–Lucas theorem asserts that the zeroes of the derivative of a polynomial have to lie in the convex hull of the zeros of the polynomial itself. On the other side,

$$\sum_{k=0}^{d} x^{k+1} = \frac{x^{d+2} - 1}{x - 1}.$$

Since the two polynomials $\sum_{k=0}^{d} x^{k+1}$ and $P_d(x, x)$ are coprime to each other, x is strictly inside the convex hull of $(d+2)$-roots of unity. Therefore, $|x| < 1$, which contradicts the fact that x is a root of unity. Hence, there is no symmetric pair (x, x) in the set of toric points of P_d. □

We are ready to prove Proposition 4.4, which asserts that the set of toric pints of $P_d(x, y)$ is $U_{d+1} \cup U_{d+2}$;

Proof. From the two previous lemma, we know that the set of toric points of P_d is included in $U_{d+1} \cup U_{d+2}$. To prove the revers we notice that for $(x, y) \in U_{d+1} \cup U_{d+2}$ we have $|x| = |y| = 1$, so we just prove $P_d(x, y) = 0$. To do so, we consider two cases:

Case 1: $(x, y) \in U_{d+1} = \{(x, y) \in \mathbb{C}^{*2} \,|\, x^{d+1} = y^{d+1} = 1, x \neq 1, y \neq 1, x \neq y\}$.

$$P_d(x, y) = \frac{(x^{d+1} + yx^d + \cdots + y^{d-1}x^2 + y^d x) - (1 + y + \cdots + y^d)}{x - 1}.$$

Because y is a $d + 1$ root of unity, so $(1 + y + \cdots + y^d)$ is equal to zero. Also, $0 = 1 - 1 = x^{d+1} - y^{d+1} = (x - y)(x^d + x^d y + \cdots + y^d)$, but $y \neq x$, so $(x^d + x^d y + \cdots + y^d) = 0$. Hence, $P_d(x, y) = 0$.

Case 2: $(x, y) \in U_{d+2} = \{(x, y) \in \mathbb{C}^{*2} \,|\, x^{d+2} = y^{d+2} = 1, x \neq 1, y \neq 1, x \neq y\}$. $P_d(x, y)$, for $d \geq 1$ is symmetric, so we have:

$$xP_d(x, y) + 1 + y + \cdots + y^{d+1} = P_{d+1}(x, y) = P_{d+1}(y, x)$$
$$= yP_d(x, y) + 1 + x + \cdots + x^{d+1}.$$

By subtracting $P_{d+1}(y, x)$ from $P_{d+1}(x, y)$, the following equation holds for any (x, y):

$$(4.4) \qquad (x - y)P_d(x, y) + \frac{y^{d+2} - 1}{y - 1} - \frac{x^{d+2} - 1}{x - 1} = 0.$$

For any toric point (x, y) we have $y^{d+2} - 1 = x^{d+2} - 1 = 0$ and since $x \neq y$, so $P_d(x, y) = 0$. □

4.2. Sign of Im$(\frac{x\partial_x P_d}{y\partial_y P_d})$ at toric points. As we explained before, we need to compute ϵ at each toric point of P_d, which is the opposite of the sign of the imaginary part of $\frac{x\partial_x P_d}{y\partial_y P_d}$. Let us define Ω, which associates each toric point with a point in $\mathbb{R}^2$. The map is defined by $\Omega : (x_i, y_i) \mapsto (l_i, k_i)$, where $(x_i, y_i) = (\omega^{l_i}, \omega^{k_i})$, with $\omega = e^{\frac{2\pi}{d+1}i}$ if $(x_i, y_i) \in U_{d+1}$, and $\omega = e^{\frac{2\pi}{d+2}i}$ if $(x_i, y_i) \in U_{d+2}$. We say $\Omega : (x, y) \mapsto (l, k)$ is above the diagonal if $l < k$ and below the diagonal if $k < l$. Note that by Lemma 4.7, $l \neq k$; In the following proposition we compute the sign of Im$(\frac{x\partial_x P_d}{y\partial_y P_d})$ and therefore ϵ at toric points.

Proposition 4.8. *Let $d \geq 1$, for the polynomial $P_d(x, y)$ the sign ϵ at each toric point is determined as follows;*

- *For $(x, y) \in U_{d+1}$:*
  - *If $\Omega(x, y)$ is above the diagonal $\epsilon(x, y) < 0$.*
  - *If $\Omega(x, y)$ is below the diagonal $\epsilon(x, y) > 0$.*
- *For $(x, y) \in U_{d+2}$:*
  - *If $\Omega(x, y)$ is above the diagonal $\epsilon(x, y) > 0$.*
  - *If $\Omega(x, y)$ is below the diagonal $\epsilon(x, y) < 0$.*

Proof. We find $\text{Sgn}(\text{Im}(\frac{x\partial_x P_d}{y\partial_y P_d})))$ at each toric point. Recall that $\epsilon(x, y)$ is its opposite! As we saw in the proof of Proposition 4.4, at each point (x, y) equation (4.4) is satisfied:

$$0 = (x - y)P_d(x, y) + \frac{y^{d+2} - 1}{y - 1} - \frac{x^{d+2} - 1}{x - 1}.$$

Let $Q(x, y) = (x - 1)(y - 1)(x - y)$. For all $(x, y) \in \mathbb{C}^2$ we have this equality of polynomials:

$$P_d(x, y)Q(x, y) = (x^{d+2} - 1)(y - 1) - (y^{d+2} - 1)(x - 1).$$

We apply ∂_x and ∂_y to the both sides of the above equality:

$$(4.5) \quad \partial_x P_d(x, y)Q(x, y) + \partial_x Q(x, y)P_d(x, y) = (d+2)(y-1)x^{d+1} - (y^{d+2}-1),$$

$$(4.6) \quad \partial_y P_d(x, y)Q(x, y) + \partial_y Q(x, y)P_d(x, y) = (x^{d+2}-1) - (d+2)(x-1)y^{d+1}.$$

We divide (4.5) by (4.6), so for all $(x, y) \in \mathbb{C}^2$ we have:

$$(4.7) \quad \frac{\partial_x P_d(x, y)Q(x, y) + \partial_x Q(x, y)P_d(x, y)}{\partial_y P_d(x, y)Q(x, y) + \partial_y Q(x, y)P_d(x, y)} = \frac{(d+2)(y-1)x^{d+1} - (y^{d+2}-1)}{(x^{d+2}-1) - (d+2)(x-1)y^{d+1}}.$$

We evaluate the previous equation at toric points and we consider two cases:

Case 1: $(x, y) \in U_{d+1}$.

$$\frac{\partial_x P_d(x, y)}{\partial_y P_d(x, y)} = -\frac{y - 1}{x - 1} \, , \quad \text{so} \quad \frac{x\partial_x P_d(x, y)}{y\partial_y P_d(x, y)} = \frac{-x(1 - y)}{y(1 - x)}.$$

Case 2: $(x, y) \in U_{d+2}$.

$$\frac{\partial_x P_d(x, y)}{\partial_y P_d(x, y)} = -\frac{x^{d+1}(y-1)}{y^{d+1}(x-1)} \text{ , so } \frac{x\partial_x P_d(x, y)}{y\partial_y P_d(x, y)} = -\frac{1-y}{1-x}.$$

To compute $\mathrm{Sgn}(\mathrm{Im}(\frac{x\partial_x P_d}{y\partial_y P_d}))$ at toric points, we first write both x and y in terms of the associated first $(d+1)$ primitive root of unity. Then, we write them in terms of the first $(d+2)$ primitive root of unity.
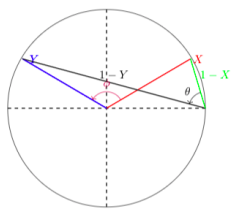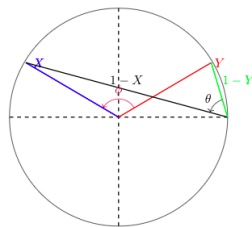

FIGURE 4.1.

FIGURE 4.2.

Case 1: $(x, y) \in U_{d+1}$. Let $\omega = e^{\frac{2\pi}{d+1}i}$, so there exist $0 < a \le d$ and $0 < b \le d$, such that $x = \omega^a, y = \omega^b$ and $a \ne b$. We have two possible cases for $\Omega(x, y)$:

(1) If $\Omega(x, y)$ is above the diagonal, or equivalently $b > a$ (see Figure 4.1), we have:

(4.8)
$$\frac{x\partial_x P_d(x, y)}{y\partial_y P_d(x, y)} = \frac{-x}{y} \frac{1-y}{1-x} = -e^{-i\phi}re^{i\theta}.$$

In the last equality in (4.8), we used the suitable polar representations according to Figure 4.1, where $\frac{x}{y} = e^{-i\phi}$, with $0 < \phi < 2\pi$ and $\frac{1-y}{1-x} = re^{i\theta}$, with $r > 0$, $0 < \theta < \pi$. We notice that ϕ and θ are respectively central and inscribed angles with the same intercepted arc in the circle, so $\phi = 2\theta$. Therefore, we have:

$$\mathrm{Sgn}\left(\mathrm{Im}\left(\frac{x\partial_x P_d(x, y)}{y\partial_y P_d(x, y)}\right)\right) = -\mathrm{Sgn}(\mathrm{Im}(re^{-i\phi/2})) = \mathrm{Sgn}\left(\sin\left(\frac{\phi}{2}\right)\right),$$

since $0 < \phi < 2\pi$, $\mathrm{Sgn}(\mathrm{Im}(\frac{x\partial_x P_d}{y\partial_y P_d}))$ is positive.

(2) If $\Omega(x, y)$ is below the diagonal, or equivalently $a > b$ (see Figure 4.2), we have:

$$\mathrm{Sgn}\left(\mathrm{Im}\left(\frac{x\partial_x P_d(x, y)}{y\partial_y P_d(x, y)}\right)\right) = -\mathrm{Sgn}(\mathrm{Im}(e^{i\phi/2})) = \mathrm{Sgn}\left(-\sin\left(\frac{\phi}{2}\right)\right),$$

so $\mathrm{Sgn}(\mathrm{Im}(\frac{x\partial_x P_d}{y\partial_y P_d}))$ is negative.

The case $(x, y) \in U_{d+2}$ is proved in the similar way as $(x, y) \in U_{d+1}$. $\square$

An immediate result from the previous proposition is that P_d is regular.

4.3. A closed formula for $m(P_d)$. In the previous sections we found a volume function and the set of the toric points associated to P_d in addition to ϵ at toric points. We are able to represent a closed formula for $m(P_d)$ in terms of the values of Dilogarithm.

Proposition 4.9. *Let $d \in \mathbb{N}$ and $d \geq 2$, so the closed formula for the Mahler measure of P_d is as follows:*

$$(4.9) \quad 2\pi m(P_d) = \frac{2}{(d+1)} \sum_{\substack{(x,y) \in U_{d+2} \\ with \ \epsilon(x,y)>0}} [D(x) - D(y) - D(x/y)]$$

$$- \frac{2}{d+2} \sum_{\substack{(x,y) \in U_{d+1} \\ with \ \epsilon(x,y)>0}} [D(x) - D(y) - D(x/y)],$$

where U_{d+1} or U_{d+2} are the set of the $d+1$ and $d+2$ toric points of P_d, computed in Proposition 4.8.

Proof. According to (4.1), We have computed the volume function in Theorem 3.9 and it is $V(x,y) = \frac{1}{(d+1)(d+2)}[D(y^{d+1}) - D(x^{d+1}) - D((y/x)^{d+1})] + \frac{1}{(d+2)}[D(x) - D(y) - D(x/y)]$. The toric points and ϵ evaluated at toric points are computed respectively in Proposition 4.4 and Proposition 4.8. Then, thanks to the properties of dilogarithm which $D(z) = -D(\bar{z})$ and $D(1) = 0$ we have $V(x,y) = \frac{1}{(d+2)}[D(x) - D(y) - D(x/y)]$ at $(x,y) \in U_{d+1}$ also $V(x,y) = \frac{1}{(d+1)}[D(x) - D(y) - D(x/y)]$ at $(x,y) \in U_{d+2}$. $\qquad \square$

We use the closed formula to compute $m(P_d)$, for arbitrary values of $d \geq 2$. The case of P_1 was first computed by Smyth [17] and it is $m(P_1) = \frac{1}{\pi}D(e^{\frac{\pi}{3}i})$, which is approximately 0.32. By using the above formula for $d = 2$ we have $m(P_2) = \frac{1}{2\pi}\left(\frac{3}{2}D(e^{i\frac{4\pi}{3}}) + 4D(e^{i\frac{\pi}{2}})\right)$, which is approximately 0.421. These computations may be automated and we get an algorithm to compute the Mahler measure of any P_d, as a combination of dilogarithm at roots of unity. This can be computed with arbitrary precision in a very efficient way. For $1 \leq d \leq 1000$ the graph of $m(P_d)$, implemented in SageMath, is shown in Figure 4.3.

The figure hints to the existence of a limit for $m(P_d)$. In the next section we study the properties of the volume function to compute the limit.

5. Convergence of $m(P_d)$, computational proof

In this section using the Riemann sum technics and error estimation we prove that $m(P_d)$ converges to $\frac{9}{2\pi^2}\zeta(3)$ which remind us of the important examples computed by Smith.
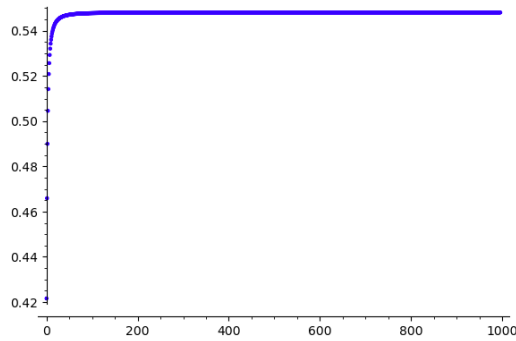
FIGURE 4.3. The graph of $m(P_d)$, for $1 \leq d \leq 1000$.

5.1. Representing $m(P_d)$ in terms of Riemann sums of the function vol.

In the previous section in (4.9) we introduce the closed formula for $m(P_d)$. However, we can not directly use that formula to compute the limit of $m(P_d)$. In this section we introduce a new function, called vol and the values of the volume function at toric points will be replaced by the values of vol at certain points. The advantage of replacing volume function by vol is that it is a concave function in a part of its domain and this is the key point to find the limit of $m(P_d)$.

Definition 5.1. *The function* vol : $[0, 2\pi] \times [0, 2\pi] \mapsto \mathbb{R}$ *is defined by* $\mathrm{vol}(\theta, \alpha) := D(e^{i\theta}) - D(e^{i(\theta+\alpha)}) + D(e^{i\alpha})$.

We have the following links between vol and volume function at toric points:

(1) For $(x, y) \in U_{d+1}$, with $x = e^{\frac{2k\pi i}{d+1}}$, $y = e^{\frac{2k'\pi i}{d+1}}$, where $0 < k < k' < d + 1$ we have:

$$V(e^{\frac{2k\pi i}{d+1}}, e^{\frac{2k'\pi i}{d+1}}) = \frac{1}{d+2} \mathrm{vol}\left(\frac{2k\pi}{d+1}, \frac{2(k'-k)\pi}{d+1}\right) = -V(e^{\frac{2k'\pi i}{d+1}}, e^{\frac{2k\pi i}{d+1}}).$$

(2) For $(x, y) \in U_{d+2}$, with $x = e^{\frac{2k\pi i}{d+2}}$ and $y = e^{\frac{2k'\pi i}{d+2}}$, where $0 < k < k' < d + 2$ we have:

$$V(e^{\frac{2k\pi i}{d+2}}, e^{\frac{2k'\pi i}{d+2}}) = \frac{1}{d+1} \mathrm{vol}\left(\frac{2k\pi}{d+2}, \frac{2(k'-k)\pi}{d+2}\right) = -V(e^{\frac{2k'\pi i}{d+2}}, e^{\frac{2k\pi i}{d+2}}).$$

According too the above equation we can recompute $m(P_d)$ in terms of the sum of the values of vol;

$$(5.1) \quad 2\pi m(P_d) = \frac{2}{d+1} \sum_{0<k<k'\leq d+1} \text{vol}\left(\frac{2k\pi}{d+2}, \frac{2(k'-k)\pi}{d+2}\right)$$

$$- \frac{2}{d+2} \sum_{0<k<k'\leq d} \text{vol}\left(\frac{2k\pi}{d+1}, \frac{2(k'-k)\pi}{d+1}\right).$$

Let us define $S_d := \frac{4\pi^2}{d^2} \sum_{0<k<k'\leq d-1} \text{vol}\left(\frac{2k\pi}{d}, \frac{2(k'-k)\pi}{d}\right)$, which is a Riemann sum of vol over the triangle with vertices $\{(0,0),(0,2\pi),(2\pi,0)\}$, denoted by T. Thus, the series appear on the R.H.S of (5.1) are respectively $\frac{(d+2)}{(d+1)^2}S_{d+1}$ and $\frac{(d+1)}{(d+2)^2}S_{d+2}$. Since vol is continues the Riemann sums S_{d+1} and S_{d+2} converge to the integral of vol over T and the sequence $E(d) = |\iint_T \text{vol} - S_d|$ goes to zero when d goes to infinity. However, the coefficients of the Riemann sums in (5.1) depend on d and when d goes the errors $\frac{(d+2)}{(d+1)^2}E(d+1)$ and $\frac{(d+1)}{(d+2)^2}E(d+2)$ may not converge to zero anymore. To prove that $(m(P_d))_{d\in\mathbb{N}}$ converges we first, introduce the properties of vol. Then, using these properties we find an upper and a lower bound for S_d in terms of the integral of vol. Finally by studying the error terms E_d we will prove that if d goes to infinity, $E(d)$ goes to zero faster than $1/d$. This completes the argument of the convergence of $m(P_d)$. The following lemma introduces some important properties of vol:

Lemma 5.2. *The function,* $\text{vol}(\theta, \alpha)$*, is positive inside of T and equals zero on its boundary. Moreover, it is concave on T.*

Proof. Since $D(1) = 0$ and $D(\bar{z}) = -D(z)$ it is easy to verify that vol is equal to zero at each boundary point of T. Moreover, vol is continuous everywhere and real analytic everywhere except at (θ, α) where $e^{i\theta} = 1$, $e^{i\alpha} = 1$ or $e^{i(\theta+\alpha)} = 1$. Thus, we check the sign of vol, at inner points of T, where the function is differentiable. To do so, first, we find the critical points of vol. Hence, we search for (θ_0, α_0), which satisfies $\frac{\partial \text{vol}}{\partial\theta}|_{(\theta_0,\alpha_0)} = \frac{\partial \text{vol}}{\partial\alpha}|_{(\theta_0,\alpha_0)} = 0$. After using the link between Dilogarithm and η, which is $-dD(z) = \eta(z, 1-z)$ and a simple computation we have:

$$\log|1 - e^{i(\theta+\alpha)}| - \log|1 - e^{i\alpha}| = \log|1 - e^{i(\theta+\alpha)}| - \log|1 - e^{i\theta}| = 0.$$

We assume that $0 < \theta < 2\pi$, $0 < \alpha < 2\pi$ and $0 < \alpha + \theta < 2\pi$, since we search for the solutions of the system inside T. Hence, the unique critical point correspond to $\theta = \alpha = 2\pi/3$. Note that $\text{vol}(2\pi/3, 2\pi/3) = 3D(e^{\frac{2\pi}{3}i})$ is approximately $2,03$. Hence, vol is positive inside T.

To prove the concavity of vol on T, we compute the Hessian matrix of vol, then we prove it is negative definite. After computing all the partial

derivatives the Hessian matrix of vol is:

$$\mathbf{H} = \begin{bmatrix} \frac{\partial^2 \text{vol}}{\partial \theta^2} & \frac{\partial^2 \text{vol}}{\partial \theta \partial \alpha} \\[2mm] \frac{\partial^2 \text{vol}}{\partial \alpha \partial \theta} & \frac{\partial^2 \text{vol}}{\partial \alpha^2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\cot(\frac{\theta+\alpha}{2}) - \frac{1}{2}\cot(\frac{\theta}{2}) & \frac{1}{2}\cot(\frac{\theta+\alpha}{2}) \\[2mm] \frac{1}{2}\cot(\frac{\theta+\alpha}{2}) & \frac{1}{2}\cot(\frac{\theta+\alpha}{2}) - \frac{1}{2}\cot(\frac{\alpha}{2}) \end{bmatrix}.$$

The symmetric (2×2) Hessian matrix is negative definite if and only if $D_1 < 0$ and $D_2 > 0$, where $D_i, (i = 1, 2)$ are leading principal minors. Then, after a computation on the minors (inside T), we have $D_1 = \frac{1}{2}\cot(\frac{\theta+\alpha}{2}) - \frac{1}{2}\cot(\frac{\theta}{2}) < 0$, since $\cot x$ is a decreasing function on $[0, \pi]$. Moreover, $D_2 = \text{Det}(\mathbf{H}) = \frac{1}{4}\left(\cot(\frac{\theta}{2})\cot(\frac{\alpha}{2}) - \frac{\cot(\frac{\theta}{2})\cot(\frac{\alpha}{2})-1}{(\cot(\frac{\theta}{2})+\cot(\frac{\alpha}{2}))}.(\cot(\frac{\theta}{2}) + \cot(\frac{\alpha}{2}))\right) = \frac{1}{4} > 0$. Therefore $\text{vol}(\theta, \alpha)$ is concave inside T. $\square$

The Figure 5.1 illustrates the properties of vol mentioned in the previous lemma. In fact, the reason for which we replaced the volume function by vol in the formula of $m(P_d)$ is to take advantage of these properties of vol.
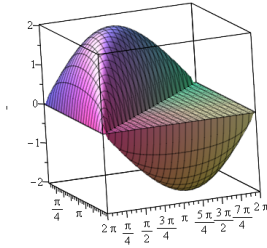

FIGURE 5.1. The graph of $\text{vol}(\theta, \alpha)$.

As we have mentioned, for computing the limit of $m(P_d)$ the second step is to estimate the error between the Riemann sums and the integral of vol. In the following lemma we compute the integral and later in Section 5.2 and Section 5.3, we bound the errors between Riemann sums and this integral.

Lemma 5.3. *We have $\iint_T \text{vol}(\theta, \alpha)\mathrm{d}A = 6\pi\zeta(3)$, where $\mathrm{d}A$ is the euclidean measure on T.*

Proof. In this proof, we use the formula, $D(e^{i\theta}) = \sum_{n=1}^{\infty} \frac{\sin(n\theta)}{n^2}$.

$$\iint_T \mathrm{vol}(\theta, \alpha)\mathrm{d}A$$

$$= \int_0^{2\pi} \int_0^{2\pi-\alpha} \mathrm{vol}(\theta, \alpha)\mathrm{d}\theta\mathrm{d}\alpha$$

$$= \int_0^{2\pi} \int_0^{2\pi-\alpha} D(e^{i\theta}) - D(e^{i(\theta+\alpha)}) + D(e^{i\alpha})\mathrm{d}\theta\mathrm{d}\alpha$$

$$= \int_0^{2\pi} \int_0^{2\pi-\alpha} \sum_{n=1}^{\infty} \frac{\sin(n\theta)}{n^2} + \sum_{n=1}^{\infty} \frac{\sin(n\alpha)}{n^2} - \sum_{n=1}^{\infty} \frac{\sin(n(\theta+\alpha))}{n^2} \, \mathrm{d}\theta\mathrm{d}\alpha$$

$$\overset{[1]}{=} \sum_{n=1}^{\infty} \int_0^{2\pi} \int_0^{2\pi-\alpha} \frac{\sin(n\theta) + \sin(n\alpha) - \sin(n(\theta+\alpha))}{n^2}\mathrm{d}\theta\mathrm{d}\alpha$$

$$= 6\pi \sum_{n=1}^{\infty} \frac{1}{n^3} = 6\pi\zeta(3).$$

We notice that in [1] the summation and the integration is commuted, since the series converges uniformly. $\qquad\square$

5.2. A lower bound for Riemann sums of vol. As we mentioned to compute the limit of $m(P_d)$ we need to study the error between the Riemann sums of vol and its integral. In this section, using affine functions we exhibit a lower bound for the Riemann sum of vol in terms of the integral of $\mathrm{vol}(\theta, \alpha)$. Then, in the next section we compute an upper bound for the Riemann sum of vol in terms of the integral. Combining these two results we have information about the error terms. Before starting the computation, in the following observation we illustrate the intuition behind the definition of S_d.

Observation 5.4 (Square subpartition)**.** *Consider the set of the points $(\frac{2k\pi}{d+1}, \frac{2(k'-k)\pi}{d+1})$ with $0 < k < k' < d+1$ inside T. For (x,y) in the set, consider the square with side $\frac{2\pi}{d+1}$ such that (x,y) is at the center of the square. The union of the squares is called* (d+1)-square subpartition *of T which does not cover all T. The set difference of T and the (d+1)-square subpartition is called* Blue part*. The 8-square subpartition (for $d = 7$) of T is shown in Figure 5.2.*

As we have already introduced we define

$$S_{d+1} := \sum_{0 < k < k' < d+1} \frac{4\pi^2}{(d+1)^2} \mathrm{vol}(\frac{2k\pi}{d+1}, \frac{2(k'-k)\pi}{d+1}),$$

where $\frac{4\pi^2}{(d+1)^2}$ is the area of each square in $(d+1)-$square subpartition. We can repeat the same process, by choosing the points $(\frac{2k\pi}{d+2}, \frac{2(k'-k)\pi}{d+2})$, for
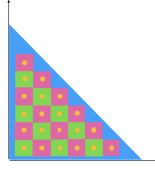
FIGURE 5.2. The figure of 8-square subpartition of T.

$0 < k < k' < d+2$. *Similarly, we have $d+2$-square subpartition of T, which leads to S_{d+2}. The difference between the value of the integral and S_d for a fixed d, is denoted by $E(d)$.*

Since our computation of $m(P_d)$ concern S_{d+1} and S_{d+2}, in the sequel we do all computations for the case $d+1$ and the case $d+2$ is concluded in the same way. Let us introduce another notation, which later gives us the upper bound for E_d:

$$\mathcal{E}(d+1) := \iint_{\text{Blue part}} \mathrm{vol}(\theta, \alpha)\mathrm{d}A.$$

In the following, using the fact that any tangent plane to the graph of a concave function is above the graph, we find a lower bound for S_d.

Lemma 5.5. *We have $E(d+1) \leq \mathcal{E}(d+1)$. Moreover,*

$$(5.2) \quad \iint_T \mathrm{vol}(\theta, \alpha)\mathrm{d}A \leq \mathcal{E}(d+1) + \frac{4\pi^2}{(d+1)^2} \sum_{0 < k < k' \leq d} \mathrm{vol}\left(\frac{2k\pi}{d+1}, \frac{2(k'-k)\pi}{d+1}\right).$$

Proof. According to Observation 5.4, for a fixed d, T is partitioned into $\frac{(d-1)(d-2)}{2}$ squares and the blue part. The function vol is concave and differentiable inside T, especially on each square. Let us focus on arbitrary and fixed square and denote its central point by (θ^*, α^*). The tangent plane to the graph of vol at (θ^*, α^*) denoted by $\mathrm{Tang}_{\mathrm{vol}}(\theta^*, \alpha^*)$, is located above the graph for all (θ, α) in the square, so we have:

$$(5.3) \qquad\qquad \mathrm{vol}(\theta, \alpha) \leq \mathrm{Tang}_{\mathrm{vol}}(\theta^*, \alpha^*).$$

The volume of the rectangular cuboid with the square as its base and bounded above by the tangent plan of $\mathrm{vol}(\theta, \alpha)$, at (θ^*, α^*), is greater than $\iint_\square \mathrm{vol}(\theta, \alpha)\mathrm{d}A$. Hence, we have:

$$\iint_\square \mathrm{vol}(\theta, \alpha)\mathrm{d}A \leq \iint_\square \mathrm{Tang}_{\mathrm{vol}}(\theta^*, \alpha^*)\mathrm{d}A = \frac{4\pi^2}{(d+1)^2} \mathrm{vol}(\theta^*, \alpha^*).$$

Therefore, we have:

$$\sum_{\text{all squares inside T}} \iint_{\square} \text{vol}(\theta, \alpha) \mathrm{d}A$$

$$\leq \sum_{0 < k < k' \leq d} \frac{4\pi^2}{(d+1)^2} \text{vol}\left(\frac{2k\pi}{d+1}, \frac{2(k'-k)\pi}{d+1}\right),$$

Thus, $E(d+1) \leq \mathcal{E}(d+1)$, moreover, we have:

$$(5.4) \quad \iint_T \text{vol}(\theta, \alpha) \mathrm{d}A$$

$$\leq \mathcal{E}(d+1) + \frac{4\pi^2}{(d+1)^2} \sum_{0 < k < k' \leq d} \text{vol}\left(\frac{2k\pi}{d+1}, \frac{2(k'-k)\pi}{d+1}\right). \quad \square$$

5.3. An upper bound for Riemann sums of vol. In this section, we define a partition of T, which leads to an upper bound for the Riemann sum.

Observation 5.6 (Triangular partition)**.** *The triangle T is partitioned into the smaller triangles belong to $T_1 \cup T_2$, where T_1 and T_2 define as follows:*

$$T_1 := \bigcup_{i=0}^{d+1} \bigcup_{j=0}^{d+1-i} \left\{ \left[\left(\frac{2\pi i}{d+1}, \frac{2\pi j}{d+1}\right), \left(\frac{2\pi i}{d+1}, \frac{2\pi(j+1)}{d+1}\right), \left(\frac{2\pi(i+1)}{d+1}, \frac{2\pi j}{d+1}\right) \right] \right\},$$

$$T_2 := \bigcup_{i=1}^{d} \bigcup_{j=1}^{d+1-i} \left\{ \left[\left(\frac{2\pi(i-1)}{d+1}, \frac{2\pi j}{d+1}\right), \left(\frac{2\pi i}{d+1}, \frac{2\pi j}{d+1}\right), \left(\frac{2\pi i}{d+1}, \frac{2\pi(j-1)}{d+1}\right) \right] \right\}.$$

In the definition of T_1 and T_2, $[(i_1, j_1), (i_2, j_2), (i_3, j_3)]$ denotes the triangle with vertices $(i_1, j_1), (i_2, j_2)$, and (i_3, j_3). The figure for the 2-triangular partition is shown in Figure 5.3; indeed, the pink and green triangles respectively belong to T_1 and T_2.

FIGURE 5.3. The figure of 2-triangular partitions of T.

Definition 5.7. *The vertices of small triangles, defined in Observation 5.6, not located on the boundary of T are called* inner *vertices. The set of all these inner vertices is denoted by* $\text{In}(T)$.

The following fact leads to an important correspondence between the triangular partition, and the square subpartition. The proof is elementary.

Fact 5.8. *Each inner vertex of a small triangle, in the d-triangular partition, is a central point of a unique square in the d-square subpartition.*

If we restrict vol to the triangle $[a, b, c]$, since it is concave there exists a unique affine function called χ, such that $\mathrm{vol}(a) = \chi(a), \mathrm{vol}(b) = \chi(b)$, $\mathrm{vol}(c) = \chi(c)$ and for any (θ, α) in the triangle we have $\chi(\theta, \alpha) \leq \mathrm{vol}(\theta, \alpha)$. The following easily verified lemma helps us to compute an upper bound for the Riemann sums.

Lemma 5.9. *Let $[a, b, c]$ be an arbitrary triangle in $T_1 \cup T_2$, introduced in Observation 5.6 and χ denotes the affine function such that $\mathrm{vol}(a) = \chi(a), \mathrm{vol}(b) = \chi(b), \mathrm{vol}(c) = \chi(c)$ and for any (θ, α) in the triangle we have $\chi(\theta, \alpha) \leq \mathrm{vol}(\theta, \alpha)$. Thus we have:*

$$(5.5) \quad \iint_{[a,b,c]} \chi(\theta, \alpha) \mathrm{d}A = \mathrm{area}[a, b, c] \left(\frac{1}{3} \mathrm{vol}(a) + \frac{1}{3} \mathrm{vol}(b) + \frac{1}{3} \mathrm{vol}(c) \right)$$

$$\leq \iint_{[a,b,c]} \mathrm{vol}(\theta, \alpha) \mathrm{d}A.$$

We are able to present an upper bound for the Riemann sum.

Lemma 5.10. *We have the following upper bound for the Riemann sum:*

$$\frac{4\pi^2}{(d+1)^2} \sum_{0<k<k'<d} \mathrm{vol}\left(\frac{2k\pi}{d+1}, \frac{2(k'-k)\pi}{d+1} \right) \leq \iint_T \mathrm{vol}(\theta, \alpha) \mathrm{d}A.$$

Proof. Let $[a, b, c]$ and $[b, c, d]$ denotes respectively an arbitrary triangle in T_1 and T_2 which both have the area $\frac{2\pi}{(d+1)^2}$, so we have:

$$\sum_{[b,c,d]\in T_2} \iint_{[b,c,d]} \mathrm{vol}(\theta, \alpha) \mathrm{d}A + \sum_{[a,b,c]\in T_1} \iint_{[a,b,c]} \mathrm{vol}(\theta, \alpha) \mathrm{d}A = \iint_T \mathrm{vol}(\theta, \alpha) \mathrm{d}A.$$

By applying Lemma 5.9 to the last equality we have:

$$\sum_{[b,c,d]\in T_2} \mathrm{area}[b, c, d] \left(\frac{1}{3} \mathrm{vol}(d) + \frac{1}{3} \mathrm{vol}(b) + \frac{1}{3} \mathrm{vol}(c) \right)$$

$$+ \sum_{[a,b,c]\in T_1} \mathrm{area}[a, b, c] \left(\frac{1}{3} \mathrm{vol}(a) + \frac{1}{3} \mathrm{vol}(b) + \frac{1}{3} \mathrm{vol}(c) \right) \leq \iint_T \mathrm{vol}(\theta, \alpha) \mathrm{d}A.$$

As we already mentioned, for every a on the boundary of T we have $\mathrm{vol}(a) = 0$. Let $a \in \mathrm{In}(T)$, so it appears in 6 small triangles, inside T. Therefore, we

have:

$$\sum_{[b,c,d] \in T_2} \text{area}[b,c,d] \left(\frac{1}{3} \text{vol}(d) + \frac{1}{3} \text{vol}(b) + \frac{1}{3} \text{vol}(c) \right)$$

$$+ \sum_{[a,b,c] \in T_1} \text{area}[a,b,c] \left(\frac{1}{3} \text{vol}(a) + \frac{1}{3} \text{vol}(b) + \frac{1}{3} \text{vol}(c) \right)$$

$$= \frac{4\pi^2}{(d+1)^2} \sum_{a \in \text{In}(T)} \frac{6}{3} \text{vol}(a) = \frac{4\pi^2}{(d+1)^2} \sum_{0<k<k'<d} \text{vol} \left(\frac{2k\pi}{d+1}, \frac{2(k'-k)\pi}{d+1} \right).$$

In the last equality we used Fact 5.8. □

5.4. Computing the limit of $(m(P_d))_{d\in\mathbb{N}}$. In this section we complete the computational proof for finding the limit of $(m(P_d))_{d\in\mathbb{N}}$, which was announced in the introduction. The only missed information is the asymptotic behavior of the error sequence $E(d)$, which is another essential tool to find the limit. Using the triangular partition, and square subpartition we prove that when d goes to infinity, $E(d)$ goes to zero faster than $1/d$.

Lemma 5.11. $E(d) = o(\frac{1}{d})$.

Proof. We use the bounds, computed in Lemma 5.5 and Lemma 5.10 and we conclude that:

$$0 \le \iint_T \text{vol}(\theta, \alpha) \mathrm{d}A - \frac{4\pi^2}{(d+1)^2} \sum_{0<k<k'\le d} \text{vol} \left(\frac{2k\pi}{d+1}, \frac{2(k'-k)\pi}{d+1} \right)$$

$$\le \mathcal{E}(d+1) \le \text{Max} \cdot \text{area}(\text{Blue part}),$$

where Max is the maximum of vol on the Blue part of the triangle. The area of the blue part is $2\pi^2 \frac{3d+1}{(d+1)^2}$, so by the definition of $E(d+1)$ we have:

$$E(d+1) = \iint_T \text{vol}(\theta, \alpha) \mathrm{d}A - \frac{4\pi^2}{(d+1)^2} \sum_{0<k<k'\le d} \text{vol} \left(\frac{2k\pi}{d+1}, \frac{2(k'-k)\pi}{d+1} \right)$$

$$\le 2\pi^2 \frac{3d+1}{(d+1)^2} \text{Max} .$$

If d goes to infinity the points inside the blue part are approaching the boundary of T, where the values of vol are zero. Hence, the Maximum of vol in the blue part goes to zero as well. Therefore, we have $dE(d+1) \xrightarrow{d\to\infty} 0$. In other words $E(d+1) = o(\frac{1}{d})$. □

We have computed $m(P_d)$ as subtraction of coefficients of two Riemann sums of vol and estimated the error terms. Thus, we have all the information to compute the limit and prove Theorem 1.2, announced in the introduction.

Theorem. *The* $\lim_{d\to\infty} m(P_d)$ *exists and it is:*

$$\lim_{d\to\infty} m(P_d) = \frac{9}{2\pi^2}\zeta(3) \simeq 0.548.$$

Proof. By using (5.1) we have:

$$2\pi m(P_d) = \frac{2}{d+1}\sum_{0<k<k'\leq d+1} \mathrm{vol}\left(\frac{2k\pi}{d+2}, \frac{2(k'-k)\pi}{d+2}\right)$$
$$- \frac{2}{d+2}\sum_{0<k<k'\leq d} \mathrm{vol}\left(\frac{2k\pi}{d+1}, \frac{2(k'-k)\pi}{d+1}\right).$$

In order to find $\lim_{d\to\infty} m(P_d)$ we compute the limit of the R.H.S. We notice that, for each d we have:

$$\iint_T \mathrm{vol}(\theta,\alpha)\mathrm{d}A = \frac{4\pi^2}{d^2}\sum_{0<k<k'\leq d+1} \mathrm{vol}\left(\frac{2k\pi}{d}, \frac{2(k'-k)\pi}{d}\right) + E(d).$$

We recompute $m(P_d)$ by using the previous information;

$$2\pi m(P_d) = \frac{2}{d+1}\sum_{0<k<k'\leq d+1} \mathrm{vol}\left(\frac{2k\pi}{d+2}, \frac{2(k'-k)\pi}{k}d+2\right)$$
$$- \frac{2}{d+2}\sum_{0<k<k'\leq d} \mathrm{vol}\left(\frac{2k\pi}{d+1}, \frac{2(k'-k)\pi}{d+1}\right)$$
$$= \frac{3d^2+8d+7}{4\pi^3(d^2+3d+2)}\iint_T \mathrm{vol}(\theta,\alpha)\mathrm{d}A + \frac{(d+1)^2}{2\pi^2(d+2)}E(d+1)$$
$$- \frac{(d+2)^2}{2\pi^2(d+1)}E(d+2).$$

According to Lemma 5.11, $E(d) = o(\frac{1}{d})$. Hence,

$$\lim_{d\to\infty} \frac{(d+1)^2}{2\pi^2(d+2)}E(d+1) = \lim_{d\to\infty} \frac{(d+2)^2}{2\pi^2(d+1)}E(d+2) = 0.$$

Therefore, thanks to Lemma 5.3 we have:

$$\lim_{d\to\infty} m(P_d) = \frac{3}{4\pi^3}\iint_T \mathrm{vol}(\theta,\alpha)\mathrm{d}A = \frac{9}{2\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^3} = \frac{9}{2\pi^2}\zeta(3). \qquad \square$$

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