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Rational points on symmetric squares of constant algebraic curves over function fields

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RÉSUMÉ. On considère des courbes projectives lisses $C/\mathbb{F}$ sur un corps fini et leurs carrés symétriques $C^{(2)}$. Pour un corps de fonctions global $K/\mathbb{F}$, nous étudions les points K-rationnels de $C^{(2)}$. Nous décrivons les points adéliques de $C^{(2)}$ survivant à la descente de Frobenius et décrivons comment les points K-rationnels y sont situés. Nos méthodes conduisent également à une borne explicite pour le nombre de points K-rationnels de $C^{(2)}$ satisfaisant une condition supplémentaire. Certains de nos résultats s'appliquent à des sous-variétés constantes arbitraires de variétés abéliennes, mais nous produisons des exemples qui montrent que certaines des nos conclusions les plus fortes ne s'étendent pas.

ABSTRACT. We consider smooth projective curves $C/\mathbb{F}$ over a finite field and their symmetric squares $C^{(2)}$. For a global function field $K/\mathbb{F}$, we study the K-rational points of $C^{(2)}$. We describe the adelic points of $C^{(2)}$ surviving Frobenius descent and how the K-rational points fit there. Our methods also lead to an explicit bound on the number of K-rational points of $C^{(2)}$ satisfying an additional condition. Some of our results apply to arbitrary constant subvarieties of abelian varieties, however we produce examples which show that not all of our stronger conclusions extend.

1. Introduction

Let C be a smooth, geometrically irreducible, proper curve over a finite field $\mathbb{F}$, and let $K/\mathbb{F}$ be a global function field. In this paper, we study the set of K-rational points of the symmetric square $X = C^{(2)}$ and explore the information that can be obtained about $X(K)$ from knowledge of only certain finite coverings of X. We also discuss to what extent our results extend to the case of a constant subvariety of a constant abelian variety.

To provide some context, we briefly recall what is known about the K-rational points of a nice curve C/K, assumed to be of genus $g \geq 2$,

embedded in its Jacobian J, and with $\text{III}(J)$ finite. In [12, Theorem 4.4] the Brauer–Manin obstruction is shown to be the only obstruction to weak approximation for curves over K, i.e., $\overline{C(K)} = C(\mathbb{A}_K)^{\text{Br}}$, with C satisfying additional but general hypotheses which exclude the case of constant curves.

However, when C is a nice constant curve (or abelian variety), the Brauer–Manin obstruction is equivalent to descent under finite abelian group schemes, and in particular $C(\mathbb{A}_K)^{\text{Br}} = C(\mathbb{A}_K)^{\text{ab}} = C(\mathbb{A}_K)^{\text{isog}}$ [5, Section 2]. Thus in [6], Creutz and Voloch study $C(K)$ via descent under pull-backs of étale isogenies on $\text{Jac}(C)$ defined over $\mathbb{F}$ together with Frobenius descent. It is shown that $C(\mathbb{A}_K)^{F^\infty} = C(K) \cup C(\mathbb{A}_{K,\mathbb{F}})$, that is, any adelic point of C unobstructed by descent under the nth iterate of the $\mathbb{F}$-Frobenius isogeny on $F\colon J \to J$ for all $n \geq 1$ is global, unless it is arbitrarily divisible by Frobenius [6, Theorem 1.2] and the latter consists of adelic points with constant components. Moreover, writing $K = \mathbb{F}(D)$ for a nice curve D over $\mathbb{F}$, if the genus of D is less than that of C, then $C(\mathbb{A}_K)^{\text{Br}} = C(K) = C(\mathbb{F})$ [6, Theorem 1.5].

Less is known about the precise relationship between finite descent obstructions and the Brauer–Manin obstruction for higher dimensional constant subvarieties X of abelian varieties, (see Section 3.5 for further remarks). Still, it is reasonable to ask what information about $X(K)$ can be deduced from descent under torsors arising as pullbacks from isogenies on the abelian variety, and specifically from Frobenius descents. Moreover, Frobenius descent is interesting in its own right since it can be viewed in analogy to the differential descent obstruction in characteristic 0 [10, 15].

1.1. Results for symmetric squares of constant curves over function fields.
Let C and D be smooth, proper curves over a finite field $\mathbb{F}$ and set $K = \mathbb{F}(D)$. We note that C may be embedded into its Jacobian J since, by the Hasse–Weil bounds, C has a 0-cycle of degree 1 defined over $\mathbb{F}$. We study the K-rational points on the image of the symmetric square $C^{(2)}$ in J, parametrizing effective divisors of degree 2 on C.

An adelic point on a variety X surviving Frobenius descent defines a class in $H^1(K, \ker(F))$ which can be related to tangent spaces of the corresponding variety X, see Section 3.1. Thus the setting of symmetric powers of a curve is particularly amenable to studying the information captured by the Frobenius descent obstruction since the local geometry of $X = C^{(2)}$ is well understood. In particular, there is an explicit description of the projectivized tangent space to points of X in terms of secant lines on the canonically embedded curve in $\mathbb{P}_{\mathbb{F}}^{g-1}$. This perspective allows us to give a precise description of the points of $C^{(2)}$ surviving Frobenius descent in terms of geometric information about C. Specifically, we prove:

Theorem 1.1 (Theorem 3.5). *Let C be a smooth curve of genus g and assume that C has no g_2^1, g_3^1, nor g_4^1. Let $Z \subset X(\mathbb{A}_K)$ be the subset of adelic points represented by degree 2 effective divisors on C of the form $y + (Q_v)$ where $y \in C(K)$ and $(Q_v) \in C(\mathbb{A}_{K,\mathbb{F}})$. Then the subset of adelic points of X unobstructed by Frobenius descent is*

$$X(\mathbb{A}_K)^{F^\infty} = X(K) \cup X(\mathbb{A}_{K,\mathbb{F}}) \cup Z.$$

The first two sets in the description of the points unobstructed by Frobenius descent, namely $X(K)$ and $X(\mathbb{A}_{K,\mathbb{F}})$, are entirely analogous to the corresponding sets in the case of curves, proved in [6] and discussed above. The set Z is a mix between the two, that is, adelic divisors of degree two which are a sum of a global point on C and an adelic point of C with constant components. The reason for the assumption on the non-existence of special divisors of low degree will become clear in the discussion of the proof of the Theorem.

The proof of Theorem 3.5 allows us to determine an explicit bound on the number of K-rational points of X satisfying certain geometric conditions, namely that the corresponding curves are not everywhere tangent to the "horizontal" direction. This is made precise in the discussion immediately preceding Theorem 3.6, which gives the explicit statement. As far as the authors know, even the corresponding finiteness statement is new.

While a few questions remain, the situation for the symmetric square is fairly clear. In contrast, for arbitrary constant surfaces (or higher dimensional varieties) contained in abelian varieties, the situation is murkier. We give constructions in Section 3.3 of non-global points unobstructed by Frobenius descent.

2. Notation and Background

Let $\mathbb{F}$ be a finite field of characteristic p and let D be a smooth, proper curve over $\mathbb{F}$. Set $K = \mathbb{F}(D)$. The places of K are in bijection with the set D^1 of closed points of D. Given a closed point $v \in D^1$, let $K_v, \mathcal{O}_v$, and $\mathbb{F}_v$ denote the corresponding completion of K, ring of integers, and residue field, respectively.

Throughout this note we shall let X denote a proper geometrically integral $\mathbb{F}$-variety and $X_K := X \otimes_K \mathbb{F}$. Recall that a variety over K is called *constant* if it is isomorphic to the base change of a variety defined over $\mathbb{F}$ and is called *isotrivial* if it becomes constant over some finite extension of K. Note that we can make the identifications

$$X_K(K) = X(K) = \text{Mor}_\mathbb{F}(\text{Spec}(K), X) = \text{Mor}_\mathbb{F}(D, X),$$

therefore when we consider a K-rational point of X, it will be convenient to also keep in mind the existence of a morphism $f \colon D \to X$.

2.1. Adelic and reduced adelic points. The *adele ring* of K is the K-algebra defined by the restricted direct product $\mathbb{A}_K := \prod_{v \in D^1} (K_v : \mathcal{O}_v)$, where the product runs over the closed points of D. Since X is proper, we may identify $X(\mathbb{A}_K) = X_K(\mathbb{A}_K) = \prod_{v \in D^1} X(K_v)$.

The *reduced adele ring* of K is the $\mathbb{F}$-algebra $\mathbb{A}_{K,\mathbb{F}} = \prod_{v \in D^1} \mathbb{F}_v$, which is an $\mathbb{F}$-subalgebra of $\mathbb{A}_K$. The set $X(\mathbb{A}_{K,\mathbb{F}}) := \mathrm{Mor}_\mathbb{F}(\mathrm{Spec}(\mathbb{A}_{K,\mathbb{F}}), X)$ of reduced adelic points on X is a closed subset of $X(\mathbb{A}_K)$ which can be identified with $\prod_{v \in D^1} X(\mathbb{F}_v)$. When the latter is endowed with the product of the discrete topologies it coincides with the subspace topology determined by $X(\mathbb{A}_{K,\mathbb{F}}) \subset X(\mathbb{A}_K)$. Such points play an important role in the description of various descent obstructions on constant curves and are still necessary to consider for higher dimensional subvarieties of constant abelian varieties.

2.2. Descent Obstructions. Consider the category $\mathrm{Cov}(X_K)$ of X_K-torsors under finite group schemes G over K for the flat topology; for a detailed account, see [14, Section 4]. We say that an adelic point $P \in X_K(\mathbb{A}_K) = X(\mathbb{A}_K)$ is *unobstructed by*, or *survives*, the torsor $(X', G) \in \mathrm{Cov}(X_K)$ if it lifts to an adelic point on some twist of (X', G). Equivalently, P survives (X', G) if the element of $\prod_v H^1(K_v, G)$ given by evaluating (X', G) at P lies in the image of $H^1(K, G)$ under the usual diagonal map in bottom row of the following diagram.

$$
\begin{array}{ccc}
X(K) & \longrightarrow & H^1(K, G) \\
\downarrow & & \downarrow \\
X(\mathbb{A}_K) & \longrightarrow & \prod_v H^1(K_v, G)
\end{array}
$$

Throughout this note, we restrict our attention to subvarieties X of an abelian variety $A/\mathbb{F}$. Furthermore, we will consider the subset of torsors in $\mathrm{Cov}(X_K)$ which arise as pullbacks of isogenies $\phi \colon A' \to A$ defined over $\mathbb{F}$.

The Frobenius isogeny on A will be of particular importance in subsequent sections. Since A is defined over $\mathbb{F}$, recall that there exists an $\mathbb{F}$-Frobenius morphism $A \to A$ defined by the n-fold composition (where $\mathbb{F} = \mathbb{F}_{p^n}$) of $\mathbb{F}_p$-Frobenius morphism with itself. Since we will be interested in isogenies with target A, it is convenient to consider the $\mathbb{F}_p$-Frobenius morphism denoted $F \colon A^{(1/p)} \to A^{(p^n)} \cong A$, given by raising coordinates to their pth powers and, Zariski locally, defining equations for A are obtained from those of $A^{(1/p)}$ by taking pth powers.

Specifically, we will consider the torsor $(X', \ker(F))$ under the finite abelian $\mathbb{F}$-group scheme $G = \ker(F)$. Recall that for a separable extension L/K, the long exact sequence in cohomology arising from the short exact sequence of K-group schemes

$$
0 \to \ker(F) \to A^{(1/p)} \xrightarrow{F} A \to 0
$$

gives rise to a homomorphism $\mu : A(L)/F(A^{(1/p)}(L)) \to H^1(L, \ker F)$. Thus, to say an adelic point $(x_v) \in X(\mathbb{A}_K)$ survives F-descent means there exists a global element in $H^1(K, \ker F)$ such that its image in $H^1(K_v, \ker F)$ coincides with the image of x_v under μ. To discern further information about adelic points unobstructed by Frobenius descent, we make use of the following lemma, originally due to Artin and Milne, [3], which describes μ explicitly.

Lemma 2.1 ([6, Lemma 3.1]). *There is a functorial (on separable extensions L/K) injection $H^1(L, \ker F) \to \Omega_{L/\mathbb{F}}^{\oplus g}$ such that the induced map $A(L)/F(A^{(1/p)}(L)) \to \Omega_{L/\mathbb{F}}^{\oplus g}$, obtained by composing with μ above, is given by $x \mapsto x^*(\omega_1, \ldots, \omega_g)$ where $\omega_1, \ldots, \omega_g$ is a basis of holomorphic differentials on A.*

We note that if L/K is a finite separable extension, then $\Omega_{L/\mathbb{F}}$ is a one-dimensional vector space over L and we can then identify the projective space $\mathbb{P}(\Omega_{L/\mathbb{F}}^{\oplus g})$ with $\mathbb{P}^{g-1}(L)$.

3. Frobenius Descent Obstruction

In this section, we consider subvarieties X of constant abelian varieties A containing adelic points $(x_v) \in X(\mathbb{A}_K)$ unobstructed by Frobenius descent. When X is a smooth, proper, geometrically integral constant curve over K embedded in its Jacobian J, Creutz and the second author proved that any adelic point on X surviving a torsor $(X', \ker(F))$ must be global, except possibly if the torsor is $F^{-1}(X) \subset J$, where F denotes the $\mathbb{F}_p$-Frobenius isogeny on J [6, Lemma 3.2]. We show that this need not be true for higher dimensional subvarieties. To exhibit such counterexamples, we must first consider a modified Gauss map for subvarieties of abelian varieties.

3.1. The Gauss map. Suppose that X is a smooth subvariety of an abelian variety A of dimension g. The Frobenius descent map, denoted by μ in [6, Lemma 3.1], assigns to a point in $A(K)$, interpreted as a map $D \to A$ over $\mathbb{F}$, a map $\Omega_{A/K} \to \Omega_{K/\mathbb{F}}^{\oplus g}$. As before, to say that a point $(x_v) \in X(\mathbb{A}_K)$ survives Frobenius descent means that there exists an element ξ of $\Omega_{K/\mathbb{F}}^{\oplus g}$ such that for each place v of K, one has $\mu_{K_v}(x_v) = \xi \in \Omega_{K_v/\mathbb{F}}^{\oplus g}$ for all v. In this case, if $\xi \neq 0$ we let $\gamma \colon D \to \mathbb{P}_{\mathbb{F}}^{g-1}$ denote the map corresponding to the point $[\xi] \in \mathbb{P}_{\mathbb{F}}^{g-1}$. If $\xi = 0$, the adelic point factors through Frobenius.

If X has finite stabilizer in A, Abramovich proved that the Gauss map $X \to \mathrm{Gr}(\dim X, \dim A)$ is finite [1, Theorem 4]. In this case, the Gauss map is the usual one, defined by sending a smooth point $x \in X$ to the point on the Grassmannian representing the tangent space of X at x inside of the tangent space to A. This result holds in arbitrary characteristic. However,

in the present context, we are impelled to use a variant of the Gauss map h which has target $\mathbb{P}_{\mathbb{F}}^{g-1}$ in order to meaningfully compare with the Frobenius descent map μ. As a result, degeneracies of h arise in positive characteristic that are not present for the usual Gauss map. These lead to interesting examples, see Section 3.4.

In order to define h, we let P be the projectivized tangent bundle of X, the variety with a map $P \to X$ such that the fiber at x in X is the set of 1-dimensional subspaces of $T_x X$. Note that $\dim P = 2 \dim X - 1$. By translating such subspaces to the tangent space to A at the origin, we obtain a map $h\colon P \to \mathbb{P}_{\mathbb{F}}^{g-1} = \mathbb{P}(T_0 A)$. If A is the Jacobian of a non-hyperelliptic curve C, we identify the projective space associated to the tangent space of A at the origin with the ambient $\mathbb{P}_{\mathbb{F}}^{g-1}$ in which C is canonically embedded. Each $x_v \in X(K_v)$ gives a map $\mathrm{Spec}(\mathcal{O}_v) \to X$. If this map has a non-zero derivative, it induces a map $\mathrm{Spec}(\mathcal{O}_v) \to P$. That is, in favorable circumstances, from an adelic point of X one obtains an adelic point on P, and hence an adelic point on $\mathbb{P}_{\mathbb{F}}^{g-1}$ by composing with the map h above for each place v of K.

Remark 3.1. In the case when $X = C$ is a non-hyperelliptic curve with an embedding into its Jacobian, the map h coincides with the canonical embedding $\phi\colon C \to \mathbb{P}_{\mathbb{F}}^{g-1}$. The existence of a point $(x_v) \in C(\mathbb{A}_K)$ unobstructed by Frobenius descent gives that $\gamma(D) = C$. When X is a surface or higher dimensional variety, we shall see that it is often less straightforward to relate $\gamma(D)$ to the image of h in $\mathbb{P}_{\mathbb{F}}^{g-1}$.

3.2. Frobenius descent on the symmetric square of a curve. In the context of higher dimensional subvarieties of constant abelian varieties, a reasonable setting in which to examine the Frobenius descent obstruction is the image of symmetric powers of a curve in its Jacobian. Let C be a smooth non-hyperelliptic curve of genus $g > 3$, and let X be the image in $J := \mathrm{Jac}(C)$ of $C^{(2)}$, parametrizing effective divisors of degree 2 on C.

As above, let P be the projectivized tangent bundle of X. The Riemann–Kempf Singularity Theorem [9],[7, Section 2.7] relates the local geometry of X, specifically the tangent cones at various points, to the corresponding geometry of the linear systems on the canonical curve of C in $\mathbb{P}_{\mathbb{F}}^{g-1}$. In particular, the projectivized tangent space to $C^{(2)}$ at $P_0 + Q_0$ is identified with the linear span of P_0 and Q_0 in $\mathbb{P}_{\mathbb{F}}^{g-1}$, i.e., the secant line between P_0 and Q_0 on the canonical curve. Thus, we have a map of threefolds, $h\colon P \to S \subset \mathbb{P}_{\mathbb{F}}^{g-1}$ from P to the secant variety S of the canonically embedded curve and we view $C \subset S$. Note that this is the same map h as above, but with the target restricted to S.

We will show below in Lemma 3.2 that the map h is birational, hence there exists an open set $U \subset S$ where h has an inverse h^{-1}. The set U can

be interpreted as the set of points R in S for which there exists a *unique* pair $\{Q, Q'\}$ of distinct points of C with $R \in \overline{QQ'}$, the secant line between Q and Q'. The map h fails to be bijective at the locus of points where two secant lines intersect. Observe that after fixing a point $P_0 \in C$ and varying another point Q, the line $\overline{P_0Q}$ passes through P_0, hence $h^{-1}(P_0)$ contains a curve. By varying P_0, we see that $h^{-1}(C)$ is a surface in P. Moreover $C \cap U = \emptyset$.

Lemma 3.2. *Let C be a smooth non-hyperelliptic curve of genus $g > 3$. Suppose that P_1, P_2, P_3, P_4 are distinct points on C. If two secant lines between disjoint pairs of the images of these points on the canonical curve in $\mathbb{P}_{\mathbb{F}}^{g-1}$ intersect, then $P_1 + P_2 + P_3 + P_4$ is a g_4^1. Moreover, if $g > 4$, the map h above is birational and if, in addition, C has no g_4^1, then h has an inverse in $S \setminus C$.*

Proof. Suppose that the lines $\overline{P_1P_2}$ and $\overline{P_3P_4}$ intersect at a point R in $\mathbb{P}_{\mathbb{F}}^{g-1}$. Then the geometric form of Riemann Roch [2, p. 12] implies that the divisor $P_1 + P_2 + P_3 + P_4$ on C is a g_4^1. This proves the first part of the lemma.

By Martens's theorem ([11], [2, Theorem IV.5.1]) under our assumptions, the set of g_4^1's of C is at most one dimensional, so the set U above is open and dense in S and h is invertible on U, hence birational. Finally, if C has no g_4^1 then $U = S \setminus C$. $\square$

Remark 3.3. A more general statement that implies that h is birational as in Lemma 3.2 is [2, Exercise VIII.A.2]

We shall give conditions on C that determine when an adelic point of X surviving Frobenius descent is a global point.

Lemma 3.4. *Let C be a smooth non-hyperelliptic curve of genus g and let X be the image of $C^{(2)}$ in $J = \mathrm{Jac}(C)$. Let $(x_v) \in X(\mathbb{A}_K)$ be an adelic point with $\mu_{K_v}(x_v) \neq 0$ for some v which survives Frobenius descent, and write $\mu(x_v) = \xi$ for some $\xi \in H^1(K, \ker F)$. Let $\gamma \colon D \to \mathbb{P}_{\mathbb{F}}^{g-1}$ be the map corresponding to $[\mu_{K_v}(x_v)] = [\xi] \in \mathbb{P}_{\mathbb{F}}^{g-1}(K)$. If $\gamma(D) \cap U \neq \emptyset$, then $(x_v) \in X(K)$.*

Proof. Since any adelic point $(x_v) \in X(\mathbb{A}_K)$ lifts to an adelic point of P, by composing with the map $h \colon P \to S \subset \mathbb{P}_{\mathbb{F}}^{g-1}$, we obtain an adelic point of S. By assumption we have $\mu(x_v) = [\xi]$, so this implies that $\gamma(D) \subset S$.

If $\gamma(D) \subset S$ meets U, then we can form the map $\pi \circ h^{-1} \circ \gamma \colon D \to X$, where $\pi \colon P \to X$ is the projection. So γ comes from a point of $X(K)$. Indeed, this follows from the uniqueness condition in the description of U given above, which guarantees that h^{-1} is well defined. $\square$

If we further assume C has no g_3^1, then the condition that $\gamma(D)$ meets U is enough to conclude that the adelic point is global. Together with this

observation, we give the following description of adelic points surviving Frobenius descent.

Theorem 3.5. *Let C be a smooth curve of genus g and assume that C has no g_2^1, g_3^1, nor g_4^1. Let $Z \subset X(\mathbb{A}_K)$ be the subset of adelic points represented by degree 2 effective divisors on C of the form $y + (Q_v)$ where $y \in C(K)$ and $(Q_v) \in C(\mathbb{A}_{K,\mathbb{F}})$. Then the subset of adelic points of X unobstructed by Frobenius descent is*

$$X(\mathbb{A}_K)^{F^\infty} = X(K) \cup X(\mathbb{A}_{K,\mathbb{F}}) \cup Z.$$

Proof. Let $(x_v) \in X(\mathbb{A}_K)^{F^\infty}$ so that in particular, it is an adelic point surviving $(X', \ker F)$. Suppose first that $\mu_{K_v}(x_v) \neq 0$ for some v. As before, we write $\mu(x_v) = \xi \in H^1(K, \ker F)$ and let $\gamma \colon D \to \mathbb{P}_{\mathbb{F}}^{g-1}$ be the corresponding map.

If $\gamma(D) \neq C$, then by Lemma 3.2, since C contains no g_4^1, we have $\gamma(D) \cap U \neq \emptyset$. Thus by Lemma 3.4, we find $(x_v) \in X(K)$. If instead $\gamma(D) = C$ so that $\gamma(D)$ is disjoint from U, we claim that $(x_v) \in Z$. Since $\gamma(D) = C$, it in fact defines a point Q of $C(K)$. Therefore, if we write $(x_v) = (Q_v + Q_v')$, then the observation that $\mu_{K_v}(x_v) = \xi \neq 0$ tells us that Q is contained in the secant line $\overline{Q_v Q_v'}$. Since C has no g_3^1, C does not have any trisecant lines and hence no point of C is on a secant line through two other points of C. Thus, we recover from ξ a point Q of $C(K)$ which occurs in $\{Q_v, Q_v'\}$ for all v. One possibility is that $Q_v = Q_v' = Q$ for all v and the point $Q_v + Q_v'$ of X is global. Let us fix $Q_v = Q$ and let Q_v' be the other point, which may or may not be Q. Then we have an adelic point (Q_v') of C with a global value of μ. It is therefore either global itself (hence is Q), or is in the image of Frobenius (when $\mu(Q_v') = 0$) by [6]. In the latter case, we can write $Q_v' = F^n(R_v)$ for some $n \geq 1$ and $(R_v) \in C(\mathbb{A}_K)$. Since we assume that $(x_v) \in X(\mathbb{A}_K)^{F^\infty}$, we get that $(R_v) \in C(\mathbb{A}_K)^F$ so by [6], either (R_v) is global and, in turn, so are Q_v', x_v or (R_v) is in the image of Frobenius. This shows that either x_v is global or in Z, by [6, Theorem 1.2].

Otherwise, $\mu_{K_v}(x_v) = 0$ for all v, which implies $(x_v) \in F(J^{(1/p)}(\mathbb{A}_K))$ and thus lifts to a point $(y_v) \in X^{(1/p)}(\mathbb{A}_K)$. The proof that any non-global such (x_v) is in $X(\mathbb{A}_{K,\mathbb{F}})$ follows in the same manner as that of [6, Theorem 1.2], but we record it here for completeness. Since $\Sha(K, \ker(F)) = 0$, (x_v) does not lift to any nontrivial twist of $(X', \ker F)$. Hence $(y_v) \in X^{(1/p)}(\mathbb{A}_K)^{F^\infty}$. Since we can iterate this argument, either we find that (x_v) is global, or $(x_v) \in F^n(X(\mathbb{A}_K))$ for all $n \geq 1$, and hence in $X(\mathbb{A}_{K,\mathbb{F}})$. Thus, we have $X(\mathbb{A}_K)^{F^\infty} \subset X(K) \cup X(\mathbb{A}_{K,\mathbb{F}}) \cup Z$.

For the reverse containment, if $(x_v) \in Z$, then by definition $(x_v) = y + F(z_v)$ for some $y \in J(K)$ and $(z_v) \in F(J^{(1/p)}(\mathbb{A}_K))$. Hence the class of (x_v) in $\prod_v H^1(K_v, \ker F)$ is in the image of $H^1(K, \ker F)$ and thus survives F-descent. Finally, $X(\mathbb{A}_{K,\mathbb{F}}) \subset X(\mathbb{A}_{K,\mathbb{F}})^{F^\infty}$. $\qquad\square$

Let us call a point $y \in X(K)$ *horizontal* if the curve $D \to X$ corresponding to y is everywhere tangent to the horizontal curves $C \to X, P \mapsto P + P_0$ with P_0 fixed. The following estimate is a consequence of the above proof:

Theorem 3.6. *Let C be a smooth curve of genus g and assume that C has no g_2^1, g_3^1, nor g_4^1. The number of points of $X(K)$ which are neither horizontal nor in the image of Frobenius is bounded by $(p^r - 1)/(p - 1)$ where r is the Mordell–Weil rank of $J(K)$.*

Proof. Let $y \in X(K)$ with $\mu(y) \neq 0$ and suppose that the image of the corresponding map $\gamma \colon D \to \mathbb{P}_{\mathbb{F}}^{g-1}$ is not contained in C. Then the proof of Theorem 3.5 shows that y can be uniquely recovered from γ. Indeed, for generic $P \in D, \gamma(P)$ is in an unique secant line $\overline{QQ'}$ to C and thus $y(P) = \{Q, Q'\} \in X$.

We note that the group $J(K)/F(J^{(1/p)}(K))$ is a quotient of $J(K)/pJ(K)$ which is a $\mathbb{F}_p$-module. Given $y \in X(K) \subset J(K)$ and $c \in \mathbb{F}_p$, we have $\mu(cy) = c\mu(y)$. Thus for $c \neq 0$, we have $[\mu(cy)] = [\mu(y)] \in \mathbb{P}_{\mathbb{F}}^{g-1}(K)$ and we find that γ depends only on $\mu(y)$ up to $\mathbb{F}_p$-scalars in the image of $(J(K)/F(J^{(1/p)}(K))) \setminus \{0\}$. Otherwise, again from the proof of Theorem 3.5, if the image of $\gamma \colon D \to \mathbb{P}_{\mathbb{F}}^{g-1}$ is contained in C, then this gives rise to a point $Q \in C(K)$ and either $y = 2Q$, which can also be recovered from $\mu(y)$ up to $\mathbb{F}_p$-scalars, or Q is in the image of Frobenius which implies y is horizontal.

In the former case, we can also uniquely recover the point from the class in the image of $(J(K)/F(J^{(1/p)}(K))) \setminus \{0\}$ but the classes we obtain are distinct from those of the first part of the argument as these have $\gamma(D) \subset C$, while for the others, $\gamma(D) \not\subset C$.

Since, as noted above, $J(K)/F(J^{(1/p)}(K))$ is a quotient of $J(K)/pJ(K)$ and Frobenius is an isomorphism on p-torsion, we have that the quotient $J(K)/F(J^{(1/p)}(K))$ has order at most p^r. Thus since we have shown that any point of y of $X(K)$ which is neither horizontal nor in the image of Frobenius is completely determined up to $\mathbb{F}_p$-scalars by the corresponding class in $J(K)/F(J^{(1/p)}(K))$, we find that that the total number of such points is bounded by $(p^r - 1)/(p - 1)$. $\qquad\square$

We note that there can be infinitely many horizontal points, since if $P, Q \in C(K), P + F_q^n(Q) \in X$ is horizontal for all $n \geq 1$, where $F_q \colon C \to C$ is the $\mathbb{F}_q$-Frobenius. Also, some hypothesis is needed in both Theorems 3.5 and 3.6. If D is an elliptic curve and $C \to D$ is a map of degree 2 (i.e. C is bielliptic and has a g_4^1), then the fibers of the map $C \to D$ define a map $D \to X$. This map can be composed with isogenies of D to give rise to infinitely many maps $D \to X$, that is, points of $X(K)$. These points are not horizontal. We can also construct adelic points, unobstructed by

Frobenius descent, by taking suitable varying local components from this infinite collection of global points. See Example 3.12.

3.3. Frobenius descent on arbitrary subvarieties. We now consider subvarieties X of constant abelian varieties A with $\dim A \geq 3$, and no longer assume that X is the symmetric square of a curve. We will construct X in such a way that guarantees the existence of non-global adelic points unobstructed by Frobenius descent. As before, we fix a smooth curve D over $\mathbb{F}$, and let $K = \mathbb{F}(D)$.

Lemma 3.7. *Let C_1 and C_2 be two curves contained in A. Then there exists an irreducible divisor X of A containing both C_1 and C_2.*

Proof. Such an X exists since if V is a closed subset of codimension at least 2 in A containing C_1 and C_2, then by [8, Theorem 1.1] (and independently [16, Theorem 1.2]), a finite field analogue of a Bertini-type theorem for hypersurface sections containing a given subscheme, one can find an irreducible divisor of A containing V. $\qquad\square$

Proposition 3.8. *Assume there exist points $P_1, Q \in A(K)$ and let $P_2 := P_1 + F(Q)$, where F denotes the Frobenius isogeny on A. Let D_1 and D_2 be the images of the corresponding maps in $\mathrm{Mor}_{\mathbb{F}}(D, A)$. Then there exists a subvariety X of A containing D_1 and D_2. Moreover, there exist adelic points $(P_v) \in X(\mathbb{A}_K)$ unobstructed by Frobenius descent.*

Proof. We apply Lemma 3.7 to A and the curves D_1 and D_2 to obtain such an X. To exhibit adelic points on X_K unobstructed by Frobenius descent, we separate the places of K into two disjoint nonempty sets $S_1 \sqcup S_2$. Define the adelic point $(P_v) \in X(\mathbb{A}_K)$ by $P_v := P_i$ if $v \in S_i$. Let $z_v \in A(K_v)$ be such that $z_v = 0$ when $v \in S_1$ and $z_v = Q$, defined above, when $v \in S_2$. Then, for each place v we have $P_v = P_1 + F(z_v)$, since if $v \in S_1$ then $P_v = P_1 + F(0)$ and if $v \in S_2$ then $P_v = P_2 = P_1 + F(Q)$. Since both S_1 and S_2 are nonempty, $(P_v) \in X(\mathbb{A}_K)$ is not a global point. Moreover, by construction, (P_v) is unobstructed by Frobenius descent. $\qquad\square$

However, we claim that any such adelic point is obstructed by a finite étale torsor.

Proposition 3.9. *The point $(P_v) \in X(\mathbb{A}_K)$ from Proposition 3.8 is obstructed by a torsor obtained from the pullback of the multiplication-by-n isogeny on A for some integer n.*

Proof. Assume for the sake of contradiction that (P_v) is unobstructed for each integer n. Then $(P_v) = R + n(y_v)$ for some $R \in A(K)$ and $(y_v) \in X(\mathbb{A}_K)$. Since P_v is either P_1 or P_2, both of which are global points in $J(K)$, this equality implies that for each v, the point $y_v \in A(K)$ as long as n and $p = \mathrm{char}(K)$ are coprime. This would imply that $P_1 - P_2$ is in the

image of the multiplication by n map for each such n. But the Mordell–Weil group of A is finitely generated, so this cannot occur. $\qquad\square$

Proposition 3.10. *Assume that the map $h\colon P \to \mathbb{P}_{\mathbb{F}}^{g-1}$ is generically finite. Suppose that there exists $(x_v) \in X(\mathbb{A}_K)$ which survives Frobenius descent and such that $\mu(x_v) \neq 0$ for some v. Let $\gamma\colon D \to \mathbb{P}_{\mathbb{F}}^{g-1}$ be the map arising from the corresponding global point on A. If $\gamma(D)$ is contained in the open set where h is finite and $h^{-1}(\gamma(D))$ is irreducible, then $h^{-1}(\gamma(D))$ gives rise to a point in $X(K) \setminus X(\mathbb{F})$.*

Proof. As in the statement of Lemma 3.4 we have $\mu(x_v) = \mu(y) \neq 0$ for some $y \in J(K)$ and y gives us a map $D \to \mathbb{P}_{\mathbb{F}}^{g-1}$ and we let C denote the image of D in $\mathbb{P}_{\mathbb{F}}^{g-1}$ under this map. Under the stated hypotheses, $C' := h^{-1}(C)$ is an irreducible curve in P. Their fiber product $D' := C' \times_C D$ is curve which covers D and since C' is irreducible, so too is D'. As in the beginning of this section, the existence of the adelic point (x_v) gives us an adelic point $(\widetilde{x}_v)$ of P which then lands in D' by our construction. It follows that $D' \to D$ is a cover of curves such that almost all primes split, thus by the Chebotarev density theorem, we must have that D' is birational to D. So we have a map $D \to D'$ and we also have maps $D' \to C' \to P \to X$ by construction, which compose to give a map $D \to X$, hence a point in $X(K)$. $\qquad\square$

Remark 3.11. Under the assumptions that h is generically finite, $(x_v) \in X(\mathbb{A}_K)$ survives Frobenius descent, and $\gamma(D)$ is contained in the open set where h is finite, we conclude that the only way $X(K) \setminus X(\mathbb{F}) = \emptyset$ is if D' is reducible. An interesting question is whether the examples provided above are typical.

3.4. Degeneracy of the Gauss map. When X is a surface contained in an abelian threefold, we find that $h\colon P \to \mathbb{P}^2$, thereby mapping the three dimensional projectivized tangent bundle P to the projective plane and hence cannot be finite. The following example illustrates that in this context, there exist non-global adelic points on such a surface X unobstructed by Frobenius descent.

Example 3.12. Assume $\mathbb{F}$ has odd characteristic. Let C be the quartic curve with affine equation $x^4 + y^4 = 1$, D the curve $x^4 + z^2 = 1$, and let X be the image of $C^{(2)}$ in the Jacobian J of C. Consider the map $C \to D$ given by $(x, y) \mapsto (x, y^2)$. By taking the fibers of this map, we obtain a map $f\colon D \to C^{(2)}$. Let $g\colon D \to C^{(2)}$ be the composition of the map $D \to D$ taking $(x, z) \mapsto (-x, z)$ with f, and let P_f and P_g represent the points of $C^{(2)}(K)$ corresponding to f and g, respectively.

By [6, Lemma 3.1], the Frobenius descent map μ induced on J is given by $P \mapsto P^*(\omega_1, \ldots, \omega_g)$ where the ω_i form a basis of holomorphic differentials of C. In this case such a basis is given by $\{\frac{dx}{y^3}, \frac{x\,dx}{y^3}, \frac{dx}{y^2}\}$, and we find $\mu(P_f) = (0, 0, \frac{2dx}{z}) = \mu(P_g)$. This is independent of the characteristic, hence the difference of these two points is constant and thus in the image of Frobenius, i.e., $P_f - P_g = F(z)$ for some $z \in J(K)$.

Following the same construction as in the proof of Proposition 3.8, we define an adelic point (P_v) of X by splitting the places of K into $S_1 \sqcup S_2$ and alternating between P_f for $v \in S_1$ and P_g for $v \in S_2$. Let $z_v = z$ when $v \in S_1$ and $z_v = 0$ when $v \in S_2$. Then by construction P_v is not global but survives Frobenius descent.

As noted above, the Gauss map in the previous example is degenerate for dimension reasons. We owe the following example to Dan Abramovich. It has a degenerate Gauss map with 2-dimensional image but not due to dimension reasons, as in the previous example.

Recall that the a-number of an abelian variety $A/\mathbb{F}$ can be defined as $\dim_{\mathbb{F}} \mathrm{Hom}(\alpha_p, A[p])$, where α_p is the group-scheme $\mathrm{Spec}(\mathbb{F}[x]/(x^p))$. It is an integer between 0 and $\dim A$ and all values can be attained for a suitable choice of A (see [13]). We start with an abelian variety A_0 with a-number equal to $\dim A_0 = g \geq 4$. The condition on the a-number implies that, for any subspace V of $T_0 A_0$, there is a unique sub-group-scheme of $\ker F$ whose tangent space is V. We now construct a smooth surface $X_0 \subset A_0$, with $0 \in X_0$ (say, by intersecting with hyperplanes in a suitable projective embedding of A_0 and applying Bertini's theorem). We now select a subspace V of $T_0 A_0$ of codimension 3 which meets $T_0 X_0$ only at 0.

Consider the quotient map $\rho : A_0 \to A$ by the sub-group-scheme of $\ker F$ with tangent space V and let $X = \rho(X_0)$. By the transversality of V and $T_0 X_0$, the map $\rho|_{X_0}$ is a local isomorphism at 0. Now, since ρ is purely inseparable, X is birational to X_0. However, the Gauss map $h : P \to \mathbb{P}_{\mathbb{F}}^{g-1}$ corresponding to X has image in $\mathbb{P}(T_0 A/V)$ which is a projective plane. On the other hand, a general choice of X_0 would ensure that X generates A as a group and therefore is not contained in a lower-dimensional abelian subvariety, so the degeneracy of h is not due to dimension reasons for $g \geq 4$.

We have not been able to construct a failure of the Frobenius descent obstruction in this setting but expect that there might be such examples.

3.5. Descent and the Brauer–Manin obstruction. At present, there are several difficulties in extending the result that the Brauer–Manin obstruction is equivalent to finite abelian descent for curves to arbitrary constant subvarieties of abelian varieties. The proof of [12, Proposition 4.6] uses the fact that $\mathrm{Br}(\overline{X}) = 0$ when X is a curve; for higher dimensional

X it is possible for transcendental Brauer elements to exist. In the number fields case, Creutz proves that there are no transcendental Brauer–Manin obstructions on abelian varieties [4], but it is unknown whether this extends to subvarieties of abelian varieties over a global function field K. Additionally, if the Neron–Severi group $\mathrm{NS}(\overline{X}) \not\cong \mathbb{Z}$ (which occurs e.g., for $X = C^{(2)}$ as follows from the corresponding classical result for C^2) and moreover if $H^1(K, \mathrm{NS}(\overline{X}))$ is non-trivial, then there may be additional Brauer obstructions not coming from finite abelian descent.

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