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Classical forms of weight one in ordinary families

par Eric STUBLEY

Résumé. Nous développons une nouvelle stratégie pour étudier les spécialisations en petits poids des familles p-adiques de formes modulaires ordinaires. Dans le cas elliptique, nous donnons une nouvelle preuve d'un résultat de Ghate–Vatsal qui énonce qu'une famille de Hida contient une infinité de formes propres classiques de poids un si et seulement si elle est à multiplication complexe. Notre stratégie est conçue afin d'éviter explicitement l'utilisation de certains faits connexes, à savoir que la représentation galoisienne attachée à une forme propre classique de poids un est d'image finie et que les formes propres classiques de poids un satisfont la conjecture de Ramanujan. Nous indiquons comment utiliser cette stratégie pour démontrer des énoncés similaires dans le cas de formes modulaires Hilbert de poids partiel un, en supposant un développement approprié de la théorie de Hida dans ce contexte.

Abstract. We develop a new strategy for studying low weight specializations of p-adic families of ordinary modular forms. In the elliptic case, we give a new proof of a result of Ghate–Vatsal which states that a Hida family contains infinitely many classical eigenforms of weight one if and only if it has complex multiplication. Our strategy is designed to explicitly avoid use of the related facts that the Galois representation attached to a classical weight one eigenform has finite image, and that classical weight one eigenforms satisfy the Ramanujan conjecture. We indicate how this strategy might be used to prove similar statements in the case of partial weight one Hilbert modular forms, given a suitable development of Hida theory in that setting.

1. Introduction

The goal of this article is to provide a new proof of a theorem of Ghate and Vatsal, which states that a p-adic family of p-ordinary modular forms contains infinitely many classical forms of weight one if and only if the family has complex multiplication. Our reason for seeking out a new proof of this result is to avoid the use of the related facts that classical eigenforms of weight one satisfy the Ramanujan conjecture and their associated Galois representations have finite image. We wish to avoid using these facts so as

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to provide a proof strategy which is likely to generalize to the case of partial weight one Hilbert modular forms, where the Ramanujan conjecture is as yet unknown.

Let us formulate this result precisely. We fix an odd prime number p, and let $\Lambda = \mathbf{Z}_p[\![1 + p\mathbf{Z}_p]\!]$ denote the Iwasawa algebra, which we think of as parametrizing continuous p-adic characters of the multiplicative group $1 + p\mathbf{Z}_p$.

We say that a classical modular eigenform is said to be p-ordinary if its U_p-eigenvalue is a p-adic unit. We will be interested in p-adic families of p-ordinary modular eigenforms. Pick a natural number N coprime to p, which will serve as the prime-to-p level of the modular forms we study. Hida's theory of p-ordinary families produces a finite free Λ-module $\mathbf{H}^{\mathrm{ord}}(N; \mathbf{Z}_p)$ containing elements T_ℓ for every prime $\ell \nmid Np$ and U_ℓ for each prime $\ell | Np$, which is universal in the following sense. For any normalized modular eigenform f which is of weight $k \geq 2$, level of the form Np^r for some $r \geq 0$, and which is p-ordinary, there is a unique homomorphism

$$\mathbf{H}^{\mathrm{ord}}(N; \mathbf{Z}_p) \longrightarrow \overline{\mathbf{Q}}_p$$

determined by sending the elements T_ℓ (or U_ℓ if $\ell | Np$) to the ℓ-th Hecke eigenvalue of f. For such f, we say that $\mathbf{H}^{\mathrm{ord}}(N; \mathbf{Z}_p)$ specializes to f.

The ordinary Hecke algebra $\mathbf{H}^{\mathrm{ord}}(N; \mathbf{Z}_p)$ is constructed as a Hecke algebra acting on a large space of p-adic modular forms. It is known that any eigensystem $\mathbf{H}^{\mathrm{ord}}(N; \mathbf{Z}_p) \to \overline{\mathbf{Q}}_p$ which is in arithmetic weight $k \geq 2$ (meaning the composite map $\Lambda \to \mathbf{H}^{\mathrm{ord}}(N; \mathbf{Z}_p) \to \overline{\mathbf{Q}}_p$ is a finite order character times the $(k-1)$-st power of the cyclotomic character) is the eigensystem of a classical, rather than just a p-adic, modular form. However this is no longer true for eigensystems in weight one (where the corresponding map $\Lambda \to \overline{\mathbf{Q}}_p$ is a finite order character): such an eigensystem may or may not be that of a classical modular form.

The following theorem of Ghate and Vatsal characterizes exactly when a component of $\mathbf{H}^{\mathrm{ord}}(N; \mathbf{Z}_p)$ admits infinitely many classical weight one specializations.

Theorem 1.1 (Ghate–Vatsal, cf. [12, Proposition 14])**.** *Let* $\mathbf{I}$ *be a reduced irreducible component of the ordinary* Λ*-adic Hecke algebra* $\mathbf{H}^{\mathrm{ord}}(N; \mathbf{Z}_p)$*. Then* $\mathbf{I}$ *specializes to infinitely many classical eigenforms of weight one if and only if* $\mathbf{I}$ *has complex multiplication.*

In this article we give a new proof of this result. The method of Ghate–Vatsal relies crucially on the fact that the Galois representations associated to classical eigenforms of weight one have finite image. As we are interested in generalizing this result to the situation of Hilbert modular forms of partial weight one, where the associated Galois representations have infinite image and the Ramanujan conjecture is still open, our new proof of this

result in the elliptic case avoids both of these known facts for classical eigenforms of weight one. We believe that an analogous result to Theorem 1.1 holds for certain "partially ordinary" families of Hilbert modular forms, which we lay out as Conjecture 7.1 in Section 7.

1.1. Overview of the strategy. Our strategy is to utilize a result of Hida from [15] which characterizes whether or not a p-ordinary family has complex multiplication (hereafter abbreviated as "CM") by the arithmetic complexity of the Hecke fields of the classical forms it interpolates. A version of this result is restated in this article as Theorem 6.1 and its proof is sketched in Section 6.

In order to apply this characterization of CM families, we rely on a crucial fact: Hecke fields of p-ordinary classical weight one eigenforms cannot be too complicated. This is shown in Lemma 3.5 where we prove that for a p-ordinary classical weight one eigenform $f \in S_1(N, \epsilon; \mathbf{Q}_p(\epsilon))$ we have that

$$[\mathbf{Q}(f) : \mathbf{Q}(\epsilon)] \leq \operatorname{rank}_\Lambda(\mathbf{H}^{\mathrm{ord}}(N; \mathbf{Z}_p)) < \infty.$$

This is a consequence of the fact that any classical weight one eigenform either has U_p eigenvalue equal to 0 or is p-ordinary.

If we were willing to apply Hida's characterization of CM families directly in weight one this bound on Hecke fields would suffice to prove that a p-ordinary family with infinitely many classical weight one specializations has CM. However, Hida's characterization crucially uses that the Frobenius eigenvalues in the Galois representations attached to elliptic modular forms are Weil numbers, which is a consequence of the Ramanujan conjecture. Since we seek a proof which avoids using the Ramanujan conjecture in weight one, we cannot directly apply Hida's characterization in weight one.

Instead, we propagate information about the boundedness of Hecke fields along the family from weight one into higher weights, where we do allow ourselves to make use of the Ramanujan conjecture and hence apply Hida's characterization. The main idea is to analyze the structure of the "algebraic power series" (elements of finite extensions of Λ) which define Frobenius eigenvalues across the family. The prototypical example of the type of rigidity result we employ is the following, which is also used by Hida in establishing the characterization of CM families: a power series $F(T) \in \Lambda$ for which $F(\zeta - 1)$ is a power of ζ for infinitely many $\zeta \in \mu_{p^\infty}$ must be of the form $\zeta'(1+T)^e = \zeta' \sum_{n=0}^{\infty} \binom{n}{e} T^n$, where $\zeta' \in \mu_{p^\infty}$ and $e \in \mathbf{Z}_p$. Our situation requires controlling algebraic power series with many specializations which are a sum of a bounded number of roots of unity rather than just a single root of unity, to account for the fact that Hecke fields of our p-ordinary weight one forms are uniformly bounded over $\mathbf{Q}(\mu_{p^\infty})$, but may not necessarily be cyclotomic themselves. Once these algebraic power series have been controlled using the boundedness of Hecke fields in weight one,

we use our exact knowledge of the shape of these power series to establish bounds on Hecke fields in higher weight, from whence we may apply Hida's characterization.

1.2. Outline. In Section 2 we recall the key facts about modular forms which are used in our arguments. We discuss Galois conjugation of modular forms and its relation to Hecke fields. Our arguments rely on two types of bounds on the Hecke eigenvalues of modular forms: Archimedean bounds (as embodied by bounds coming from the holomorphicity of the L-series of f) and non-Archimedean bounds (as embodied by the slope bounds $0 \leq \mathrm{ord}_p(a_p(f)) \leq k-1$ for classical forms of weight k).

Section 3 covers the statements we need from the theory of ordinary families of p-adic modular forms. We cover what we need in the elliptic case, paying particular attention to the inclusion of forms of weight one into ordinary families, as the literature often works only with forms of weight $k \geq 2$.

Section 4 is where new results start appearing. This chapter focuses on rigidity principles for p-adic power series and integral extensions of power series rings. After recalling in Subsection 4.1 some facts about Weierstrass preparation and Newton polygons to set the stage for how we approach thinking about elements of integral extension of power series rings, we prove our main rigidity result in Subsection 4.2. We conclude this chapter with tools for studying the fields of definition of the values of algebraic power series of the form determined by our rigidity results.

Section 5 works throughout with a component $\mathbf{I}$ of the ordinary Λ-adic Hecke algebra, which is assumed to admit infinitely many classical weight one specializations. This chapter covers the construction of a high-dimensional Galois representation whose characteristic polynomials of Frobenius provide a way to propagate information about Hecke fields from low weight into regular weight. Subsection 5.1 carries out the actual construction, which is a careful selection of components of the Hecke algebra by an extended pigeonhole principle argument; the Galois representation we want is the direct sum of the representations attached to a well-chosen set of such components. In Subsection 5.2 we show that the characteristic polynomials of Frobenius of our high-dimensional Galois representation satisfy the conditions necessary to apply the rigidity principles of Section 4.

Section 6 sketches a proof of Hida's characterization of CM families in Subsection 6.1 and then assembles the ingredients from Section 5 in order to apply Hida's theorem to families containing infinitely many classical forms of low weight, proving our main theorem.

We end in Section 7 with a brief section discussing how the strategy in this article might be applied to p-adic families containing Hilbert modular forms of partial weight one.

1.3. Notation. Throughout the article we work with a fixed odd prime p. We fix an algebraic closure $\overline{\mathbf{Q}}_p$ of $\mathbf{Q}_p$, and completion $\mathbf{C}_p$ of $\overline{\mathbf{Q}}_p$. We fix the p-adic valuation ord_p on $\mathbf{C}_p$, normalized so that $\mathrm{ord}_p(p) = 1$.

For any field F we let G_F denote the absolute Galois group of F, $\mathrm{Gal}(\overline{F}/F)$. We will only deal with absolute Galois groups of finite extensions of $\mathbf{Q}$ and $\mathbf{Q}_\ell$ for primes ℓ.

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2. Modular forms

In this section we collect some facts about elliptic modular eigenforms, their Hecke fields, and bounds on their Hecke eigenvalues. Everything in this section is either already known or easily deduced from known results, we are simply collecting the key facts for our arguments to have them in one place.

2.1. Galois conjugates. Acting on the space $S_k(N, \epsilon; \mathbf{C})$ of cuspidal modular forms of weight k, level $\Gamma_0(N)$, Nebentypus character ϵ, with complex coefficients is a commutative algebra $H_k(N, \epsilon; \mathbf{C})$, called the Hecke algebra. The generators of this algebra and their action on q-expansions are:

- for each prime $\ell \nmid N$ a Hecke operator T_ℓ acting by
$$T_\ell\left(\sum_{n=1}^{\infty} a_n q^n\right) = \sum_{n=1}^{\infty} a_{n\ell} q^n + \epsilon(\ell)\ell^{k-1} \sum_{n=1}^{\infty} a_n q^{n\ell},$$
- for each prime $\ell | N$ an Atkin-Lehner operator U_ℓ acting by
$$U_\ell\left(\sum_{n=1}^{\infty} a_n q^n\right) = \sum_{n=1}^{\infty} a_{n\ell} q^n.$$

Throughout this article we will mostly work with Hecke algebras rather than directly with spaces of forms. Fundamentally the two are equivalent thanks to the following proposition.

Proposition 2.1. *The pairing*
$$\begin{array}{ccccc} H_k(N, \epsilon; \mathbf{C}) & \times & S_k(N, \epsilon; \mathbf{C}) & \to & \mathbf{C} \\ (T & , & f) & \mapsto & a_1(T(f)) \end{array}$$
is a perfect pairing of complex vector spaces.

Thanks to this perfect pairing, we have that $S_k(N, \epsilon; \mathbf{C})$ admits a basis of (simultaneous) eigenvectors for the action of $H_k(N, \epsilon; \mathbf{C})$. If f is such an eigenvector, we have that there is a homomorphism $\psi(f) : H_k(N, \epsilon; \mathbf{C}) \to \mathbf{C}$ sending T_ℓ (resp. S_ℓ or U_ℓ) to its eigenvalue on f. If we scale the eigenvector f so that $a_1(f) = 1$, we call f a normalized eigenform, and we have that $T_\ell(f) = a_\ell(f)$, the ℓ-th Fourier coefficient of f.

Moreover we can consider the $\mathbf{Q}(\epsilon)$ or $\mathbf{Z}[\epsilon]$-algebra generated by these same operators as a Hecke algebra $H_k(N, \epsilon; \mathbf{Q}(\epsilon))$ or $H_k(N, \epsilon; \mathbf{Z}[\epsilon])$. Since the integral Hecke algebra is finitely generated over $\mathbf{Z}[\epsilon]$ (it is a sub-algebra of the finitely generated $\mathrm{End}_{\mathbf{Z}}(S_k(N, \epsilon; \mathbf{Z}[\epsilon])))$ each such homomorphism $\psi(f)$ must have image in the ring of integers of a finite extension of $\mathbf{Q}$. With this structure in place we can define Galois conjugation on these spaces of modular forms. Given a normalized eigenform $f \in S_k(N, \epsilon; \mathbf{C})$ and an element σ of the absolute Galois group $G_{\mathbf{Q}}$ we define the Galois conjugate f^σ by letting $\psi_f : H_k(N, \epsilon; \mathbf{Z}[\epsilon]) \to \mathbf{C}$ be the homomorphism corresponding to f by the duality between Hecke algebras and modular forms, and letting f^σ be the normalized eigenform corresponding to $\sigma \circ \psi_f : H_k(N, \epsilon^\sigma; \mathbf{Z}[\epsilon^\sigma]) \to \mathbf{C}$. Note that this is well-defined as we can either choose an extension of σ to all of $\mathbf{C}$ or observe that the image of ψ_f lands in a number field.

Remark 2.2. We could define the Galois action on modular forms directly on q-expansions by acting with any field automorphism of $\mathbf{C}/\mathbf{Q}$, but with that definition it is entirely unclear why the Galois conjugate of a modular form should still be a modular form! The given definition makes it clear that Galois conjugate of a Hecke eigenform is still an eigenform.

See Section 6.5 of [10] for a more in depth discussion of Galois conjugates of eigenforms.

2.2. Hecke fields.

Definition 2.3. Given a normalized eigenform $f \in S_k(N, \epsilon; \mathbf{C})$ we define its character field

$$\mathbf{Q}(\epsilon) = \mathbf{Q}(\{\epsilon(x) : x \in (\mathbf{Z}/N\mathbf{Z})^\times\})$$

and its Hecke field

$$\mathbf{Q}(f) = \mathbf{Q}(\{a_n(f) : n \in \mathbf{N}\}).$$

Remark 2.4. We make several remarks about character and Hecke fields of modular forms. First off, the relation between Fourier coefficients

$$a_{\ell^2}(f) = a_\ell(f)^2 - \epsilon(\ell)\ell^{k-1}a_\ell(f)$$

for almost all primes ℓ shows that $\mathbf{Q}(f) \supseteq \mathbf{Q}(\epsilon)$.

Second, since we know that the Fourier coefficient $a_\ell(f)$ is equal to the eigenvalue of the operator T_ℓ acting on f (similarly $\epsilon(\ell)$ is the eigenvalue of S_ℓ acting on f) we have that $\mathbf{Q}(f)$ is the image of the homomorphism $H_k(N, \epsilon; \mathbf{Q}(\epsilon)) \to \mathbf{C}$ sending T_ℓ to $a_\ell(f)$. Since the Hecke algebra is finitely integrally generated we have that $\mathbf{Q}(f)$ is a finitely generated extension of $\mathbf{Q}$. Moreover the finiteness of the Hecke algebra shows that each $a_\ell(f)$ must be algebraic (integral even) and so we conclude that $\mathbf{Q}(f)$ is of finite degree over $\mathbf{Q}$.

We record here the important principle that the degree of the Hecke field tells us the number of Galois conjugates of a form.

Lemma 2.5. *Let $f \in S_k(N, \epsilon; \mathbf{C})$ be a normalized eigenform. Then the number of Galois conjugates of f over $\mathbf{Q}$ is equal to the degree $[\mathbf{Q}(f) : \mathbf{Q}]$ of the Hecke field of f over $\mathbf{Q}$.*

Proof. This is more generally just a fact about algebraic field extensions. The degree $[\mathbf{Q}(f) : \mathbf{Q}]$ is equal to the size of the orbit of the generating set $\{a_n(f) : n \in \mathbf{N}\}$ under the action of the absolute Galois group of $\mathbf{Q}$. But we also have that the size of this orbit is the number of Galois conjugates of f itself, since for $\sigma, \tau \in G_{\mathbf{Q}}$ we have that $\sigma(a_n(f)) = \tau(a_n(f))$ for all n if and only if $f^\sigma = f^\tau$. $\square$

2.3. Modular forms with complex multiplication.

Definition 2.6. Let E be an imaginary quadratic field. We say that a modular form f has complex multiplication (or CM for short) by E if $a_p(f) = 0$ whenever p is inert in the extension $E/\mathbf{Q}$. We say that a modular form has CM if it has CM by some imaginary quadratic field.

A reference for basic facts about modular forms with complex multiplication is Sections 3 and 4 of [29]. Of interest to us is the fact that eigenforms with CM can be constructed using algebraic Hecke characters, which we recall here. Let E be an imaginary quadratic field with a chosen embedding $\sigma : E \to \mathbf{C}$, and let ψ be an algebraic Hecke character of infinity-type σ^{k-1} and conductor $\mathfrak{m}$. Given such a Hecke character one can construct a weight k eigenform given by the series

$$g = \sum_{\mathfrak{a}} \psi(\mathfrak{a}) q^{\mathrm{Norm}_{\mathbf{Q}}^{E}(\mathfrak{a})}$$

where the sum is over all integral ideals $\mathfrak{a}$ of $\mathcal{O}_E$ which have $(\mathfrak{a}, \mathfrak{m}) = 1$. The modular form g is called the θ-series attached to ψ. The eigenform g thus constructed has level DM where D is the discriminant of $E/\mathbf{Q}$ and $M = \mathrm{Norm}_{\mathbf{Q}}^{E}(\mathfrak{m})$, and character $\epsilon = \chi\eta$ where χ is the quadratic Dirichlet character attached to E and η is the "finite order" part of ψ on the integers, given by $\eta(n) = \psi((n))/\sigma(n)^{k-1}$ for $n \in \mathbf{Z}$. Note that it is immediate from this definition that g has CM by E; if p is inert in $E/\mathbf{Q}$ then there are no ideals of $\mathcal{O}_E$ having norm p, so $a_p(g) = \sum_{\mathfrak{a}, \mathrm{Norm}_{\mathbf{Q}}^{E}(\mathfrak{a})=p} \psi(\mathfrak{a}) = 0$. This construction is studied in Section 3 of [29], and Section 4 of [29] deals with basic facts about Galois representations attached to CM eigenforms.

Of fundamental importance for our method is the characterization of CM families by the arithmetic complexity of their Hecke fields. This characterization due to Hida is recalled in Section 6. The key philosophy is that Hecke fields attached to CM eigenforms are much simpler than those attached to non-CM eigenforms. We begin this study here with a description of the Hecke field of a CM eigenform.

Lemma 2.7. *Suppose that $f \in S_k(N, \epsilon; \mathbf{C})$ be a normalized eigenform with CM by the imaginary quadratic field E. Suppose that f is realized as a theta series by an algebraic Hecke character ψ of conductor $\mathfrak{m}$, and let $M = \mathrm{Norm}_{\mathbf{Q}}^{E}(\mathfrak{m})$. Let h be the class number of E. Then there are elements $a_1, \ldots, a_h$ in E such that*

$$\mathbf{Q}(f) \subseteq E(\mu_{h \cdot M}, a_1^{1/h}, \ldots, a_h^{1/h}).$$

In particular, the degree of the Hecke field over $\mathbf{Q}$ is bounded solely in terms of the CM field E and the conductor $\mathfrak{m}$ of the character ψ.

Proof. We know that our eigenform f has q-expansion given by

$$f = \sum_{\mathfrak{a}} \psi(\mathfrak{a}) q^{\mathrm{Norm}_{\mathbf{Q}}^{E}(\mathfrak{a})}$$

where the sum is over all integral ideals $\mathfrak{a}$ of $\mathcal{O}_E$ which have $(\mathfrak{a}, \mathfrak{m}) = 1$. We certainly have that $\mathbf{Q}(f) = \mathbf{Q}(\{\psi(\mathfrak{a})\})$, so it suffices to understand the field of definition of ψ.

If (a) is a principal ideal with generator $a \equiv 1 \mod \mathfrak{m}$, we have that $\psi((a)) = a^{k-1}$. If the generator a is not necessarily $\equiv 1 \mod \mathfrak{m}$, then we know that $a^{hM} \equiv 1 \mod \mathfrak{m}$ and hence $\frac{\psi((a))}{a^{k-1}} \in \mu_{hM}$.

Let $\mathfrak{a}_1, \ldots, \mathfrak{a}_h$ be a complete set of representatives of the ideal class group of E. For each i we have that $\psi(\mathfrak{a}_i^h) \in E$ since $\mathfrak{a}_i^h$ is principal and ψ necessarily sends principal ideals to elements of E (using that the infinity-type of ψ is σ^{k-1} for $\sigma : E \to \mathbf{C}$). Let a_i be an element of E such that $\psi(\mathfrak{a}_i^h) = a_i$. We know that every ideal $\mathfrak{a}$ is equal to a principal ideal times one of our representatives $\mathfrak{a}_i$, hence we have that $\psi(\mathfrak{a})$ must be a root of unity of order hM times $a_i^{1/h}$. □

The preceding lemma is the first instance of the phenomenon that Hecke fields of CM forms are "uniform", here seen in the uniform bound on degrees of Hecke fields with a given CM field and conductor. This idea will be developed further in Subsection 3.4, where we show that there is a uniform description of the Hecke fields themselves on all forms in an ordinary family which has CM, not just a uniform bound on the degrees of the Hecke fields.

2.4. Archimedean bounds. In this section we state bounds on the (Archimedean) absolute value of Hecke eigenvalues of modular forms.

Theorem 2.8 (The Ramanujan conjecture, due to Deligne [8])**.** *Let $f \in S_k(N, \epsilon; \mathbf{C})$ be a normalized eigenform. Then for all primes ℓ we have that*

$$|a_\ell(f)|_{\mathbf{C}} \leq 2\ell^{\frac{k-1}{2}}.$$

Remark 2.9. Let $\rho_{f,p}$ be the p-adic Galois representation attached to f. Let α_ℓ, β_ℓ be the eigenvalues of $\rho_{f,p}(\mathrm{Frob}_\ell)$; these are the roots of $x^2 - a_\ell(f)x + \epsilon(\ell)\ell^{k-1}$. The bound of Theorem 2.8 then gives that α_ℓ, β_ℓ are *ℓ-Weil numbers*, in other words their complex absolute values are exactly $\ell^{\frac{k-1}{2}}$. This fact is key to the characterization of CM ordinary families due to Hida that we employ in Section 6.

Theorem 2.8 also holds in weight one, where it is due to Deligne–Serre in [9]. However, we wish to avoid using it in low weight in order to provide a proof technique which has the possibility of applying to the case of Hilbert modular forms of partial weight one. To get around this we will only use a weaker bound in the case $k = 1$, the analog of which is comparatively

easy to establish for all forms. Though this weaker bound can doubtless be extracted from the literature, we provide a proof here. We note that all our method requires is an upper bound on $|a_\ell(f)|_{\mathbf{C}}$ which is independent of f (but may depend on the weight k), so the exact exponent $\frac{k}{2}+1$ that appears in our bound is unimportant.

Theorem 2.10. *Let $f \in S_k(N, \epsilon; \mathbf{C})$ be a normalized eigenform. Then for all primes $\ell \nmid N$ we have that*

$$|a_\ell(f)|_{\mathbf{C}} \leq 2\ell^{\frac{k}{2}+1}.$$

Proof. The proof is a combination of two facts. First, for any cusp form $f(q) = \sum_{n=1}^{\infty} a_n q^n$ of weight k we know that there is some constant C such that $|a_n|_{\mathbf{C}} \leq Cn^{\frac{k}{2}}$ and so the L-series

$$L(s, f) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

associated to f converges absolutely in the right half-plane $\mathrm{Re}(s) > \frac{k}{2}+1$. This bound on coefficients does not suffice for our purposes, as the constant C may depend on the form f. The second fact that we need is that since f is a normalized eigenform its L-series admits an Euler product expansion,

$$L(s, f) = \prod_p \frac{1}{1 - a_p p^{-s} + \epsilon(p) p^{k-1} p^{-2s}}.$$

Both of these facts can be found in Section 5.9 of [10].

Consider the series $L_p(s, f) = \sum_{r=0}^{\infty} \frac{a_{p^r}}{p^{-rs}}$. On one hand we know that this series converges absolutely on the right half-plane $\mathrm{Re}(s) > \frac{k}{2}+1$, since it is just a sum over fewer terms of the series $L(s, f)$. We also know that this is equal to the geometric series $\frac{1}{1-a_p p^{-s}+\epsilon(p)p^{k-1}p^{-2s}}$ after some rearranging of terms, and a geometric series $\frac{1}{1-x}$ converges exactly when $|x| < 1$. So we get that

$$|a_p p^{-s} - \epsilon(p) p^{k-1} p^{-2s}|_{\mathbf{C}} < 1$$

whenever $\mathrm{Re}(s) > \frac{k}{2}+1$. Taking a limit as $\mathrm{Re}(s) \to \frac{k}{2}+1$ and juggling the inequality around yields the desired bound

$$|a_p|_{\mathbf{C}} \leq 2p^{\frac{k}{2}+1}.$$

□

2.5. Non-Archimedean bounds. In this section we discuss non-Archimedean bounds on the Hecke eigenvalues of modular forms. The main result is a classical bound on the U_p eigenvalue of classical forms: if f has weight k and its U_p eigenvalue is non-zero, then that eigenvalue has valuation bounded between 0 and $k-1$. This valuation is often referred to as the slope of f. This bound appears throughout the literature, and some cases can

be easily proved using the Galois representations attached to eigenforms, but we provide a proof using the automorphic representations attached to eigenforms that covers all cases of interest. Following that we derive a key consequence for weight one forms.

Theorem 2.11. *Let $f \in S_k(N, \epsilon; \mathbf{C}_p)$ be a normalized eigenform. Suppose that p divides N, so we have that the U_p eigenvalue of f is $a_p(f)$. If $a_p(f) \neq 0$ then*

$$0 \leq \mathrm{ord}_p(a_p(f)) \leq k - 1.$$

Proof. We know from the integrality of the Hecke algebra that $a_p(f)$ will be integral, i.e. $\mathrm{ord}_p(a_p(f)) \geq 0$. Let π_f be the automorphic representation attached to f. We choose to normalize π_f so that the U_p eigenvalue of the modular form f is $\sqrt{p}$ times the U_p eigenvalue of $\pi_{f,p}$. Thus we wish to show that the U_p eigenvalue of $\pi_{f,p}$ has valuation bounded above by $k - \frac{3}{2}$. Under the assumption that the U_p eigenvalue is non-zero, there are three possible cases for what $\pi_{f,p}$ can be: an irreducible principal series representation with both characters unramified, an irreducible principal series representation with one character unramified, or an unramified twist of the Steinberg representation. We treat each case separately to establish the upper bound.

Case 1. $\pi_{f,p}$ is an irreducible principal series representation $PS(\chi_1, \chi_2)$ where both characters χ_i are unramified. In this case we know that the U_p eigenvalue is either $\chi_1(p)$ or $\chi_2(p)$. Since the central character of π_f has weight $k-2$, we know that the product $\chi_1(p)\chi_2(p)$ has valuation $k-2$. Since we know that $\mathrm{ord}_p(\chi_i(p)) \geq -\frac{1}{2}$, we get that each must have valuation at most $k - 2 + \frac{1}{2} = k - \frac{3}{2}$. Thus the U_p eigenvalue of $\pi_{f,p}$ has valuation at most $k - \frac{3}{2}$.

Case 2. $\pi_{f,p}$ is an irreducible principal series representation $PS(\chi_1, \chi_2)$ where only χ_1 is unramified. Let $\alpha_1 = \chi_1(p)$, which is the U_p eigenvalue of $\pi_{f,p}$, hence it has valuation at least $-\frac{1}{2}$. Let χ be the character of $\prod_\ell \mathbf{Z}_\ell^\times$ which is equal to χ_2 on the p-component and trivial on all others; we can view this as a finite order Dirichlet character. Take g to be the eigenform $f \otimes \chi^{-1}$. Thus we have that $\pi_{g,p} = \pi_{f,p} \otimes \chi^{-1}|_{\mathbf{Z}_p^\times}$, which is the principal series representation $PS(\chi_1\chi^{-1}, \chi_2\chi^{-1})$. Note that our choice of χ means that $\chi_1\chi^{-1}$ is ramified and $\chi_2\chi^{-1}$ is unramified. Let $\alpha_2 = \chi_2(p)$, which is the U_p eigenvalue of g, hence it has valuation at least $-\frac{1}{2}$. Since we've only twisted f by a finite order character that weight of the central character remains unchanged. From this we conclude that $\mathrm{ord}_p(\alpha_1\alpha_2) \leq k - 2$, and since each has valuation at least $-\frac{1}{2}$ we conclude that each has valuation at most $k - \frac{3}{2}$.

Case 3. $\pi_{f,p}$ is an unramified twist of the Steinberg representation $S(\chi)$. In this case the U_p eigenvalue of $\pi_{f,p}$ is $\chi(p)$. We know that the central character evaluated at p is equal to $p\chi(p)^2$. Since this must have valuation equal to $k-2$, we see that $\chi(p)$ has valuation equal to $\frac{k-3}{2}$ which is certainly less than $k-\frac{3}{2}$. Note in particular that this cannot occur when $k=1$ since $\frac{k-3}{2}=-1$ is less than $-\frac{1}{2}$, which we already know to be a lower bound on the valuation.

Thus in all cases we have the desired bounds on the U_p eigenvalue of $\pi_{f,p}$, which gives us the desired bounds on the U_p eigenvalue of f itself. □

Using this bound on the valuation of the U_p eigenvalue of an eigenform, we prove the "automatically ordinary" property for weight one eigenforms alluded to in Subsection 1.1. This is simply a matter of applying the bound from Theorem 2.11 in the case $k=1$.

Corollary 2.12. *Let $f \in S_1(N,\epsilon;\mathbf{C}_p)$ be a normalized eigenform of weight one. If $a_p(f) \neq 0$ then*

$$\mathrm{ord}_p(a_p(f^\sigma)) = 0$$

for all $\sigma \in G_{\mathbf{Q}}$.

Proof. A Galois conjugate f^σ of f will be a normalized eigenform in the space $S_1(N,\epsilon^\sigma;\mathbf{C}_p)$. In particular Theorem 2.11 still applies to f^σ, since $a_p(f^\sigma)=\sigma(a_p(f)) \neq 0$. So we conclude that

$$0 \leq \mathrm{ord}_p(a_p(f^\sigma)) \leq 1-1=0. \qquad \square$$

3. Families of modular forms

3.1. Ordinary families of elliptic modular forms. In this section we summarize the elements of Hida's theory of ordinary families of elliptic modular forms which we will use in later sections. The main idea is that for any space of forms with level divisible by p we have an action of the U_p operator. Hida's key realization was that the U_p-ordinary subspace of a space of modular forms has bounded dimension as we vary the weight and Nebentypus character. As a consequence of this the U_p-ordinary subspaces of forms in a fixed tame level can be interpolated into a single family, finite over a weight space Λ parametrizing the weight-character (viewed as a p-adic character of $\mathbf{Z}_p^\times$). Whenever we discuss ordinarity (of a space of modular forms, or Hecke algebra, etc.) from now we always mean ordinarity with respect to the U_p operator, so we will say "ordinary" rather than "U_p-ordinary". For elliptic forms all the statements we need can be found in Hida's original papers [13] and [14] together with Wiles' work on Galois representations attached to ordinary eigenforms [33].

We work with Λ-adic Hecke algebras as our main objects. We fix the following notation for use in this section.

- An odd prime number p.
- A positive integer N coprime to p, which will be the prime-to-p part of the level of our forms.
- K a finite extension of $\mathbf{Q}_p$, with ring of integers $\mathcal{O}_K$.
- $\Lambda = \mathcal{O}_K[\![T]\!]$, the ring of formal power series in one variable over $\mathcal{O}_K$.
- For any positive integer k and p-power root of unity ζ, let $P_{k,\zeta}$ be the kernel of the homomorphism

$$\begin{aligned} \Lambda &\longrightarrow \overline{\mathbf{Q}}_p \\ T &\longmapsto \zeta(1+p)^{k-1} - 1. \end{aligned}$$

We view Λ as the $\mathcal{O}_K$ group ring of the torsion-free part of the Galois group

$$\mathrm{Gal}(\mathbf{Q}(\mu_{p^\infty})/\mathbf{Q}) \cong \mathbf{Z}_p^\times.$$

The torsion-free part is $(1+p)^{\mathbf{Z}_p}$; the isomorphism with $\mathcal{O}_K[\![T]\!]$ is realized by sending $1+p$ to T. While we could view all of our Hecke algebras as living over the larger group ring $\mathcal{O}_K[\![\mathbf{Z}_p^\times]\!]$, the $(\mathbf{Z}/p\mathbf{Z})^\times$ part of the character plays no role in our arguments so we will work solely with Hecke algebras as Λ-modules. Given the above setup, Hida's theory asserts the existence of a "universal" ordinary Hecke algebra.

Theorem 3.1. *There exists a Hecke algebra $\mathbf{H}^{\mathrm{ord}}(N;\mathcal{O}_K)$ with the following properties.*

(1) *$\mathbf{H}^{\mathrm{ord}}(N;\mathcal{O}_K)$ is a finitely generated free Λ-module.*

(2) *(Base Change) If L is a finite extension of K, with ring of integers $\mathcal{O}_L$, we have that*

$$\mathbf{H}^{\mathrm{ord}}(N;\mathcal{O}_L) \cong \mathbf{H}^{\mathrm{ord}}(N;\mathcal{O}_K) \otimes_{\mathcal{O}_K} \mathcal{O}_L.$$

(3) *$\mathbf{H}^{\mathrm{ord}}(N;\mathcal{O}_K)$ is generated as a Λ-module by a collection of elements T_ℓ, S_ℓ for each prime $\ell \nmid Np$, and elements U_ℓ for each prime $\ell | Np$.*

(4) *(Control Theorem) Let $k \geq 2$ be an integer, and ζ a p^r-th root of unity for some $r \geq 0$. Suppose that K is large enough to contain ζ. Let $\epsilon : (1+p)^{\mathbf{Z}_p} \to \mathcal{O}_K^\times$ be the character taking $1+p$ to ζ. Then the natural map*

$$\mathbf{H}^{\mathrm{ord}}(N;\mathcal{O}_K)/P_{k,\zeta}\mathbf{H}^{\mathrm{ord}}(N;\mathcal{O}_K) \xrightarrow{\cong} H_k^{\mathrm{ord}}(\Gamma_1(Np) \cap \Gamma_0(p^r), \epsilon; \mathcal{O}_K)$$

sending the abstract elements T_ℓ, S_ℓ, U_ℓ on the left to the equivalently named Hecke operators on the right is an isomorphism of Λ-modules. In other words $\mathbf{H}^{\mathrm{ord}}(N;\mathcal{O}_K)$ interpolates the ordinary subspaces of all spaces of forms with prime to p level $\Gamma_1(N)$ and weight $k \geq 2$.

Proof. The fact that $\mathbf{H}^{\mathrm{ord}}(N;\mathcal{O}_K)$ is finite free over Λ is Theorem 3.1 of [14]. The control theorem is Theorem 1.2 of [13]. The other statements are all consequences of the definition of $\mathbf{H}^{\mathrm{ord}}(N;\mathcal{O}_K)$, and can be found in [14]. $\square$

There are two constructions of this universal ordinary Hecke algebra $\mathbf{H}^{\mathrm{ord}}(N;\mathcal{O}_K)$. The first, using Katz's theory of geometric p-adic modular forms, appears in [14]. The second, based on Betti cohomology of modular curves and group cohomology of congruence subgroups of $\mathrm{SL}_2(\mathbf{Z})$, appears in [13]. We refer to these approaches as the *geometric* and *cohomological* approaches to Hida theory. Each approach has benefits and drawbacks, and both are necessary in order to develop all facets of the theory which we use in this work. The geometric approach is crucial to understanding how forms of weight one fit into $\mathbf{H}^{\mathrm{ord}}(N;\mathcal{O}_K)$, a topic which we explore in Subsection 3.3. If one wants freeness over Λ of $\mathbf{H}^{\mathrm{ord}}(N;\mathcal{O}_K)$ rather than just torsion-freeness this is provided only by the geometric approach. A downside of the geometric approach is that, at least in Hida's original work, it only deals with the case $r = 0$ of the control theorem, i.e. forms with trivial Nebentypus character. While this is sufficient to uniquely determine $\mathbf{H}^{\mathrm{ord}}(N;\mathcal{O}_K)$ it is not enough for our application, as we will need to specialize at infinitely many different Nebentypus characters in a single weight. The cohomological approach is comparatively simpler as it does not require the algebraic geometry machinery of the geometric approach. Proving the control theorem for all characters ϵ is much more straightforward under the cohomological framework than the geometric one. The downsides of the cohomological approach are that it only produces torsion-freeness of $\mathbf{H}^{\mathrm{ord}}(N;\mathcal{O}_K)$ over Λ (as opposed to freeness) and that it gives no information about the weight one specializations of the Hecke algebra.

Since we are interested in maps from $\mathbf{H}^{\mathrm{ord}}(N;\mathcal{O}_K)$ to rings of integers, we will focus our attention on the quotients $\mathbf{I} = \mathbf{H}^{\mathrm{ord}}(N;\mathcal{O}_K)/\mathfrak{P}$ by minimal primes $\mathfrak{P}$ of $\mathbf{H}^{\mathrm{ord}}(N;\mathcal{O}_K)$. We refer to these rings $\mathbf{I}$ as components of $\mathbf{H}^{\mathrm{ord}}(N;\mathcal{O}_K)$. Since $\mathbf{H}^{\mathrm{ord}}(N;\mathcal{O}_K)$ is noetherian we have that there are finitely many minimal primes $\mathfrak{P}_1,\ldots,\mathfrak{P}_n$ and thus we have an injection

$$\mathbf{H}^{\mathrm{ord}}(N;\mathcal{O}_K)^{\mathrm{red}} \hookrightarrow \prod_{i=1}^{n} \mathbf{I}_i$$

where $\mathbf{I}_i = \mathbf{H}^{\mathrm{ord}}(N;\mathcal{O}_K)/\mathfrak{P}_i$.

Given a normalized eigenform $f \in S_k^{\mathrm{ord}}(\Gamma_1(Np)\cap\Gamma_0(p^r),\epsilon;\mathcal{O}_L)$ for some weight $k \geq 2$, character ϵ of conductor p^r, and finite extension L of K, we say that f arises from $\mathbf{H}^{\mathrm{ord}}(N;\mathcal{O}_K)$ or f arises as a specialization of $\mathbf{H}^{\mathrm{ord}}(N;\mathcal{O}_K)$. If f arises from $\mathbf{H}^{\mathrm{ord}}(N;\mathcal{O}_K)$ and $\mathbf{I}$ is a component of $\mathbf{H}^{\mathrm{ord}}(N;\mathcal{O}_K)$, we say that f arises from $\mathbf{I}$ if the homomorphism $\mathbf{H}^{\mathrm{ord}}(N;\mathcal{O}_K) \to \mathcal{O}_K$ realizing the eigensystem of f factors through the surjective map $\mathbf{H}^{\mathrm{ord}}(N;\mathcal{O}_K) \to \mathbf{I}$.

With this notion of forms arising from components, we can state an important uniqueness property of the Λ-adic Hecke algebra.

Theorem 3.2. *Let $f \in S_k^{\mathrm{ord}}(\Gamma_1(Np) \cap \Gamma_0(p^r), \epsilon; \mathcal{O}_K)$ be a normalized eigenform of weight $k \geq 2$. Then there is a unique component $\mathbf{I}$ of the Λ-adic Hecke algebra $\mathbf{H}^{\mathrm{ord}}(N; \mathcal{O}_K)$ such that f arises from $\mathbf{I}$.*

Proof. This is Corollary 1.5 of [13]. □

3.2. Galois representations attached to ordinary families. Each of the normalized eigenforms which $\mathbf{H}^{\mathrm{ord}}(N; \mathcal{O}_K)$ interpolates has an attached 2-dimensional p-adic Galois representation. It should thus not be surprising that these Galois representations also interpolate into a single Λ-adic Galois representation. These Λ-adic representations were first studied by Hida in [13] and Wiles in [33]. We record here the minimal properties of these representations that we use in later sections.

Theorem 3.3. *Suppose that $\mathbf{I}$ is a reduced, irreducible component of $\mathbf{H}^{\mathrm{ord}}(N; \mathcal{O}_K)$. Then there exists a continuous 2-dimensional Galois representation*

$$\rho_{\mathbf{I}} : G_{\mathbf{Q}} \longrightarrow \mathrm{GL}_2(\mathrm{Frac}(\mathbf{I}))$$

which has the following properties.

(1) *$\rho_{\mathbf{I}}$ is absolutely irreducible.*
(2) *$\rho_{\mathbf{I}}$ is unramified away from Np, and the characteristic polynomial of a Frobenius element at a prime $\ell \nmid Np$*

$$X^2 - T_\ell X - \ell S_\ell.$$

(3) *When restricted to a decomposition group at p, $\rho_{\mathbf{I}}$ is of the form*

$$\rho_{\mathbf{I}}|_{G_{\mathbf{Q}_p}} \cong \begin{bmatrix} * & * \\ 0 & \lambda \end{bmatrix}$$

where $\lambda : G_{\mathbf{Q}_p} \to \mathbf{I}^\times$ is the unramified character sending Frob_p to U_p.
(4) *For almost all primes P of $\mathbf{I}$, the representation $\rho_{\mathbf{I}}$ can be taken to have values in the localization $\mathbf{I}_P$. In particular for almost all normalized eigenforms f arising from $\mathbf{I}$ we have that the p-adic Galois representation $\rho_{f,p}$ attached to f is equal to the composition of $\rho_{\mathbf{I}} : G_{\mathbf{Q}} \to \mathrm{GL}_2(\mathbf{I}_P)$ combined with the quotient map $\mathrm{GL}_2(\mathbf{I}_P) \to \mathrm{GL}_2(\mathbf{I}_P/P\mathbf{I}_P)$ for some prime P of $\mathbf{I}$.*

Proof. This is Theorem 2.1 of [13]. □

3.3. Weight one forms in ordinary families. While Hida's articles are very precise about the specialization of ordinary families in weights $k \geq 2$, eigenforms of weight one are not discussed directly in these articles. It does follow from Hida's first construction of ordinary families, using geometric p-adic modular forms, that every classical p-ordinary weight one eigenform arises as the specialization of an ordinary family. As this is not obviously stated in the literature, we discuss this explicitly here, along with a key consequence for the Hecke fields of p-ordinary weight one eigenforms.

Proposition 3.4. *Given a character $\epsilon : (1+p)^{\mathbf{Z}_p} \to \mu_{p^r}$ sending $1+p$ to a generator ζ of μ_{p^r}, there is a natural surjective homomorphism*

$$\mathbf{H}^{\mathrm{ord}}(N;\mathcal{O}_K)/P_{1,\zeta}\mathbf{H}^{\mathrm{ord}}(N;\mathcal{O}_K) \twoheadrightarrow H_1^{\mathrm{ord}}(\Gamma_1(Np)\cap\Gamma_0(p^r),\epsilon;\mathcal{O}_K).$$

sending the abstract elements T_ℓ, S_ℓ, U_ℓ on the left to the equivalently named Hecke operators on the right. Put differently, every p-ordinary weight one eigenform arises as the specialization of an ordinary family.

Proof. In Hida's first article [14], the universal ordinary Hecke algebra $\mathbf{H}^{\mathrm{ord}}(N;\mathcal{O}_K)$ is constructed as a limit of Hecke algebras acting on the spaces $S_k^{\mathrm{ord}}(\Gamma_1(N);K/\mathcal{O}_K)$. These spaces of forms (or a suitable direct sum of these spaces allowing for divided congruences) are dense in the space $\mathcal{S}$ of all ordinary geometric p-adic modular forms, and so $\mathbf{H}^{\mathrm{ord}}(N;\mathcal{O}_K)$ can also be viewed as the Hecke algebra acting on this single large space of p-adic modular forms. Any space of forms $S_k^{\mathrm{ord}}(\Gamma_1(Np^r);\mathcal{O}_K)$ can be viewed as a subspace of $\mathcal{S}$ by interpreting these classical forms of higher level as p-adic modular forms; in particular this holds for $k=1$. At the level of Hecke algebras, this means that we can realize the Hecke algebra of $S_k^{\mathrm{ord}}(\Gamma_1(Np^r);\mathcal{O}_K)$ as a quotient of the Hecke algebra on $\mathcal{S}$. Decomposing the Hecke algebra on $S_k^{\mathrm{ord}}(\Gamma_1(Np^r);\mathcal{O}_K)$ as a direct sum corresponding to the various possible Nebentypus characters, we get the desired result. □

Lemma 3.5. *Suppose that $f \in S_1(\Gamma_1(Np)\cap\Gamma_0(p^r),\epsilon;\mathcal{O}_K)$ is a classical eigenform of weight one arising as a specialization of $\mathbf{H}^{\mathrm{ord}}(N;\mathcal{O}_K)$. Recall the finite extensions of $\mathbf{Q}$ defined using the Hecke eigenvalues of f:*

$$\mathbf{Q}(\epsilon) = \textit{the character field of } f$$
$$\mathbf{Q}(f) = \textit{the Hecke field of } f.$$

Then we have that

$$[\mathbf{Q}(f):\mathbf{Q}(\epsilon)] \leq \mathrm{rank}_\Lambda(\mathbf{H}^{\mathrm{ord}}(N;\mathcal{O}_K)).$$

Proof. The degree $[\mathbf{Q}(f):\mathbf{Q}(\epsilon)]$ is equal to the number of distinct Galois conjugates of f by the absolute Galois group $G_{\mathbf{Q}(\epsilon)}$ of $\mathbf{Q}(\epsilon)$. Let us assume that our local coefficient field K is large enough to contain $\mathbf{Q}(f)$ and all of its Galois conjugates. Call these Galois conjugates $f_1 = f, f_2, \ldots, f_n$. Each

f_i is a classical weight one eigenform of the same level and character as f, i.e. each $f_i \in S_1(\Gamma_1(Np) \cap \Gamma_0(p^r), \epsilon; \mathcal{O}_K)$.

Crucially, we know that f is ordinary since it is a specialization of an ordinary family $\mathbf{H}^{\mathrm{ord}}(N; \mathcal{O}_K)$. As Galois conjugates of an eigenform with finite slope, each f_i necessarily has finite slope. But since the slope of a finite slope classical weight k eigenform must be between 0 and $k-1$ by Theorem 2.11, we conclude that each f_i is in fact ordinary, since $k-1=0$ when $k=1$. So each f_i is in the ordinary subspace $S_1^{\mathrm{ord}}(\Gamma_1(Np) \cap \Gamma_0(p^r), \epsilon; \mathcal{O}_K)$. This entire space is a quotient of $\mathbf{H}^{\mathrm{ord}}(N; \mathcal{O}_K)/P_{1,\zeta}\mathbf{H}^{\mathrm{ord}}(N; \mathcal{O}_K)$ for some height one prime ideal $P_{1,\zeta}$ of Λ by Proposition 3.4, so in total we have that

$$\begin{aligned}
[\mathbf{Q}(f) : \mathbf{Q}(\epsilon)] &= \text{the number of distinct Galois conjugates of } f \text{ by } G_{\mathbf{Q}(\epsilon)} \\
&\leq \mathrm{rank}_{\mathcal{O}_K}(S_1^{\mathrm{ord}}(\Gamma_1(Np) \cap \Gamma_0(p^r), \epsilon; \mathcal{O}_K) \\
&\leq \mathrm{rank}_{\mathcal{O}_K}(\mathbf{H}^{\mathrm{ord}}(N; \mathcal{O}_K)/P_{1,\zeta}\mathbf{H}^{\mathrm{ord}}(N; \mathcal{O}_K)) \\
&\leq \mathrm{rank}_{\Lambda}(\mathbf{H}^{\mathrm{ord}}(N; \mathcal{O}_K)).
\end{aligned}$$

Note that the first inequality holds since distinct Galois conjugates of f are linearly independent, as they lie in distinct eigenspaces for the action of the Hecke algebra. □

Remark 3.6. If we are willing to use the Ramanujan conjecture for classical weight one eigenforms, then it is likely that Lemma 3.5 already provides a sufficient input to prove our main result without appealing to the constructions of Section 4 and Section 5. The goal of Section 4 and Section 5 is to find a method by which the Hecke field bound of Lemma 3.5 can be propagated into regular weight, where Theorem 6.1 may be applied. We expect that Theorem 6.1 can be adapted to require only that the forms in question satisfy the Ramanujan conjecture; see Remark 6.3 for a discussion of adapting Hida's result to the weight one case.

Note that the Ramanujan conjecture *is* known for weight one forms, having been proved by Deligne–Serre as a consequence of their construction of the Galois representations attached to weight one forms in [9]); our interest in finding a method which avoids using the Ramanujan conjecture is so that this strategy also applies to the case of partial weight one Hilbert modular forms, where the Ramanujan conjecture is still open.

Remark 3.7. We remark that the principle encapsulated by Lemma 3.5 is unique to weight one. For forms of weight $k \geq 2$ it is frequently the case that not all Galois conjugates of a given p-ordinary form are p-ordinary. Indeed, one may think of Hida's characterization of CM families Theorem 6.1 as saying that for non-CM ordinary eigenforms, the proportion of Galois conjugates which are also ordinary goes to 0 as we increase the level.

3.4. Components with complex multiplication. In this section we briefly recall properties of the CM components of the Λ-adic ordinary Hecke algebra. The interested reader should consult Section 7 of [14] for a more thorough account of CM components.

We sketch the construction of CM components outlined by Hida in [14]. Let E be an imaginary quadratic field in which our fixed prime p splits as $(p) = \mathfrak{p}\bar{\mathfrak{p}}$. Fix an integral ideal $\mathfrak{m}$ of E which is coprime to $\mathfrak{p}$; this $\mathfrak{m}$ will serve as the tame conductor of our CM components. Let W be the idèle class group of E of conductor $\mathfrak{m}\mathfrak{p}^\infty$, that is

$$W = \mathbf{A}_E^\times / \overline{U E_\infty^\times E^\times}$$

where $U = \prod_{\ell \neq \mathfrak{p}} U_\ell$, with U_ℓ the entire group of integral units of the completion E_ℓ if ℓ is coprime to $\mathfrak{m}$, and U_ℓ being those integral units which are congruent to 1 mod $\mathfrak{m}$ if ℓ divides $\mathfrak{m}$. The group $\Gamma = 1 + p\mathbf{Z}_p$ injects into $U_\mathfrak{p}$ which itself injects into W. Moreover since the idèle class group of conductor $\mathfrak{m}$ is finite we have that Γ has finite index in W.

Let us assume that our coefficient ring $\mathcal{O}_K$ is large enough to contain the values of all characters of the finite group W/Γ. Define $A = \mathcal{O}_K[\![W]\!]$ to be the $\mathcal{O}_K$ group ring of W. Then the inclusion $\Gamma \to W$ gives a map on group rings $\Lambda \to A$ which realizes A as a finite free Λ-module.

Attached to an algebraic Hecke character ψ on E of conductor $\mathfrak{m}\mathfrak{p}^r$ for some $r \geq 0$ is a compatible system of p-adic Hecke characters $\{\psi_p\}$; the p-adic avatar ψ_p of ψ can be though of as continuous p-adic character of W. If ψ is an algebraic Hecke character inducing the CM eigenform f_ψ, we have that the map $A \to \mathcal{O}_K$ corresponding to $\psi_p : W \to \mathcal{O}_K^\times$ realizes the Hecke eigensystem of f_ψ inside $\mathcal{O}_K$. Let $M = \mathrm{Norm}_{\mathbf{Q}}^K(\mathfrak{m})$, and let $-d$ be the discriminant of $E/\mathbf{Q}$. In particular we have that

$$a_\ell(f) = \begin{cases} 0 & \ell \text{ is inert in } E/\mathbf{Q} \\ \psi(\mathfrak{l}) + \psi(\bar{\mathfrak{l}}) & \ell \text{ splits as } \mathfrak{l}\bar{\mathfrak{l}} \text{ in } E/\mathbf{Q} \end{cases}$$

for primes $\ell \nmid dMp$. Since these quantities vary continuously with the character ψ, we can patch them together into a single map with coefficients in A, which interpolates (the p-stabilizations of) the modular forms f_ψ. Letting $\Psi : W \to A^\times$ be the tautological character, there is a map which is defined at least on the T_ℓ Hecke operators as

$$\begin{aligned} \mathbf{H}^{\mathrm{ord}}(dM; \mathcal{O}_K) &\longrightarrow A \\ T_\ell &\longmapsto \begin{cases} 0 & \ell \text{ is inert in } E/\mathbf{Q} \\ \Psi(\mathfrak{l}) + \Psi(\bar{\mathfrak{l}}) & \ell \text{ splits as } \mathfrak{l}\bar{\mathfrak{l}} \text{ in } E/\mathbf{Q}. \end{cases} \end{aligned}$$

The full details of this construction are presented in Theorem 7.1 of [14]. The components $\mathbf{I}$ of $\mathbf{H}^{\mathrm{ord}}(dM; \mathcal{O}_K)$ which have CM by E are those for which the map $\mathbf{H}^{\mathrm{ord}}(dM; \mathcal{O}_K) \to \mathbf{I}$ factors through this map to A.

Proposition 3.8. *Suppose that* $\mathbf{I}$ *is a reduced, irreducible component of* $\mathbf{H}^{\mathrm{ord}}(N;\mathcal{O}_K)$*. If a CM eigenform of weight* $k \geq 2$ *arises as a specialization of* $\mathbf{I}$*, then* $\mathbf{I}$ *is a CM component, and in particular every specialization of* $\mathbf{I}$ *has CM by the same imaginary quadratic field.*

Proof. Let f be a CM eigenform arising from $\mathbf{I}$. We know by Theorem 3.2 that there is a unique component of $\mathbf{H}^{\mathrm{ord}}(N;\mathcal{O}_K)$ giving rise to f. By assumption this component is $\mathbf{I}$, however the construction above produces a CM component which specializes to any given CM eigenform. Thus we must have that $\mathbf{I}$ itself is one of the CM components constructed above. □

We return to the description of Hecke fields, building an explicit description of the Hecke fields of CM components. This is essentially a combination of Lemma 2.7 with the explicit description of CM components given above.

Lemma 3.9. *Suppose that* $\mathbf{I}$ *is a reduced, irreducible CM component of* $\mathbf{H}^{\mathrm{ord}}(N;\mathcal{O}_K)$*. Let* E *be the imaginary quadratic field by which* $\mathbf{I}$ *has CM, and let* h *be the class number of* E*. Fix a weight* $k \geq 2$*. There are elements* $a_1, \ldots, a_h$ *of* K *such that any weight* k *specialization* f *of* $\mathbf{I}$ *has*

$$\mathbf{Q}(f) \subseteq E(\mu_{hNp^\infty}, a_1^{1/h}, \ldots, a_h^{1/h}).$$

In particular the Hecke field of each weight k *form arising from* $\mathbf{I}$ *has its Hecke field contained within a fixed finite extension of the* p*-th cyclotomic field* $\mathbf{Q}(\mu_{p^\infty})$*.*

Proof. Suppose that f_1, f_2 are any two (CM) forms of weight k arising from $\mathbf{I}$, each as the theta series attached to an algebraic Hecke character ψ_1, ψ_2. We know by the construction of $\mathbf{I}$ that the character $\psi_1\psi_2^{-1}$ has finite p-power order (it is an algebraic Hecke character of trivial infinity-type). Pick $a_1, \ldots, a_h$ as in Lemma 2.7 as applied to the form f_1. Since the character $\psi_1\psi_2^{-1}$ is finite order and moreover has p-power order, we see that in the presence of all p-power roots of unity (and the required "tame" roots of unity of order hN) these same a_i generate over E a field containing the Hecke field of f_2. Since we could take f_2 to be any form of weight k arising from $\mathbf{I}$, we see that the field $E(\mu_{hNp^\infty}, a_1^{1/h}, \ldots, a_h^{1/h})$ contains the Hecke field of any weight k specialization of $\mathbf{I}$. □

Remark 3.10. Hida's characterization of CM families, stated in this article as Theorem 6.1, can be interpreted as a converse of Lemma 3.9. Lemma 3.9 shows that the Hecke fields of forms arising from a CM component are uniformly controlled. Hida's result Theorem 6.1 shows that any component of $\mathbf{H}^{\mathrm{ord}}(N;\mathcal{O}_K)$ which has sufficiently controlled Hecke fields in a single weight must be a CM component. It is worth noting that Theorem 6.1 is much weaker than requiring that a component has uniformly controlled Hecke fields; rather it only requires that for a density 1 set of primes ℓ, the

degree of the "ℓ-Hecke field" $\mathbf{Q}(a_\ell(f))$ remains bounded over $\mathbf{Q}(\mu_{p^\infty})$ as one varies over forms f of a fixed weight which arise from $\mathbf{I}$.

Under the assumption that the family in question has infinitely many classical forms of low weight, we establish this boundedness of Hecke fields across the entire family using the special properties of Hecke fields in low weight as embodied by Lemma 3.5, along with the rigidity principles of Section 4 and construction of Section 5 to extend from low weight to regular weight.

4. Rigidity principles for p-adic power series

In this section we prove rigidity results for integral extensions of p-adic power series rings. These results will be used to propagate the boundedness of Hecke fields in low weight to regular weight, where the Ramanujan conjecture is known and Hida's theorem (stated as Theorem 6.1) relating boundedness of Hecke fields and complex multiplication may be applied. In particular the boundedness of Hecke fields in low weight is what motivates the conditions of Theorem 4.12; see Subsection 5.2 for the application of this theorem to the coefficients of the characteristic polynomial of Frobenius elements in a high-dimensional representation of the absolute Galois group of $\mathbf{Q}$.

We fix the following notation for use in this section.

- K is a finite extension of $\mathbf{Q}_p$, with ring of integers $\mathcal{O}$, uniformizer π, and residue field $\mathbf{F}$.
- $\mathbf{C}_p$ is the completion of an algebraic closure of K, $\mathcal{O}_{\mathbf{C}_p}$ is the integral closure of $\mathcal{O}$ within $\mathbf{C}_p$, and $\mathfrak{m}_{\mathbf{C}_p}$ is the maximal ideal of $\mathcal{O}_{\mathbf{C}_p}$.
- As in Subsection 1.3 ord_p is the valuation on K and extensions thereof, normalized so that $\mathrm{ord}_p(p) = 1$.
- $\Lambda = \mathcal{O}[\![T]\!]$ is the ring of formal power series in one variable with coefficients in $\mathcal{O}$.
- M is the integral closure of Λ in some finite extension of $\mathrm{Frac}(\Lambda)$.

4.1. Weierstrass preparation and Newton polygons. The Weierstrass preparation theorem and the theory of Newton polygons will be the basic tools we use to describe the behaviour of elements of Λ and M. Recall that a distinguished polynomial $f(T) \in \mathcal{O}[T] \subset \Lambda$ is a monic polynomial such that every coefficient other than the leading one is divisible by the uniformizer π. With this notion we can state the p-adic Weierstrass preparation theorem.

Theorem 4.1 (p-adic Weierstrass preparation)**.** *If $F(T) \in \Lambda$ is non-zero there is a unique way to write it as*

$$F(T) = \pi^k f(T) u(T)$$

where $k \geq 0$ is an integer, $f(T)$ is a distinguished polynomial, and $u(T)$ is a unit in Λ (in other words, the constant term of u is an element of $\mathcal{O}^\times$).

For a proof of the p-adic Weierstrass preparation theorem, see Theorem 2.2 in Chapter 5, Section 2 of [24].

Lemma 4.2. *If $F(T) \in \Lambda$ is non-zero, then for any $t \in \mathfrak{m}_{\mathbf{C}_p}$ (i.e. $t \in \mathbf{C}_p$ and $\mathrm{ord}_p(t) > 0$) the series $F(t)$ converges in $\mathbf{C}_p$. Moreover such $F(T)$ have only finitely many roots $t \in \mathfrak{m}_{\mathbf{C}_p}$.*

Proof. Let $F(T) = \pi^k f(T)u(T)$ in Weierstrass preparation. If $\mathrm{ord}_p(t) > 0$, then $u(t)$ converges since $\mathrm{ord}_p(t^n) = n \cdot \mathrm{ord}_p(t)$ goes to infinity with n, and $\mathrm{ord}_p(u(t)) = 0$ since the unit constant term of $u(T)$ dominates the norm of any term involving t. Thus $F(t) = \pi^k u(t) f(t)$ converges since f is a polynomial and u converges at t. Finally we see that since $u(t)$ is always a unit, we have $F(t) = 0$ if and only if $f(t) = 0$, and f necessarily has finitely many roots in $\mathbf{C}_p$ as it is a polynomial. □

Lemma 4.3. *Let $F(T)$ have Weierstrass preparation $F(T) = \pi^k f(T)u(T)$ where $f(T)$ has degree d. If $t \in \mathfrak{m}_{\mathbf{C}_p}$ with $0 < \mathrm{ord}_p(t) < \frac{\mathrm{ord}_p(\pi)}{d}$, then*

$$\mathrm{ord}_p(F(t)) = k \cdot \mathrm{ord}_p(\pi) + d \cdot \mathrm{ord}_p(t).$$

Proof. We compute the valuation of $F(t)$ using its Weierstrass preparation

$$\mathrm{ord}_p(F(t)) = \mathrm{ord}_p(\pi^k) + \mathrm{ord}_p(u(t)) + \mathrm{ord}_p(f(t)).$$

We have that $\mathrm{ord}_p(\pi^k) = k \cdot \mathrm{ord}_p(\pi)$, and $\mathrm{ord}_p(u(t)) = 0$ since the unit constant term dominates the norm. Finally we have that $\mathrm{ord}_p(f(t)) = d \cdot \mathrm{ord}_p(t)$ since the leading term t^d has smaller valuation than any of the other terms of $f(t)$, as $d \cdot \mathrm{ord}_p(t) < \mathrm{ord}_p(\pi)$ and every other term has valuation at least $\mathrm{ord}_p(\pi)$ since $f(T)$ is a distinguished polynomial. □

We recall the construction of Newton polygons for polynomials over $\mathbf{C}_p$. Suppose that $f(X) = \sum_{i=0}^{d} a_i X^i$ in $\mathbf{C}_p[X]$. We plot the points $(d-i, \mathrm{ord}_p(a_i))$ in the plane (allowing points "at ∞" if some of the coefficients a_i are 0 and hence have infinite valuation), and form their lower convex hull. The resulting set of line segments in the plane is called the *Newton polygon* of f. The usefulness of Newton polygons lies in the fact that this simple combinatorial construction gives us total knowledge of the valuations of the roots of f.

Theorem 4.4. *Suppose that the Newton polygon of f consists of n line segments, with the i-th segment having horizontal length ℓ_i and slope m_i. If there is a line segment of infinite slope it must occur at the end, and in that case we consider the length ℓ_i to be such that X^{ℓ_i} divides f exactly. Then for each i in the range $1 \leq i \leq n$ there are ℓ_i roots of f which have valuation equal to m_i.*

See Chapter 3, Section 3 of [23] for more information on the theory of Newton polygons. Note that Koblitz's convention for Newton polygons is slightly different from ours; his Newton polygons are vertical reflections of ours. Koblitz's convention has the benefit of also applying easily to power series, at the drawback that the slopes of the polygon correspond to inverses of the valuations of the roots. Our convention is chosen so that the slopes are the valuations, and we won't need to use Newton polygons for non-polynomial power series.

We will use Newton polygons to study specializations of elements of M, where M is the integral closure of Λ in a finite extension of $\mathrm{Frac}(\Lambda)$. Suppose that we have a ring homomorphism $P : M \to \mathbf{C}_p$ which extends the ring homomorphism $P_t : \Lambda \to \mathbf{C}_p$ given by $T \mapsto t \in \mathfrak{m}_{\mathbf{C}_p}$. In a slight abuse of notation we call P a $\mathbf{C}_p$-valued point of M (rather than of $\mathrm{Spec}(M)$). Given $F \in M$, we write $F(P)$ rather than $P(F)$, thinking of F as an "algebraic" analytic function, to align with how we think of elements of Λ as analytic functions. Note that if $F \in \Lambda$, $F(P_t)$ is simply the power series F evaluated at t.

Remark 4.5. Suppose that we're given $F \in M$, and P is a $\mathbf{C}_p$ point of M, extending the $\mathbf{C}_p$ point P_t of Λ. If $R(T, X)$ is a monic irreducible polynomial satisfied by F, we have that $R(T, F) = 0$ in M, and so also $R(t, F(P)) = 0$ in $\mathbf{C}_p$. By computing the Newton polygon of $R(t, X)$ we can obtain the valuation of $F(P)$; in particular for $t \in \mathfrak{m}_{\mathbf{C}_p}$ of sufficiently small valuation we get that each coefficient of $R(t, X)$ has valuation of the form $d_i \cdot \mathrm{ord}_p(t) + k_i \cdot \mathrm{ord}_p(\pi)$ as in Lemma 4.3. We then have that for $\mathrm{ord}_p(t)$ sufficiently small, $\mathrm{ord}_p(F(P)) = a \cdot \mathrm{ord}_p(t) + b$ for some positive rational a, b. Of course since all of the valuations involved are rational there is always some choice of a and b making the above statement true; the point is that the Newton polygon produces such a choice for us, and those a and b can computed from the Weierstrass preparations of the coefficients of $R(T, X)$.

Lemma 4.6. *Let F be an element of M. If there is an infinite set of S of $\mathbf{C}_p$ points of M such that $F(P) = 0$ for all $P \in S$ and each $P \in S$ is of the form $T \mapsto t_P \in \mathfrak{m}_{\mathbf{C}_p}$ when restricted to Λ, then $F = 0$.*

Proof. Let $R(T, X) \in \Lambda[X]$ be a monic irreducible polynomial which F satisfies. Since $R(t, F(P)) = 0$ for any $P \in S$, the constant term $a_0(T)$ of $R(T, X)$ must satisfy $a_0(t_P) = 0$ for each of the infinitely many points P. Since each point $T \mapsto t_P$ of Λ extends to at most finitely many points of M, there must be infinitely many such t. By Lemma 4.2 since the constant term of $R(T, X)$ has infinitely many roots t in $\mathfrak{m}_{\mathbf{C}_p}$ it must be 0. Since $R(T, X)$ is irreducible by assumption, we must have that $R(T, X) = X$, and hence $F = 0$. $\square$

Lemma 4.7. *Let $R(T,X) \in \Lambda[X]$. For all $t \in \mathfrak{m}_{\mathbf{C}_p}$ with $\mathrm{ord}_p(t)$ sufficiently small, the vertices of the Newton polygon of $R(t,X)$ occur at the same indices.*

Proof. The Newton polygon of a monic degree d polynomial is completely determined by the set of valuations of the coefficients. Thinking about the set of valuations as living in $\mathbf{R}^d$, we have a stratification of $\mathbf{R}^d$ according to which vertices lie in the Newton polygon. The condition of an index contributing a vertex to the Newton polygon is given by a collection of linear inequalities; in other words the set of valuations having a given vertex in the Newton polygon is a finite intersection of half-spaces in $\mathbf{R}^d$. The boundary of these half-spaces correspond to multiple vertices lying on the same line segment of the Newton polygon. Note that even though the valuation map takes values in $\mathbf{Q} \cup \{\infty\}$ rather than $\mathbf{R}$ we are simply working with the defining inequalities over $\mathbf{R}$, and if needs be maybe we may replace any infinite valuations with sufficiently large non-infinite valuations without affecting any of the arguments.

We know that for $t \in \mathbf{C}_p$ with $\mathrm{ord}_p(t)$ sufficiently small, the coefficients of $R(t,X)$ have valuation of the form $a \cdot \mathrm{ord}_p(t) + b$ by Lemma 4.3. Say that the i-th coefficient of $R(T,X)$ has valuation $a_i \cdot \mathrm{ord}_p(t) + b_i$ for $\mathrm{ord}_p(t)$ sufficiently small. We consider the curve in $\mathbf{R}^d$ given by $s \mapsto (a_1 s + b_1, \ldots, a_d s + b_d)$. Since the image of this curve is an affine line, we have that for a half-space in $\mathbf{R}^d$ the curve must satisfy one of the following three possibilities:

- the curve is contained entirely within the interior of either the half-space or its complement,
- the curve is contained entirely in the boundary of the half-space,
- the curve intersects the boundary of the half-space exactly once.

Since there are only finitely many affine conditions involved in defining the stratification, we see that the curve will intersect the boundaries of strata transversally only finitely many times. Therefore for $s \in (0, \epsilon)$ for a sufficiently small ϵ the image of the curve will be entirely contained within a single stratum (moving from $s = 0$ to $s > 0$ may change strata, but the curve cannot encounter a boundary within a sufficiently small interval above 0). Since the valuations of the coefficients of $R(t,X)$ for $\mathrm{ord}_p(t)$ sufficiently small land on this curve, we see that the vertices of the Newton polygon occur at the same indices for any t with $0 < \mathrm{ord}_p(t) < \epsilon$ and $\mathrm{ord}_p(t)$ small enough for each coefficient of $R(T,X)$ to satisfy Lemma 4.3. $\square$

4.2. Bounded sums of roots of unity. In this section we prove our main result on the rigidity of algebraic power series. By algebraic power series we mean elements of integral extensions M of Λ. Our main result (Theorem 4.12) is the following: if an algebraic power series is a sum of at most B roots of unity when specialized at infinitely many points which

extend points of the form $T \mapsto \zeta - 1$ for p-power roots of unity ζ, then it is a power series which is a linear combination of at most B terms of the form

$$(1+T)^e = \sum_{n=0}^{\infty} \binom{e}{n} T^n$$

where for $e \in \mathbf{Z}_p$, $\binom{e}{n}$ is the usual binomial coefficient $\binom{e}{n} = \frac{e(e-1)\ldots(e-n+1)}{n!}$. We call power series of the form $(1+T)^e$ "exponential" power series. This result is inspired by the rigidity results used by Hida in his work on the relationship between Hecke fields and complex multiplication for ordinary families. For example see Lemma 5.1 and Proposition 5.2 of [17], see also [18], [19], and [20] for variations on these statements. In a different context, similar results are also used in [30] and [31] in studying p-adic families of automorphic forms over imaginary quadratic fields.

In Hida's work the need for these rigidity lemmas arises in the following way. Given infinitely many ℓ-Weil numbers of bounded degree over $\mathbf{Q}(\mu_{p^\infty})$ (ℓ a prime different from p), there are only finitely many up to equivalence (two Weil numbers are equivalent if their quotient is a root of unity, see Corollary 2.2. of [19]). Hence if F is an algebraic power series specializing to ℓ-Weil numbers at roots of unity, it must be the case that after dividing out by some Weil number we have a power series which takes values in μ_{p^∞} infinitely often. The algebraic power series F in question are those interpolating Frobenius eigenvalues across a p-ordinary family of modular forms. Applying the rigidity statement allows us to produce forms in this family with controlled Hecke fields, and from there use those forms to establish that the family has complex multiplication.

We are interested in rigidity statements that apply to algebraic power series specializing to a bounded number of roots of unity infinitely often. The main difficulty in establishing rigidity statements for algebraic power series specializing to a bounded number of roots of unity, rather than a single root of unity, is that cancellation between different terms can interfere with precise control of valuations. The following facts about quotients of rings of integers are crucial to putting limits on the possible cancellations that can occur among sums of roots of unity.

Lemma 4.8. *Suppose that $\mathcal{O}/\pi^n$ has characteristic p, i.e. n is less than or equal to the ramification index $e = [\mathcal{O} : W(\mathbf{F})]$. Then $\mathbf{F}[x]/x^n \cong \mathcal{O}/\pi^n$, with the isomorphism given by $x \mapsto \pi$.*

Proof. Since $\mathcal{O}/\pi^n$ has characteristic p, the Teichmüller lift map $\mathbf{F} \to \mathcal{O}/\pi^n$ given by $a \mapsto \lim_{m\to\infty} \widetilde{a}^{p^m}$, where $\widetilde{a}$ is any lift of a, is an algebra homomorphism. Consider the map $\mathbf{F}[x] \to \mathcal{O}/\pi^n$ given by $x \mapsto \pi$. This map is surjective since $\mathcal{O}/\pi^n$ is generated by Teichmüller lifts and π. The kernel of this map is (x^n), and so we have the claimed isomorphism. $\square$

Remark 4.9. Suppose that ζ is a primitive p^n-th root of unity, and $\mathcal{O} = W(\mathbf{F})$ is the ring of integers of an unramified extension of $\mathbf{Q}_p$. Then for $m < n$ we have that $(\zeta^{p^m} - 1) = (\zeta - 1)^{p^m}$ as ideals of $\mathcal{O}[\zeta]$ by comparing valuations. Since $\zeta^{p^m} - 1$ has positive valuation less than 1, the quotient $\mathcal{O}[\zeta]/(\zeta^{p^m} - 1)$ has characteristic p, and is a polynomial ring $\mathbf{F}[x]/x^{p^m}$ by Lemma 4.8. Moreover, by changing variables to $y = x + 1$ we see that

$$\begin{aligned} \frac{\mathbf{F}[y]}{y^{p^m} - 1} &\longrightarrow \frac{\mathcal{O}[\zeta]}{\zeta^{p^m} - 1} \\ y &\longmapsto \zeta \end{aligned}$$

is an isomorphism.

We begin by proving our main result in the special case of an algebraic power series which takes values in μ_{p^∞} infinitely often. This proof serves as a good introduction to the ideas in the proof of Theorem 4.12 while being less technical. The first appearance of this result is in [17], where two proofs are given; our strategy builds off of the second proof in [17] which Hida credits to Kiran Kedlaya.

Theorem 4.10 (Lemma 5.1 in [17])**.** *Suppose that we are given the following data:*

- *an element F in an integral extension M of Λ*
- *an infinite set $S \subset \mu_{p^\infty}$*
- *for each $\zeta \in S$, a $\overline{\mathbf{Q}}_p$ point P_ζ of M which extends the point $T \mapsto \zeta - 1$ of Λ*

with the property that for each $\zeta \in S$, $F(P_\zeta)$ is a power of ζ. Then there is a root of unity ξ' and exponent $e \in \mathbf{Z}_p$ such that $F \in \Lambda[\xi']$ and $F = \xi'(1+T)^e$.

Remark 4.11. In Hida's formulation of this result (which is stated for power series only rather than elements of integral extensions of Λ), it is only required that $F(P_\zeta) \in \mu_{p^\infty}$ for infinitely many ζ. While this may seem more general in that $F(P_\zeta)$ could potentially be a p-th root of ζ for all ζ, the control of valuations as in Remark 4.5 and Lemma 4.7 is enough to show that if we have such an $F \in \Lambda$, then in fact $F(P_\zeta)$ is a power of ζ for all ζ of sufficiently large order.

Proof. Let e_ζ be any integer such that $F(P_\zeta) = \zeta^{e_\zeta}$. Since $\mathbf{Z}_p$ is compact, the infinite set of e_ζ must have a limit point $e \in \mathbf{Z}_p$. We restrict S to a subset such that $e_\zeta \to e$ as the multiplicative order of ζ goes to ∞. Define $G(T) = (1+T)^e$. Define $H = F - G$, and let $R(T, X)$ be a monic irreducible polynomial in $\Lambda[X]$ which H satisfies.

On one hand we know from Remark 4.5 and Lemma 4.7 that there are positive rational numbers a, b such that for $\zeta \in S$ of sufficiently large order we have

$$\mathrm{ord}_p(H(P_\zeta)) = a \cdot \mathrm{ord}_p(\zeta - 1) + b.$$

On the other hand we can compute directly that if $\zeta \in S$ is of order p^n, and $e_\zeta \equiv e \bmod p^m$ for $n \geq m$, then

$$\begin{aligned}\mathrm{ord}_p(H(P_\zeta)) &= \mathrm{ord}_p(\zeta^{e_\zeta} - \zeta^e) \\ &= \mathrm{ord}_p(\zeta^{e_\zeta - e} - 1) \\ &\geq \mathrm{ord}_p(\zeta^{p^m} - 1) \\ &= p^m \cdot \mathrm{ord}_p(\zeta - 1).\end{aligned}$$

since $\zeta^{e_\zeta - e}$ has multiplicative order at most p^{n-m}. Choosing our ζ of large enough order so that $p^m > a$, we see by comparing our two expressions for $\mathrm{ord}_p(H(P_\zeta))$ that we must have $b > 0$.

Fix a k such that $b > \frac{1}{\varphi(p^k)}$ (note that $\frac{1}{\varphi(p^k)} = \mathrm{ord}_p(\zeta^{p^{n-k}} - 1)$ for ζ of order p^n). Then if $\zeta \in S$ is a primitive p^n-th root of unity for $n > k$ and e_ζ is sufficiently p-adically close to e, we have that $H(P_\zeta) = 0$ in the quotient ring $R_\zeta = \mathbf{Z}_p[\zeta]/(\zeta^{p^{n-k}} - 1)$; this follows by computing valuations since $\mathrm{ord}_p(H(P_\zeta)) \geq \mathrm{ord}_p(\zeta^{p^{n-k}} - 1)$ by the above choices. As in Remark 4.9 we have that R_ζ is isomorphic to a truncated polynomial ring $\mathbf{F}_p[y]/(y^{p^{n-k}} - 1)$ where the isomorphism sends $y \mapsto \zeta$. In order for $y^{e_\zeta} - y^e = 0$ in R_ζ it must be the case that $e_\zeta \equiv e \bmod p^{n-k}$ for all such ζ. However there are only finitely many values that $\zeta^{e_\zeta - e}$ can take if $e_\zeta \equiv e \bmod p^{n-k}$ as this must be a p^k-th root of unity. So if we choose ξ' such that $\zeta^{e_\zeta - e} = \xi'$ for infinitely many ζ, we see that $F - \xi'(1+T)^e$ is 0 when specialized at infinitely many of the P_ζ. Therefore Lemma 4.6 shows that $F = \xi'(1+T)^e$. □

We are now in place to prove the main result of this section. Before doing so we sketch the idea of the proof, which follows the same strategy as Theorem 4.10. Given an F as in Theorem 4.12, we use the density of the exponents appearing to produce a guess $G(T)$ for the form of F which is a linear combination of exponential power series. We can show that the difference $F - G$ is p-adically close to 0 under many specializations; the challenge is to show that this is because the terms of F match up with the terms of G to cancel out, rather than the terms of F cancelling out with each other. By working in the quotient ring by an appropriate power of $(\zeta - 1)$ as in Remark 4.9 we are in a polynomial ring, where we can ensure that unexpected cancellations are limited. Some cancellation between terms may still occur, but we can classify such cancellations into groups of terms which are consistently close to each other p-adically. This grouping allows us to refine our guess G, possibly reducing the value B, and repeat until we've ruled out all possible unexpected cancellations.

Theorem 4.12. *Suppose that we are given the following data:*

- *an element F in an integral extension M of Λ*

- *a constant* $B \in \mathbf{Z}_{\geq 0}$
- *an infinite set* $S \subset \mu_{p^\infty}$
- *for each* $\zeta \in S$, *a* $\overline{\mathbf{Q}}_p$ *point* P_ζ *of* M *which extends the point* $T \mapsto \zeta - 1$ *of* Λ
- *a root of unity* ξ
- *coefficients* $c_1, \ldots, c_B \in \mathbf{Z}[\xi]$

with the property that for each $\zeta \in S$, $F(P_\zeta) \in \mathbf{Z}[\xi, \zeta]$ *and* $F(P_\zeta)$ *can be written in the form*

$$F(P_\zeta) = \sum_{i=1}^{B} c_i \zeta^{e_{\zeta,i}}$$

for some exponents $e_{\zeta,i}$. *Then there is a root of unity* ξ', *coefficients* $d_i \in \mathbf{Z}[\xi']$, *and exponents* $e_i \in \mathbf{Z}_p$ *such that* $F \in \Lambda[\xi']$ *and*

$$F = \sum_{i=1}^{B} d_i (1+T)^{e_i}.$$

Proof. The proof proceeds by induction on B. If $B = 0$, we have that $F(P_\zeta) = 0$ for infinitely many points P_ζ, and Lemma 4.6 allows us to conclude that $F = 0$, which is of the desired form. The bulk of the proof is therefore to show that given such an F as in the theorem statement, we may write F in the form $G + F_1$, where G is a power series of the desired form (a linear combination of terms of the form $(1+T)^e$) and F_1 satisfies the assumptions of the theorem with a smaller value of B than that of F.

The first step is to construct a candidate expression G, and then to show that the specializations of $H = F - G$ at many of the points P_ζ are p-adically close to 0. Considering the $e_{\zeta,i}$ as integers, we have an infinite set of points in $\mathbf{Z}_p^B$. Since $\mathbf{Z}_p^B$ is compact, the set of tuples $e_{\zeta,i}$ must have a limit point $(e_1, \ldots, e_B) \in \mathbf{Z}_p^B$. Define

$$H = F - \sum_{i=1}^{B} c_i (1+T)^e$$

as an element of $M[\xi]$, and let $R(T, X) \in \Lambda[X]$ be a monic irreducible polynomial satisfied by H.

Let us restrict ourselves to an infinite subset of S such that as the multiplicative order of $\zeta \in S$ goes to infinity we have that $e_{\zeta,i} \to e_i$ for each i. We know from Lemma 4.7 that for ζ of sufficiently large multiplicative order the Newton polygon of $R(\zeta - 1, X)$ is stable, and hence the specialization $H(P_\zeta)$ for $\zeta \in S$ must have valuation determined by one of the slopes of this polygon. Passing to a further infinite subset of S we may assume that the specialization has valuation determined by a single line segment in the

stable Newton polygon, and hence for ζ of large multiplicative order we must have that

$$\mathrm{ord}_p(H(P_\zeta)) = a \cdot \mathrm{ord}_p(\zeta - 1) + b$$

for some fixed rational a and b. Since we know that the $e_{\zeta,i} \to e_i$, let us restrict to ζ of sufficiently large multiplicative order so that $e_{\zeta,i} \equiv e_i \bmod p^m$ for m chosen large enough to ensure that $p^m > a$. Then we have that

$$\begin{aligned}\mathrm{ord}_p(H(P_\zeta)) &\geq \min_i \left(\mathrm{ord}_p(c_i(\zeta^{e_{\zeta,i}} - \zeta^{e_i}))\right)\\ &= \min_i \left(\mathrm{ord}_p(c_i\zeta^{e_i}(\zeta^{e_{\zeta,i}-e_i} - 1))\right)\\ &\geq \min_i \left(\mathrm{ord}_p(c_i)\right) + \mathrm{ord}_p(\zeta^{p^m} - 1)\\ &\geq \min_i \left(\mathrm{ord}_p(c_i)\right) + p^m \cdot \mathrm{ord}_p(\zeta - 1)\end{aligned}$$

using the construction of H and our choice of ζ large enough ensuring that $e_{\zeta,i} - e_\zeta$ is divisible by p^m. Combining these two perspectives on $\mathrm{ord}_p(H(P_\zeta))$ we get that

$$a \cdot \mathrm{ord}_p(\zeta - 1) + b \geq p^m \cdot \mathrm{ord}_p(\zeta - 1) + \min_i \mathrm{ord}_p(c_i).$$

Since $p^m > a$ by construction, we see that $b > \min_i \mathrm{ord}_p(c_i)$. In particular let c be one of the coefficients achieving the minimum valuation, then for every $\zeta \in S$ we have that $c^{-1}H(P_\zeta)$ has valuation at least $v > 0$ for some constant v. Note that $c^{-1}H(P_\zeta)$ is still an element of $\mathbf{Z}_p[\xi, \zeta]$ rather than $\mathbf{Q}_p(\xi, \zeta)$ since each coefficient c_i has valuation at least that of c.

Pick a k such that $v > \frac{1}{\varphi(p^k)}$. If $\zeta \in S$ is a primitive p^n-th root of unity for $n > k$, we know that $\zeta^{p^{n-k}}$ is a primitive p^k-th root of unity, and by valuations we have that $c^{-1}H(P_\zeta) = 0$ in the quotient ring $R_\zeta = \mathbf{Z}_p[\xi, \zeta]/(\zeta^{p^{n-k}} - 1)$. As in Remark 4.9 we have that this quotient ring R_ζ is isomorphic to a truncated polynomial ring $\mathbf{F}[y]/(y^{p^{n-k}} - 1)$ where $\mathbf{F}$ is the residue field of $\mathbf{Z}_p[\xi, \zeta]$ and the isomorphism sends y to ζ. We restrict S to those ζ of large enough multiplicative order p^n so that

- if $e_i \neq e_j$, then $e_i \not\equiv e_j \bmod p^d$, and $n - k > d$,
- if $e_{\zeta,i} \equiv e_i \bmod p^m$ for each $i = 1, \ldots, B$, and $e_i \not\equiv e_j \bmod p^d$, then $n - k > m > d$. In particular this forces $e_{\zeta,i} \not\equiv e_{\zeta,j} \bmod p^{n-k}$ if $e_i \neq e_j$.

We know that $c^{-1}H(P_\zeta) = 0$ as an element of R_ζ; we also have by simply reducing the expression that

$$c^{-1}H(P_\zeta) = c^{-1}\sum_{i=1}^{B} c_i(y^{e_{\zeta,i}} - y^{e_i})$$

in R_ζ. Since this expression is equal to 0 in R_ζ and ζ is chosen large enough to ensure that the powers of y associated to $e_i \neq e_j$ cannot interact in

R_ζ (as these powers are distinct mod p^{n-k}), there must be cancellation occurring among the terms corresponding to each of the values e_i. These cancellations must be some combination of the following three possibilities:

- the coefficients $c^{-1}c_i$ are 0 in R_ζ; this cannot happen for all coefficients, as at least one of these is equal to 1 since $c = c_i$ for some $i = 1, \ldots, B$.
- a set of the coefficients c_i sums to 0 in $\mathbf{F}$, and the corresponding terms $y^{e_{\zeta,i}}$ have exponents which are congruent $\bmod p^{n-k}$ (similarly the corresponding terms y^{e_i} have exponents congruent $\bmod$ p^{n-k} which in fact implies they are equal).
- $e_{\zeta,i} \equiv e_i \bmod p^{n-k}$.

There are finitely many patterns that such cancellations can occur in, so restrict to an infinite set of S such that the same cancellation pattern occurs for each ζ in the restricted S. If two terms ζ^x and ζ^y have exponents that agree $\bmod p^{n-k}$, then $\zeta^y = \zeta^x\zeta_0$ for a root of unity ζ_0 of order dividing p^k. Since there are finitely many such ζ_0, we restrict to an infinite subset of S where, after collapsing down terms in the cancellation pattern with exponents congruent $\bmod p^{n-k}$, the pattern of p^k-th roots of unity appearing is the same. Note that this collapsing must occur at least once since not all of the coefficients $c^{-1}c_i$ are 0 in $\mathbf{F}$.

We are now in the situation where for an infinite subset of S we have that

$$F(P_\zeta) = \sum_{i=1}^{B} c_i' \zeta^{e'_{\zeta,i}}$$

where the c_i' are in $\mathbf{Z}[\xi, \zeta_0]$ for a primitive p^k-th root of unity ζ_0, and for at least some indices i the $e'_{\zeta,i}$ are either equal to $e_i \bmod p^n$ for all ζ or there are several i for which the $e_{\zeta,i}$ are equal for all ζ. Thus through a combination of combining coefficients with equal $e'_{\zeta,i}$ or subtracting off a term of the form $c_i'(1+T)^{e_i}$ we have a new $F_1 \in M[\xi, \zeta_0]$ which satisfies the assumptions of the theorem (with an enlarged integral extension M and base ring $\mathbf{Z}[\xi]$ and) with a smaller value of B. $\square$

4.3. Sums of exponential power series. In this section we collect several results on power series of the form $F(T) = \sum_{i=1}^{n} d_i(1+T)^{e_i}$. Given such a power series F, define $\pi_i = (1+p)^{e_i}$. The use of π_i is that the specializations of F that we are interested in (namely at points extending $P_{k,\zeta} : T \mapsto \zeta(1+p)^{k-1} - 1$) can all be expressed using the π_i:

$$F(P_{k,\zeta}) = \sum_{i=1}^{n} d_i \zeta^{e_i} \pi_i^{k-1}.$$

In particular $F(P_{k,\zeta}) \in \mathbf{Q}(\zeta, d_i, \pi_i)$, so if we control the field of definition of the d_i and π_i, we have control of the field of definition of $F(P_{k,\zeta})$.

The F that we will use arise from families of modular forms; in particular $F(P_{k,\zeta})$ will be related to Hecke eigenvalues of classical modular forms of weight k and character coming from ζ, and will be algebraic. Our goal in this section is to show that under assumptions of the algebraicity of d_i and $F(P_{k,\zeta})$ we have that the π_i are algebraic. We begin with the following combinatorial lemma.

Lemma 4.13. *Suppose that we are given non-zero algebraic numbers $d_1, \ldots, d_n$ and distinct p-adic integers $e_1, \ldots, e_n$. Then there are distinct p-power roots of unity $\zeta_1, \ldots, \zeta_n$ such that the matrix with entries $x_{i,j} = d_i \zeta_j^{e_i}$ has non-zero determinant.*

Proof. We proceed by induction on n, with the base case $n = 1$ being satisfied by the choice of $\zeta = 1$ since $d_1 \neq 0$.

Assume by induction that we've chosen $\zeta_1, \ldots, \zeta_m$ for some $m < n$ such that the $m \times m$ matrix with entries $x_{i,j} = d_i \zeta_j^{e_i}$ for $1 \leq i, j \leq m$ has non-zero determinant. Choose k large enough so that $e_1, \ldots, e_{m+1}$ are distinct modulo p^k and such that $\zeta_1, \ldots, \zeta_m$ are all in μ_{p^k}. Let $\widetilde{e}_i$ be the unique integer which satisfies $0 \leq \widetilde{e}_i < p^k$ and $e_i \equiv \widetilde{e}_i \bmod p^k$. Consider the $(m+1) \times (m+1)$ matrix with entries

$$y_{i,j} = \begin{cases} x_{i,j} & j \leq m \\ d_i X^{\widetilde{e}_i} & j = m+1 \end{cases}$$

where X is a formal variable. The determinant of this matrix is thus a polynomial in X which is necessarily non-zero as each matrix entry containing X appears with a different power of X so no cancellation can occur between them, and there is at least one term (the $X^{\widetilde{e}_{m+1}}$ term) which appears with a non-zero coefficient, by the inductive hypothesis guaranteeing that the upper left minor of the matrix has non-zero determinant. The degree of this determinant polynomial is one of the $\widetilde{e}_i$, hence it is strictly less than p^k. Since there are p^k roots of unity in μ_{p^k}, not every element of μ_{p^k} can be a root of this polynomial, hence there is a choice of $\zeta_{m+1} \in \mu_{p^k}$ which produces a non-zero determinant when we set $X = \zeta_{m+1}$. Note that $\zeta_{m+1} \neq \zeta_j$ for any $1 \leq j \leq m$, as that would cause two columns of the matrix to be equal (and the determinant to be 0).

Finally we conclude that this choice of ζ_{m+1} satisfies the original claim with e_i instead of $\widetilde{e}_i$: since $e_i \equiv \widetilde{e}_i \bmod p^k$, we have that $\zeta_{m+1}^{e_i} = \zeta_{m+1}^{\widetilde{e}_i}$ for each i. $\square$

This lemma is used in the following proposition to show that the π_i are algebraic, given algebraicity assumptions on the d_i and specializations of

F. The strategy of the proof is to realize (powers of) the π_i as solutions to a system of linear equations in the d_i and specializations of F.

Proposition 4.14. *Suppose that $F(T) \in \mathcal{O}[\![T]\!]$ is of the form $\sum_{i=1}^{n} d_i(1+T)^{e_i}$ for some non-zero $d_i \in \mathcal{O}$ and distinct $e_i \in \mathbf{Z}_p$. Assume further that the d_i are algebraic, and that there is an integer $k \geq 2$ such that $F(P_{k,\zeta})$ is algebraic for almost all $\zeta \in \mu_{p^\infty}$. Then $\pi_i = (1+p)^{e_i}$ is algebraic for each i.*

Proof. Applying Lemma 4.13 we see that there exists distinct p-power roots of unity $\zeta_1, \ldots, \zeta_n$ such that the system of equations

$$\begin{array}{ccccccc} F(P_{k,\zeta_1}) & = & d_1\zeta_1^{e_1}\pi_1^{k-1} & + & \ldots & + & d_n\zeta_1^{e_n}\pi_n^{k-1} \\ \vdots & = & \vdots & & \vdots & & \vdots \\ F(P_{k,\zeta_n}) & = & d_1\zeta_n^{e_1}\pi_1^{k-1} & + & \ldots & + & d_n\zeta_n^{e_n}\pi_n^{k-1} \end{array}$$

having "coefficients" $d_i\zeta_j^{e_i}$ has a unique solution (the matrix of these coefficients is invertible), the solution being the π_i^{k-1}. Note that we may choose our ζ_j of sufficiently large multiplicative order to guarantee that the $F(P_{k,\zeta})$ are algebraic.

By Cramer's rule the solutions π_i^{k-1} to the system of equations above have polynomial expressions in terms of the quantities $F(P_{k,\zeta})$ and $d_i\zeta_j^{e_i}$. As all of these quantities are algebraic, we conclude that the π_i^{k-1} are algebraic, and hence the π_i themselves are algebraic. □

5. Construction of large Galois representations

In this section we perform the construction which will allow us to propagate information about the degrees of Hecke fields between different weights in our ordinary families.

The idea behind this construction is to essentially take the trace over a character field of the Galois representations attached to a component of the ordinary Hecke algebra which contains many weight one specializations. We do this to put ourselves in a situation where the characteristic polynomials of Frobenius will have cyclotomic integer coefficients at many weight one specializations, so that the results of Subsection 4.2 apply. This will allow us to propagate the fact that we have bounded Hecke fields in weight one to higher weights, where we are allowing ourselves to utilize the Ramanujan conjecture and we may apply results of Hida to deduce that our component of the ordinary Hecke algebra has complex multiplication. This link with higher weight will occur in the next section; in this section we content ourselves with performing this trace construction and showing that the results of Subsection 4.2 apply to the resulting characteristic polynomials of Frobenius.

We set up the following notation for use in this section.

- Fix a prime p.
- For some large enough finite extension $\mathcal{O}$ of $\mathbf{Z}_p$, $\Lambda = \mathcal{O}[\![T]\!]$ is the weight space for p-ordinary Hecke algebras. We use the notation $P_{k,\zeta}$ for the map $\Lambda \to \overline{\mathbf{Q}}_p$ given by $T \mapsto \zeta(1+p)^{k-1} - 1$; where it will not cause confusion we also use $P_{k,\zeta}$ as notation for the kernel of this map.
- Fix a tame level $N \nmid p$.
- We let $\mathbf{H}^{\mathrm{ord}}$ be the Λ-adic ordinary Hecke algebra $\mathbf{H}^{\mathrm{ord}}(N;\mathcal{O})$ with tame level N.

5.1. Selecting components. Our basic assumption will be that we have a component $\mathbf{I}$ of $\mathbf{H}^{\mathrm{ord}}$ which specializes to infinitely many classical weight one eigenforms. By an extended pigeonhole principle argument, we select several more components of $\mathbf{H}^{\mathrm{ord}}$ with the property that together these components see all Galois conjugates of the classical weight one forms arising from $\mathbf{I}$.

Theorem 5.1. *Suppose that $\mathbf{I} = \mathbf{H}^{\mathrm{ord}}/\mathfrak{P}$ is a reduced, irreducible component of $\mathbf{H}^{\mathrm{ord}}$, with the property that there are infinitely many classical weight one eigenforms arising as specializations of I. Then there is an infinite set R of classical weight one eigenforms arising from $\mathbf{I}$ such that the following hold.*

(1) *There is an integer $m \leq \mathrm{rank}_\Lambda(\mathbf{H}^{\mathrm{ord}})$ such that each $f \in R$ has exactly m Galois conjugates over its character field.*
(2) *There are reduced, irreducible components $\mathbf{I}_i = \mathbf{H}^{\mathrm{ord}}/\mathfrak{P}_i$ of $\mathbf{H}^{\mathrm{ord}}$ for $i = 1, \ldots, m$ and for each $f \in R$ there is a $\overline{\mathbf{Q}}_p$-point $P_{f,i}$ of $\mathbf{I}_i$ with the following property. In some ordering $f_1, \ldots, f_m$ of the m Galois conjugates of f over its character field, the system of eigenvalues of f_i arises as the specialization of $\mathbf{I}_i$ at the point $P_{f,i}$. We may take $f_1 = f$ and $\mathbf{I}_1 = \mathbf{I}$.*
(3) *There exists a finite Galois extension $\mathrm{Frac}(M)$ of $\mathrm{Frac}(\Lambda)$ with M the integral closure of Λ, together with fixed embeddings $e_i : \mathbf{I}_i \to M$.*
(4) *For each $f \in F$ there is a $\overline{\mathbf{Q}}_p$-point P_f of M with the property that for each $i = 1, \ldots, m$, we have that*
$$P_f|_{e_i(\mathbf{I}_i)} = P_{f,i}.$$
(5) *Define $r : G_{\mathbf{Q}} \to \bigoplus_{i=1}^m \mathrm{GL}_2(\mathrm{Frac}(\mathbf{I}_i))$ to be the direct sum of the Galois representations attached to each of the $\mathbf{I}_i$. Using the embeddings e_i we can think of r as having image in $\mathrm{GL}_{2m}(\mathrm{Frac}(M))$. Denote M_{P_f} the localization of M at the kernel of P_f; then for each $f \in R$, the image of r lands in $\mathrm{GL}_{2m}(M_{P_f})$, and so can be pushed forward through P_f to obtain a representation $r_f : G_{\mathbf{Q}} \to \mathrm{GL}_{2m}(\overline{\mathbf{Q}}_p)$.*

Proof. The theorem is an extended application of the pigeonhole principle. We show the conclusions of the theorem one at a time; at each stage we refine the results from the previous step while maintaining an infinite set of weight one forms.

We know from Lemma 3.5 that any weight one form which is parameterized by $\mathbf{H}^{\mathrm{ord}}$ has at most $\mathrm{rank}_\Lambda(\mathbf{H}^{\mathrm{ord}})$ Galois conjugates over its character field. Thus from our initial set of infinitely many classical weight one forms there must be an $m \leq \mathrm{rank}_\Lambda(\mathbf{H}^{\mathrm{ord}})$ which occurs infinitely often as the number of Galois conjugates over the character field. Restrict to only those f having exactly m Galois conjugates over their character field, and call this set R_1. This shows (1).

To show (2), we use that $\mathbf{H}^{\mathrm{ord}}$ has only finitely many irreducible components since it is finite over Λ, which is irreducible. For a given $f \in R_1$, its m Galois conjugates over its character field arise from some set of m components of $\mathbf{H}^{\mathrm{ord}}$. Since there are only finitely many possible such sets of components, one must occur for infinitely many $f \in R_1$. Let R_2 be an infinite subset of R_1 for which all $f \in R_2$ have their Galois conjugates arising from the same set of m components. Pick this set of components and label them $\mathbf{I}_1, \ldots, \mathbf{I}_m$ such that (in some ordering) the conjugates $f_1, \ldots, f_m$ satisfy that f_i arises from $\mathbf{I}_i$. Without loss of generality we may assume that $\mathbf{I} = \mathbf{I}_1$ and $f = f_1$. Note that for each $f \in R_2$ we have a $\overline{\mathbf{Q}}_p$-point $P_{f,i}$ of $\mathbf{I}_i$ with the property that the specialization of $\mathbf{I}_i$ at $P_{f,i}$ is the system of Hecke eigenvalues of f_i.

The integral extension M/Λ as in (3) can be constructed as follows. Each $\mathbf{I}_i$ is an integral domain finite over Λ, so $\mathrm{Frac}(\mathbf{I}_i)$ is a finite extension of $\mathrm{Frac}(\Lambda)$. Take the Galois closure of the compositum of these fields $\mathrm{Frac}(\mathbf{I}_i)$; this is some finite extension of $\mathrm{Frac}(\Lambda)$, and we take M as the integral closure of Λ inside that field. If P is the point of $\mathbf{I}$ giving rise to f, choose any extension of P to a point P_f of M.

For a given $f \in R_2$, we have points $P_{f,i}$ of $\mathbf{I}_i$ for $i = 1, \ldots, m$. Since the Galois action is transitive on points in fibers of M/Λ, there is some embedding of $\mathbf{I}_i$ into M such that $P_{f,i}$ is the restriction of P_f to the image of $\mathbf{I}_i$. As there are only finitely many embeddings $\mathbf{I}_i \to M$ for each i, there are only finitely many possible choices total. Since for each $f \in R_2$ there is at least one choice of embeddings $\mathbf{I}_i \to M$ with the desired compatibility between points, and R_2 is infinite, there must be some choice of embeddings $e_i : \mathbf{I}_i \to M$ such that for an infinite subset $R_3 \subseteq R_2$ we have the desired compatibility of points. This shows (3) and (4) of the theorem.

As in the statement of the theorem we define r to be the representation

$$r : G_{\mathbf{Q}} \longrightarrow \bigoplus_{i=1}^{m} \mathrm{GL}_2(\mathrm{Frac}(\mathbf{I}_i)) \overset{\bigoplus_{i=1}^{m} e_i}{\hookrightarrow} \mathrm{GL}_{2m}(\mathrm{Frac}(M))$$

obtained as the direct sum of the Galois representations attached to each $\mathbf{I}_i$, viewed as having coefficients in $\mathrm{Frac}(M)$. We would like to be able to specialize $r : G_{\mathbf{Q}} \to \mathrm{GL}_{2m}(\mathrm{Frac}(M))$ through the map $P_f : M \to \overline{\mathbf{Q}}_p$ in order to recover the representations attached to each f_i; however without knowing that M is a unique factorization domain it may not be possible to find a basis of $V = \mathrm{Frac}(M)^{2m}$ in which r takes values in $\mathrm{GL}_{2m}(M)$. For a given P_f we can always extend the point P_f to have domain M_{P_f} (localization of M at the kernel of P_f), so it will suffice to show that r has coordinates in M_{P_f} for an infinite subset of R_3.

The group $G_{\mathbf{Q}}$ is compact, so there is a lattice $\mathcal{L} \subset \mathrm{Frac}(M)^{2m}$ which is stable under the action of $G_{\mathbf{Q}}$ through r. Each element of r can thus be though of equally well as an element of $\mathrm{End}_M(\mathcal{L})$. We claim that $\mathrm{End}_M(\mathcal{L})$ is a finitely generated M-module. Let $a_1, \ldots, a_n$ be a set of generators for $\mathcal{L}$ as an M-module. Given an $n \times n$ matrix $X = \{x_{i,j}\}$ with coefficients in M we say that X descends to $\mathcal{L}$ if the map given by $a_i \mapsto \sum_{j=1}^n x_{i,j} a_j$ defines an endomorphism of $\mathcal{L}$; note that for any endomorphism of $\mathcal{L}$ there is at least one such matrix. The subset of $\mathrm{End}_M(M^n)$ of those matrices which descend to $\mathcal{L}$ is an M-submodule. Since M is noetherian (it is a finite extension of the noetherian ring Λ) we have that any submodule of the finitely generated $\mathrm{End}_M(M^n)$ is finitely generated. In particular one such submodule surjects onto $\mathrm{End}_M(\mathcal{L})$, proving that it is finitely generated.

Viewing each generator of $\mathrm{End}_M(\mathcal{L})$ as an element of $\mathrm{GL}_{2m}(\mathrm{Frac}(M))$, we see that it has at most finitely many "poles", where by "pole" we mean a $\overline{\mathbf{Q}}_p$ point P of M such that the entries of that element of $\mathrm{GL}_{2m}(\mathrm{Frac}(M))$ are not in M_P. Since there are finitely many generators of $\mathrm{End}_M(\mathcal{L})$, each with finitely many poles, we see that there are at most finitely many $\overline{\mathbf{Q}}_p$ points of M which can arise as poles of an element $r(g)$. After removing finitely many of the points in R_3 in order to avoid these poles, we obtain a set R and have that the image of r lands in $\mathrm{GL}_{2m}(M_{P_f})$ for each $f \in R$. $\square$

The representation r defined in Theorem 5.1 is what we'll use to propagate control of Hecke fields from weight one into regular weight. Since our main focus will be on the specializations of r through the primes P_f of M, we set r_f to be that specialization

$$r_f : G_{\mathbf{Q}} \longrightarrow \mathrm{GL}_{2m}(\mathrm{Frac}(M)) \xrightarrow{P_f} \mathrm{GL}_{2m}(\overline{\mathbf{Q}}_p)$$

which is well-defined by part (5) of Theorem 5.1. We begin with some basic properties of the representations r and r_f.

Lemma 5.2. *The representation $r : G_{\mathbf{Q}} \to GL_{2m}(\mathrm{Frac}(M))$ satisfies the following properties.*

(1) *For primes $\ell \nmid Np$, r is unramified at ℓ.*

(2) *For every $g \in G_{\mathbf{Q}}$, the characteristic polynomial of $r(g)$ has coefficients in M rather than* $\mathrm{Frac}(M)$.
(3) *For every $f \in R$, the representation $r_f : G_{\mathbf{Q}} \to \mathrm{GL}_{2m}(\overline{\mathbf{Q}}_p)$ is equal to the direct sum of the p-adic Galois representations attached to the conjugates $f_1, \ldots, f_m$ of f.*

Proof. (1) is immediate as r is constructed as a direct sum of representations which are unramified at primes $\ell \nmid Np$. Part (2) is a consequence of continuity of the representation as we now show. The characteristic polynomial map $G_{\mathbf{Q}} \to \mathrm{Frac}(M)[X]$ given by $g \mapsto \det(XI - r(g))$ is continuous since r itself is continuous and taking characteristic polynomials is continuous. For primes ℓ as above, each of the direct summands of r has the property that characteristic polynomials of Frob_ℓ land in $\mathbf{I}_i$; the trace and determinant are Hecke operators and so are in $\mathbf{I}_i$ rather than $\mathrm{Frac}(\mathbf{I}_i)$. Since the characteristic polynomial of the direct sum is simply the product of the characteristic polynomials, we see that the characteristic polynomial of $r(\mathrm{Frob}_\ell)$ has coefficients in M. Finally since the Frob_ℓ are topologically dense in $G_{\mathbf{Q}}$ by the Cebotarev density theorem, we see that every element in the image of r must have characteristic polynomial in $M[X]$ since it is a closed subset of $\mathrm{Frac}(M)[X]$.

Part (3) of this lemma is a consequence of parts (3) and (4) of Theorem 5.1. By our choice of embeddings we have that P_f restricted to $\mathbf{I}_i$ produces the system of Hecke eigenvalues of f_i; hence $r : G_{\mathbf{Q}} \to \bigoplus_{i=1}^{2m} \mathrm{GL}_2(\mathrm{Frac}(\mathbf{I}_i))$ will specialize to the direct sum of the $\rho_{f_i,p}$. □

Since the representation r is unramified at primes $\ell \nmid Np$, we introduce notation for the characteristic polynomial of $r(\mathrm{Frob}_\ell)$. Let

$$A_\ell(X) = \sum_{j=0}^{2m} A_{\ell,j} X^j = \det(XI - r(\mathrm{Frob}_\ell)).$$

Note that the coefficients $A_{\ell,j}$ of $A_\ell(X)$ lie in M as per Lemma 5.2. The key property of the representation r is contained in the following theorem, where we show how the choices in the construction of r lead to control over the field of definition of the specializations r_f. In particular studying the specializations r_f will amount to studying the specialized characteristic polynomials of Frobenius $A_\ell(P_f)(X) = \det(XI - r_f(\mathrm{Frob}_\ell))$.

Theorem 5.3. *Suppose that $f \in R$ and P_f is the corresponding $\overline{\mathbf{Q}}_p$-point of M as in Theorem 5.1. Then for a prime $\ell \nmid Np$ we have that the characteristic polynomial $A_\ell(P_f)(X)$ of $r_f(\mathrm{Frob}_\ell)$ has coefficients in the character field of f.*

Proof. Let $f_1, f_2, \ldots, f_m$ be the m Galois conjugates of $f = f_1$. Let $\mathbf{Q}(\epsilon)$ and $\mathbf{Q}(f)$ be, respectively, the character and Hecke fields of f. For each f_i,

let α_i, β_i be the eigenvalues of $\rho_{f_i,p}(\mathrm{Frob}_\ell)$ (so that the T_ℓ-eigenvalue of f_i is equal to $\alpha_i+\beta_i$). We then have by Lemma 5.2 that $A_\ell(P_f)(X)$ factors as

$$A_\ell(P_f)(X) = \prod_{i=1}^{m}(X-\alpha_i)(X-\beta_i)$$

since r_f is the direct sum of the representations $\rho_{f_i,p}$. The coefficients $A_{\ell,j}(P_f)$ of $A_\ell(P_f)(X)$ are thus (up to sign) the elementary symmetric polynomials of degree $2m$ evaluated at the α_i and β_i.

We know that α, β are in a degree at most 2 extension of $\mathbf{Q}(f)$ since they satisfy a degree 2 polynomial with coefficients in $\mathbf{Q}(f)$ (the characteristic polynomial of $\rho_{f,p}(\mathrm{Frob}_\ell)$). For any positive integer n, the expression $\alpha^n+\beta^n$ is invariant under switching these two roots, hence it must itself be in $\mathbf{Q}(f)$. Therefore we have that the power sum $\sum_{i=1}^m \alpha_i^n+\beta_i^n$ is in $\mathbf{Q}(\epsilon)$, since it is a field trace:

$$\sum_{i=1}^{m}\alpha_i^n+\beta_i^n = \mathrm{Trace}_{\mathbf{Q}(\epsilon)}^{\mathbf{Q}(f)}(\alpha^n+\beta^n).$$

Since both the power sums and the elementary symmetric polynomials are generating sets (over $\mathbf{Q}$ or extensions thereof) for the space of symmetric polynomials, all of the elementary symmetric polynomials can be expressed as polynomial combinations with rational coefficients of the power sums. We've shown that the power sums of the α_i, β_i all lie in the character field $\mathbf{Q}(\epsilon)$, so the coefficients of $A_\ell(P_f)(X)$ will also lie in $\mathbf{Q}(\epsilon)$. □

5.2. Rigidity of Frobenius characteristic polynomials. The goal is this section is to show that the coefficients of the characteristic polynomial of $r(\mathrm{Frob}_\ell)$ are constrained as in Theorem 4.12. We've seen in the previous section that the specialized characteristic polynomials take values in cyclotomic fields (the characteristic polynomial of $r_f(\mathrm{Frob}_\ell)$ has coefficients in the character field of f). Since these characteristic polynomials are also necessarily integral, the coefficients must be *cyclotomic integers*, that is, elements of $\mathbf{Z}[\zeta]$ for some root of unity ζ. Following Cassels [6] and Loxton [25] we define the following quantities for a cyclotomic integer α:

- $N(\alpha)$ = the minimal natural number n such that α can be written as a sum of n roots of unity.
- $\overline{|\alpha|}$ = the maximum of the complex absolute values of α. This is called the *house* of α.

The following theorem of Loxton relates these two measures of the size of α; this is the key link between the bounds we know on the sizes of Hecke eigenvalues and the rigidity results of Subsection 4.2.

Theorem 5.4 (Loxton, Theorem 1 of [25])**.** *Choose a real number d with $d > \log(2)$. Then there is a positive constant c depending only on d such that if α is a cyclotomic integer with*

$$N(\alpha) = n$$

then

$$\overline{|\alpha|}^2 \geq c \cdot n \cdot \exp(-d \log(n)/\log(\log(n))).$$

As explained in the introduction of [25], this theorem allows us to bound $N(\alpha)$ if we have a bound on $\overline{|\alpha|}$. In particular if we know that

$$\overline{|\alpha|}^2 < c \cdot n \cdot \exp(-d \log(n)/\log(\log(n)))$$

then it must be the case that $N(\alpha) < n$. Since the expression $c \cdot n \cdot \exp(-d \log(n)/\log(\log(n)))$ is increasing in n, if we have an absolute bound $\overline{|\alpha|}^2$ for some collection of cyclotomic integers α, then it forces an absolute bound on the $N(\alpha)$.

Lemma 5.5. *There is a constant $C_{\ell,j}$, depending only on ℓ and j, such that for each $f \in R$ we have*

$$\overline{|A_{\ell,j}(P_f)|} \leq C_{\ell,j}.$$

Proof. This follows from the fact that the Frobenius eigenvalues α and β of $\rho_{f,p}(\mathrm{Frob}_\ell)$ have house bounded by a polynomial in ℓ by Theorem 2.10. The construction in the proof of Theorem 5.3 shows that $A_{\ell,j}(P_f)$ is a polynomial expression with rational coefficients in α and β and their Galois conjugates, which all satisfy the same Archimedean bound coming from Theorem 2.10. Applying the triangle inequality liberally to the expression for $A_{\ell,j}(P_f)$ we obtain a bound on $\overline{|A_{\ell,j}(P_f)|}$ which is polynomial in the bound on $\overline{|\alpha|}, \overline{|\beta|}$. Since this polynomial expression in α and β and their conjugates is the same for all f in R, and the bound on $\overline{|\alpha|}, \overline{|\beta|}$ is the same for all f in R, we obtain a uniform (in f) upper bound on $\overline{|A_{\ell,j}(P_f)|}$. $\square$

Lemma 5.6. *Fix a prime ℓ of F such that $\ell \nmid Np$. Then for each j in the range $0 \leq j \leq 2m$ the coefficient $A_{\ell,j}$ of the characteristic polynomial of $r(\mathrm{Frob}_\ell)$ satisfies the assumptions of Theorem 4.12.*

Proof. For any $f \in R$, we have by Theorem 5.3 that $A_{\ell,j}(P_f)$ is in the character field $\mathbf{Q}(\epsilon)$ which is a cyclotomic field. Since $A_{\ell,j}(P_f)$ is an elementary symmetric polynomial evaluated at integral inputs (the eigenvalues of $\rho_{f,p}(\mathrm{Frob}_\ell)$ are integral), it is integral, and hence is a cyclotomic integer.

Fix j, and let $C_{\ell,j}$ be the upper bound on all $\overline{|A_{\ell,j}(P_f)|}$ established in Lemma 5.5. Choose n sufficiently large so that

$$C_{\ell,j}^2 < c \cdot n \cdot \exp(-\log(n)/\log(\log(n)))$$

where c is the constant associated to $d = 1 > \log(2)$ of Theorem 5.4. Choosing such an n is possible since the function on the right is unbounded in n. Theorem 5.4 then guarantees for us that $N(A_{\ell,j}(P_f)) < n$, i.e. each $A_{\ell,j}(P_f)$ can be written as a sum of fewer than n roots of unity.

Let ξ be a root of unity that generates the tame (i.e. prime to p) part of the character fields $\mathbf{Q}(\epsilon)$ (changing f only changes the p-power roots of unity present). Among the finitely many combinations of $B < n$ and possible powers of ξ used to write an integral element of $\mathbf{Q}(\epsilon)$ as a sum of B roots of unity, we pick one that occurs infinitely often among the $A_{\ell,j}(P_f)$ for $f \in R$. For the subset R' of R where this combination occurs, we let B and $c_1, \ldots, c_B$ be the chosen values, and

$$S = \{\zeta \in \mu_{p^\infty} : \text{there is } f \in R' \text{ such that } P_f \text{ extends the point } P_{1,\zeta} \text{ of } \Lambda\}.$$

With this data $A_{\ell,j}$ satisfies the assumptions of Theorem 4.12 and hence is a linear combination of exponential power series. □

6. Bounded Hecke fields

Now that we have constructed our representation r and controlled the form of its characteristic polynomials of Frobenius using the rigidity results of Section 4, we are in a good position to specialize in regular weight. The advantage of regular weight is that we can apply results of Hida on the complexity of Hecke fields associated to non-CM ordinary families in order to conclude that our family has CM. The flavour of Hida's results is that if the Hecke fields of the forms in an ordinary family are sufficiently complicated (as measured by their degree relative to the p-cyclotomic extension $\mathbf{Q}(\mu_{p^\infty})$), then the family cannot have complex multiplication. Put another way, if the Hecke fields of an ordinary family are sufficiently bounded then that family has CM.

Hida has published several variations on theorems of this flavour. In Subsection 6.1 we sketch the proof of the version of this result that we use. We then assemble the results of Section 5 together with Hida's Hecke field result to prove our main theorem in Subsection 6.2.

We keep the notation introduced at the start of Section 5. Most importantly $\mathbf{H}^{\mathrm{ord}}$ is a Λ-adic Hecke algebra parametrizing p-ordinary modular forms having a fixed tame level and character.

6.1. Hida's results on bounded Hecke fields. Hida has proven several variations on theorems of this flavour (see [17, 18, 19, 20]). In this section we build off of [19] as it works with Hecke fields away from p (i.e. degrees of a_ℓ for $\ell \nmid Np$ rather than a_p). We found it to be more convenient to take this approach rather an approach that relies on bounding a_p, although in principle such a strategy could also work in our situation. We offer a sketch of the proof of this theorem; the main idea is to use Theorem 4.10 to

find two eigenforms f and g whose p-adic Galois representations are "too similar" unless $\mathbf{I}$ is a CM family.

Of course we could also attempt to use this result directly for elliptic modular forms of weight one, as the Ramanujan conjecture is known for these forms! However, our goal is to access our main result without utilizing the Ramanujan conjecture in low weight, so that there is the possibility of our strategy generalizing to the case of Hilbert modular forms of partial weight one.

Theorem 6.1 (Hida, Theorem 3.1 of [19])**.** *Suppose that we are given the following data:*

(1) *a set Σ of primes of $\mathbf{Q}$ of positive density*
(2) *for each $\ell \in \Sigma$ a constant $C_\ell > 0$*
(3) *an infinite set of p-power roots of unity S*
(4) *a fixed integer $k \geq 2$*
(5) *a reduced, irreducible component $\mathbf{I} = \mathbf{H}^{\mathrm{ord}}/\mathfrak{P}$ of $\mathbf{H}^{\mathrm{ord}}$*
(6) *for each $\zeta \in S$, a point $P_{k,\zeta}$ of $\mathbf{I}$ extending the point $P_{k,\zeta}$ of Λ*

with the property that for each $\zeta \in S$, the specialization $\mathbf{I}(P_{k,\zeta})$ (which is a classical modular eigenform f_ζ) has its Hecke fields satisfying the following bounds

$$[\mathbf{Q}(\mu_{p^\infty}, a_\ell(f_\zeta)) : \mathbf{Q}(\mu_{p^\infty})] \leq C_\ell$$

for each $\ell \in \Sigma$. Then $\mathbf{I}$ has complex multiplication.

Proof Sketch. For $\ell \in \Sigma$ let α_ℓ be a choice of root of the characteristic polynomial of $\rho_{\mathbf{I}}(\mathrm{Frob}_\ell)$ where $\rho_{\mathbf{I}}$ is the p-adic Galois representation attached to $\mathbf{I}$. We assume that $\mathbf{I}$ is large enough to contain α_ℓ, extending it and the points $P_{k,\zeta}$ if necessary. Since $k \geq 2$ the Ramanujan conjecture is known for the specializations f_ζ of $\mathbf{I}$ under $P_{k,\zeta}$. As a consequence of the Ramanujan conjecture we have that $\alpha_\ell(P_{k,\zeta})$ is an ℓ-Weil number. Our condition bounding the degrees of Hecke fields yields that $\alpha_\ell(P_{k,\zeta})$ has degree at most $2C_\ell$ over $\mathbf{Q}(\mu_{p^\infty})$. As there are only finitely many such ℓ-Weil numbers up to equivalence (see Corollary 2.2 of [19]), we pick π_ℓ which occurs infinitely often up to equivalence as the specialization $\alpha_\ell(P_{k,\zeta})$ for $\zeta \in S$. Thus we have that $\pi_\ell^{-1}\alpha_\ell$ specializes to a p-power root of unity infinitely often, hence by a generalization of Theorem 4.10 which allows for p-power roots of $(1+T)$(see Proposition 4.1 of [19]), α_ℓ is of the form $\pi_l(1+T)^{e_\ell}$ with $e_\ell \neq 0 \in \mathbf{Q}_p$ for each $\ell \in \Sigma$.

Choose a p-power root of unity $\zeta \neq 1$, and consider the forms f and g which arise as specialization of $\mathbf{I}$ by $P_{k,1}$ and $P_{k,\zeta}$. We assume for a contradiction that neither f nor g has complex multiplication.

We let $\alpha_\ell(f) = \alpha_\ell(P_{k,1})$ be the Frobenius eigenvalue of f produced by α_ℓ, and similarly $\alpha_\ell(g) = \alpha_\ell(P_{k,\zeta})$. By our control of α_ℓ we know that $\alpha_\ell(g) = \zeta^{e_\ell}\alpha_\ell(f)$. Since the characteristic polynomial of $\rho_f(\mathrm{Frob}_\ell)$ (with any choice

of coefficients) has constant term a power of ℓ times a root of unity, we see that a similar relationship holds with the second eigenvalue of Frobenius of each form f, g. Choosing a prime q which splits completely in $\mathbf{Q}(f, g)$ (the compositum of the Hecke fields $\mathbf{Q}(f)$ and $\mathbf{Q}(g)$) for convenience, we see that if $\zeta^m = 1$ that

$$\mathrm{Trace}(\rho_{f,q}(\mathrm{Frob}_\ell)^m) = \mathrm{Trace}(\rho_{g,q}(\mathrm{Frob}_\ell)^m)$$

for each $\ell \in \Sigma$. Using that

$$\mathrm{Trace}(\rho^m) = \mathrm{Trace}(\mathrm{Sym}^m(\rho)) - \mathrm{Trace}(\mathrm{Sym}^{m-2}(\rho) \otimes \det(\rho))$$

for any 2-dimensional representation ρ, we obtain

$$\begin{aligned}\mathrm{Trace}\Big(\mathrm{Sym}^m(\rho_{f,q}) \oplus \Big(\mathrm{Sym}^{m-2}(\rho_{g,q}) \otimes \det(\rho_{g,q})\Big)\Big)\\ = \mathrm{Trace}\Big(\mathrm{Sym}^m(\rho_{g,q}) \oplus \Big(\mathrm{Sym}^{m-2}(\rho_{f,q}) \otimes \det(\rho_{f,q})\Big)\Big)\end{aligned}$$

when evaluated on any Frob_ℓ for $\ell \in \Sigma$.

Since f and g are not CM forms by assumption, we claim that for q sufficiently large the image of their q-adic Galois representations contains an open subgroup of $\mathrm{SL}_2(\mathbf{Z}_q)$. The residual representations contain $\mathrm{SL}_2(\mathbf{F}_q)$ for large enough q (see Section 0.1 of [11]), and the classification of compact subgroups of $\mathrm{SL}_2(\mathbf{Z}_q)$ shows that we must therefore have an open subgroup of $\mathrm{SL}_2(\mathbf{Z}_q)$ in the image. In particular since the representations in question are irreducible and Σ has positive density, a result of Rajan (see Theorem 2 of [28]) guarantees that we have an equality of representations

$$\begin{aligned}\mathrm{Sym}^m(\rho_{f,q}) \oplus \Big(\mathrm{Sym}^{m-2}(\rho_{g,q}) \otimes \det(\rho_{g,q})\Big)\\ = \mathrm{Sym}^m(\rho_{g,q}) \oplus \Big(\mathrm{Sym}^{m-2}(\rho_{f,q}) \otimes \det(\rho_{f,q})\Big)\end{aligned}$$

when restricted to a finite index subgroup G_K of $G_{\mathbf{Q}}$. We also have that $\mathrm{Sym}^m(\rho_{f,q}) = \mathrm{Sym}^m(\rho_{g,q}) \otimes \chi$ for some finite order character χ, using the same result of Rajan.

Since the representations $\rho_{f,q}$ and $\rho_{g,q}$ are members of compatible systems, so are their symmetric powers. Since one member of the compatible system $\mathrm{Sym}^m(\rho_f)$ agrees with one member of the compatible system $\mathrm{Sym}^m(\rho_g) \otimes \chi$, the whole systems agree; thus we conclude that for the prime p

$$\mathrm{Sym}^m(\rho_{f,p}) = \mathrm{Sym}^m(\rho_{g,p}) \otimes \chi.$$

We know that the p-adic Galois representation of a p-ordinary form is upper triangular when restricted to the decomposition group at p. In particular we know that $\rho_{f,p}|G_{\mathbf{Q}_p}$ has the form

$$\begin{bmatrix} \omega_p^{k-1}\psi_f & * \\ 0 & \lambda_f \end{bmatrix}$$

for some characters ψ_f, λ_f; similarly $\rho_{g,p}|G_{\mathbf{Q}_p}$ is of that form with characters ψ_g, λ_g. Since the symmetric powers agree up to twist, we have an equality of sets of characters

$$\{\omega_p^{i(k-1)}\psi_f^i\lambda_f^{m-i} : i = 0, \ldots, m\} = \{\omega_p^{i(k-1)}\psi_g^i\lambda_g^{m-i}\chi : i = 0, \ldots, m\}.$$

Note that λ_f, λ_g are unramified, and ψ_f, ψ_g have finite order on inertia with $\psi_f \neq \psi_g$ on inertia by the choice of $\zeta \neq 1$. By comparing powers of the cyclotomic character which appear in the above equality of sets of characters, we conclude that $\psi_f^i\lambda_f^{m-i} = \psi_g^i\lambda_g^{m-i}\chi$ for each $i = 0, \ldots, m$. Rearranging we get that

$$\frac{\psi_f^i}{\psi_g^i} = \frac{\lambda_g^{m-i}}{\lambda_f^{m-i}}\chi$$

for each $i = 0, \ldots, m$; in particular when restricted to inertia this yields $\frac{\psi_f^i}{\psi_g^i} = \chi$ for each i. Taking $i = 0$ we see that χ is unramified, but taking $i = 1$ shows that χ must be non-trivial on inertia. This is a contradiction.

The only assumption that we made outside of the original hypotheses was that neither f nor g has CM, in order to use the fact that Galois representations attached to non-CM forms have large image. Since we arrived at a contradiction it must be the case that at least one of them has CM, and hence the whole component $\mathbf{I}$ has CM by Proposition 3.8. □

Remark 6.2. While Hida's proof is written in the case of parallel weight $k \geq 2$ Hilbert modular forms, and we've only sketched the argument in the case of elliptic modular forms, the argument applies almost verbatim to the partially ordinary families and fixed regular weights $(\underline{k}, w)$ discussed in Section 7. The only necessary difference in the partially ordinary cases comes right at the end when extracting a contradiction from the equality of sets of characters. In the partially ordinary case it is natural to work with Galois representations having a fixed determinant, rather than determinant varying with the weight as is the common choice for elliptic modular forms. Since we're working with a fixed weight and varying the Nebentypus character, there's no obstruction to still matching up characters based on the power of the cyclotomic character that appears (i.e. based on their Hodge–Tate weights). From there a slightly modified argument from the elliptic case provides a contradiction, so long as the chosen ζ has sufficiently large order.

Remark 6.3. One might ask if Theorem 6.1 can be applied to a set of classical weight one eigenforms arising from $\mathbf{I}$ and having appropriately bounded Hecke fields. There are two places where the regular weight assumption is used in the proof. First, the fact that the Frobenius eigenvalues in the Galois representation attached to a regular weight form are Weil numbers, which is a consequence of the Ramanujan conjecture. Second, the

fact that the ℓ-adic Galois representation attached to a non-CM eigenform of regular weight has large image for sufficiently large ℓ.

If we are willing to use the (known!) Ramanujan conjecture for weight one eigenforms then the first use of the regular weight assumption can be taken care of. The second presents more difficulty in generalizing directly to the weight one case. However, we expect that given the strong control over Frobenius eigenvalues across the entire component $\mathbf{I}$ provided by the Ramanujan conjecture it should be possible to start with bounded Hecke fields in weight one to establish rigidity of the Frobenius eigenvalues, and then carry out the rest of Hida's argument in regular weight.

We remark once again that the role of Section 4 and Section 5 is to provide a method by which bounds on Hecke fields may be propagated from low weight into regular weight.

6.2. Proof of the main theorem. We are finally in a position to assemble all the ingredients of the previous two sections in order to prove our main theorem. We remind the reader that the main goal is to prove that a component $\mathbf{I}$ of $\mathbf{H}^{\mathrm{ord}}$ has CM if it admits infinitely many classical weight one specializations, which we do by propagating information about Hecke fields from low weight into regular weight so that we many apply Hida's result Theorem 6.1.

Theorem 6.4. *A reduced irreducible component* $\mathbf{I}$ *of* $\mathbf{H}^{\mathrm{ord}}$ *has CM if and only if it admits infinitely many classical weight one specializations.*

Proof. If $\mathbf{I}$ has CM, then any specialization in a classical weight is a classical CM form; in particular there will be infinitely many classical weight one eigenforms arising as specializations of a CM component $\mathbf{I}$. Thus the real task in this proof is to show that if $\mathbf{I}$ admits infinitely many classical weight one specializations, then $\mathbf{I}$ has CM.

We show that the conditions of Theorem 6.1 apply if we are given such an infinite set of classical weight one specializations, which is now just a matter of assembling the ingredients of Section 4 and Section 5. In fact we'll show a stronger statement than necessary to apply Theorem 6.1. For each $\ell \nmid Np$ we produce a constant C_ℓ such that for almost all classical specializations f of $\mathbf{I}$ we have that

$$[\mathbf{Q}(\mu_{p^\infty}, a_\ell(f)) : \mathbf{Q}(\mu_{p^\infty})] \leq C_\ell.$$

We recall that the main result of Section 5 was the construction of a Galois representation $r : G_{\mathbf{Q}} \to \mathrm{GL}_{2m}(\mathrm{Frac}(M))$ for some integral extension M of Λ. The key property of r is that for some infinite set R of classical weight one specializations of $\mathbf{I}$ we have for $f \in R$ that

$$r : G_{\mathbf{Q}} \longrightarrow \mathrm{GL}_{2m}(\mathrm{Frac}(M)) \xrightarrow{P_f} \mathrm{GL}_{2m}(\overline{\mathbf{Q}}_p)$$

is well-defined, and equal to the direct sum of the p-adic Galois representations attached to the external Galois conjugates $f_1, \ldots, f_m$ of f over its character field. In Lemma 5.6 we showed that each coefficient $A_{\ell,j}$ of the characteristic polynomial $A_\ell(X) = \sum_{j=0}^{2m} A_{\ell,j} X^j$ of $r(\mathrm{Frob}_\ell)$ is controlled by the rigidity results of Section 4. Namely, for each $j = 0, \ldots, 2m$ we have an expression

$$A_{\ell,j} = \sum_{i=1}^{n_j} d_{i,j}(1+T)^{e_{i,j}}$$

for some algebraic $d_{i,j}$ and p-adic integers $e_{i,j}$.

A final key feature of the representation r is that it "sees" almost all classical specializations of $\mathbf{I}$. If f is a classical eigenform arising as the specialization of $\mathbf{I}$ through a $\overline{\mathbf{Q}}_p$ point P, denote by P again an extension of this point to M. Then for almost all P we have that the image of r lands in $\mathrm{GL}_{2m}(M_P)$ (as in part (5) of Theorem 5.1), so we can push forward r through P to obtain a representation into $\mathrm{GL}_{2m}(\overline{\mathbf{Q}}_p)$. By the construction of r as a direct sum of representations attached to components of $\mathbf{H}^{\mathrm{ord}}$, we see that $\rho_{f,p}$ is a direct summand of this specialization of r. In particular this shows that the Frobenius eigenvalues α_f, β_f of $\rho_{f,p}(\mathrm{Frob}_\ell)$ are roots of $A_\ell(P)(X)$. Since the other direct summands of this specialization of r are the Galois representations attached to other classical forms, we may also conclude that all coefficients of $A_\ell(P)(X)$ are algebraic.

We now make use of our exact formula for the $A_{\ell,j}$, as explained in Subsection 4.3. Define $\pi_{i,j} = (1+p)^{e_{i,j}}$. Since almost all specializations of $\mathbf{I}$ in weights $k \geq 2$ are classical and are witnessed by r, we have that almost all specializations $A_{\ell,j}(P_{k,\zeta})$ are algebraic, as these specializations are polynomial combinations of Frobenius eigenvalues of classical forms. Thus Proposition 4.14 shows that the $\pi_{i,j}$ are all algebraic.

We conclude that for almost all points $P_{k,\zeta}$ of $\mathbf{I}$ for $k \geq 2$ satisfy that $A_\ell(P_{k,\zeta})(X)$ has coefficients in $L_\ell(\zeta)$, where

$$L_\ell = \mathbf{Q}(\{d_{i,j}\}, \{\pi_{i,j}\}).$$

Note that L_ℓ is a finite extension of $\mathbf{Q}$ since we've adjoined finitely many algebraic quantities to $\mathbf{Q}$. Therefore for almost all f arising as specializations of $\mathbf{I}$ we have that the eigenvalues of $\rho_{f,p}(\mathrm{Frob}_\ell)$, and hence also the T_ℓ-eigenvalue $a_\ell(f)$, lie in a degree at most $2m$ extension of $L_\ell(\zeta)$, since they are roots the degree $2m$ polynomial $A_\ell(P)(X)$ which has coefficients in $L_\ell(\zeta)$. Adjoining all p-power roots of unity, we see that $a_\ell(f)$ has degree at most $2m[L_\ell : \mathbf{Q}]$ over $\mathbf{Q}(\mu_{p^\infty})$.

Define C_ℓ to be this constant

$$C_\ell = 2m[L_\ell : \mathbf{Q}]$$

which depends only on $\mathbf{I}$ and ℓ, and not on our choice of regular weight specialization.

We are now ready to apply Theorem 6.1. Fix any weight $k \geq 2$, and let S be the complement in μ_{p^∞} of the finite set of ζ where the representation r has a pole at a point above $P_{k,\zeta}$. We've established that for all primes $\ell \nmid Np$ the classical eigenforms f arising from $\mathbf{I}$ above points $P_{k,\zeta}$ for $\zeta \in S$ satisfy

$$[\mathbf{Q}(\mu_{p^\infty}, a_\ell(f)), \mathbf{Q}(\mu_{p^\infty})] \leq C_\ell.$$

This is the condition under which Theorem 6.1 applies to show that $\mathbf{I}$ has CM. $\square$

7. Hilbert modular forms of partial weight one

In this section we discuss extensions of our method to the case of partial weight one Hilbert modular forms. Our result doesn't immediately generalize to the commonly studied families of Hilbert modular forms, as one of our key tools (Corollary 2.12) breaks down for these families. However we will see that by choosing the right type of families (the "partially ordinary families") the same tools are available. We formulate several conjectures in Subsection 7.3 and Subsection 7.4 which cover what our techniques should prove given a suitable development of these partially ordinary families.

Fix a totally real field F of degree $d = [F : \mathbf{Q}]$ in which our prime p splits completely. We will discuss Hilbert modular forms for the field F.

Appearing in the transformation law for a Hilbert modular form are d weights $k_1, \ldots, k_d$ indexed by the real embeddings of F. By fixing an isomorphism $\mathbf{C} \to \mathbf{C}_p$, we get a bijection between the real and p-adic embeddings of f, so we can equally well think of the first d weights as being indexed by the p-adic places of f. It is common to introduce another weight w which serves at the level of Galois representations to fix what the determinant character should be. For elliptic modular forms this choice is usually implicitly made so that the determinant character is a finite order character times the $k-1$st power of the cyclotomic character; when considering families of Hilbert modular forms it is often convenient to be able to vary this extra weight w with the other weights to get a desired behaviour of the determinant of the associated Galois representations.

For each of the p-adic places $v_1, \ldots, v_d$ of F, there is an operator U_{v_i} acting on Hilbert modular forms of level divisible by v_i. Suppose that f is an eigenform for the operators U_{v_i}, of weight $(k_1, \ldots, k_d, w)$. Under a suitable choice of normalization for these operators depending on w, we have for a_{v_i} the eigenvalue of U_{v_i} acting on f that

$$0 \leq \mathrm{ord}_p(a_{v_i}) \leq k_i - 1.$$

For a p-adic place v of F we say that a Hilbert modular eigenform is v-ordinary if its U_v eigenvalue is a p-adic unit; we say that f is p-ordinary if it is v-ordinary for each $v|p$. Since the operators U_{v_i} commute we can equally define a p-ordinary Hilbert modular form to be one whose $U_p = \prod_{i=1}^{d} U_{v_i}$ eigenvalue is a p-adic unit.

Hida has constructed families of p-ordinary Hilbert modular forms. The weight space for these families of forms is now $d+1$-dimensional, and much of the theory that is familiar in the elliptic case is also known in the Hilbert case. See [15] and [16] for the construction of these families.

7.1. Background on partial weight one forms. Compared to the case of elliptic modular forms of weight one, very little is known about Hilbert modular forms of partial weight one. It is expected that compared to Hilbert modular forms of regular weight (i.e. those with all weights at least 2) there are relatively few classical Hilbert modular forms of partial weight one except those having CM. There are a few results known which establish the non-existence of non-CM classical partial weight one eigenforms in very specific settings, chiefly the work of Moy in [26] where it is proved that there are no non-CM classical Hilbert modular forms of weight $[k, 1]$, level $N = 1$, over $F = \mathbf{Q}(\sqrt{7})$ with quadratic Nebentypus character arising from $F(\sqrt{-1})$.

One may ask whether or not any non-CM classical Hilbert modular forms of partial weight one even exist! There is only a single known example of a classical partial weight one which does not have complex multiplication, constructed by Moy–Specter in [27].

It is reasonable to expect that finiteness results analogous to those known for elliptic modular forms of weight one should hold for Hilbert modular forms of partial weight one as well, as both of these types of objects fall under the $\ell_0 = 1$ case of the framework developed by Calegari–Geraghty in [5] for approaching modularity lifting through coherent cohomology. There is a growing body of evidence for various types of $\ell_0 = 1$ situations that forms not arising through base-change should be rare. For elliptic modular forms, finiteness of non-base-change (i.e. non-CM) forms in a fixed tame level is the result of Ghate–Vatsal which has been the main focus of this article. Regular algebraic cuspidal automorphic forms for GL_3 over $\mathbf{Q}$ also fall under the $\ell_0 = 1$ case of Calegari–Geraghty's framework; the analogous question here is whether in a given tame level all but finitely many such forms arise as symmetric squares of forms for GL_2 over $\mathbf{Q}$. This question for level 1 forms was studied by Ash–Pollack in [1], where they formulate precise conjectures and provide computational evidence in low weights. Regular algebraic cuspidal automorphic forms for GL_2 over imaginary quadratic fields provide another example of the $\ell_0 = 1$ case; see the work of Serban in [31] for some analogous results in that case.

7.2. Galois conjugates of partial weight one forms. We lay out here why the main strategy of this article does not immediately produce results for partial weight one forms. The issue lies in generalizing Corollary 2.12.

Suppose that we have a Hilbert modular eigenform f, of partial weight one. Explicitly let us say that the first r weights $k_1, \ldots, k_r$ are equal to 1, with the others $k_{r+1}, \ldots, k_d$ being greater than 1. Consider a Galois conjugate f^σ of f for some σ in the absolute Galois group of the character field of f over F. This form f^σ has the same weights as f, and its Hecke eigenvalues are the Galois conjugates by σ of those of f.

Let us assume now that our starting form f is p-ordinary. By the same argument as applied in Corollary 2.12, we have that f^σ is still v_i-ordinary at those i for which $k_i = 1$. However, all we know at the other p-adic places is that the U_{v_i} eigenvalue has valuation bounded between 0 and $k_i - 1 > 0$. This is why our method does not produce results immediately for partial weight one forms living in Hida's full p-ordinary families: Galois conjugation may ruin ordinarity at non-weight-1 places.

The fact that Galois conjugates of elliptic p-ordinary eigenforms of weight one remain p-ordinary was crucial to our strategy of characterizing families containing many of these by studying their Hecke fields. In order to apply Hida's characterization of CM families as those having bounded Hecke fields, we need some input to bound our Hecke fields in low weight, so analogs of Corollary 2.12 and Lemma 3.5 are necessary for our strategy to function.

7.3. Partially ordinary families. However, all is not lost in generalizing Corollary 2.12 to the partial weight one case. The argument outlined above does show that if f has its first r weights equal to 1, and is v_i-ordinary at those corresponding p-adic places, then appropriate Galois conjugates of f will still be v_i-ordinary for each $i = 1, \ldots, r$. Thus if we choose to work with families of forms which are v_i-ordinary for $i = 1, \ldots, r$, we can hope to recover a uniform bound on Hecke fields as in Lemma 3.5 for the forms in this family which are weight one at $v_1, \ldots, v_r$. The setback here is that these "partially ordinary" families are not at all well-studied!

In [34], a candidate for a partially ordinary Hecke algebra is constructed using the algebraic approach to Hida theory: Wilson works with algebraic automorphic forms in the Betti cohomology of quaternionic Shimura varieties of dimension 0 or 1, depending on the parity of $d = [F : \mathbf{Q}]$. This construction produces an algebra partially ordinary Hecke algebra which is torsion-free over an appropriate Iwasawa algebra, and for which the control theorem is known only up to a finite kernel. Hida's article [18] also works with partially ordinary families. Hida sketches some parts of their construction, but references [34] for many of their properties. In this article

Hida states without proof the existence of the Galois representations attached to these partially ordinary families. Partially ordinary families are also discussed briefly in [32], again working cohomologically with quaternionic automorphic forms. We should mention that similar families have been constructed independently by Yamagami in [35] and by Johansson–Newton in [21]. Both Yamagami and Johansson–Newton work with more general v-finite-slope rather than v-ordinary families, though one can think of the $\mathrm{ord}_p(U_v) = 0$ locus in the v-finite-slope eigenvariety as being the v-ordinary families that we study.

Worse than the fact that the literature on partially ordinary families is not as mature as that of fully ordinary families is the fact that all existing constructions of partially ordinary families are algebraic, rather than geometric. The algebraic description has the downside that it does not give any information about the inclusion of forms of partial weight one into these families. Since it is crucial for us to know that every suitably v-ordinary partial weight one form is included in such a family, we believe that a geometric construction of these families is needed in order to apply them in our situation.

The strategy of this article is adapted to deal with 1-dimensional families. We state below what we expect our strategy to prove given a suitable construction of 1-dimensional partially ordinary families, and in the next section we outline how one might combine the result for 1-dimensional families with a putative classicality result for partial weight one forms into a statement about families of any dimension.

Fix weights $k_2, \ldots, k_d, w$, all of which are odd and at least 3. We will consider forms of weight $(k, k_2, \ldots, k_d, w)$, and we want the fixed weights to be odd so that the partial weight one forms we work with are paritious (all of the weights have the same parity, which guarantees algebraicity of Hecke eigenvalues and GL_2 rather than PGL_2 valued Galois representations). The first weight will be the one that varies across the family while the others remain constant. We also fix a tame level $\mathfrak{N}$, which may be divisible by any of $v_2, \ldots, v_d$ but is not divisible by v_1. We suppose that we have constructed a partially ordinary Hecke algebra $\mathbf{H}$ satisfying the following properties:

- For each prime $\mathfrak{l} \nmid \mathfrak{N}v_1$ of F, there is an element $T_\mathfrak{l} \in \mathbf{H}$.
- For each prime $\mathfrak{l} | \mathfrak{N}v_1$ of F, there is an element $U_\mathfrak{l} \in \mathbf{H}$.
- $\mathbf{H}$ is a finite free $\Lambda = \mathbf{Z}_p[\![T]\!]$ module.
- $\mathbf{H}$ is the "universal Hecke algebra" parametrizing v_1-ordinary Hilbert modular eigenforms of weight $(k, k_2, \ldots, k_d, w)$ and levels $\mathfrak{N}v_1^r$. More precisely, an analog of Theorem 3.1 holds for $\mathbf{H}$.
- Associated to any component $\mathbf{I}$ of $\mathbf{H}$ is a Galois representation $G_F \to \mathrm{GL}_2(\mathrm{Frac}(\mathbf{I}))$ interpolating the Galois representations of the

eigenforms which **I** specializes to. An analog of Theorem 3.3 holds for these representations.

- Any v_1-ordinary Hilbert modular eigenform of weight $(1, k_2, \ldots, k_d, w)$ arises as a specialization of **H**. An analog of Proposition 3.4 is what is required.

Given such a family **H** we expect that the strategy of this article immediately adapts to prove the following conjecture, made in analogy with the elliptic case.

Conjecture 7.1. *Suppose that* **H** *is a partially ordinary Hecke algebra as outlined above. Then a component* **I** *of* **H** *contains infinitely many classical eigenforms of partial weight one if and only if that component has complex multiplication.*

7.4. Returning to full families. We turn now to the question of when Hida's full p-ordinary families contain many classical forms of low weight. We note that the original technique of Ghate–Vatsal does adapt to the case of families containing a Zariski dense set of classical parallel weight one forms, the details of which are worked out in [2]. We therefore focus on explaining how to recover a result characterizing full families which contain a Zariski dense set of classical partial weight one forms under the assumption of the result from the previous section for 1-dimensional families along with a suitable classicality theorem.

By analogy with elliptic forms of weight one, we expect that forms which are weight one at v and v-ordinary in fact have Galois representations which are split, rather than just upper triangular, on a decomposition group at v. It may be possible to extract this fact from the literature, though this has not been clearly stated anywhere to the author's knowledge. Moreover we expect that this splitting of the local Galois representation characterizes classical forms of partial weight one among p-adic forms. To be more precise, we expect that "classicality" results of the following type hold: suppose that f is a p-adic Hilbert modular eigenform for which

- the weight $(k_1, \ldots, k_d, w)$ of f is arithmetic (the k_i are paritious positive integers),
- for those i with $k_i > 1$, f has finite U_{v_i} slope in the range $[0, k_i - 1)$ (analog of the classical Coleman classicality condition [7]),
- and for those i with $k_i = 1$, the Galois representation attached to f is split when restricted to a decomposition group at v_i;

then (perhaps assuming some technical conditions on f and its Galois representation) f is classical.

The literature on classicality theorems for (Hilbert) modular forms is well-developed, though as yet no results have appeared for forms of partial weight one. The basic techniques used are analytic continuation and gluing

of overconvergent eigenforms, as pioneered by Buzzard–Taylor in [4], Buzzard in [3], and Kassaei in [22]. Much work has been done on extending these techniques to the Hilbert setting; the simplest case is when p splits in the extension $F/\mathbf{Q}$ where one can essentially apply the arguments of the elliptic case "prime by prime" to establish analytic continuation over a large enough region of the relevant Hilbert modular variety. We expect that once the ecosystem for dealing with partial weight one forms is sufficiently developed this prime by prime approach and the standard techniques for establishing classicality results should prove such a result for partial weight one forms.

Suppose that we have such a classicality result for partial weight one forms and a construction of partially ordinary families as outlined in the previous section. If we have a component of a full p-ordinary family which has a Zariski dense set of classical specializations which are weight one at one of the primes $v|p$, then by the Zariski density of these specializations the Galois representation attached to the component must split on a decomposition group at v. By the classicality result, we then get that *any* point of the family which is weight one v and regular weight elsewhere must be classical. In particular if we restrict to a 1-dimensional slice of our component where only the weight at v is allowed to vary, our conjecture for 1-dimensional families predicts that that slice of the full family has complex multiplication, as that 1-dimensional slice is nothing other than a component of a v-ordinary family. Since there are only finitely many possible imaginary quadratic extensions of F which a family with a given tame level could have complex multiplication by, we see that infinitely many of our 1-dimensional slices must have CM by the same imaginary quadratic E/F. But then we have that our full family must overlap with a CM family at a Zariski dense set of these 1-dimensional slices, meaning our full p-ordinary family must in fact have CM.

We note a pleasing consequence of this at the level of Galois representations: assuming the classicality criterion described above and Conjecture 7.1 the argument in the previous paragraph shows that if the Galois representation attached to a component of a full p-ordinary family splits on a decomposition group at one place $v|p$, then that component must have CM. But CM families which are p-ordinary have Galois representations which are split at *each* place above p! So if the Galois representation of a component splits at one place above p we expect that it must split at all of them.

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