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Marwa ALA EDDINE

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Explicit Reciprocity Laws for Formal Drinfeld Modules

par Marwa ALA EDDINE

Résumé. Dans cet article, nous prouvons des lois de réciprocité explicites pour une classe de modules de Drinfeld formels ayant une réduction stable de hauteur 1, dans l'esprit de celles existant en caractéristique zéro (cf. le travail de Wiles [13]). Nous commençons par définir l'accouplement de Kummer dans le langage des modules de Drinfeld formels définis sur des corps locaux de caractéristique positive. Nous prouvons ensuite des formules explicites pour cet accouplement en termes du logarithme du module de Drinfeld formel, d'une certaine série de Coleman, de points de torsion et de la trace. Nos résultats étendent les formules explicites déjà prouvées par Anglès [1] pour les modules de Carlitz, et par Bars et Longhi [4] pour les modules de Drinfeld de rang un signe-normalisés. L'approche suivie est similaire à celle des articles précédemment mentionnés [1, 4, 13], en tenant compte des subtilités découlant du fait que les modules de Drinfeld formels considérés sont des séries formelles, et ne sont plus des polynômes.

Abstract. In this paper, we prove explicit reciprocity laws for a class of formal Drinfeld modules having stable reduction of height one, in the spirit of those existing in characteristic zero (cf. the work of Wiles [13]). We begin by defining the Kummer pairing in the language of formal Drinfeld modules defined over local fields of positive characteristic. We then prove explicit formulas for this pairing in terms of the logarithm of the formal Drinfeld module, a certain Coleman power series, torsion points and the trace. Our results extend the explicit formulas already proved by Anglès [1] for Carlitz modules, and by Bars and Longhi [4] for sign-normalized rank one Drinfeld modules. The approach followed is similar to the ones followed in the previously mentioned papers [1, 4, 13], taking into account the subtleties derived from the fact that the formal Drinfeld modules considered are formal power series, and are no longer polynomials.

1. Introduction

Explicit reciprocity laws have a long history. In 1928, Artin and Hasse [2] proved explicit formulas in characteristic zero for the multiplicative group. These formulas were completed by Iwasawa [8] in 1968. In 1978, Wiles [13]

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proved an important generalization to the case of general Lubin–Tate formal groups. Soon after, Kolyvagin [10] extended all these results to formal groups of finite height. In the present paper, we place ourselves in positive characteristic, and we consider formal Drinfeld modules as defined by Rosen in [12]. Let K be a local field of positive characteristic. We know that formal Drinfeld modules can be seen as homomorphisms from the valuation ring of K to the endomorphism ring of the formal additive group. Moreover, torsion points of such modules generate abelian towers of K. The maximal abelian extension of K is equal to the compositum of the maximal unramified extension of K and the union of these abelian towers. Therefore, there should be an analogue of the reciprocity laws in our settings. In [1], Bruno Anglès considered a special class of formal Drinfeld modules, which he called Carlitz polynomials, and for which he proved explicit reciprocity laws in the spirit of those proved in characteristic zero. Later in 2009, Francesc Bars and Ignacio Longhi [4] proved similar formulas for formal Drinfeld modules obtained from sign-normalized rank 1 Drinfeld modules.

Let p be the characteristic of the local field K, and let μ_K be its normalized discrete valuation. We denote $\mathcal{O}$ the valuation ring of K and $\mathfrak{p}$ its maximal ideal. Let q be the order of the residue field $\mathcal{O}/\mathfrak{p}$. Then q is a power of p. Fix an algebraic closure Ω of K, and let μ be the unique extension of μ_K to Ω. Let $(\overline{\Omega}, \overline{\mu})$ be the completion of (Ω, μ). All the extensions F of K considered in this paper are supposed to be such that $F \subset \Omega$. We also denote $\mathcal{O}_F$ the valuation ring of F and $\mathfrak{p}_F$ its maximal ideal. Let π be a fixed prime of K.

Let B be an $\mathcal{O}$- algebra and let $\gamma : \mathcal{O} \to B$ be the structure map. Let $B\{\{\tau\}\}$ be the twisted power series ring where τ is the q-Frobenius element satisfying

$$\tau x = x^q \tau, \quad \forall\, x \in B. \tag{1.1}$$

Let $D : B\{\{\tau\}\} \to B$ be the ring homomorphism that assigns to a power series $\sum_{n\geq 0} b_n \tau^n$ its constant term b_0. In [12], Rosen defined a formal Drinfeld $\mathcal{O}$-module over B to be a ring homomorphism

$$\begin{aligned} \rho : \mathcal{O} &\longrightarrow B\{\{\tau\}\} \\ a &\mapsto \rho_a \end{aligned}$$

satisfying

(i) $\forall\, a \in \mathcal{O}$, $D(\rho_a) = \gamma(a)$.
(ii) $\rho(\mathcal{O}) \not\subset B$.
(iii) $\rho_\pi \neq 0$.

This definition is a special case of formal $\mathcal{O}$-modules over B defined by Drinfeld in Section 1 of [5]. Let $f = \sum_{n\geq 0} b_n \tau^n \in B\{\{\tau\}\}$. We set $\mathrm{ord}_\tau(f)$

to be the least integer n such that $b_n \neq 0$. The height of ρ is defined by $\mathrm{ht}(\rho) = \mathrm{ord}_\tau(\rho_\pi)$. Clearly, the height is independent of the choice of the prime π.

For any extension $K \subset L \subset \overline{\Omega}$, the rings $\mathcal{O}_L$ and $\mathcal{O}_L/\mathfrak{p}_L$ are naturally $\mathcal{O}$-algebras. The structure map $\gamma : \mathcal{O} \to \mathcal{O}_L$ is the inclusion map. Let ρ be a formal Drinfeld $\mathcal{O}$- module over $\mathcal{O}_L$ as defined above. We say that ρ has stable reduction if the ring homomorphism $\bar{\rho} : \mathcal{O} \to \mathcal{O}_L/\mathfrak{p}_L\{\{\tau\}\}$, obtained by reducing modulo $\mathfrak{p}_L$ the coefficients of ρ_a, for $a \in \mathcal{O}$, is also a formal Drinfeld module.

Let $K_{ur} \subset \Omega$ be the maximal unramified extension of K in Ω, and $H \subset K_{ur}$ be a finite unramified extension of K. Let ρ be a formal Drinfeld $\mathcal{O}$-module over $\mathcal{O}_H$, having stable reduction, and such that $\mathrm{ht}(\bar{\rho}) = 1$, then $\overline{\Omega}$ is an $\mathcal{O}$-module for the following action of ρ

$$a \cdot_\rho x = \rho_a(x) \quad \forall\, x \in \overline{\Omega}. \tag{1.2}$$

For an integer $n \geq 0$, let

$$V_\rho^n = \{\alpha \in \overline{\Omega};\ \rho_a(\alpha) = 0\ \ \forall\, a \in \mathfrak{p}^n\}$$

be the $\mathfrak{p}^n$ torsion submodule of $\overline{\Omega}$ for the action (1.2). Using the Weierstrass preparation theorem, we can see that $V_\rho^n \setminus V_\rho^{n-1}$ is the set of roots of a separable Eisenstein polynomial in $\mathcal{O}_H[X]$ of degree $q^{n-1}(q-1)$, whose constant term is a prime of H. Therefore, for an element $v_0 \in V_\rho^n \setminus V_\rho^{n-1}$, the extension $H(v_0)|H$ is totally ramified of degree $q^{n-1}(q-1)$. Furthermore, the kernel of $a \mapsto \rho_a(v_0)$ is $\mathfrak{p}^n$. Thus it induces an isomorphism of $\mathcal{O}$-modules

$$\mathcal{O}/\mathfrak{p}^n \cong V_\rho^n. \tag{1.3}$$

This implies that any element $v_0 \in V_\rho^n \setminus V_\rho^{n-1}$ is a generator of V_ρ^n as $\mathcal{O}$-module. This also implies that the extension $H_\rho^n = H(V_\rho^n)$ is equal to $H(v_0)$. For more details see [9, 11]. Now let m_0 be an integer dividing $[H : K]$, and $\eta \in K$ of valuation $\mu(\eta) = m_0$. Let

$$W_\rho^n = V_\rho^{nm_0} = \{\alpha \in \mathfrak{p}_{\overline{\Omega}};\ \rho_{\eta^n}(\alpha) = 0\}, \quad \text{and} \quad W_\rho = \bigcup_n V_\rho^n = \bigcup_n W_\rho^n.$$

Let

$$E_\rho^n = H(W_\rho^n) = H_\rho^{nm_0}.$$

Let $\mathcal{O}_n$ be the valuation ring of E_ρ^n and $\mathfrak{p}_n$ be its maximal ideal. If L is a finite extension of E_ρ^n, then we denote by

$$\Phi_L : L^\times \longrightarrow Gal(L^{ab}|L)$$

the norm residue map. For an $\alpha \in \mathfrak{p}_L$ we will show in Section 2 that there exists $\xi \in L^{ab}$ such that $\rho_{\eta^n}(\xi) = \alpha$. Therefore we can define the map

$(\,\cdot\,,\cdot\,)_{\rho,L,n} : \mathfrak{p}_L \times L^\times \to W_\rho^n$ such that

$$(\alpha,\beta)_{\rho,L,n} = \Phi_L(\beta)(\xi) - \xi; \quad \rho_{\eta^n}(\xi) = \alpha. \tag{1.4}$$

for $\alpha \in \mathfrak{p}_L$ and $\beta \in L^\times$. It is clear that $(\,\cdot\,,\cdot\,)_{\rho,L,n}$ is a bilinear from.

The main objective of this paper is to prove explicit reciprocity laws for formal Drinfeld modules having stable reduction of height 1. In other words, we prove explicit formulas for the map $(\,\cdot\,,\cdot\,)_{\rho,L,n}$. Now we can state our main results.

Proposition 1.1 (Proposition 3.3)**.** Fix a generator v_n of W_ρ^n as an $\mathcal{O}$-module and suppose $L|K$ is separable. There exists a unique map $\psi_{L,v_n} : L^n \to \mathfrak{X}_L/\eta^{n+1}\mathfrak{X}_L$ such that

$$(\alpha,\beta)_{\rho,L,n} = T_{L|K}(\lambda_\rho(\alpha)\psi_{L,v_n}(\beta)) \cdot_\rho v_n \tag{1.5}$$

for all $\alpha \in \mathfrak{p}_L$ and $\beta \in L^n$, where λ_ρ is the logarithm of ρ, $L^n = \{\beta \in L^\times;\ (\alpha,\beta)_{\rho,L,n} = 0 \ \ \forall\ \alpha \in L \cap W_\rho\}$ and $\mathfrak{X}_L = \{y \in L;\ \mathrm{T}_{L|K}(xy) \in \mathcal{O}\ \forall\ x \in \lambda_\rho(\mathfrak{p}_L)\}$.

Proposition 1.1 is the analogue of Proposition 14 of [8]. The map ψ_{L,v_n} is the so-called Iwasawa map introduced in loc. cit. In the case where $L = E_\rho^n$ and ρ is such that $\rho_\eta \equiv \tau^{m_0} \mod \mathfrak{p}_H$, we can give an explicit form of ψ_{L,v_n} in the following theorem.

Theorem 1.2 (Theorem 5.9)**.** Suppose that $L = E_\rho^n$ and that $\rho_\eta \equiv \tau^{m_0} \mod \mathfrak{p}_H$. This means that if $\rho_\eta = \sum b_i\tau^i$, all the coefficients b_i are in $\mathfrak{p}_H$ except for b_{m_0}. Let $\alpha \in \mathfrak{p}_L$ such that $\mu(\alpha) \geq \frac{nm_0}{q} + \frac{1}{q-1} + \frac{1}{q^{nm_0}(q-1)}$. Then for all $\beta \in L^\times$, we have

$$(\alpha,\beta)_{\rho,L,n} = \frac{1}{\eta^n} T_{L|K}(\lambda_\rho(\alpha)\delta_{v_n}(\beta)) \cdot_\rho v_n, \tag{1.6}$$

where $\delta_{v_n} : L^\times \to \mathfrak{p}_L/\mathcal{D}_n$ is a group homomorphism defined as follows: for $\beta \in L^\times$, choose a power series $f(X) \in \mathcal{O}_H((X))^\times$ such that $f(v_n) = \beta$, and set

$$\delta_{v_n}(\beta) := \frac{f'(v_n)}{\beta} \mod \mathcal{D}_n. \tag{1.7}$$

Here, $\mathcal{D}_n$ denotes the different of the extension $E_\rho^n|K$. For more details, see Lemma 4.8 and Lemma 4.9.

Let $m \geq n$, let $\alpha \in \mathfrak{p}_n$ and $\alpha_m = \rho_{\eta^{m-n}}(\alpha)$. Let $\beta_m \in E_\rho^m$ and $\beta = \mathrm{N}_{m,n}(\beta_m)$, where $\mathrm{N}_{m,n}$ is the norm of the extension $E_\rho^m|E_\rho^n$, then

$$\begin{aligned}(\alpha_m,\beta_m)_{\rho,E_\rho^m,m} &= \Phi_{E_\rho^m}(\beta_m)(\xi) - \xi \\ &= \Phi_{E_\rho^n}(\mathrm{N}_{m,n}(\beta_m))(\xi) - \xi = (\alpha,\beta)_{\rho,E_\rho^n,n},\end{aligned} \tag{1.8}$$

where ξ is a root of $\rho_{\eta^n}(X) = \alpha$, hence a root of $\rho_{\eta^m}(X) = \alpha_m$. As a consequence of this equality, we deduce that (1.6) is also valid for all $\alpha \in \mathfrak{p}_n$

and all $\beta \in \mathrm{N}_{m,n}(E_\rho^m)$ for $m \geq \frac{q}{q-1}(2n+\frac{1}{2m_0})$. This recalls both Theorem 19 of Wiles [13] and Theorem 3.12 of Anglès [1]. This also implies (1.6) for all $\alpha \in \mathfrak{p}_n$ if β is a universal norm in $(E_\rho^n)^\times$, which is the analogue of Theorem 1 of Wiles [13]. Let us consider the inverse limit $\varprojlim (E_\rho^n)^\times$ with respect to the norm maps, and the direct limit $\varinjlim \mathfrak{p}_n$ with respect to the maps

$$(1.9) \qquad \begin{aligned} \mathfrak{p}_n &\longrightarrow \mathfrak{p}_m \\ \alpha_n &\longmapsto \rho_{\eta^{m-n}}(\alpha_n). \end{aligned}$$

We can define a limit form of $(\,\cdot\,,\cdot\,)_{\rho,L,n}$ as follows

$$(1.10) \qquad (\alpha,\beta)_\rho = (\alpha_n,\beta_n)_{\rho,E_\rho^n,n}$$

for sufficiently large n, where $\alpha = (\alpha_n)_n \in \varinjlim \mathfrak{p}_n$ and $\beta = (\beta_n)_n \in \varprojlim E_\rho^n$. The limit form (1.10) is well defined due to (1.8). Moreover, we deduce from the discussion above that for all $\alpha = (\alpha_n)_n \in \varinjlim \mathfrak{p}_n$ and $\beta = (\beta_n)_n \in \varprojlim E_\rho^n$, we have

$$(1.11) \qquad (\alpha,\beta)_\rho = (\alpha_n,\beta_n)_{\rho,E_\rho^n,n} = \frac{1}{\eta^n} T_{L|K}(\lambda_\rho(\alpha_n)\delta_{v_n}(\beta_n)) \cdot_\rho v_n$$

for sufficiently large n. Here, $\delta_{v_n}(\beta_n)$ can be expressed using the Coleman power series associated to $\beta \in \varprojlim E_\rho^n$. The existence of such power series was proved by Oukhaba in [11]. This gives a generalization of Theorem 23 of Longhi–Bars [4] proved for formal Drinfeld modules obtained from sign-normalized rank 1 Drinfeld modules. To go further, one may ask if any explicit reciprocity laws can be proved for all formal Drinfeld modules having stable reduction of height 1. We plan to address this question in a future work. In another request, we are interested in considering local fields of higher dimension in the vein of the work of Jorge Florez [6, 7] and Bars–Longhi [3].

2. The Kummer pairing and first properties

In this section, we fix a positive integer n and a finite extension L of E_ρ^n. In particular, we have $W_\rho^n \subset L$.

Lemma 2.1. Let $\alpha \in \mathfrak{p}_L$. There exists an element ξ in $\mathfrak{p}_\Omega$ such that $\rho_{\eta^n}(\xi) = \alpha$. Moreover, the extension $L(\xi)|L$ is abelian, of degree $\leq q^{nm_0}$, and independent of the choice of ξ satisfying $\rho_{\eta^n}(\xi) = \alpha$.

Proof. By Section 2 of [11], we can write $\rho_{\pi^{nm_0}}$ as

$$(2.1) \qquad \rho_{\pi^{nm_0}} = U_1 U_{nm_0} Q_{nm_0} Q_{nm_0-1} \cdots Q_1,$$

where U_i are invertible elements of $\mathcal{O}_H\{\{\tau\}\}$ and $Q_i = \tau + \pi_i$, each π_i being a prime of H. Let

$$(2.2) \qquad P_{nm_0} = Q_{nm_0} Q_{nm_0-1} \cdots Q_1$$

then W_ρ^n is the set of roots of $P_{nm_0}(X)$. Let u be the unit of K such that $\eta = u\pi^{m_0}$. We denote $V_n = \rho_{u^n} U_1 U_{nm_0}$. Since V_n is invertible in $\mathcal{O}_H\{\{\tau\}\}$, we have

$$\begin{aligned}\rho_{\eta^n}(X) = \alpha &\iff V_n(P_{nm_0}(X)) = \alpha \\ &\iff P_{nm_0}(X) = V_n^{-1}(\alpha) \\ &\iff P_{nm_0}(X) - V_n^{-1}(\alpha) = 0.\end{aligned}$$

However, $V_n^{-1}(\alpha) \in \mathfrak{p}_L$, hence, $P_{nm_0}(X) - V_n^{-1}(\alpha)$ is a polynomial with coefficients in L. Therefore there exists an element ξ in Ω such that $P_{nm_0}(\xi) - V_n^{-1}(\alpha) = 0$. Furthermore, since $0 \equiv P_{nm_0}(\xi) \equiv \xi^{q^{nm_0}} \mod \mathfrak{p}_\Omega$, we have $\xi \in \mathfrak{p}_\Omega$. Moreover, the polynomial $P_{nm_0}(X) - V_n^{-1}(\alpha)$ is of degree q^{nm_0}, and all the elements of the set $\xi + W_\rho^n$, which we recall is a set of q^{nm_0} elements, are roots of this polynomial. Hence, it is separable and $L(\xi)|L$ is a Galois extension of degree $\leq q^{nm_0}$ depending only on α. Finally, to prove that it is an abelian extension, it suffices to notice that the group homomorphism $\mathrm{Gal}(L(\xi)|L) \to W_\rho^n$ defined by $\sigma \mapsto \sigma(\xi) - \xi$ is injective. □

By this Lemma, we see that the map $(\,\cdot\,,\cdot\,)_{\rho,L,n} : \mathfrak{p}_L \times L^\times \to W_\rho^n$ in (1.4) is well defined. We omit ρ in the index when there is no risk of confusion. Exactly as in [10, 13] we have

Proposition 2.2. The map $(\,\cdot\,,\cdot\,)_{L,n}$ satisfies the following properties

(i) The map $(\,\cdot\,,\cdot\,)_{L,n}$ is bilinear and $\mathcal{O}$-linear in the first coordinate for the action (1.2).

(ii) We have

$$(\alpha,\beta)_{L,n} = 0 \iff \beta \text{ is a norm from } L(\xi), \text{ where } \rho_{\eta^n}(\xi) = \alpha.$$

(iii) Let M be a finite separable extension of L, let $\alpha \in \mathfrak{p}_L$ and $\beta \in M^\times$. Then $(\alpha,\beta)_{M,n} = (\alpha, \mathrm{N}_{M|L}(\beta))_{L,n}$.

(iv) Let M be a finite separable extension of L of degree d, let $\alpha \in \mathfrak{p}_M$ and $\beta \in L^\times$. Then $(\alpha,\beta)_{M,n} = (\mathrm{T}_{M|L}(\alpha), \beta)_{L,n}$.

(v) Suppose $L \supset E_\rho^m$ for $m \geq n$. Then

$$(\alpha,\beta)_{L,n} = \rho_{\eta^{m-n}}((\alpha,\beta)_{L,m}) = (\rho_{\eta^{m-n}}(\alpha),\beta)_{L,m}.$$

(vi) Let ρ' be a formal Drinfeld $\mathcal{O}$-module isomorphic to ρ, i.e. there exists a power series t invertible in $\mathcal{O}_H\{\{\tau\}\}$ such that $\rho'_a = t^{-1} \circ \rho_a \circ t$ for all $a \in \mathcal{O}$. Then we have $(\alpha,\beta)_{\rho',L,n} = t^{-1}((t(\alpha),\beta)_{\rho,L,n})$.

Proof. The properties (i), (ii), (iii), (v) and (vi) are straightforward. The property (iv) can be proved as in [10, Section 3.3]. □

3. The Iwasawa map

In this section, we will study the so-called Iwasawa map, first introduced by Iwasawa in [8, Proposition 14] in the cyclotomic case. This map was generalized by Wiles [13, Proposition 7] in the case of Lubin–Tate formal groups, and by Kolyvagin [10, Proposition 3.2] in the case of formal groups of finite height. As in Section 2 above, we fix a positive integer n and a finite extension L of E_ρ^n. We also fix a generator v_n of the $\mathcal{O}$-module W_ρ^n and we suppose that $L|K$ is separable. First, we need to introduce the logarithm λ_ρ of ρ, defined by Rosen in [12, Section 2].

Lemma 3.1. There exists a unique power series $\lambda_\rho \in H\{\{\tau\}\}$, called the logarithm of ρ, such that $\lambda_\rho(X) \equiv X \mod \deg 2$ and $\lambda_\rho \rho_a = a\lambda_\rho$ for all $a \in \mathcal{O}$. Moreover, we have

(i) If $\lambda_\rho = \sum_{i\geq 0} c_i\tau^i$, then $\mu(c_i) \geq -i$ for all $i \geq 0$. Thus the element $\lambda_\rho(x) = \sum_{i\geq 0} c_i x^{q^i}$ is well defined in L for any $x \in \mathfrak{p}_L$.

(ii) If $x \in \mathfrak{p}_\Omega$, then $\lambda_\rho(X) = 0$ if and only if $x \in W_\rho$. Put $W_L = L \cap W_\rho \subset \mathfrak{p}_L$. Then the map $\lambda_\rho : \mathfrak{p}_L/W_L \to \lambda_\rho(\mathfrak{p}_L)$ is an isomorphism of $\mathcal{O}$-modules.

(iii) Let $\mathfrak{p}_{\Omega,1}$ denote the set of all the elements x of $\mathfrak{p}_\Omega$ such that $\mu(x) > 1/(q-1)$. The logarithm λ_ρ gives an isomorphism of $\mathcal{O}$-modules from $\mathfrak{p}_{\Omega,1}$, viewed as an $\mathcal{O}$-module under the action (1.2), to itself, viewed as an $\mathcal{O}$-module under the multiplication in Ω. If we denote $\mathfrak{p}_{L,1} = \mathfrak{p}_L \cap \mathfrak{p}_{\Omega,1}$, the logarithm λ_ρ also induces an isomorphism from $\mathfrak{p}_{L,1}$ to itself.

(iv) The $\mathcal{O}$-module $\lambda_\rho(\mathfrak{p}_L)$ is free of rank $[L : K]$ and we have $L = K\lambda_\rho(\mathfrak{p}_L)$.

Proof. The first three properties are proved by M. Rosen in [12]. For instance, the property (i) is a part of the proof of Proposition 2.1 of loc. cit. The property (ii) is exactly Proposition 2.4 of [12]. Finally, (iii) corresponds to Proposition 2.3 of [12]. Let us give a sketch of the proof of (iii). Let $x \in \mathfrak{p}_\Omega$ such that $\mu(x) > \frac{1}{q-1}$. By (i), we have

$$\mu(c_i x^{q^i}) = \mu(c_i) + q^i\mu(x) \geq -i + q^i\mu(x) > \mu(x)$$

for all $i \geq 1$. Hence $\mu(\lambda_\rho(x)) = \mu(x)$ so that $\lambda_\rho(x) \in \mathfrak{p}_{\Omega,1}$. Now we consider the inverse e_ρ of λ_ρ in $H\{\{\tau\}\}$. This series is called the exponential of ρ and satisfies $e_\rho(X) \equiv X \mod \deg 2$ and $e_\rho a = \rho_a e_\rho$ for all $a \in \mathcal{O}$. By [12, Proposition 2.2], if we write $e_\rho(x) = x + \sum_{i\geq 1} d_i x^{q^i}$, we have $\mu(d_i) \geq -(1 + q + \cdots + q^{i-1})$. Thus,

$$\mu(d_i x^{q^i}) = \mu(d_i) + q^i\mu(x) \geq -\frac{q^i - 1}{q - 1} + q^i\mu(x) > -(q^i - 1)\mu(x) + \mu(x) = \mu(x)$$

for all $i \geq 1$. Hence we have $\mu(e_\rho(x)) = \mu(x)$. This completes the proof since e_ρ is the formal inverse of λ_ρ.

As for the proof of (iv), let $x \in \mathfrak{p}_L$ and e_L be the ramification index of $L|K$, then $\mu(x) \geq \frac{1}{e_L}$. By (i), we have

$$\begin{aligned}\mu(\lambda_\rho(x)) &\geq \min(\mu(x), -i + q^i\mu(x)) \\ &\geq \min(\frac{1}{e}, -i + \frac{q^i}{e}).\end{aligned}$$

Thus, for a sufficiently large integer l, we have $\lambda_\rho(\mathfrak{p}_L) \subset \frac{1}{\pi^l}\mathcal{O}_L$. Therefore $\lambda_\rho(\mathfrak{p}_L)$ is free for it is a $\mathcal{O}$-submodule of the free $\mathcal{O}$-module $\frac{1}{\pi^l}\mathcal{O}_L$. Now let us prove that $L = K\lambda_\rho(\mathfrak{p}_L)$. Clearly, we have $K\lambda_\rho(\mathfrak{p}_L) \subset L$. Let $x \in L$, then we can write $x = u\pi_L^j$, where u is a unit of L and π_L is a prime of L. Then, for a sufficiently large integer i, we have $u\pi_L^j\pi^i \in \mathfrak{p}_{L,1} = \lambda_\rho(\mathfrak{p}_{L,1}) \subset \lambda_\rho(\mathfrak{p}_L)$. Therefore $x = \frac{1}{\pi^i}u\pi_L^j\pi^i \in K\lambda_\rho(\mathfrak{p}_L)$. □

Since the extension $L|K$ is supposed to be separable, the bilinear map $\langle\cdot,\cdot\rangle_L : L \times L \to K$ defined by $< x, y >_L = \mathrm{T}_{L|K}(xy)$ is non degenerate. This gives us the classical isomorphism from L to the space of K-linear forms from L to K. The pairing $\langle\cdot,\cdot\rangle_L$ also induces the following $\mathcal{O}$-linear map

$$\begin{aligned} L &\longrightarrow \mathrm{Hom}_{\mathcal{O}}(\lambda_\rho(\mathfrak{p}_L), K/\mathcal{O}) \\ y &\longmapsto \begin{cases}\lambda_\rho(\mathfrak{p}_L) \longrightarrow K/\mathcal{O} \\ \qquad x \longmapsto \langle x, y\rangle_L \mod \mathcal{O}\end{cases}\end{aligned} \tag{3.1}$$

Lemma 3.2. The map (3.1) is a surjective homomorphism of $\mathcal{O}$-modules, with kernel

$$\mathfrak{X}_L := \{y \in L;\ \langle x, y\rangle_L \in \mathcal{O}\ \ \forall\, x \in \lambda_\rho(\mathfrak{p}_L)\}. \tag{3.2}$$

Proof. It is clear that $\mathfrak{X}_L$ is the kernel of this map. Let us prove that the map is surjective. To do so, let $\gamma : \lambda_\rho(\mathfrak{p}_L) \to K/\mathcal{O}$ be an $\mathcal{O}$-linear map.

Let $\{e_1, \ldots, e_d\}$ be a basis of L as a K-vector space. Since $L = K\lambda_\rho(\mathfrak{p}_L)$ by Lemma 3.1 (iv), we can choose the e_i to be in $\lambda_\rho(\mathfrak{p}_L)$. Choose elements $\widetilde{\gamma}(e_i)$ in K such that $\gamma(e_i)$ is the class of $\widetilde{\gamma}(e_i)$ modulo $\mathcal{O}$. Define the K-linear map $\widetilde{\gamma} : L \to K$ by $\widetilde{\gamma}(\sum a_ie_i) = \sum a_i\widetilde{\gamma}(e_i)$ where $a_i \in K$. Thus we obtain the following commutative diagram

$$\begin{array}{ccc} L & \xrightarrow{\tilde{\gamma}} & K \\ \uparrow & & \downarrow \\ \lambda_\rho(\mathfrak{p}_L) & \xrightarrow{\gamma} & K/\mathcal{O} \end{array}$$

the right hand arrow being the canonical projection and the left hand arrow being the inclusion. However, the K-linear form $\widetilde{\gamma}$ is induced by some element $y \in L$ satisfying $\widetilde{\gamma}(x) = \mathrm{T}_{L|K}(xy)$ for all $x \in \lambda_\rho(\mathfrak{p}_L)$. Therefore we have $\gamma(x) \equiv \widetilde{\gamma}(x) =< x, y >_L \mod \mathcal{O}$. $\square$

Now, we give the construction of the so-called Iwasawa map. As mentioned in the introduction (1.3), the map

$$\begin{aligned} \mathcal{O}/\eta^n\mathcal{O} &\longrightarrow W_\rho^n \\ a &\longmapsto \rho_a(v_n) \end{aligned} \tag{3.3}$$

is an isomorphism of $\mathcal{O}$-modules because v_n is a generator of W_ρ^n. We denote by ι_1 its inverse. We define the $\mathcal{O}$-linear map

$$\begin{aligned} \iota : W_\rho^n &\xrightarrow{\iota_1} \mathcal{O}/\eta^n\mathcal{O} \longrightarrow K/\mathcal{O} \\ \rho_a(v_n) &\longmapsto a \longmapsto \tfrac{a}{\eta^n} \end{aligned}$$

Let

$$L^n = \{\beta \in L^\times;\ (\alpha, \beta)_{L,n} = 0 \ \ \forall\, \alpha \in W_L\}, \tag{3.4}$$

where we recall $W_L = L \cap W_\rho$. Any $\beta \in L^n$ defines an $\mathcal{O}$-linear map

$$h_\beta : \begin{cases} \mathfrak{p}_L/W_L \longrightarrow K/\mathcal{O} \\ \qquad \alpha \longmapsto \iota((\alpha,\beta)_{L,n}) \end{cases}$$

where the action of $\mathcal{O}$ on $\mathfrak{p}_L/W_L$ is given by (1.2). The map $\beta \mapsto h_\beta$ gives a group homomorphism from L^n to $\mathrm{Hom}_{\mathcal{O}}(\mathfrak{p}_L/W_L, K/\mathcal{O})$. The isomorphism of Lemma 3.1 (ii) induces the following isomorphism of $\mathcal{O}$-modules

$$\mathrm{Hom}_{\mathcal{O}}(\mathfrak{p}_L/W_L, K/\mathcal{O}) \cong \mathrm{Hom}_{\mathcal{O}}(\lambda_\rho(\mathfrak{p}_L), K/\mathcal{O}). \tag{3.5}$$

Let $\beta \in L^n$ and let g_β be the image of h_β by the isomorphism (3.5). Then g_β is defined by $g_\beta(\lambda_\rho(\alpha)) = \iota((\alpha,\beta)_{L,n})$. However g_β is an $\mathcal{O}$-linear map from $\lambda_\rho(\mathfrak{p}_L)$ to $K/\mathcal{O}$. Thus, by Lemma 3.2, there exists a unique $y \in L/\mathfrak{X}_L$ satisfying $g_\beta(\lambda_\rho(\alpha)) = \mathrm{T}_{L|K}(\lambda_\rho(\alpha)y) \mod \mathcal{O}$ for all $\alpha \in \mathfrak{p}_L$. It is easy to see that $y \in \eta^{-n}\mathfrak{X}_L/\mathfrak{X}_L$. We set

$$\psi_{L,v_n}(\beta) = \eta^n y \mod \eta^n\mathfrak{X}_L. \tag{3.6}$$

Proposition 3.3. We have

$$(\alpha,\beta)_{L,n} = T_{L|K}(\lambda_\rho(\alpha)\psi_{L,v_n}(\beta)) \cdot_\rho v_n \tag{3.7}$$

for all $\alpha \in \mathfrak{p}_L$ and $\beta \in L^n$. Furthermore, the map $\psi_{L,v_n} : L^n \to \mathfrak{X}_L/\eta^n\mathfrak{X}_L$ is a group homomorphism.

Proof. The Proposition follows immediately from the construction. $\square$

Exactly as in [10, Section 3.5], our ψ_{L,v_n} satisfies the properties φ_1, φ_2, φ_3, φ_4, φ_5 and φ_6 of loc. cit.

4. More properties of the pairing $(\,\cdot\,,\cdot\,)_{L,n}$

As above, we continue to fix a positive integer n, a finite extension L of E_ρ^n and a generator v_n of W_ρ^n.

Lemma 4.1. There exists a constant $c_{L,n}$, dependant only on L and n, such that for $\alpha \in \mathfrak{p}_L$, if we set $\alpha_m = \rho_{\eta^{m-n}}(\alpha)$ for $m \geq n$, we get $\mu(\alpha_m) \geq mm_0 - c_{L,n}$. Furthermore, the map $(\,\cdot\,,\cdot\,)_{L,n}$ is continuous, and $(\alpha,\cdot)_{L,n} = 0$ for all $\alpha \in \mathfrak{p}_L$ such that $\mu(\alpha) > nm_0 + \frac{1}{q-1}$.

Proof. We follow [4, Lemma 15]. Let $\alpha \in \mathfrak{p}_L$ and set $\mu_j := \frac{1}{q^{j-1}(q-1)}$ for $j \geq 1$ and $\mu_0 := \infty$. Choose ξ a root of $\rho_{\eta^n}(X) = \alpha$ of maximal valuation. This is possible because the equation $\rho_{\eta^n}(X) = \alpha$ has a finite set of solutions: $\xi + W_\rho^n$. We have

$$\alpha = \rho_{\eta^n}(\xi) = V_n(P_{nm_0}(\xi)),$$

where $V_n(X)$ and $P_{nm_0}(X) = \Pi_{w \in W_\rho^n}(X - w)$ are defined in the proof of Lemma 2.1. Therefore, we get

$$\mu(\alpha) = \mu(P_{nm_0}(\xi)) = \sum_{w \in W_\rho^n} \mu(\xi - w)$$

because V_n is invertible in $\mathcal{O}_H\{\{\tau\}\}$. Let $w \in W_\rho^n$. If $\mu(\xi) \neq \mu(w)$, then $\mu(\xi - w) = \min\{\mu(\xi), \mu(w)\}$. If $\mu(\xi) = \mu(w)$, then

$$\mu(\xi) = \min\{\mu(\xi), \mu(w)\} \leq \mu(\xi - w) \leq \mu(\xi),$$

the last inequality being a consequence of the maximality hypothesis on $\mu(\xi)$. Hence we have $\mu(\xi - w) = \min\{\mu(\xi), \mu(w)\}$ for all $w \in W_\rho^n$ and

$$\mu(\alpha) = \sum_{w \in W_\rho^n} \min\{\mu(\xi), \mu(w)\}. \tag{4.1}$$

Let $j \geq 0$ be such that $\mu_{j+1} < \mu(\xi) \leq \mu_j$. If $0 \leq j \leq nm_0$, the equality (4.1) yields

$$\mu(\alpha) = \sum_{w \in V_\rho^j} \mu(\xi) + \sum_{w \in W_\rho^n \setminus V_\rho^j} \mu(w) = q^j \mu(\xi) + nm_0 - j$$

so that $nm_0 - j + \frac{1}{q-1} < \mu(\alpha) \leq nm_0 - j + 1 + \frac{1}{q-1}$. Now if $j > nm_0$, by (4.1) we get $\mu(\alpha) = q^{nm_0}\mu(\xi)$ so that

$$nm_0 - j + \frac{1}{q-1} \leq 0 < \frac{1}{q^{j-nm_0}(q-1)} < \mu(\alpha) \leq \frac{1}{q^{j-nm_0-1}(q-1)}.$$

Since ξ is also a root of $\rho_{\eta^m}(X) = \alpha_m$ for all $m \geq n$, we deduce by the same arguments that $\mu(\alpha_m) \geq mm_0 - j + \frac{1}{q-1}$. Considering the degree of the extension $L(\xi)|K$, we see that $j \leq 2nm_0 + \log_q(e)$, where e is the ramification index of $L|E^n_\rho$. Hence, we get

$$\mu(\alpha_m) \geq mm_0 - 2nm_0 - \log_q(e) + \frac{1}{q-1}.$$

Finally, if we suppose $\mu(\alpha) > nm_0 + \frac{1}{q-1}$, we get $j = 0$, which implies that $\mu((\alpha, \beta)_{L,n}) \geq \mu(\xi) > \frac{1}{q-1}$ for all $\beta \in L^\times$. It follows that $(\alpha, \beta)_{L,n} = 0$ for all $\beta \in L^\times$, because $(\alpha, \beta)_{L,n}$ belongs to W^n_ρ, and the elements of $W^n_\rho \setminus \{0\}$ are of valuation less or equal to $\frac{1}{q-1}$. The fact that the map $(\,\cdot\,,\cdot\,)_{L,n}$ is continuous follows immediately since the reciprocity map Φ_L is continuous. □

Remark 4.2. Let e be the ramification index of $L|E^n_\rho$, then the constant $c_{L,n}$ from Lemma 4.1 is bounded as follows

$$\frac{-1}{q-1} \leq c_{L,n} \leq 2nm_0 + \log_q(e) - \frac{1}{q-1}. \tag{4.2}$$

Proposition 4.3. There exists a unique power series $r = r_n \in \mathcal{O}_H\{\{\tau\}\}$ such that

$$\prod_{\omega \in W^n_\rho} (X - \omega) = r \circ \rho_{\eta^n}(X).$$

Furthermore, the power series r is invertible in $\mathcal{O}_H\{\{\tau\}\}$ and satisfies

$$(x, r(x))_{L,n} = 0, \quad \forall\, x \in \mathfrak{p}_L \setminus \{0\}.$$

Proof. As in the proof of Lemma 2.1, we can write

$$\rho_{\eta^n}(X) = \rho_{u^n} \circ U_1 \circ U_{nm_0} \circ P_{nm_0}(X).$$

Thus for $r = (\rho_{u^n} \circ U_1 \circ U_{nm_0})^{-1}$ we get $P_{nm_0}(X) = \prod_{\omega \in W^n_\rho}(X - \omega) = r \circ \rho_{\eta^n}(X)$. It remains to show that $(x, r(x))_{L,n} = 0$ for all $x \in \mathfrak{p}_L \setminus \{0\}$. Take $x \in \mathfrak{p}_L \setminus \{0\}$ and ξ such that $\rho_{\eta^n}(\xi) = x$. Then,

$$r(x) = (r \circ \rho_{\eta^n})(\xi) = \prod_{\omega \in W^n_\rho} (\xi - \omega) = \prod_i \mathrm{N}_{L(\xi)|L}(\xi_i)$$

where ξ_i are the pairwise distinct roots of $\rho_{\eta^n}(X) = x$. It follows that $(x, r(x))_{L,n} = 0$ by Proposition 2.2 (ii). □

Lemma 4.4. Let $r = r_n$ be the power series defined in Proposition 4.3. Let ρ' be defined by

$$\rho'_a = r \circ \rho_a \circ r^{-1} \tag{4.3}$$

for all $a \in \mathcal{O}$. Then ρ' is a formal Drinfeld module having a stable reduction of height 1, and we have $(x, x)_{\rho', L, n} = 0$ for all $x \in \mathfrak{p}_L \setminus \{0\}$.

Proof. That ρ' is a formal Drinfeld module having a stable reduction of height 1 follows from the fact that ρ itself is supposed to be a formal Drinfeld module having a stable reduction of height 1. It follows from Proposition 2.2 (vi) that $(x,x)_{\rho',L,n} = r((r^{-1}(x),x)_{\rho,L,n}) = r(0) = 0$. ☐

Lemma 4.5. If ρ is such that $(x,x)_{\rho,L,n} = 0$ for all $x \in \mathfrak{p}_L \setminus \{0\}$, then we have

$$(c, 1-b)_{L,n} = (\frac{bc}{1-b}, b^{-1})_{L,n}$$

for all $b \in \mathfrak{p}_L \setminus \{0\}$ and $c \in \mathfrak{p}_L$.

Proof. See [4, Lemma 18]. ☐

For a finite extension $F'|F$ of local fields, let $\mathfrak{m}_{F'|F}$ be the fractional ideal of $\mathcal{O}_{F'}$ defined by

$$\mathfrak{m}_{F'|F} = \{x \in F';\ \mathrm{T}_{F'|F}(x\mathcal{O}_{F'}) \subset \mathcal{O}_F\} \supset \mathcal{O}_{F'}$$

As usual, the different $\mathcal{D}_{F'|F}$ of $F'|F$ is the inverse ideal of $\mathfrak{m}_{F'|F}$

$$\mathcal{D}_{F'|F} := \mathfrak{m}_{F'|F}^{-1}.$$

If $F'|F$ is unramified, then $\mathcal{D}_{F'|F} = \mathcal{O}_{F'}$, and if $F'|F$ is totally ramified, then $\mathcal{D}_{F'|F} = h'(w)\mathcal{O}_{F'}$, where w is a prime element of F' and $h(X)$ is the minimal polynomial of w over F. Moreover, if $F''|F$ is a finite extension of local fields such that $F \subset F' \subset F''$, we have

$$\mathcal{D}_{F''|F} = \mathcal{D}_{F''|F'}\mathcal{D}_{F'|F}.$$

For more details, the reader may check [9, Section 2.4].

Lemma 4.6. Let $\mathcal{D}_n$ be the different of the extension $E_\rho^n|K$, then $\mathcal{D}_n$ is generated by an element of valuation $nm_0 - \frac{1}{q-1}$.

Proof. The proof of [4, Lemma 3] is suitable for our case. ☐

Lemma 4.7. Let $x \in E_\rho^n$ and denote by T_n the trace of the extension $E_\rho^n|K$. Then,

$$\mu(\mathrm{T}_n(x)) \geq \lfloor \mu(x) + nm_0 - \frac{1}{q-1} \rfloor,$$

where $\lfloor a \rfloor$ is the integral part of $a \in \mathbb{R}$. Furthermore, for $m \leq n$, we have

$$\mu(\mathrm{T}_{n,m}(x)) > \mu(x) + (n-m)m_0 - \mu(v_m),$$

where $\mathrm{T}_{n,m}$ is the trace of the extension $E_\rho^n|E_\rho^m$.

Proof. See [4, Lemma 4]. ☐

For the rest of the paper, we suppose $L = E_\rho^n$.

Lemma 4.8. (i) The map $\delta_{v_n} : L^\times \to \mathfrak{p}_L^{-1}/\mathcal{D}_n$ defined by

$$\delta_{v_n}(\beta) := \frac{f'(v_n)}{\beta} \mod \mathcal{D}_n, \tag{4.4}$$

where $f \in \mathcal{O}_H((X))^\times$ is such that $f(v_n) = \beta$, is a group homomorphism.

(ii) For $m \geq n$, let v_m be a generator of W_ρ^m such that $v_m = \rho_{\eta^{m-n}}(v_n)$ and let $\beta \in L^\times$. If we define $\delta_{v_m}(\beta)$ as in (4.4), we get

$$\delta_{v_m}(\beta) \equiv \eta^{m-n}\delta_{v_n}(\beta) \mod \mathcal{D}_m.$$

Proof. This lemma is easy to prove, the interested reader may check [13, Lemma 10]. □

Lemma 4.9. The map

$$[\alpha, \beta]_{\rho,L,n} := \frac{1}{\eta^n}\mathrm{T}_{L|K}(\lambda_\rho(\alpha)\delta_{v_n}(\beta)) \cdot_\rho v_n$$

is well defined for all $\alpha \in \mathfrak{p}_L$ of valuation $\mu(\alpha) \geq \frac{2}{q-1}$, and all $\beta \in L^\times$. We drop ρ in the index when there is no risk of confusion.

Proof. We need to show that $\frac{1}{\eta^n}\mathrm{T}_{L|K}(\lambda_\rho(\alpha)b) \in \mathcal{O}$ for every $b \in \mathfrak{p}_L^{-1}$ and that

$$\mu(\frac{1}{\eta^n}\mathrm{T}_{L|K}(\lambda_\rho(\alpha)d)) \geq nm_0$$

for all $d \in \mathcal{D}_n$. Using (i) of Lemma 3.1, we can deduce that $\mu(\lambda_\rho(\alpha)) = \mu(\alpha)$. Thus the result follows from Lemma 4.7. □

Proposition 4.10. The map $[\cdot,\cdot]_{L,n}$ satisfies the following properties

(i) The map $[\cdot,\cdot]_{L,n}$ is bilinear and $\mathcal{O}$-linear in the first coordinate for the action (1.2).

(ii) Let ρ' be a formal Drinfeld $\mathcal{O}$-module isomorphic to ρ, i.e. there exists a power series t invertible in $\mathcal{O}_H\{\{\tau\}\}$ such that $\rho'_a = t^{-1} \circ \rho_a \circ t$ for all $a \in \mathcal{O}$. Then we have $[\alpha, \beta]_{\rho',L,n} = t^{-1}([t(\alpha), \beta]_{\rho,L,n})$.

Proof. The property (i) is clear, so we will only prove (ii). To do so, let $v'_n = t^{-1}(v_n)$ be a generator of the $\mathcal{O}$-module $W_{\rho'}^n$. Then, if $f \in \mathcal{O}_H((X))^\times$ is such that $f(v_n) = \beta$, we have $f \circ t(v'_n) = f(v_n) = \beta$ so that

$$\delta'_{v_n}(\beta) = \frac{t'(v'_n)f'(v_n)}{\beta} = t'(0)\delta_{v_n}(\beta),$$

where δ'_{v_n} is the map defined in Lemma 4.8 corresponding to ρ'. Furthermore, we have $\lambda_{\rho'} \circ t^{-1} = (t^{-1})'(0)\lambda_\rho$. The result follows immediately since $(t^{-1})'(0) = \frac{1}{t'(0)}$. □

Lemma 4.11. Let $\alpha \in \mathfrak{p}_L$ such that $\mu(\alpha) \geq \frac{nm_0}{q} + \frac{1}{q-1} + \frac{1}{q^{nm_0}(q-1)}$ and let $\beta \in L^\times$. We have

$$[\alpha, \beta]_{L,n} = \frac{1}{\eta^n} \mathrm{T}_{L|K}(\alpha \delta_{v_n}(\beta)) \cdot_\rho v_n.$$

Proof. We need to prove that

$$\frac{1}{\eta^n} \mathrm{T}_{L|K}(\lambda_\rho(\alpha)\delta_{v_n}(\beta)) \cdot_\rho v_n = \frac{1}{\eta^n} \mathrm{T}_{L|K}(\alpha \delta_{v_n}(\beta)) \cdot_\rho v_n,$$

i.e. that

$$\mu(\mathrm{T}_{L_K}(\lambda_\rho(\alpha) - \alpha)\delta_{v_n}(\beta)) \geq 2nm_0.$$

We have

$$\mu((\lambda_\rho(\alpha) - \alpha)\delta_{v_n}(\beta)) \geq \min_i \{q^i \mu(\alpha) - i\} - \frac{1}{q^{nm_0-1}(q-1)}.$$

The hypothesis implies that $\min_i\{q^i\mu(\alpha) - i\} = q\mu(\alpha) - 1$ so that $\mu(\lambda_\rho(\alpha) - \alpha)\delta_{v_n}(\beta)) \geq nm_0 + \frac{1}{q-1}$. Finally, we conclude using Lemma 4.7. ☐

Lemma 4.12. Let $\beta \in E_\rho^n$ and $\beta' \in E_\rho^m$ such that $\mathrm{N}_{m,n}(\beta') = \beta$. We have

$$\mathrm{T}_{m,n}(\delta_{v_m}(\beta')) = \eta^{m-n} \delta_{v_n}(\beta).$$

Proof. This lemma is the analogue of Lemma 8.9 in [9], whose proof is adaptable to our case. The main ingredient used is the Coleman norm operator associated to ρ, defined by Oukhaba in [11, Section 5]. ☐

5. Explicit reciprocity laws

In this section, we assume that $\rho_\eta \equiv \tau^{m_0} \mod \mathfrak{p}_H$. We fix a positive integer n and a generator v_n of W_ρ^n, and we set $L = E_\rho^n$.

As in the classical case of Lubin–Tate formal groups, we have

Proposition 5.1. For every unit u of K, we have

$$\Phi_K(u)(\omega) = \rho_{u^{-1}}(\omega) \tag{5.1}$$

for all $\omega \in W_\rho$.

Proof. Let $f(X) = \pi X + X^q$. As

$$f(X) \equiv \pi X \mod \deg 2 \quad \text{and} \quad f(X) \equiv X^q \mod \mathfrak{p}_H,$$

then by Lubin–Tate theory (see for instance [9, Proposition 4.2]), there exists a unique formal group $F_f(X, Y)$ such that

$$f \circ F_f = F_f^\phi \circ f. \tag{5.2}$$

It is easy to see that $F_f(X, Y) = X + Y$. Consider the rinf of endomorphsims of F_f

$$\mathrm{End}(F_f) := \{g \in \mathcal{O}_H[\![X]\!]; \quad g(F_f(X, Y)) = F_f(g(X), g(Y))\}.$$

By [9, Proposition 4.4], there exists an injective ring homomorphism $\mathcal{O} \to \mathrm{End}(G_a)$ which associates for each $a \in \mathcal{O}$, the unique power series $[a]_f$ satisfying

$$[a]_f(X) \equiv aX \mod \deg 2 \quad \text{and} \quad f \circ [a]_f = [a]_f^{\phi} \circ f,$$

where ϕ is the Frobenius automorphism of $K_{ur}|K$. Clearly, we have $f(X) = [\pi]_f(X)$. Let $d \in \mathbb{N}$ be such that $[H : K] = dm_0$. Since

$$\rho_{\eta^d}(X) \equiv \eta^d X, \quad [\pi^{dm_0}]_f(X) \equiv \pi^{dm_0} X \mod \deg 2$$
$$\text{and} \quad \rho_{\eta^d}(X) \equiv [\pi^{dm_0}]_f(X) \equiv X^{q^{dm_0}} \mod \mathfrak{p}_H,$$

then by [11, Proposition 3.1], there exists a unique power series $\theta \in \mathcal{O}_{\overline{K}_{ur}}[\![X]\!]$ such that

$$\theta(X) \equiv u_0 X \mod \deg 2 \quad \text{and} \quad \rho_{\eta^d} \circ \theta = \theta^{\phi^{dm_0}} \circ [\pi^{dm_0}]_f$$

where $\overline{K}_{ur}$ is the completion of K_{ur} and u_0 is a unit of $\overline{K}_{ur}$. We deduce that for all $m \geq 1$, we have

$$\rho_{\eta^{md}} \circ \theta = \theta^{\phi^{dm_0}} \circ [\pi^{dmm_0}]_f,$$

and therefore we have an isomorphism of $\mathbb{F}_q$-vector spaces

$$\theta : W_f \longrightarrow W_\rho.$$

Here $W_f = \bigcup W_f^m$, where W_f^m is the set of roots of $[\pi^m]_f$. Now let u be a unit of K and consider $\Phi_K(u) \in Gal(K^{ab}|K_{ur})$. By [9, Chapter 6], we have

$$\Phi_K(u)(\omega') = [u^{-1}]_f(\omega') \quad \forall\, \omega' \in W_f.$$

However, since $\Phi_K(u)_{|H(W_\rho)} \in \mathrm{Gal}(H(W_\rho)|H)$, then by [11, Proposition 2.5], there exists a unit $v \in K$ such that

$$\Phi_K(u)(\omega) = \rho_{v^{-1}}(\omega), \quad \forall\, \omega \in W_\rho.$$

Let $\omega' \in W_f$ and $\omega = \theta(\omega') \in W_\rho$, then $\rho_{v^{-1}} \circ \theta(\omega') = \Phi_K(u)(\theta(\omega'))$. However, $\Phi_K(u)$ is an automorphism of $K^{ab} = K_{ur}(W_\rho)$ over K_{ur}. Hence, we can extend it to an automorphism of $\overline{K^{ab}}(W_\rho)$ over $\overline{K}_{ur}$ so that

$$\begin{aligned}\Phi_K(u)(\theta(\omega')) &= \theta(\Phi_K(u)(\omega')) \\ &= \theta \circ [u^{-1}]_f(\omega').\end{aligned}$$

Therefore we have $\rho_{v^{-1}} \circ \theta(\omega') = \theta \circ [u^{-1}]_f(\omega')$ for all $\omega' \in W_f$. Then, reasoning as in [9, Lemma 8.1], we can prove that

$$\rho_{v^{-1}} \circ \theta = \theta \circ [u^{-1}]_f.$$

We deduce by identification that $u = v$. This concludes the proof. □

Lemma 5.2. Let $\alpha \in \mathfrak{p}_L$. For $m \geq n$, we set $\alpha_m = \rho_{\eta^{m-n}}(\alpha)$ and $b_m = \alpha_m v_m^{-1}$. Then, there exists an integer $N(\rho, \alpha) \in \mathbb{N}$ such that for all $m \geq N(\rho, \alpha)$, we have

$$(\alpha, \mathrm{N}_{m,n}(1 + b_m))_{L,n} = 0 \tag{5.3}$$

and

$$\mathrm{N}_m(1 + b_m)^{-1} \equiv 1 - \mathrm{T}_m(b_m) \mod \mathfrak{p}_K^{2mm_0}, \tag{5.4}$$

where T_m and N_m denote respectively the trace and the norm of the extension $E_\rho^m|K$ and $N_{m,n}$ denotes the norm of the extension $E_\rho^m|E_\rho^n$.

Proof. We first prove (5.3). Let $m \geq n$. By Lemma 4.1, there exists a constant c depending only on n such that $\mu(b_m) \geq mm_0 - c$. Thus $1 + b_m$ tends to 1 as m tends to ∞. Moreover,

$$\mathrm{N}_{m,n}(1 + b_m) = \prod (1 + \sigma(b_m)) = 1 + y, \tag{5.5}$$

where σ varies among the automorphisms in $\mathrm{Gal}(E_\rho^m|E_\rho^n)$ and $\mu(y) \geq \mu(b_m)$. Thus, $\mathrm{N}_{m,n}(1 + b_m)$ also tends to 1 as m tends to ∞. Furthermore,

$$(\alpha, \mathrm{N}_{m,n}(1 + b_m))_{L,n} = \Phi_L(\mathrm{N}_{m,n}(1 + b_m))(\xi) - \xi, \tag{5.6}$$

where $\rho_{\eta^n}(\xi) = \alpha$. But Φ_L is continuous. Hence, for the neighborhood $\mathrm{Gal}(L^{ab}|L(\xi))$ of $\Phi_L(1)$, there exists $N_1 \in \mathbb{N}$ such that if $m \geq N_1$, then $\Phi_L(\mathrm{N}_{m,n}(1 + b_m)) \in \mathrm{Gal}(L^{ab}|L(\xi))$. Thus, for all $m \geq N_1$, we have (5.3). Now let us prove (5.4). Let $k \leq m$ be an integer. Let $x = \mathrm{T}_{m,m-k}(b_m)$, then it is easy to check that $\mathrm{N}_{m,m-k}(1+b_m)^{-1} = 1-x+y$, where $\mu(y) \geq 2\mu(b_m)$. Therefore, we have

$$\mathrm{N}_m(1 + b_m)^{-1} = \mathrm{N}_{m-k}(1 - x + y) = 1 - \mathrm{T}_{m-k}(x - y) + z, \tag{5.7}$$

$\mu(z) \geq \mu(x - y)$. If k and m are such that $km_0 \geq c + 1$ and $mm_0 \geq km_0 + 2c + \frac{1}{(q-1)}$, then, by Lemma 4.7 we get $\mu(\mathrm{T}_{m-k}(x-y)) \geq 2mm_0$ and $\mu(z) \geq \mu(x - y) \geq 2mm_0$. Thus, (5.4) follows. Finally, we set $N(\rho, \alpha) = \max\{N_1, \lfloor k + \frac{2c}{m_0} + \frac{1}{m_0(q-1)} \rfloor + 1\}$. □

Remark 5.3. Let $\alpha \in \mathfrak{p}_L$ and let ρ' be a formal Drinfeld $\mathcal{O}_K$-module isomorphic to ρ, i.e. there exists a power series t invertible in $\mathcal{O}_H\{\{\tau\}\}$ such that $\rho'_a = t^{-1} \circ \rho_a \circ t$ for all $a \in \mathcal{O}_K$. It is easy to prove that $E_\rho^m = E_{\rho'}^m$ for all $m \geq 0$. Moreover, by Proposition 2.2 (vi) we have $N(\rho, \alpha) = N(\rho', t^{-1}(\alpha))$.

Lemma 5.4. Let $\alpha \in \mathfrak{p}_L$ and suppose that there exists $m \geq \max\{N(\rho, \alpha), \frac{q}{q-1}(2n + \frac{1}{2m_0})\}$ such that $(x, x)_{E_\rho^m, m} = 0$ for all $x \in \mathfrak{p}_{E_\rho^m} \setminus \{0\}$, where $N(\rho, \alpha)$ is defined in Lemma 5.2. Then, there exists a prime π_n of L such that

$$(\alpha, \pi_n)_{L,n} = [\alpha, \pi_n]_{L,n} = \frac{1}{\eta^n} \mathrm{T}_{L|K}(\lambda_\rho(\alpha)\delta_{v_n}(\pi_n)) \cdot_\rho v_n. \tag{5.8}$$

Proof. We prove the Lemma following the steps of [4, Proposition 23], which were essentially used by Wiles [13, Lemma 8]. Let $\alpha \in \mathfrak{p}_L$.

Step 1. For $m \geq n$, let $\alpha_m = \rho_{\eta^{m-n}}(\alpha)$ and $b_m = \alpha_m v_m^{-1}$. If we suppose $(x,x)_{E_\rho^m,m} = 0$ for all $x \in \mathfrak{p}_{E_\rho^m} \setminus \{0\}$, we have

$$\begin{aligned} 0 &= (\alpha_m + v_m, (1+b_m)v_m)_{E_\rho^m,m} \\ &= (\alpha_m, v_m)_{E_\rho^m,m} + (\alpha_m, 1+b_m)_{E_\rho^m,m} + (v_m, 1+b_m)_{E_\rho^m,m}, \end{aligned}$$

because $\alpha_m + v_m = (1+b_m)v_m$.

Step 2. For $m \geq N(\rho,\alpha)$, we have $(\alpha_m, 1+b_m)_{E_\rho^m,m} = (\alpha, \mathrm{N}_{m,n}(1+b_m))_{L,n} = 0$ by Lemma 5.2.

Step 3. Let $m \geq N(\rho,\alpha)$ so that $(\alpha_m, 1+b_m)_{E_\rho^m,m} = 0$ and suppose that $(x,x)_{E_\rho^m,m} = 0$ for all $x \in \mathfrak{p}_{E_\rho^m} \setminus \{0\}$. Let $\pi_n = \mathrm{N}_{m,n}(v_m)$, then π_n is a prime of L because $E_\rho^m|L$ is a totally ramified extension. Let v_{2m} be a generator of W_ρ^{2m} such that $\rho_{\eta^m}(v_{2m}) = v_m$. We have

$$(\alpha, \pi_n)_{L,n} = v_{2m} - \rho_{\mathrm{N}_m(1+b_m)^{-1}}(v_{2m}).$$

Indeed,

$$\begin{aligned} (\alpha,\pi_n)_{L,n} = (\alpha_m, v_m)_{E_\rho^m,m} &= -(v_m, 1+b_m)_{E_\rho^m,m} \qquad \text{(by Step 1 and 2)} \\ &= -(\Phi_{E_\rho^m}(1+b_m)(v_{2m}) - v_{2m}) \\ &= -(\Phi_K(\mathrm{N}_m(1+b_m))(v_{2m}) - v_{2m}). \end{aligned}$$

By Proposition 5.1 we have $\Phi_K(\mathrm{N}_m(1+b_m))(v_{2m}) = \rho_{\mathrm{N}_m(1+b_m)^{-1}}(v_{2m})$ and hence $(\alpha,\pi_n)_n = v_{2m} - \rho_{\mathrm{N}_m(1+b_m)^{-1}}(v_{2m})$.

Step 4. For $m \geq N(\rho,\alpha)$, we have $\mathrm{N}_m(1+b_m)^{-1} \equiv 1 - \mathrm{T}_m(b_m) \mod \mathfrak{p}_K^{2mm_0}$ by Lemma 5.2.

Step 5. Choose $m \geq \max\{N(\rho,\alpha), \frac{q}{q-1}(2n + \frac{1}{2m_0})\}$, then m is sufficiently large to satisfy Step 2 and Step 4. If in addition we have $(x,x)_{E_\rho^m,m} = 0$ for all $x \in \mathfrak{p}_{E_\rho^m} \setminus \{0\}$, then $(\alpha,\pi_n)_{L,n} = [\alpha,\pi_n]_{L,n}$, where $\pi_n = \mathrm{N}_{m,n}(v_m)$ as in Step 3. Indeed, by the previous steps we get

$$(\alpha,\pi_n)_{L,n} = \mathrm{T}_m(\alpha_m v_m^{-1}) \cdot_\rho v_{2m} = \frac{1}{\eta^m}\mathrm{T}_m(\alpha_m v_m^{-1}) \cdot_\rho v_m. \tag{5.9}$$

We draw the attention of the reader to the fact that m is sufficiently large so that $\mu(\alpha_m) \geq \frac{mm_0}{q} + \frac{1}{q-1} + \frac{1}{q^{mm_0}(q-1)}$. This is a consequence of Lemma 4.1 and Remark 4.2. This implies that $\frac{1}{\eta^m}\mathrm{T}_m(\alpha_m v_m^{-1}) \in \mathcal{O}_K$. Moreover, by

Lemma 4.11 and Lemma 4.12, we get

$$\begin{aligned}(\alpha,\pi_n)_{L,n} &= \frac{1}{\eta^m}\mathrm{T}_m(\alpha_m v_m^{-1})\cdot_\rho v_m \\ &= [\alpha_m, v_m]_{E_\rho^m,m} && \text{(Lemma 4.11)} \\ &= [\alpha,\pi_n]_{L,n}. && \text{(Lemma 4.12)}\end{aligned}$$ □

Remark 5.5. If $\rho_\eta(X)$ is a polynomial (as in [1, 4, 13]), the condition $(x,x)_{\rho,E_\rho^m,m}=0$ is satisfied for all $m\geq 1$, and following the same steps as in the proof of Lemma 5.4, one can prove that

$$(\alpha, v_n)_{L,n} = \frac{1}{\eta^n}\mathrm{T}_{L|K}(\lambda_\rho(\alpha)\frac{1}{v_n})\cdot_\rho v_n \tag{5.10}$$

for all $\alpha\in\mathfrak{p}_L$.

Lemma 5.6. Suppose ρ is such that $(x,x)_{L,n}=0$ for all $x\in\mathfrak{p}_L\setminus\{0\}$. Let $\alpha\in\mathfrak{p}_L$ such that $\mu(\alpha)\geq\frac{nm_0}{q}+\frac{1}{q-1}+\frac{1}{q^{nm_0}(q-1)}$ and β a unit in $L^\times$. Then

$$(\alpha,\beta)_{L,n} = [\alpha,\beta]_{L,n} = \frac{1}{\eta^n}\mathrm{T}_{L|K}(\lambda_\rho(\alpha)\delta_{v_n}(\beta))\cdot_\rho v_n. \tag{5.11}$$

Proof. We first notice that a unit $\beta\in L$ is of the form ζu_1, where ζ is a $(q-1)^{\text{th}}$ root of unity and u_1 is a principle unit in L. It is obvious that both sides of (5.11) are zero for $\beta=\zeta$. Hence, it is sufficient to prove the Lemma for the principal units $\beta=1-\zeta{\pi_L}^j$, where π_L is a prime of L, ζ is any $(q-1)^{\text{th}}$ root of unity, and j is any integer greater than 1. This goes back to the structure of the principal units as a $\mathbb{Z}_p$-module and to the continuity of the pairings. By Lemma 4.5, we have

$$(\alpha, 1-\zeta{\pi_L}^j)_{\rho,L,n} = \left(\frac{\zeta{\pi_L}^j}{1-\zeta{\pi_L}^j}\alpha, (\zeta{\pi_L}^j)^{-1}\right)_{\rho,L,n} \tag{5.12}$$

$$= -j\left(\frac{\zeta{\pi_L}^j}{1-\zeta{\pi_L}^j}\alpha, \pi_L\right)_{\rho,L,n}. \tag{5.13}$$

Let $m\geq\max\{N(\rho,\frac{\zeta{\pi_L}^j}{1-\zeta{\pi_L}^j}\alpha), \frac{q}{q-1}(2n+\frac{1}{2m_0})\}$ and let $r_m\in\mathcal{O}_H\{\{\tau\}\}$ be the invertible power series defined in Proposition 4.3. Let ρ' be the formal Drinfeld module defined by $\rho'_a=r_m\circ\rho_a\circ r_m^{-1}$ for all $a\in\mathcal{O}_K$. Hence, by Proposition 2.2 (vi), we have

$$\left(\frac{\zeta{\pi_L}^j}{1-\zeta{\pi_L}^j}\alpha,\pi_L\right)_{\rho,L,n} = r_m^{-1}\left(\left(r_m\left(\frac{\zeta{\pi_L}^j}{1-\zeta{\pi_L}^j}\alpha\right),\pi_L\right)_{\rho',L,n}\right). \tag{5.14}$$

Moreover, by Remark 5.3, we have

$$N\left(\rho,\frac{\zeta{\pi_L}^j}{1-\zeta{\pi_L}^j}\alpha\right) = N\left(\rho', r_m\left(\frac{\zeta{\pi_L}^j}{1-\zeta{\pi_L}^j}\alpha\right)\right).$$

Hence, since ρ' satisfies $(x,x)_{\rho',E_\rho^m,m}=0$, then by Lemma 5.4, there exists a prime π_n of L such that

$$(5.15)\qquad \left(r_m\left(\frac{\zeta{\pi_n}^j}{1-\zeta{\pi_n}^j}\alpha\right),\pi_n\right)_{\rho',L,n}=\left[r_m\left(\frac{\zeta{\pi_n}^j}{1-\zeta{\pi_n}^j}\alpha\right),\pi_n\right]_{\rho',L,n}.$$

Hence, if we put $\pi_L=\pi_n$, we get

$$(5.16)\qquad (\alpha,1-\zeta{\pi_n}^j)_{\rho,L,n}=-jr_m^{-1}\left(\left[r_m\left(\frac{\zeta{\pi_n}^j}{1-\zeta{\pi_n}^j}\alpha\right),\pi_n\right]_{\rho',L,n}\right).$$

By Proposition 4.10 (ii), (5.16) is equal to

(5.17)
$$-j\left[\frac{\zeta{\pi_n}^j}{1-\zeta{\pi_n}^j}\alpha,\pi_n\right]_{\rho,L,n}=\frac{-j}{\eta^n}\mathrm{T}_{L|K}\left(\frac{\zeta{\pi_n}^j}{1-\zeta{\pi_n}^j}\times\alpha\times\delta_{v_n}(\pi_n)\right)\cdot_\rho v_n$$

$$(5.18)\qquad =\frac{1}{\eta^n}\mathrm{T}_{L|K}\left(\frac{-j\zeta{\pi_n}^j}{1-\zeta{\pi_n}^j}\times\alpha\times\frac{t'(v_n)}{\pi_n}\right)\cdot_\rho v_n,$$

where (5.17) is deduced from Lemma 4.11, and $t(X)\in\mathcal{O}_H((X))$ satisfies $t(v_n)=\pi_n$. Since $1-\zeta(t(v_n))^j=1-\zeta{\pi_n}^j$, we have

$$(5.19)\qquad \delta_{v_n}(1-\zeta{\pi_n}^j)=\frac{-j\zeta{\pi_n}^{j-1}t'(v_n)}{1-\zeta{\pi_n}^j},$$

and thus, (5.18) is equal to $\frac{1}{\eta^n}\mathrm{T}_{L|K}(\alpha\delta_{v_n}(1-\zeta{\pi_n}^j))\cdot_\rho v_n$ which is equal to $\frac{1}{\eta^n}\mathrm{T}_{L|K}(\lambda_\rho(\alpha)\delta_{v_n}(1-\zeta{\pi_n}^j))\cdot_\rho v_n$ by Lemma 4.11. Hence,

$$(5.20)\qquad (\alpha,1-\zeta{\pi_n}^j)_{\rho,L,n}=[\alpha,1-\zeta{\pi_n}^j]_{\rho,L,n}.\qquad\square$$

Proposition 5.7. Let $\alpha\in\mathfrak{p}_L$ such that $\mu(\alpha)\geq\frac{nm_0}{q}+\frac{1}{q-1}+\frac{1}{q^{nm_0}(q-1)}$ and β a unit in $L^\times$. Then

$$(\alpha,\beta)_{L,n}=[\alpha,\beta]_{L,n}=\frac{1}{\eta^n}\mathrm{T}_{L|K}(\lambda_\rho(\alpha)\delta_{v_n}(\beta))\cdot_\rho v_n.$$

Proof. By Proposition 4.3, there exists an invertible power series $r\in\mathcal{O}_H\{\{\tau\}\}$ such that

$$\prod_{\omega\in W_\rho^n}(X-\omega)=r\circ\rho_{\eta^n}(X).$$

Let ρ' be the formal Drinfeld module defined by $\rho'_a=r\circ\rho_a\circ r^{-1}$ for all $a\in\mathcal{O}$. Then, by Lemma 4.4 we have $(x,x)_{\rho',E_\rho^n,n}=0$. Hence, by Lemma 5.6 for ρ', we have

$$(\alpha,\beta)_{\rho,L,n}=r^{-1}((r(\alpha),\beta)_{\rho',L,n})=r^{-1}([r(\alpha),\beta]_{\rho',L,n})=[\alpha,\beta]_{\rho,L,n}.\qquad\square$$

Proposition 5.8. Let $\alpha \in \mathfrak{p}_L$ such that $\mu(\alpha) \geq \frac{nm_0}{q} + \frac{1}{q-1} + \frac{1}{q^{nm_0}(q-1)}$ and let β be a prime of L, then

$$(\alpha, \beta)_{L,n} = [\alpha, \beta]_{L,n} = \frac{1}{\eta^n} \mathrm{T}_{L|K}(\lambda_\rho(\alpha)\delta_{v_n}(\beta)) \cdot_\rho v_n.$$

Proof. Let $m \geq \max\{N(\rho, \alpha), \frac{q}{q-1}(2n + \frac{1}{2m_0})\}$ and let $r_m \in \mathcal{O}_H\{\{\tau\}\}$ be the invertible power series defined in Proposition 4.3. Let ρ' be the formal Drinfeld module defined by $\rho'_a = r_m \circ \rho_a \circ r_m^{-1}$ for all $a \in \mathcal{O}_K$. Thus by Lemma 4.4, we have $(x, x)_{\rho', E_\rho^m, m} = 0$. Hence, by Lemma 5.4, there exists a prime π_n of L satisfying $(r_m(\alpha), \pi_n)_{\rho', L, n} = [r_m(\alpha), \pi_n]_{\rho', L, n}$. Then we can write $\beta = u\pi_n$ for a unit $u \in L$. Hence,

$$(\alpha, \beta)_{\rho,L,n} = (\alpha, u\pi_n)_{\rho,L,n} = (\alpha, u)_{\rho,L,n} + (\alpha, \pi_n)_{\rho,L,n}. \tag{5.21}$$

By Proposition 5.7, we have $(\alpha, u)_{\rho,L,n} = [\alpha, u]_{\rho,L,n}$. On the other hand, by Proposition 2.2 (vi), we have

$$(\alpha, \pi_n)_{\rho,L,n} = r_m^{-1}((r_m(\alpha), \pi_n)_{\rho',L,n}) = r_m^{-1}([r_m(\alpha), \pi_n]_{\rho',L,n}), \tag{5.22}$$

the last equality being deduced from Remark 5.3 and Lemma 5.4. Hence, by Proposition 4.10 (ii), we have

$$(\alpha, \beta)_{\rho,L,n} = [\alpha, u]_{\rho,L,n} + [\alpha, \pi_n]_{\rho,L,n} = [\alpha, \beta]_{\rho,L,n}. \tag{5.23}$$

□

Combining Proposition 5.7 and Proposition 5.8, we obtain

Theorem 5.9. Let $\alpha \in \mathfrak{p}_L$ such that $\mu(\alpha) \geq \frac{nm_0}{q} + \frac{1}{q-1} + \frac{1}{q^{nm_0}(q-1)}$ and $\beta \in L^\times$. We have

$$(\alpha, \beta)_{\rho,L,n} = [\alpha, \beta]_{\rho,L,n} = \frac{1}{\eta^n} T_{E_\rho^n|K}(\lambda_\rho(\alpha)\delta_{v_n}(\beta)) \cdot_\rho v_n.$$

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References

[1] B. Anglès, "On explicit reciprocity laws for the local Carlitz–Kummer symbols", *J. Number Theory* **78** (1999), no. 2, p. 228-252.

[2] E. Artin & H. Hasse, "Die beiden Ergänzungssätze zum reziprozitätsgesetz der l^n-ten potenzreste im körper der l^n-ten Einheitswurzeln", *Abh. Math. Semin. Univ. Hamb.* **6** (1928), no. 1, p. 146-162.

[3] F. Bars & I. Longhi, "Reciprocity laws à la Iwasawa–Wiles", in *Proceedings of the "Segundas Jornadas de Teoría de Números"*, Biblioteca de la Revista Matemática Iberoamericana, Revista Matemática Iberoamericana, 2008, p. 39-62.

[4] ———, "Coleman's power series and Wiles' reciprocity for rank 1 Drinfeld modules", *J. Number Theory* **129** (2009), no. 4, p. 789-805.

[5] V. G. Drinfeld, "Elliptic modules", *Mat. Sb., N. Ser.* **94(136)** (1974), p. 594-627, 656.

[6] J. Flórez, "Explicit reciprocity laws for higher local fields", *J. Number Theory* **213** (2020), p. 400-444.

[7] ———, "The norm residue symbol for higher local fields", *J. Number Theory* (2022), p. 157-218.

[8] K. Iwasawa, "On some modules in the theory of cyclotomic fields", *J. Math. Soc. Japan* **16** (1964), p. 42-82.

[9] ———, *Local class field theory*, Oxford Science Publications, Clarendon Press, 1986, viii+155 pages.

[10] V. A. Kolyvagin, "Formal groups and the norm residue symbol", *Izv. Akad. Nauk SSSR, Ser. Mat.* **43** (1979), no. 5, p. 1054-1120, 1198.

[11] H. Oukhaba, "On local fields generated by division values of formal Drinfeld modules", *Glasg. Math. J.* **62** (2020), no. 2, p. 459-472.

[12] M. Rosen, "Formal Drinfeld modules", *J. Number Theory* **103** (2003), no. 2, p. 234-256.

[13] A. Wiles, "Higher explicit reciprocity laws", *Ann. Math.* **107** (1978), no. 2, p. 235-254.

Marwa Ala Eddine
Université de Franche-Comté
Laboratoire de Mathématique de Besançon
25030 Besançon, France
E-mail: `marwaalaeddine1@gmail.com`