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Birational Nevanlinna Constants, Beta Constants, and Diophantine Approximation to Closed Subschemes

par Paul VOJTA

Résumé. Dans un article antérieur (en commun avec Min Ru), nous avons prouvé un résultat sur l'approximation diophantienne relativement aux diviseurs de Cartier, en généralisant un résultat de 2011 de P. Autissier. Cela a été récemment étendu à certains sous-schémas fermés (à la place de diviseurs) par Ru et Wang. Dans cet article, nous étendons ce résultat à une classe de sous-schémas fermés plus large. Nous montrons également que certaines notions de $\beta(\mathscr{L}, D)$ coïncident, et qu'elles peuvent toutes être évaluées comme des limites.

Abstract. In an earlier paper (joint with Min Ru), we proved a result on diophantine approximation to Cartier divisors, extending a 2011 result of P. Autissier. This was recently extended to certain closed subschemes (in place of divisors) by Ru and Wang. In this paper we extend this result to a broader class of closed subschemes. We also show that some notions of $\beta(\mathscr{L}, D)$ coincide, and that they can all be evaluated as limits.

1. Introduction

Let k be either a number field or the field $\mathbb{C}$ of complex numbers, and let X be a complete variety over k (see Section 2 for detailed definitions). We recall the following from our earlier joint work [9] with M. Ru.

Definition 1.1 ([9, Def. 1.9])**.** Let $\mathscr{L}$ be a big line sheaf on X and let D be a nonzero effective Cartier divisor on X. Then

$$\beta(\mathscr{L}, D) = \liminf_{N\to\infty} \frac{\sum_{m=1}^{\infty} h^0(X, \mathscr{L}^N(-mD))}{N h^0(X, \mathscr{L}^N)} . \tag{1.1.1}$$

(In this paper $\mathscr{L}^N$ always means $\mathscr{L}^{\otimes N}$, the tensor power of N copies of $\mathscr{L}$.)

Theorem 1.2 ([9, "General Theorem"])**.** *Let k and X be as above, let $\mathscr{L}$ be a big line sheaf on X, and let $D_1, \dots, D_q$ be nonzero effective Cartier divisors on X that intersect properly (i.e., for any nonempty $I \subseteq \{1, \dots, q\}$*

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and any $x \in \bigcap_{i\in I} \operatorname{Supp} D_i$*, the divisors* D_i*,* $i \in I$ *are locally generated near* x *by a regular sequence in* $\mathscr{O}_{X,x}$*).*

(a) (Arithmetic part) *Assume that* k *is a number field, and let* S *be a finite set of places of* k*. Then, for all* $\epsilon > 0$*, there is a proper Zariski-closed subset* Z *of* X *such that the inequality*

$$\sum_{i=1}^{q} \beta(\mathscr{L}, D_i) m_S(D_i, x) \le (1+\epsilon) h_{\mathscr{L}}(x) + O(1) \tag{1.2.1}$$

holds for all points $x \in X(k) \setminus Z$*.*

(b) (Analytic part) *Assume that* $k = \mathbb{C}$*. Then, for all* $\epsilon > 0$*, there is a proper Zariski-closed subset* Z *of* X *such that the inequality*

$$\sum_{i=1}^{q} \beta(\mathscr{L}, D_i) m_f(D_i, r) \le_{\text{exc}} (1+\epsilon) T_{f,\mathscr{L}}(r) \tag{1.2.2}$$

holds for all holomorphic mappings $f \colon \mathbb{C} \to X$ *whose image is not contained in* Z*. The subscript "exc" means that the inequality holds for all* $r \in (0, \infty)$ *outside of a set of finite Lebesgue measure.*

Remark 1.3. Part (b) in this theorem has been strengthened so that it applies to all f whose image is not contained in Z, whereas in [9] f was required to have Zariski-dense image. This version can be obtained by revising the statements of Theorem 2.7 (see Remark 2.8), Theorem 2.11, Theorem 1.4, and the Main Theorem of [9] accordingly, where Z depends on ϵ only in the last two theorems.

The main purpose of this paper is to generalize Theorem 1.2 to replace the divisors D_i with proper closed subschemes Y_i.

Upon circulating an early version of this paper, I was informed that Ru and Wang [12] had already proved a version of Theorem 1.2 for closed subschemes. However, the version presented here is somewhat more general.

For both the work of Ru and Wang and the present paper, extending Theorem 1.2 to closed subschemes involves defining what it means for the subschemes Y_i to intersect properly. In both cases this is done using regular sequences—see Remark 3.11 and Definition 4.1. However, the details of this definition are different in the two papers, and this is the main difference between them.

For example, if X is Cohen–Macaulay (e.g., if it is nonsingular), then the Y_i intersect properly, in the sense of the present paper, if and only if (i) at each intersection point x, each of the Y_i passing through x is generated by monomials in the elements of some regular sequence in the local ring, and (ii) the subschemes Y_i are in general position (in other words, they intersect properly in the sense of intersection theory). See Definitions 3.4 and 3.9. This condition is only needed at points where two or more of the Y_i

intersect, leading to a definition that they "weakly intersect properly" (Definition 4.1 (c)). The definition of Ru and Wang uses the stronger condition that the ideals are generated by the actual elements of a regular sequence. In particular, (in the Cohen–Macaulay case) their result requires each Y_i to be a local complete intersection as a scheme, but this paper relaxes this condition somewhat—see Remark 3.11.

The generalization of Theorem 1.2 to be proved here is stated below as Theorem 1.9. This statement also describes the main theorem of Ru and Wang [12], except that it is relative to the notion of proper intersection described in Remark 3.11 instead of Definition 4.1.

Heier and Levin [3] have also proved a diophantine theorem on approximation to proper closed subschemes. In their theorem, closed subschemes of codimension r may be repeated up to r times. In this paper, as well as in the paper of Ru and Wang, however, subschemes may not be repeated. Instead, $\beta(\mathscr{L}, Y_i)$ is usually larger for such subschemes, as is the case for linear subspaces of projective space (Proposition 13.2).

The theorem of Heier and Levin is stated and discussed more thoroughly later in this Introduction (see Theorem 1.11).

The definition of $\beta(\mathscr{L}, Y)$ for a proper closed subscheme Y of X is a straightforward extension of (1.1.1):

Definition 1.4 (Ru and Wang [12, Def. 1.2])**.** Let $\mathscr{L}$ be a big line sheaf on X, let Y be a nonempty proper closed subscheme of X, and let $\mathscr{I}$ be the sheaf of ideals corresponding to Y. Then

$$\beta(\mathscr{L}, Y) = \liminf_{N\to\infty} \frac{\sum_{m=1}^{\infty} h^0(X, \mathscr{L}^N \otimes \mathscr{I}^m)}{N h^0(X, \mathscr{L}^N)} \;. \tag{1.4.1}$$

Remark 1.5. A closely related definition was given by Ru and Wang [11, Def. 1.1]:

$$\beta_{\mathscr{L},Y} = \liminf_{N\to\infty} \frac{\sum_{m=1}^{\infty} h^0(W, \pi^*\mathscr{L}^N(-mE))}{N h^0(X, \mathscr{L}^N)} \;, \tag{1.5.1}$$

where $\pi\colon W \to X$ is the blowing-up of X along Y and E is the exceptional divisor, so in particular the two definitions coincide when Y is an effective Cartier divisor. In fact, they coincide for all Y; see Ru and Wang [12, Rem. 1.3] when X is Cohen–Macaulay and Y is a local complete intersection, or Corollary 6.9 for the general case.

Remark 1.6. There is a "birational" version of Definition 1.1, in which D is replaced by a Cartier b-divisor $\mathbf{D}$. This constant is denoted $\beta(\mathscr{L}, \mathbf{D})$; see Definition 6.5. Since a proper closed subscheme is a special case of a b-divisor (see Definition 6.7), this leads to a constant $\beta(\mathscr{L}, \mathbf{Y})$ defined by a slightly different limit. This, as it turns out, has the same value as $\beta(\mathscr{L}, Y)$ and $\beta_{\mathscr{L},Y}$—see Corollary 6.9. Note also that in Definitions 1.1, 1.5, 6.5, and 1.4, the lim inf can be replaced by a limit whenever $\operatorname{char} k = 0$. This is proved in Section 7.

Another major goal of this paper is to show that these three beta constants all coincide (Corollary 6.9).

Remark 1.7. The proof of Theorem 1.9 uses $\beta(\mathscr{L}, Y)$. This is because the Autissier property (see below) is not preserved by blowing up, so we work on X (not on a model W).

Remark 1.8. It is possible to let $\mathbf{D}$ be an $\mathbb{R}$-Cartier b-divisor in the definition of $\beta(\mathscr{L}, \mathbf{D})$. We have not done that here, though, as it would not provide any benefit (so far), but would involve additional complexity.

One defines Weil functions relative to proper closed subschemes Y on X by blowing up X along Y to obtain a Cartier divisor on the blow-up; see for example Silverman [13, 2.2] or Yamanoi [18, 2.2], in combination with [13, Thm. 2.1 (h)]. These can then be used to define proximity and counting functions for Y (see Section 12).

Another goal of this paper (suggested by the referee, to whom I am very thankful) stems from the following fact. In Schmidt's Subspace Theorem, it is essential for many applications that the set of hyperplanes be allowed to vary with the places $v \in S$. In Nevanlinna theory, though, where the set corresponding to S is infinite, it is more natural to take finitely many collections of hyperplanes, and then take the maximum of the corresponding Weil functions at each value of $\theta \in [0, 2\pi]$ when defining the proximity function. This is also the case for Roth's theorem over arithmetic function fields [17, Thm. 4.5], where the set of archimedean places is infinite (unless the field is a number field). Of course, when S is finite, the two formulations are equivalent, by a standard pigeonhole argument.

Having provided this background, the main theorem of this paper is as follows.

Theorem 1.9. *Let X be a complete variety over a field k, let $\mathscr{L}$ be a big line sheaf on X, and let $p \in \mathbb{Z}_{>0}$. For each $i = 1, \dots, p$ let $Y_{i,1}, \dots, Y_{i,q_i}$ be proper closed subschemes of X that weakly intersect properly (see Definition 4.1).*

(a) (Arithmetic part) *Assume that k is a number field, let S be a finite set of places of k, and for all i and j and all $v \in S$ let $\lambda_{Y_{i,j},v}$ be a Weil function for $Y_{i,j}$ at v. Then, for all $\epsilon > 0$ and all $C \in \mathbb{R}$, there is a proper Zariski-closed subset Z of X such that the inequality*

$$\frac{1}{[k:\mathbb{Q}]} \sum_{v \in S} \max_i \sum_j \beta(\mathscr{L}, Y_{i,j}) \lambda_{Y_{i,j},v}(x) \le (1+\epsilon) h_{\mathscr{L}}(x) + C \tag{1.9.1}$$

holds for all points $x \in X(k) \setminus Z$.

(b) (Analytic part) *Assume that* $k = \mathbb{C}$. *For all* i *and* j *let* $\lambda_{Y_{i,j}}$ *be a Weil function for* $Y_{i,j}$ *on* $X(\mathbb{C})$. *Then, for all* $\epsilon > 0$, *there is a proper Zariski-closed subset* Z *of* X *such that the inequality*

$$\int_0^{2\pi} \max_i \sum_j \beta(\mathscr{L}, Y_{i,j}) \lambda_{Y_{i,j}}(f(re^{\sqrt{-1}\theta})) \frac{\mathrm{d}\theta}{2\pi} \le_{\text{exc}} (1+\epsilon) T_{f,\mathscr{L}}(r) \tag{1.9.2}$$

holds for all holomorphic mappings $f \colon \mathbb{C} \to X$ *whose image is not contained in* Z.

This theorem reduces to Theorem 1.2 when $p = 1$ and $Y_{1,1}, \dots, Y_{1,q_1}$ are Cartier divisors $D_1, \dots, D_q$, respectively.

Diophantine inequalities for closed subschemes have already been obtained by other authors. For example, Ru and Wang [11] proved the inequality

$$\sum_{i=1}^{q} m_S(Y_i, x) \le \left(\ell \max_i \beta(\mathscr{L}, Y_i)^{-1} + \epsilon \right) h_{\mathscr{L}}(x) ,$$

where at most ℓ of the Y_i have nonempty intersection. This overlaps with our results here, but is not fully implied by our Theorem 1.9 because the latter theorem requires that the Y_i weakly intersect properly, but Ru and Wang only need the condition involving ℓ.

As noted earlier, Heier and Levin [3] also have an inequality involving closed subschemes. Their theorem again has weaker conditions on the Y_i (and in fact it allows some Y_i of codimension > 1 to be repeated).

Their theorem relies on a notion of "general position" for closed subschemes which is different from that used here.

Definition 1.10. Let X be a projective variety of dimension n. We say that closed subschemes $Y_1, \dots, Y_q$ of X are in *HL-general position* if $\operatorname{codim} \bigcap_{i \in I} Y_i \ge |I|$ for all $I \subseteq \{1, \dots, q\}$ with $|I| \le n+1$, using the convention that $\operatorname{codim} \emptyset = n+1$.

Theorem 1.11 (Heier and Levin [3, Thm. 1.3])**.** *Let* X *be a projective variety of dimension* n *over a number field* k, *and let* S *be a finite set of places of* k. *For each* $v \in S$ *let* $Y_{0,v}, \dots, Y_{n,v}$ *be closed subschemes of* X *in HL-general position. Let* $\mathscr{A}$ *be an ample line sheaf on* X, *and let* $\epsilon > 0$. *Then there exists a proper Zariski-closed subset* Z *of* X *such that for all points* $x \in X(k) \setminus Z$,

$$\sum_{v \in S} \sum_{i=0}^{n} \epsilon_{Y_{i,v},\mathscr{A}} \lambda_{Y_{i,v},v}(x) < (n+1+\epsilon) h_{\mathscr{A}}(x) . \tag{1.11.1}$$

Here $\epsilon_{Y,\mathscr{A}}$ denotes the Seshadri constant of the closed subscheme Y with respect to $\mathscr{A}$ (see [3, Def. 2.3]) and $\lambda_{Y,v}$ denotes the local Weil function for Y at v (see our earlier joint work with M. Ru [9, §2.3]).

For simplicity of notation, we compare Theorems 1.9 and 1.11 only in the special case $p = 1$. For Theorem 1.11, this means that $Y_1, \dots, Y_q$ are closed subschemes of X in HL-general position, and (1.11.1) reduces to

$$\sum_{i=1}^{q} \epsilon_{Y_i,\mathscr{A}} m_S(Y_i, x) < (n+1+\epsilon) h_{\mathscr{A}}(x) \ . \tag{1.12}$$

For Theorem 1.9, $Y_1, \dots, Y_q$ are closed subschemes of X that weakly intersect properly, and the left-hand side of (1.9.1) can be replaced by $\sum_{i=1}^{q} \beta(\mathscr{L}, Y_i) m_S(Y_i, x)$.

On the one hand, Theorem 1.11 is much stronger than Theorem 1.9, because HL-general position is more lenient (for example, some of the Y_i may be proper subschemes of others).

On the other hand, assume that X is nonsingular, that each Y_i occurs exactly $\operatorname{codim} Y_i$ times in the list $Y_1, \dots, Y_q$, and that each Y_i is a locally complete intersection. Then Theorem 1.11 is a consequence of Theorem 1.9 (a). Indeed, by Heier and Levin [3, Thm. 4.2], if Y is a closed subscheme of codimension $r > 0$, then

$$\frac{r}{n+1} \epsilon_{Y,\mathscr{A}} \le \beta_{\mathscr{A},Y} \ . \tag{1.13}$$

(This is proved using Autissier's method.) Let $Z_1, \dots, Z_{q'}$ be the distinct elements of $\{Y_1 \dots, Y_q\}$. Then $Z_1, \dots, Z_{q'}$ intersect properly. By (1.13), Corollary 6.9, and Theorem 1.9 (a), we then have

$$\begin{aligned} \sum_{i=1}^{q} \epsilon_{Y_i,\mathscr{A}} m_S(Y_i, x) &= \sum_{j=1}^{q'} (\operatorname{codim} Z_j) \epsilon_{Z_j,\mathscr{A}} m_S(Z_j, x) \\ &\le (n+1) \sum_{j=1}^{q'} \beta_{\mathscr{A},Z_j} m_S(Z_j, x) \\ &\le (n+1+\epsilon) h_{\mathscr{A}}(x) + 1 \\ &< (n+1+\epsilon) h_{\mathscr{A}}(x) \end{aligned}$$

for all $x \in X(k)$ outside of a proper Zariski-closed subset.

Of course, this is a very special case of Theorem 1.11.

See also Theorem 13.3.

Theorem 1.9 will be proved by splitting it up into two theorems, involving a property due originally to Autissier [1, Lem. 3.3]; see also Lemma 3.3. This will be expressed by saying that closed subschemes $Y_1, \dots, Y_q$ have the *Autissier property*; see Definition 4.2.

These two theorems are the following.

Theorem 1.14. *Let X be a complete variety over a field k, and let $Y_1, \dots, Y_q$ be proper closed subschemes of X. If $Y_1, \dots, Y_q$ weakly intersect properly, then they have the Autissier property.*

Theorem 1.15. *Let k be either a number field or the field $\mathbb{C}$, let X be a complete variety over k, let $\mathscr{L}$ be a big line sheaf on X, let $p > 0$, and for each $i = 1, \dots, p$ let $Y_{i,1}, \dots, Y_{i,q_i}$ be proper closed subschemes of X that have the Autissier property. Then the conclusion of part* (a) *or* (b) *of Theorem 1.9 holds, if k is a number field or if $k = \mathbb{C}$, respectively.*

It is clear that the conjunction of these theorems implies Theorem 1.9.

The outline of the paper is as follows. Section 2 briefly gives some fundamental definitions used in the paper. Sections 3 and 4 give a version of Autissier's lemmas on ideals associated to saturated subsets of $\mathbb{N}^r$, in the local and global cases, respectively, leading up to the proof of Theorem 1.14 (see Proposition 4.3) and the proof in Section 5 that his function $N(\mathbf{t})$ is convex. These constitute the key insight of the paper. Section 6 develops the machinery that will be used to work with the ideal sheaves associated to the closed subschemes Y_i in the paper. Section 7 gives a detailed proof of the fact that the limits infima in (1.1.1), (1.4.1), (1.5.1), and (6.5.1) (the definitions of $\beta(\mathscr{L}, D)$, $\beta(\mathscr{L}, Y)$, $\beta_{\mathscr{L},Y}$, and $\beta(\mathscr{L}, \mathbf{D})$, respectively) can be replaced by limits (in characteristic 0). This is needed in order to prove Theorem 1.15 (and therefore Theorem 1.9) when $p > 1$. In particular, the fact that the definition of $\beta(\mathscr{L}, Y)$ converges as a limit (Corollary 7.4) is needed in order to prove Corollary 9.12, which in turn is used in the proof of Theorem 10.4. Section 8 adapts work of Autissier [1], as modified in joint work [9] with M. Ru, to the current context, finishing the technical parts of the proof. Section 9 gives some more information on the structure of the group of $\mathbb{R}$-Cartier b-divisors on a variety, and then restates the main result of Section 8 in these terms. Sections 10 and 11 define "multidivisor" versions of the birational Nevanlinna constant and the proximity function for $\mathbb{R}$-Cartier b-divisors, extending the definitions (for $\mathbb{R}$-Cartier divisors) of Ru and the author [10]. Section 12 then finishes the proof of Theorem 1.9. Finally, Section 13 explores the special case of linear subvarieties of $\mathbb{P}^n$.

I thank Min Ru for suggesting the idea of extending the Main Theorem of [9] to subschemes. I also thank the referee for many suggestions, including a much shorter proof of Proposition 13.2 and the idea of allowing $p > 1$ in Theorem 1.9.

2. Basic Notation and Conventions

The basic notations of this paper follow those of our earlier work with M. Ru [9, 10].

In this paper $\mathbb{N} = \{0, 1, 2, \dots\}$. Also $\mathbb{Z}_{>0} = \{1, 2, 3, \dots\}$, $\mathbb{R}_{\geq 0} = \{x \in \mathbb{R} : x \geq 0\}$, etc.

A *variety* over a field k is an integral scheme, separated and of finite type over k. A *morphism* of varieties over k is a morphism of schemes over k.

Subschemes will always be assumed to be closed and proper (i.e., not the whole scheme).

3. A Property of Autissier

This section extends [1, Lem. 3.3] to accommodate subschemes of higher codimension.

This lemma motivates a definition of a property of subschemes, which basically says that they satisfy the conclusion of this lemma. This property will be called the *Autissier property*; see Definitions 3.12 and 4.2. The entire remainder of the proof of Theorem 1.9 hinges on this property.

Throughout this section, A is a noetherian local ring.

We start by recalling some definitions and a lemma of Autissier [1].

Definition 3.1. Let $r \in \mathbb{Z}_{>0}$. A subset N of $\mathbb{N}^r$ is *saturated* if it is nonempty and if $N \supseteq \mathbf{a} + \mathbb{N}^r$ for all $\mathbf{a} \in N$.

Definition 3.2. Let $\phi_1, \dots, \phi_r \in A$ with $r > 0$, and let N be a saturated subset of $\mathbb{N}^r$. Then $\mathscr{I}(N)$ is the ideal of A generated by the set $\{\phi_1^{b_1} \dots \phi_r^{b_r} : \mathbf{b} \in N\}$.

The key fact about this definition is the following lemma due to Autissier.

Lemma 3.3 ([1, Lem. 3.3])**.** *Let $\phi_1, \dots, \phi_r$ ($r > 0$) be a regular sequence in A, and let N_1 and N_2 be saturated subsets of $\mathbb{N}^r$. Then*

$$\mathscr{I}(N_1 \cap N_2) = \mathscr{I}(N_1) \cap \mathscr{I}(N_2)\ .$$

Now we carry the above over to the situation of ideals in A.

Definition 3.4. Let I be an ideal of A and let $\phi_1, \dots, \phi_r$ be a sequence of elements of A. Then I is of *monomial type* with respect to $\phi_1, \dots, \phi_r$ if $r > 0$ and $I = \mathscr{I}(N)$ (taken relative to $\phi_1, \dots, \phi_r$) for some saturated subset N of $\mathbb{N}^r$.

Note that if I is of monomial type with respect to some sequence $\phi_1, \dots, \phi_r$, then so is I^n for all $n \in \mathbb{N}$. This is immediate from the following lemma.

Lemma 3.5. *Let $r \in \mathbb{Z}_{>0}$ and let N be a saturated subset of $\mathbb{N}^r$. For all $n \in \mathbb{N}$ let*

$$nN = \begin{cases} \mathbb{N}^r & \text{if } n = 0; \\ \{\mathbf{b}_1 + \dots + \mathbf{b}_n : \mathbf{b}_1, \dots, \mathbf{b}_n \in N\} & \text{if } n > 0\ . \end{cases} \tag{3.5.1}$$

(When $n > 0$ this is the Minkowski sum of N with itself n times.) Then

(a) *nN is saturated for all n;*
(b) *$\mathscr{I}(N)^n = \mathscr{I}(nN)$ for all n; and*
(c) *$nN \subseteq mN$ for all $n \ge m \ge 0$.*

Proof. Left to the reader. □

As a counterpart to Definition 3.2, but with closed subschemes in place of Cartier divisors, we have the following.

Definition 3.6. Let $q \in \mathbb{Z}_{>0}$, let $I_1, \dots, I_q$ be ideals in A, and let N be a saturated subset of $\mathbb{N}^q$. Then $\mathscr{J}(N)$ is the ideal of A defined by

$$\mathscr{J}(N) = \sum_{\mathbf{b} \in N} I_1^{b_1} \cdots I_q^{b_q} \,.$$

This can be expressed in terms of $\mathscr{I}(\,\cdot\,)$ as follows.

Definition 3.7. Let $q \in \mathbb{Z}_{>0}$. For each $i = 1, \dots, q$ let M_i be a saturated subset of $\mathbb{N}^{r_i}$ with $r_i \in \mathbb{Z}_{>0}$. For all saturated subsets N of $\mathbb{N}^q$, we then define

$$M(N) = \bigcup_{\mathbf{c} \in N} c_1 M_1 \times \cdots \times c_q M_q \,. \tag{3.7.1}$$

This is a saturated subset of $\mathbb{N}^r$, where $r = r_1 + \cdots + r_q$.

Lemma 3.8. *Let $q \in \mathbb{Z}_{>0}$. For each $i = 1, \dots, q$, let M_i be a saturated subset of $\mathbb{N}^{r_i}$ and let $I_i = \mathscr{I}(M_i)$, taken relative to a nonempty sequence $\phi_{i1}, \dots, \phi_{ir_i}$ in A. Let N be a saturated subset of $\mathbb{N}^q$. Then*

$$\mathscr{J}(N) = \mathscr{I}(M(N)) \,,$$

where $\mathscr{J}(N)$ is taken with respect to $I_1, \dots, I_q$ and $\mathscr{I}(M(N))$ is taken with respect to the sequence

$$\phi_{11}, \dots, \phi_{1r_1}, \dots, \phi_{q1}, \dots, \phi_{qr_q} \,. \tag{3.8.1}$$

Proof. This is immediate from Definitions 3.2, 3.6, and 3.7. See also Ru and Wang [12, Lem. 3.3]. □

We can now state the main definitions and main result of this section.

Definition 3.9. Let $I_1, \dots, I_q$ be ideals of A, with $q \in \mathbb{N}$. Then $I_1, \dots, I_q$ *intersect properly* if (i) for each $i = 1, \dots, q$ there is a nonempty regular sequence $\phi_{i1}, \dots, \phi_{ir_i}$ in A such that I_i is of monomial type with respect to $\phi_{i1}, \dots, \phi_{ir_i}$; and (ii) the combined sequence (3.8.1) is a regular sequence.

Remark 3.10. Since the length of the sequence (3.8.1) is at most $\dim A$, we must have $q \le \dim A$ whenever $I_1, \dots, I_q$ intersect properly.

Remark 3.11. Ru and Wang [12] say that $I_1, \dots, I_q$ intersect properly if, in the notation of Definition 3.9, I_i is generated by $\phi_{i1}, \dots, \phi_{ir_i}$ for all i (and (3.8.1) is a regular sequence). In other words, this is the special case of Definition 3.9 in which the subset N of Definition 3.4 equals $\mathbb{N}^r \setminus \{\mathbf{0}\}$. One then obtains their notion of subschemes $Y_1, \dots, Y_q$ intersecting properly by using this definition in place of Definition 3.9 in Definition 4.1.

As an example, assume that X contains an open subset isomorphic to $\mathbb{A}_k^2 = \operatorname{Spec} k[x,y]$. Then x, y is a regular sequence in the local ring at $(0,0)$, so $I = (x, y)$ satisfies the hypotheses of Ru and Wang's theorem (and it is also of monomial type with respect to x and y). The ideal (x^3, xy, y^2), however, does not satisfy their condition, but it is of monomial type in x, y, so it can be handled by Theorem 1.9.

Definition 3.12. Let $I_1, \dots, I_q$ be ideals in A. We say that they have the *Autissier property* if

$$\mathscr{I}(N \cap N') = \mathscr{I}(N) \cap \mathscr{I}(N') \tag{3.12.1}$$

for all saturated subsets N and N' of $\mathbb{N}^q$.

Proposition 3.13. *Let $I_1, \dots, I_q$ be ideals in A. If they intersect properly, then they have the Autissier property.*

Proof. By Lemmas 3.8 and 3.3, we immediately reduce to showing that

$$M(N \cap N') = M(N) \cap M(N') \,. \tag{3.13.1}$$

To prove this, we first need some basic facts on the product ordering on $\mathbb{N}^q$.

Recall that the product ordering on $\mathbb{N}^q$ is defined by $\mathbf{a} \le \mathbf{b}$ if and only if $a_i \le b_i$ for all $i = 1, \dots, q$. This ordering is a lattice; in particular, for any $\mathbf{a}, \mathbf{b} \in \mathbb{N}^q$, the join, or least upper bound, of $\mathbf{a}$ and $\mathbf{b}$ is the element $\mathbf{a} \vee \mathbf{b} = \mathbf{c} \in \mathbb{N}^q$ defined by $c_i = \max\{a_i, b_i\}$ for all i.

Now we note that if N and N' are saturated subsets of $\mathbb{N}^q$, then

$$\{\mathbf{c} \vee \mathbf{c}' : \mathbf{c} \in N, \mathbf{c}' \in N'\} = N \cap N' \,. \tag{3.13.2}$$

Indeed, the inclusion "$\supseteq$" is immediate by taking $\mathbf{c}' = \mathbf{c}$ for all $\mathbf{c} \in N \cap N'$. Conversely, if $\mathbf{c}'' = \mathbf{c} \vee \mathbf{c}'$ with $\mathbf{c} \in N$ and $\mathbf{c}' \in N'$, then $\mathbf{c}'' \in N$ and $\mathbf{c}'' \in N'$ because N and N' are saturated (respectively); hence $\mathbf{c}'' \in N \cap N'$.

Then, by (3.7.1), distributivity of $\cap$ over $\cup$, compatibility of intersection and product, Lemma 3.5 (c), (3.13.2), and (3.7.1) again, we have

$$\begin{aligned} M(N) \cap M(N') &= \left(\bigcup_{\mathbf{c} \in N} c_1 M_1 \times \cdots \times c_q M_q \right) \cap \left(\bigcup_{\mathbf{c}' \in N'} c'_1 M_1 \times \cdots \times c'_q M_q \right) \\ &= \bigcup_{\substack{\mathbf{c} \in N \\ \mathbf{c}' \in N'}} \left((c_1 M_1 \times \cdots \times c_q M_q) \cap (c'_1 M_1 \times \cdots \times c'_q M_q) \right) \\ &= \bigcup_{\mathbf{c}, \mathbf{c}'} \left((c_1 M_1 \cap c'_1 M_1) \times \cdots \times (c_q M_q \cap c'_q M_q) \right) \end{aligned}$$

$$\begin{aligned}
&= \bigcup_{\mathbf{c},\mathbf{c}'} \max\{c_1, c_1'\}M_1 \times \cdots \times \max\{c_q, c_q'\}M_q \\
&= \bigcup_{\mathbf{c}''\in N\cap N'} c_1''M_1 \times \cdots \times c_q''M_q \\
&= M(N\cap N')\ .
\end{aligned}$$

This gives (3.13.1). ☐

Turning to consequences of the Autissier property, in the local setting we only need the following.

Proposition 3.14 ([1, Rem. 3.4] and [9, Rem. 6.3])**.** *Let $q \in \mathbb{Z}_{>0}$, let*

$$\square = \mathbb{R}^q_{\ge 0} \setminus \{\mathbf{0}\}\ , \tag{3.14.1}$$

and for all $\mathbf{t} \in \square$ and all $x \in \mathbb{R}_{\ge 0}$ let

$$N(\mathbf{t}, x) = \{\mathbf{b} \in \mathbb{N}^q : t_1b_1 + \cdots + t_qb_q \ge x\}\ . \tag{3.14.2}$$

Let $I_1, \dots, I_q$ be ideals in A that have the Autissier property. Then

$$\mathscr{I}(N(\mathbf{t}, x)) \cap \mathscr{I}(N(\mathbf{u}, y)) \subseteq \mathscr{I}(N(\lambda\mathbf{t} + (1-\lambda)\mathbf{u}, \lambda x + (1-\lambda)y)) \tag{3.14.3}$$

for all $\mathbf{t}, \mathbf{u} \in \square$, all $x, y \in \mathbb{R}_{\ge 0}$, and all $\lambda \in [0, 1]$.

Proof. This is immediate from Definition 3.12 and the observation that

$$N(\mathbf{t}, x) \cap N(\mathbf{u}, y) \subseteq N(\lambda\mathbf{t} + (1-\lambda)\mathbf{u}, \lambda x + (1-\lambda)y)\ . \qquad \square$$

Remark 3.15. An interesting theory of regular sequences of ideals has been developed by Jothilingham, et al. [4]. In this theory, ideals $I_1, \dots, I_q$ of A are said to be a *regular sequence of ideals* if all of them are nonzero and proper, and if

$$(I_1 + \cdots + I_j) \cap I_{j+1} = (I_1 + \cdots + I_j)\,.\,I_{j+1}$$

for all $j = 1, \dots, q-1$. This extends the definition of a regular sequence of elements of a local ring, in the sense that a sequence $(x_1), \dots, (x_q)$ of principal ideals in A is regular if and only if the elements $x_1, \dots, x_q$ form a regular sequence.

Although it was very tempting to write this paper using the concept of regular sequences of ideals, ultimately we decided not to. This was because many of the results of [4] (e.g., Theorem 1) required A to be a regular local ring; in addition, there were other difficulties in trying to rewrite the proof of [1, Lem. 6.2] directly in terms of a regular sequence of ideals.

4. The Autissier Property of Subschemes

This brief section carries over Definitions 3.9 and 3.12 and Proposition 3.13 to the case of subschemes.

First we start with the definitions.

Throughout this section, X is a complete variety over a field k and $Y_1, \dots, Y_q$ are proper closed subschemes of X.

Definition 4.1. Let $\mathscr{I}_1, \dots, \mathscr{I}_q$ be the ideal sheaves that correspond to $Y_1, \dots, Y_q$, respectively.

(a) We say that $Y_1, \dots, Y_q$ *intersect properly* at a point $P \in X$ if the subsequence of proper ideals in the sequence $(\mathscr{I}_1)_P, \dots, (\mathscr{I}_q)_P$ of ideals of the local ring $\mathscr{O}_{X,P}$ intersect properly (in the sense of Definition 3.9). (If the subsequence is trivial, i.e., if $P \notin \bigcup Y_i$, then this is vacuously true.)

(b) We say that $Y_1, \dots, Y_q$ *intersect properly* if $Y_1, \dots, Y_q$ intersect properly at all points of X.

(c) We say that $Y_1, \dots, Y_q$ *weakly intersect properly* if they intersect properly at all $P \in \bigcup_{i \ne j}(Y_i \cap Y_j)$.

Clearly, if $Y_1, \dots, Y_q$ intersect properly, then they also weakly intersect properly.

Definition 4.2. Let $\mathscr{I}_1, \dots, \mathscr{I}_q$ be as in Definition 4.1.

(a) Let $P \in X$, and let $j_1, \dots, j_r$ be the subsequence of $1, \dots, q$ consisting of those j such that $P \in Y_j$. We say that $Y_1, \dots, Y_q$ have the *Autissier property* at P if

$$\mathscr{J}(N \cap N') = \mathscr{J}(N) \cap \mathscr{J}(N') \tag{4.2.1}$$

for all saturated subsets N and N' of $\mathbb{N}^r$, where $\mathscr{J}$ is taken with respect to the sequence $(\mathscr{I}_{j_1})_P, \dots, (\mathscr{I}_{j_r})_P$ of (proper) ideals of $\mathscr{O}_{X,P}$. (This is equivalent to saying that $(\mathscr{I}_{j_1})_P, \dots, (\mathscr{I}_{j_r})_P$ have the Autissier property as in Definition 3.12.)

(b) We say that $Y_1, \dots, Y_q$ have the *Autissier property* if they have the Autissier property at all $P \in X$.

Corresponding to Proposition 3.13, we then have the following, which is Theorem 1.14.

Proposition 4.3. *If $Y_1, \dots, Y_q$ weakly intersect properly, then they have the Autissier property.*

Proof. First, note that if $P \in X \setminus \bigcup_{i \ne j}(Y_i \cap Y_j)$; i.e., if $P \in X$ lies in at most one of the Y_i, then the Autissier property holds trivially at P, because (4.2.1) is trivial when $r \le 1$.

For all $P \in \bigcup_{i\ne j}(Y_i \cap Y_j)$, we then have that $Y_1,\dots,Y_q$ intersect properly at P; therefore they have the Autissier property at P by Proposition 3.13. $\square$

(Of course, if $Y_1,\dots,Y_q$ intersect properly, then the first paragraph of the above proof is unnecessary.)

5. Filtrations and Convexity

This section summarizes the core of Autissier's argument in [1], as adapted for working with subschemes.

Throughout this section, we fix a complete variety X over a field k and proper closed subschemes $Y_1,\dots,Y_q$ of X. Let $\mathscr{I}_1,\dots,\mathscr{I}_q$ be the sheaves of ideals in $\mathscr{O}_X$ corresponding to $Y_1,\dots,Y_q$, respectively.

We start with some definitions.

Definition 5.1. Let $\square$ and $N(\mathbf{t},x)$ be as in Proposition 3.14.

(a) Let N be a saturated subset of $\mathbb{N}^q$. Then

$$\mathscr{J}_X(N) = \sum_{\mathbf{b}\in N} \mathscr{I}_1^{b_1}\cdots\mathscr{I}_q^{b_q}\ . \tag{5.1.1}$$

This is a coherent ideal sheaf in $\mathscr{O}_X$.

(b) For each $\mathbf{t}\in\square$ and all $x\in\mathbb{R}_{\ge 0}$, let

$$\mathscr{J}_X(\mathbf{t},x) = \mathscr{J}_X(N(\mathbf{t},x)) = \sum_{\mathbf{b}\in N(\mathbf{t},x)} \mathscr{I}_1^{b_1}\cdots\mathscr{I}_q^{b_q}\ . \tag{5.1.2}$$

(c) Fix a line sheaf $\mathscr{L}$ on X, and let $\mathbf{t}$ and x be as above. Then we let

$$\mathscr{F}(\mathbf{t})_x = \mathscr{F}_{\mathscr{L}}(\mathbf{t})_x = H^0(X, \mathscr{L}\otimes\mathscr{J}_X(\mathbf{t},x))\ . \tag{5.1.3}$$

Then $(\mathscr{F}(\mathbf{t})_x)_{x\in\mathbb{R}_{\ge 0}}$ is a descending filtration of $H^0(X,\mathscr{L})$ that satisfies $\mathscr{F}(\mathbf{t})_x = 0$ for all $x \gg 0$.

(d) Finally, for all $\mathbf{t}\in\square$ we let

$$F(\mathbf{t}) = F_{\mathscr{L}}(\mathbf{t}) = \frac{1}{h^0(X,\mathscr{L})}\int_0^\infty (\dim \mathscr{F}(\mathbf{t})_x)\,\mathrm{d}x\ . \tag{5.1.4}$$

In terms of this definition, Proposition 3.14 gives the following.

Lemma 5.2. *Assume that $Y_1,\dots,Y_q$ have the Autissier property, and let $\mathscr{L}$ be a line sheaf on X. Let $\square$ and $N(\mathbf{t},x)$ be as in Proposition 3.14. Then*

$$\mathscr{F}(\mathbf{t})_x \cap \mathscr{F}(\mathbf{u})_y \subseteq \mathscr{F}(\lambda\mathbf{t}+(1-\lambda)\mathbf{u})_{\lambda x+(1-\lambda)y} \tag{5.2.1}$$

for all $\mathbf{t},\mathbf{u}\in\square$, all $x,y\in\mathbb{R}_{\ge 0}$, and all $\lambda\in[0,1]$.

Proof. Let $\mathbf{t},\mathbf{u},x,y,\lambda$ be as above. By Proposition 3.14 (applied at all $P\in X$),

$$\mathscr{J}_X(\mathbf{t},x)\cap\mathscr{J}_X(\mathbf{u},y) \subseteq \mathscr{J}_X(\lambda\mathbf{t}+(1-\lambda)\mathbf{u}, \lambda x+(1-\lambda)y)\ . \tag{5.2.2}$$

This remains true after tensoring with $\mathscr{L}$, and (5.2.1) then follows because the global section functor is left exact. $\square$

We then have the following concavity theorem of Autissier [1, Thm. 3.6] (see also [9, Prop. 6.7]).

Theorem 5.3. *Let $\mathscr{F}(\mathbf{t})_x$ ($\mathbf{t} \in \square$, $x \in \mathbb{R}_{\ge 0}$) and $F\colon \square \to \mathbb{R}$ be as in Definition 5.1. Let $\beta_1, \dots, \beta_q \in \mathbb{R}_{>0}$. If* (5.2.1) *holds, then the inequality*

$$F(\mathbf{t}) \ge \min_{1 \le i \le q} \left(\frac{1}{\beta_i} \sum_{m=1}^{\infty} \frac{h^0(X, \mathscr{L} \otimes \mathscr{I}_i^m)}{h^0(X, \mathscr{L})} \right) \tag{5.3.1}$$

holds for all $\mathbf{t} \in \square$ for which $\sum \beta_i t_i = 1$.

Proof. See [9, Prop. 6.7]. $\square$

The results of this section can then be summarized as follows.

Theorem 5.4. *If $Y_1, \dots, Y_q$ have the Autissier property and if $\beta_1, \dots, \beta_q \in \mathbb{R}_{>0}$, then* (5.3.1) *holds.*

This provides a slight strengthening of the "General Theorem" of [9]: in that theorem, the divisors D_i were assumed to be Cartier, but this condition has been relaxed so that they only need to be Cartier at points where they meet other divisors in the collection.

6. Ideal Sheaves and B-divisors

The remainder of the proof in [9] involves Proposition 4.18 of that paper, so it is necessary to interpret things such as $H^0(X, \mathscr{L} \otimes \mathscr{I}_1^{b_1} \dots \mathscr{I}_q^{b_q})$ in terms of Cartier b-divisors. This is quite easy, because ideal sheaves are special cases of b-divisors. That is the topic of this section.

Let X be a variety over a field k. Here X is not necessarily complete. We briefly recall that a *model* of X is a proper birational morphism $\pi\colon W \to X$ of varieties over k, and a *Cartier b-divisor* $\mathbf{D}$ on X is an equivalence class of pairs $(W, D) = (\pi\colon W \to X, D)$, where $\pi\colon W \to X$ is a model of X and D is a Cartier divisor on W; here pairs (W_1, D_1) and (W_2, D_2) are said to be equivalent if there exist a model W_3 of X and morphisms $f_i\colon W_3 \to W_i$ over X for $i = 1, 2$ such that $f_1^* D_1 = f_2^* D_2$. For more details and basic properties, see [9, §4] and note that [9, Def. 4.1] does not need to assume that X is complete.

We start with some basic results about spaces of global sections of line sheaves on projective varieties. The first result is a general result on growth of cohomology groups, and is essentially due to the Stacks Project [14, Lem. 0BEM]. This lemma says that the Euler characteristic of the sheaves $\mathscr{F} \otimes \mathscr{L}_1^{n_1} \otimes \dots \otimes \mathscr{L}_r^{n_r}$ is a numerical polynomial in $n_1, \dots, n_r$ of a certain degree. Although the lemma below gives instead a bound on the dimensions of the cohomology groups of these sheaves, the method of proof is the same. These upper bounds will only be needed for h^0, but we will prove the general case as it is no more difficult.

Lemma 6.1. *Let X be a proper scheme over a field k, let $\mathscr{F}$ be a coherent sheaf on X, let $d = \dim \operatorname{Supp} \mathscr{F}$, and let $\mathscr{L}_1, \dots, \mathscr{L}_r$ be line sheaves on X. Then*

$$h^i(X, \mathscr{F} \otimes \mathscr{L}_1^{n_1} \otimes \dots \otimes \mathscr{L}_r^{n_r}) \le O(|\mathbf{n}|^d + 1)$$

for all $\mathbf{n} = (n_1, \dots, n_r) \in \mathbb{N}^r$ and all i, where $|\mathbf{n}| = n_1 + \dots + n_r$ and the implicit constant depends only on X, k, $\mathscr{F}$, and $\mathscr{L}_1, \dots, \mathscr{L}_r$.

Proof. We give a sketch of this proof, following the Stacks Project [14], including all places where the proofs differ.

For typographical simplicity, we let $\mathscr{L}^{\mathbf{n}}$ denote $\mathscr{L}_1^{n_1} \otimes \dots \otimes \mathscr{L}_r^{n_r}$ (multiindex notation) for all $\mathbf{n} \in \mathbb{N}^r$.

The proof is by induction on d. The base case $d = 0$ (including also $\operatorname{Supp} \mathscr{F} = \emptyset$) is trivial.

First, if $\mathscr{F}$ contains embedded points, then by [14, Lem. 02OL] there is a short exact sequence

$$0 \longrightarrow \mathscr{K} \longrightarrow \mathscr{F} \longrightarrow \mathscr{F}' \longrightarrow 0 \tag{6.1.1}$$

of coherent sheaves such that $\dim \operatorname{Supp} \mathscr{K} < d$ and $\mathscr{F}'$ has no embedded points. It remains exact after tensoring with $\mathscr{L}^{\mathbf{n}}$, so by the long exact sequence in cohomology and the inductive hypothesis we have

$$\left| h^i(X, \mathscr{F} \otimes \mathscr{L}^{\mathbf{n}}) - h^i(X, \mathscr{F}' \otimes \mathscr{L}^{\mathbf{n}}) \right| \le O(|\mathbf{n}|^{d-1} + 1) .$$

Therefore it suffices to prove the lemma when $\mathscr{F}$ has no embedded points.

We may replace X with $\operatorname{Supp} \mathscr{F}$ (this does not change the cohomologies), so we may assume that $\dim X = d$ and that X has no embedded points. In this situation, by [14, Lemmas 02OZ and 02P2], there exist a coherent ideal sheaf $\mathscr{I}$ on X and short exact sequences

$$0 \longrightarrow \mathscr{I}\mathscr{F} \longrightarrow \mathscr{F} \longrightarrow \mathscr{Q} \longrightarrow 0$$

and

$$0 \longrightarrow \mathscr{I}\mathscr{F} \longrightarrow \mathscr{F} \otimes \mathscr{L}_1 \longrightarrow \mathscr{Q}' \longrightarrow 0$$

such that $\dim \operatorname{Supp} \mathscr{Q} < d$ and $\dim \operatorname{Supp} \mathscr{Q}' < d$. Again tensoring with $\mathscr{L}^{\mathbf{n}}$ and applying the long exact sequence and the inductive hypothesis, we have

$$\left| h^i(X, \mathscr{F} \otimes \mathscr{L}^{\mathbf{n}}) - h^i(X, \mathscr{I}\mathscr{F} \otimes \mathscr{L}^{\mathbf{n}}) \right| \le O(|\mathbf{n}|^{d-1} + 1)$$

and

$$\left| h^i(X, \mathscr{F} \otimes \mathscr{L}^{\mathbf{n}} \otimes \mathscr{L}_1) - h^i(X, \mathscr{I}\mathscr{F} \otimes \mathscr{L}^{\mathbf{n}}) \right| \le O(|\mathbf{n}|^{d-1} + 1)$$

for all $\mathbf{n}$ and all i. Combining these inequalities, and using a symmetrical argument, we obtain

$$\left|h^i(X,\mathscr{F}\otimes\mathscr{L}^{\mathbf{n}}\otimes\mathscr{L}_j)-h^i(X,\mathscr{F}\otimes\mathscr{L}^{\mathbf{n}})\right|\le O(|\mathbf{n}|^{d-1}+1)$$

for all $\mathbf{n}$, all i, and all $j=1,\dots,r$.

Applying this inequality $|\mathbf{n}|$ times then gives

$$\left|h^i(X,\mathscr{F}\otimes\mathscr{L}^{\mathbf{n}})-h^i(X,\mathscr{F})\right|\le O(|\mathbf{n}|^{d}+1)\ ,$$

and the result follows. □

The following lemma applies this to give bounds more directly applicable to the current situation.

Lemma 6.2. *Let $\pi\colon W'\to W$ be a proper birational morphism of complete varieties over a field k.*

(a) *Assume that W is normal. Then $\pi_*\pi^*\mathscr{L}\cong\mathscr{L}$ for all line sheaves $\mathscr{L}$ on W, and the natural map $H^0(W,\mathscr{L})\to H^0(W',\pi^*\mathscr{L})$ is an isomorphism.*

(b) *For general W, the coherent sheaf $\mathscr{F}=\pi_*\mathscr{O}_{W'}/\mathscr{O}_W$ on W is supported on a proper subset of W, and*

$$0\le h^0(W',\pi^*\mathscr{L})-h^0(W,\mathscr{L})\le h^0(W,\mathscr{F}\otimes\mathscr{L}) \tag{6.2.1}$$

for all line sheaves $\mathscr{L}$ on W.

(c) *Let $\mathscr{L}$ be a line sheaf on W, let D be a Cartier divisor on W, and let $d=\dim W$. Then*

$$0\le h^0(W',\pi^*\mathscr{L}^N(-m\pi^*D))-h^0(W,\mathscr{L}^N(-mD))\le O((N+m)^{d-1}) \tag{6.2.2}$$

for all $N\in\mathbb{Z}_{>0}$ and all $m\in\mathbb{N}$, where the implicit constant depends on π, k, $\mathscr{L}$, and D, but not on N or m.

(d) *Under the same conditions as* (c),

$$0\le\sum_{m=1}^{\infty}h^0(W',\pi^*\mathscr{L}^N(-m\pi^*D))-\sum_{m=1}^{\infty}h^0(W,\mathscr{L}^N(-mD))\le O(N^d)\ . \tag{6.2.3}$$

Proof. For part (a), we first note that $\pi_*\mathscr{O}_{W'}=\mathscr{O}_W$ (as subsheaves on the constant sheaves of the function field $K(W')\cong K(W)$) by Hartshorne [2, II Prop. 6.3A] and the fact that W is normal. Therefore the projection formula gives $\pi_*\pi^*\mathscr{L}\cong\mathscr{L}$, and taking global sections gives $H^0(W',\pi^*\mathscr{L})\cong H^0(W,\mathscr{L})$.

For (b), we have an exact sequence

$$0\longrightarrow\mathscr{O}_W\longrightarrow\pi_*\mathscr{O}_{W'}\longrightarrow\mathscr{F}\longrightarrow0$$

of sheaves on W, where $\mathscr{F}$ is supported on a proper subset of W. Tensoring each term with $\mathscr{L}$ and taking global sections then gives an exact sequence

$$(6.2.4)\qquad 0 \longrightarrow H^0(W,\mathscr{L}) \longrightarrow H^0(W',\pi^*\mathscr{L}) \longrightarrow H^0(W,\mathscr{F}\otimes\mathscr{L})\ ,$$

which gives (6.2.1).

By (b), part (c) is a matter of showing that

$$h^0(W,\mathscr{F}\otimes\mathscr{L}^N(-mD)) \le O((N+m)^{d-1})$$

for all N and m. This is immediate from Lemma 6.1 with $\mathscr{L}_1=\mathscr{L}$ and $\mathscr{L}_2=\mathscr{O}(-D)$, since $\dim\operatorname{Supp}\mathscr{F}\le d-1$.

For (d), the lower bound holds (termwise) by the first part of (6.2.2).

For the upper bound, we first note that there is a constant c (independent of N and m) such that the summands in (6.2.3) vanish for all $m>cN$. Indeed, let $\pi''\colon W''\to W'$ be a projective model of W that dominates W' and let A be an ample divisor on W''; then it suffices to take $c\ge(\pi''^*\mathscr{L}\,.\,A^{d-1})/(\pi''^*D\,.\,A^{d-1})$, where in this case A^{d-1} is meant in the sense of intersection theory.

The sums then have $O(N)$ nonzero terms with $m\le O(N)$, so the upper bound follows from (6.2.2). □

For the remainder of this section, X is a complete variety over a field k.

Definition 6.3. Let $\mathscr{L}$ be a line sheaf on X and let $\mathbf{D}$ be an effective Cartier b-divisor on X. Then

$$H^0_{\mathrm{bir}}(X,\mathscr{L}(-\mathbf{D})) = H^0(W,\pi^*\mathscr{L}(-D))\ ,$$

where $\pi\colon W\to X$ is any normal model of X on which $\mathbf{D}$ is represented by a Cartier divisor D. This is independent of the choice of W by Lemma 6.2a.

Also (as usual)

$$h^0_{\mathrm{bir}}(X,\mathscr{L}(-\mathbf{D})) = \dim_k H^0_{\mathrm{bir}}(X,\mathscr{L}(-\mathbf{D}))\ .$$

When $\mathbf{D}=0$, these are also denoted $H^0_{\mathrm{bir}}(X,\mathscr{L})$ and $h^0_{\mathrm{bir}}(X,\mathscr{L})$, respectively.

The subscript "bir" is needed because $H^0_{\mathrm{bir}}(X,\mathscr{L})$ may be larger than $H^0(X,\mathscr{L})$ if X is not normal.

Lemma 6.4. *Let $\mathscr{L}$ be a line sheaf on X, let D be a nonzero effective Cartier divisor on X, and let $d=\dim X$. Then*

$$(6.4.1)\qquad h^0_{\mathrm{bir}}(X,\mathscr{L}^N) = h^0(X,\mathscr{L}^N) + O(N^{d-1})$$

and

$$(6.4.2)\qquad \sum_{m=1}^{\infty} h^0_{\mathrm{bir}}(X,\mathscr{L}^N(-mD)) = \sum_{m=1}^{\infty} h^0(X,\mathscr{L}^N(-mD)) + O(N^d)$$

as $N \to \infty$, where the implicit constants depend only on $\mathscr{L}$ and D. In particular, if $\mathscr{L}$ is big, then

$$\beta(\mathscr{L}, D) = \liminf_{N\to\infty} \frac{\sum_{m=1}^{\infty} h^0_{\mathrm{bir}}(X, \mathscr{L}^N(-mD))}{N h^0_{\mathrm{bir}}(X, \mathscr{L}^N)} . \tag{6.4.3}$$

Proof. First of all, by Lemma 6.2 (a), $h^0_{\mathrm{bir}}(X, \mathscr{L}^N(-mD))$ for all $N, m \in \mathbb{N}$ can be computed on a fixed normal model W of X, independent of N and m.

Then (6.4.1) is immediate from Lemma 6.2 (c).

For (6.4.2), note that $h^0_{\mathrm{bir}}(X, \mathscr{L}^N(-mD)) = h^0(W, \pi^*\mathscr{L}^N(-mD))$ for any normal model $\pi\colon W \to X$. Then (6.4.2) is immediate from Lemma 6.2 (d) applied to any such model π.

Finally, since $\mathscr{L}$ is big, (6.4.3) follows easily from (6.4.1) and (6.4.2). □

Therefore we may extend Definition 1.1 as follows.

Definition 6.5. Let $\mathscr{L}$ be a big line sheaf on X and let $\mathbf{D}$ be a nonzero effective Cartier b-divisor on X. Then

$$\beta(\mathscr{L}, \mathbf{D}) = \liminf_{N\to\infty} \frac{\sum_{m=1}^{\infty} h^0_{\mathrm{bir}}(X, \mathscr{L}^N(-m\mathbf{D}))}{N h^0_{\mathrm{bir}}(X, \mathscr{L}^N)} . \tag{6.5.1}$$

Remark 6.6. As noted in our earlier joint work [9] with M. Ru (following Def. 1.9), the above lim inf is actually a limit when $\mathscr{L}$ is big and D is a Cartier divisor. A detailed proof is given in Section 7, including the case when $\mathbf{D}$ is a b-divisor.

Now we consider b-divisors associated to proper closed subschemes.

Definition 6.7. Let Y be a proper closed subscheme of X, and let $\mathscr{I}$ be the corresponding ideal sheaf. Let $\pi\colon W \to X$ be the blow-up of X along $\mathscr{I}$, and let E be the exceptional divisor of π (so that $\mathscr{O}(E) = \mathscr{O}(-1)$ for the blowing-up). Then the Cartier b-divisor $\mathbf{Y}$ associated to Y is the b-divisor represented by E on W.

Next we compare the relevant spaces of global sections.

Lemma 6.8. *Let $\mathscr{L}$ be a line sheaf on X.*

(a) *Let Y, $\mathscr{I}$, $\pi\colon W \to X$, and E be as in Definition 6.7. Let $m \in \mathbb{N}$. Then the restriction to $\mathscr{I}^m$ of the natural map $\mathscr{O}_X \hookrightarrow \pi_*\mathscr{O}_W$ gives a map*

$$\mathscr{I}^m \hookrightarrow \pi_*(\mathscr{O}_W(-mE)) . \tag{6.8.1}$$

This gives an injection

$$H^0(X, \mathscr{L} \otimes \mathscr{I}^m) \hookrightarrow H^0(W, \pi^*\mathscr{L}(-mE)) . \tag{6.8.2}$$

(b) *The map (6.8.2) is an isomorphism for all sufficiently large m, independent of $\mathscr{L}$.*

(c) *For each $i = 1, \dots, q$ let Y_i be a proper closed subscheme of X, and let $\mathbf{Y}_i$ and $\mathscr{I}_i$ be the corresponding Cartier b-divisor and ideal sheaf on X, respectively. Then, for all $n_1, \dots, n_q \in \mathbb{N}$, there is a canonical injection*

$$H^0(X, \mathscr{L} \otimes \mathscr{I}_1^{n_1} \cdots \mathscr{I}_q^{n_q}) \hookrightarrow H^0_{\text{bir}}(X, \mathscr{L}(-n_1\mathbf{Y}_1 - \dots - n_q\mathbf{Y}_q)) , \tag{6.8.3}$$

induced by the maps of part (a) *for all i.*

Proof. With notation as in part (a), let U be an open subset of X. Then any local section $s \in \Gamma(U, \mathscr{I})$ pulls back to a section of $\pi^{-1}\mathscr{I} \cdot \mathscr{O}_W = \mathscr{O}(1) = \mathscr{O}(-E)$ over $\pi^{-1}(U)$; see Hartshorne [2, II Prop. 7.13]. This gives (6.8.1).

Tensoring both sides of (6.8.1) with $\mathscr{L}$ and applying the projection formula gives an injection $\mathscr{L} \otimes \mathscr{I}^m \hookrightarrow \pi_*(\pi^*\mathscr{L}(-mE))$, which then gives (6.8.2).

For part (b), it suffices to show that the map (6.8.1) is surjective (hence an isomorphism) for all $m \gg 0$. This map can be written $\mathscr{I}^m \to \pi_*\mathscr{O}_W(m)$. The fact that it is surjective for all $m \gg 0$ is noted at the very end of the proof of [2, II Thm. 5.19]. (This is shown locally over open affines of X, but extends to all of X by a compactness argument.)

For part (c), let $\pi \colon W \to X$ be any normal model of X that dominates the blowings-up of X along Y_i for all i. Since $\mathscr{L} \otimes \mathscr{I}_1^{n_1} \cdots \mathscr{I}_q^{n_q}$ is locally generated by products of local sections of $\mathscr{L}$ and of $\mathscr{I}_1^{n_1}, \dots, \mathscr{I}_q^{n_q}$, we obtain from (6.8.1) an injection

$$\mathscr{L} \otimes \mathscr{I}_1^{n_1} \cdots \mathscr{I}_q^{n_q} \hookrightarrow \pi_*(\pi^*\mathscr{L}(-n_1E_1 - \dots - n_qE_q)) ,$$

which gives (6.8.3). □

We conclude this section by proving the assertions of Remarks 1.5 and 1.6.

Corollary 6.9. *Let Y be a proper closed subscheme of X and let $\mathbf{Y}$ be the corresponding b-divisor. Let $\mathscr{L}$ be a big line sheaf on X. Then:*

(a) *Recalling Definitions 1.4, 1.5, and 6.5,*

$$\beta(\mathscr{L}, Y) = \beta_{\mathscr{L},Y} = \beta(\mathscr{L}, \mathbf{Y}) . \tag{6.9.1}$$

(b) *If any of these three quantities can be computed by evaluating the corresponding limits, then all of them can.*

Proof. Let $\mathscr{I}$, $\pi \colon W \to X$, and E be as in Definition 6.7, and let $d = \dim X$. For all $m \in \mathbb{Z}_{>0}$ let $\mathscr{F}_m$ be the cokernel of the map (6.8.1); by Lemma 6.8 (b) there is an m_0 such that $\mathscr{F}_m = 0$ for all $m > m_0$. Tensoring the short exact sequence $0 \longrightarrow \mathscr{I}^m \longrightarrow \pi_*\mathscr{O}_W(-mE) \longrightarrow \mathscr{F}_m \longrightarrow 0$ with $\mathscr{L}^N$ and taking global sections then gives

$$\begin{aligned} 0 &\le \sum_{m=1}^{\infty} h^0(W, \pi^*\mathscr{L}^N(-mE)) - \sum_{m=1}^{\infty} h^0(X, \mathscr{L}^N \otimes \mathscr{I}^m) \\ &\le \sum_{m=1}^{m_0} h^0(X, \mathscr{F}_m \otimes \mathscr{L}^N) \\ &\le O(N^{d-1} + 1) \end{aligned}$$

for all $N > 0$, by Lemma 6.2 (c). This gives

(6.9.2)
$$\liminf_{N\to\infty} \frac{\sum_{m=1}^{\infty} h^0(X, \mathscr{L}^N \otimes \mathscr{I}^m)}{N h^0(X, \mathscr{L}^N)} = \liminf_{N\to\infty} \frac{\sum_{m=1}^{\infty} h^0(W, \pi^*\mathscr{L}^N(-mE))}{N h^0(X, \mathscr{L}^N)} ,$$

which is the first equality $\beta(\mathscr{L}, Y) = \beta_{\mathscr{L},Y}$ of (6.9.1).

The second equality $\beta_{\mathscr{L},Y} = \beta(\mathscr{L}, \mathbf{Y})$ is a matter of showing that

(6.9.3)
$$\liminf_{N\to\infty} \frac{\sum_{m=1}^{\infty} h^0(W, \pi^*\mathscr{L}^N(-mE))}{N h^0(X, \mathscr{L}^N)} = \liminf_{N\to\infty} \frac{\sum_{m=1}^{\infty} h^0_{\mathrm{bir}}(X, \mathscr{L}^N(-m\mathbf{D}))}{N h^0_{\mathrm{bir}}(X, \mathscr{L}^N)}$$

This is true by (6.4.2) and (6.4.1).

Part (b) is immediate from the fact that (6.9.2) and (6.9.3) remain true (for the same reasons) when all instances of lim inf are replaced by lim sup. $\square$

7. Proof that (1.1.1), (1.4.1), (1.5.1), and (6.5.1) Exist as Limits

This section gives a proof that the limits infima in the definitions of $\beta(\mathscr{L}, D)$ (Definition 1.1), $\beta(\mathscr{L}, Y)$ (Definition 1.4), $\beta_{\mathscr{L},Y}$ (Remark 1.5), and $\beta(\mathscr{L}, \mathbf{D})$ (Definition 6.5) can be replaced by limits.

It has already been noted that the lim inf in the definition of $\beta(\mathscr{L}, D)$ (Definition 1.1) is a limit when D is a nonzero effective Cartier divisor (see the discussion following Definition 1.9 in [9]). We extend this result to allow D to be a nonzero effective Cartier b-divisor. Since a detailed proof has not appeared before, we include here such a proof of both results. It will then be immediate from Corollary 6.9 (b) that the same is true for $\beta(\mathscr{L}, Y)$ and $\beta_{\mathscr{L},Y}$.

Recall that in all cases, $\mathscr{L}$ is assumed to be big.

This argument is based on an idea of Julie Wang to compare the limit to a Riemann sum.

We start with the proof that the limit in (1.1.1) converges (Theorem 7.2). This first requires a lemma.

Lemma 7.1. *Let $\mathscr{M}$ be a line sheaf on X. If the limit*

$$\lim_{N\to\infty} \frac{1}{N^{d+1}} \sum_{m\ge 1} h^0(X, \mathscr{L}^N(-mD))$$

converges, then so does the limit

$$\lim_{N\to\infty} \frac{1}{N^{d+1}} \sum_{m\ge 1} h^0(X, \mathscr{M}\otimes\mathscr{L}^N(-mD)) ,$$

and the two limits are equal.

Proof. First, let $\mathscr{M}_1$ and $\mathscr{M}_2$ be line sheaves on X. Since X is projective, we have $h^0(X, \mathscr{M}_1 \otimes \mathscr{M}_2^{-1} \otimes \mathscr{L}^p) \ne 0$ for some $p \in \mathbb{N}$ by a consequence of Kodaira's lemma (see Lazarsfeld [6, 2.2.7]). Therefore

$$h^0(X, \mathscr{M}_2 \otimes \mathscr{L}^N(-mD)) \le h^0(X, \mathscr{M}_1 \otimes \mathscr{L}^{N+p}(-mD))$$

for all $m, N \in \mathbb{Z}_{>0}$. Therefore we have

$$\begin{aligned}\liminf_{N\to\infty} &\frac{1}{N^{d+1}} \sum_{m\ge 1} h^0(X, \mathscr{M}_2\otimes\mathscr{L}^N(-mD)) \\ &\le \liminf_{N\to\infty} \frac{1}{(N-p)^{d+1}} \sum_{m\ge 1} h^0(X, \mathscr{M}_1\otimes\mathscr{L}^N(-mD)) \\ &= \liminf_{N\to\infty} \frac{1}{N^{d+1}} \sum_{m\ge 1} h^0(X, \mathscr{M}_1\otimes\mathscr{L}^N(-mD)) ,\end{aligned}$$

and likewise for $\limsup$.

This gives

$$\begin{aligned}\liminf_{N\to\infty} &\frac{1}{N^{d+1}} \sum_{m\ge 1} h^0(X, \mathscr{L}^N(-mD)) \\ &\le \liminf_{N\to\infty} \frac{1}{N^{d+1}} \sum_{m\ge 1} h^0(X, \mathscr{M}\otimes\mathscr{L}^N(-mD)) \\ &\le \limsup_{N\to\infty} \frac{1}{N^{d+1}} \sum_{m\ge 1} h^0(X, \mathscr{M}\otimes\mathscr{L}^N(-mD)) \\ &\le \limsup_{N\to\infty} \frac{1}{N^{d+1}} \sum_{m\ge 1} h^0(X, \mathscr{L}^N(-mD)) ,\end{aligned}$$

and this implies the lemma. □

Theorem 7.2. *Let X be a complete variety over a field F of characteristic zero, let $\mathscr{L}$ be a big line sheaf on X, and let D be a nonzero effective Cartier divisor on X. Then the limit*

$$\lim_{N\to\infty} \frac{\sum_{m=1}^\infty h^0(X, \mathscr{L}^N(-mD))}{N h^0(X, \mathscr{L}^N)} \tag{7.2.1}$$

converges. In particular, the lim inf *in Definition 1.1 can be replaced by a limit.*

Proof. We start by reducing to the projective case. Let $d = \dim X$.

By Chow's lemma and resolution of singularities there is a model $\pi\colon W \to X$, with W projective and nonsingular. Then

$$\sum_{m=1}^{\infty} h^0(X, \mathscr{L}^N(-mD)) = \sum_{m=1}^{\infty} h^0(W, \pi^*(\mathscr{L}^N(-mD))) + O(N^d)$$

by Lemma 6.2 (d), and

$$h^0(X, \mathscr{L}^N) = h^0(W, \pi^*\mathscr{L}^N) + O(N^{d-1})$$

by Lemma 6.2 (c). Therefore

$$\lim_{N\to\infty} \frac{\sum_{m=1}^{\infty} h^0(X, \mathscr{L}^N(-mD))}{N h^0(X, \mathscr{L}^N)} = \lim_{N\to\infty} \frac{\sum_{m=1}^{\infty} h^0(W, \pi^*(\mathscr{L}^N(-mD)))}{N h^0(W, \pi^*\mathscr{L}^N)} ,$$

in the sense that if one limit converges, then both do, and they are equal.

So assume now that X is projective and nonsingular.

For all line sheaves $\mathscr{L}$ on X, all effective Cartier divisors D on X, and all $x \in \mathbb{R}_{\ge 0}$, we let

$$H^0(X, \mathscr{L}(-xD)) = \{s \in H^0(X, \mathscr{L}) : \text{the } \mathbb{R}\text{-divisor } (s) - xD \text{ is effective}\}$$

and (as usual)

$$h^0(X, \mathscr{L}(-xD)) = \dim_F H^0(X, \mathscr{L}(-xD)) .$$

These coincide with the usual definitions whenever xD is an integral divisor.

Define $f\colon \mathbb{R}_{\ge 0} \to \mathbb{R}$ by

$$f(x) = \lim_{N\to\infty} \frac{h^0(X, \mathscr{L}^N(-NxD))}{N^d} ,$$

where the limit is over $N \in \mathbb{Z}_{>0}$.

Recall from Lazarsfeld [6, II, Def. 11.4.2 and Ex. 11.4.7] that

$$\operatorname{vol}(\mathscr{L}) = \lim_{N\to\infty} \frac{h^0(X, \mathscr{L}^N)}{N^d/d!} . \tag{7.2.2}$$

Then $f(x) = \operatorname{vol}(\mathscr{L}(-xD))/d!$ whenever xD is an integral Cartier divisor.

Since D is effective, f is a nonincreasing function.

We also have $f(x) = 0$ for all sufficiently large x. Indeed, given an ample divisor A on X, this is true for all $x > (\mathscr{L} \,.\, A^{d-1})/(D \,.\, A^{d-1})$. Fix some $R \in \mathbb{R}_{\ge 0}$ such that $H^0(X, \mathscr{L}^N(-NRD)) = 0$ for all $N > 0$ (and therefore $f(R) = 0$).

Let

$$I = \int_0^{\infty} f(x)\,\mathrm{d}x = \int_0^R f(x)\,\mathrm{d}x .$$

It will then suffice to prove that

$$\lim_{N\to\infty}\frac{1}{N^{d+1}}\sum_{m\ge 1}h^0(X,\mathscr{L}^N(-mD))=I\ , \tag{7.2.3}$$

since by (7.2.2) this would imply

$$\lim_{N\to\infty}\frac{\sum_{m\ge 1}h^0(X,\mathscr{L}^N(-mD))}{Nh^0(X,\mathscr{L}^N)}=\frac{d!I}{\operatorname{vol}(\mathscr{L})}\ .$$

Let $k\in\mathbb{Z}_{>0}$. We show that if

$$\lim_{N\to\infty}\frac{1}{N^{d+1}}\sum_{m\ge 1}h^0(X,\mathscr{M}\otimes\mathscr{L}^{Nk}(-mD))=k^{d+1}I \tag{7.2.4}$$

with $\mathscr{M}=\mathscr{O}_X$, then (7.2.3) is true. Indeed, if (7.2.4) is true with $\mathscr{M}=\mathscr{O}_X$, then by Lemma 7.1 it is true with $\mathscr{M}=\mathscr{L}^j$ with $j=0,1,\dots,k-1$. Therefore the limit in (7.2.3) exists for N in each congruence class modulo k, and these limits are all equal.

Thus, if (7.2.3) is true with $\mathscr{L}$ replaced by $\mathscr{L}^k$ for some $k>0$, then it is true with the original $\mathscr{L}$. In particular, choosing k such that $H^0(X,\mathscr{L}^k)\ne 0$, we may assume that $H^0(X,\mathscr{L})\ne 0$.

We then have

$$h^0(X,\mathscr{L}^N(-mD))\le h^0(X,\mathscr{L}^{N'}(-mD)) \tag{7.2.5}$$

for all $0\le N\le N'$ and all $m\in\mathbb{N}$.

We now begin the main argument of the proof.

Given $\epsilon>0$, pick $\epsilon_1>0$ and $k,l_0\in\mathbb{Z}_{>0}$ such that

$$\left(1+\frac{1}{l_0}\right)^{d+1}\left(I+\frac{f(0)}{k}+\epsilon_1\right)\le I+\epsilon \tag{7.2.6}$$

and

$$\left(1-\frac{1}{l_0}\right)^{d+1}\left(I-\frac{f(0)}{k}-\epsilon_1\right)\ge I-\epsilon\ .$$

We claim that if k and l_0 are chosen sufficiently large, then we also have

$$\frac{1}{(lk)^d}\sum_{m=0}^{\infty}h^0(X,\mathscr{L}^{lk}(-mlD))\le\sum_{m=0}^{\infty}f\left(\frac{m}{k}\right)+\epsilon_1 k \tag{7.2.7}$$

and

$$\frac{1}{(lk)^d}\sum_{m=1}^{\infty}h^0(X,\mathscr{L}^{lk}(-mlD))\ge\sum_{m=1}^{\infty}f\left(\frac{m}{k}\right)-\epsilon_1 k \tag{7.2.8}$$

for all $l\ge l_0$.

We will show this result only for (7.2.7). The argument for (7.2.8) is similar and is left to the reader.

We may assume that $R\in\mathbb{Z}$.

Choose $\epsilon_2>0$, $\epsilon_3>0$, and $\epsilon_4>0$ such that $\epsilon_2+\epsilon_3+\epsilon_4\le\epsilon_1$.

Choose $x_0,\dots,x_t\in\mathbb{R}$ such that $0=x_0<x_1<\dots<x_t=R$ and

$$\sum_{i=1}^{t}(x_i-x_{i-1})(f(x_{i-1})-f(x_i))\le\epsilon_2\ .$$

Define a function $g\colon\mathbb{R}_{\ge0}\to\mathbb{R}_{\ge0}$ by

$$g(x)=f(x_{i-1})-f(x)\qquad\text{for all } x\in[x_{i-1},x_i)\text{ and all } i$$

and by $g(x)=0$ for all $x\ge R$. Then $g(x)\le f(x_{i-1})-f(x_i)$ for all $x\in[x_{i-1},x_i)$ and all i; hence

$$\int_0^\infty g(x)\,\mathrm{d}x=\int_0^R g(x)\,\mathrm{d}x\le\epsilon_2\ .$$

By the theory of Riemann integration, since g is piecewise nondecreasing, we have

$$\frac{1}{k}\sum_{m=0}^{\infty}g\left(\frac{m}{k}\right)\le\epsilon_2+\epsilon_3$$

for all sufficiently large k. Fix such a k. Then there is an integer N_0, depending on k, such that

$$\left|\frac{h^0(X,\mathscr{L}^N(-Nx_{i-1}D))}{N^d}-f(x_{i-1})\right|\le\frac{\epsilon_4}{kR}$$

for all $1\le i\le t$ and all $N\ge N_0$. Therefore

$$\begin{aligned}\frac{h^0(X,\mathscr{L}^N(-NxD))}{N^d}&\le\frac{h^0(X,\mathscr{L}^N(-Nx_{i-1}D))}{N^d}\\&\le f(x_{i-1})+\frac{\epsilon_4}{kR}=f(x)+g(x)+\frac{\epsilon_4}{kR}\end{aligned}$$

for all i, all $x\in[x_{i-1},x_i]$, and all $N\ge N_0$. Let $l_0=\lceil N_0/k\rceil$. Then, for all $l\ge l_0$,

$$\begin{aligned}\frac{1}{k}\sum_{m=0}^{\infty}\frac{h^0(X,\mathscr{L}^{lk}(-mlD))}{(lk)^d}&-\frac{1}{k}\sum_{m=0}^{\infty}f\left(\frac{m}{k}\right)\\&=\frac{1}{k}\sum_{m=0}^{kR-1}\frac{h^0(X,\mathscr{L}^{lk}(-mlD))}{(lk)^d}-\frac{1}{k}\sum_{m=0}^{kR-1}f\left(\frac{m}{k}\right)\\&\le\frac{1}{k}\sum_{m=0}^{kR-1}g\left(\frac{m}{k}\right)+\epsilon_4\\&\le\epsilon_2+\epsilon_3+\epsilon_4\\&\le\epsilon_1\ .\end{aligned}$$

This concludes the proof of the claim.

By elementary facts about Riemann sums for monotone functions, we have

$$\frac{1}{k}\sum_{m=1}^{kR} f\left(\frac{m}{k}\right) \le I \le \frac{1}{k}\sum_{m=0}^{kR-1} f\left(\frac{m}{k}\right) .$$

Since $f(R)=0$ and since the two sums differ by $f(0)/k$, we have

$$\text{(7.2.9)}\qquad \frac{1}{k}\sum_{m=0}^{kR} f\left(\frac{m}{k}\right) \le I + \frac{f(0)}{k} \quad\text{and}\quad \frac{1}{k}\sum_{m=1}^{kR} f\left(\frac{m}{k}\right) \ge I - \frac{f(0)}{k} .$$

Now let any $N \ge l_0 k$ be given. Let $l = \left\lceil \frac{N}{k} \right\rceil$. Then $lk \ge N$ and $l \ge l_0$; hence

$$\text{(7.2.10)}\qquad \frac{lk}{N} < \frac{N+k}{N} \le 1 + \frac{1}{l_0} .$$

By (7.2.5), effectivity of D, (7.2.10), (7.2.7), (7.2.9), and (7.2.6),

$$\begin{aligned}
&\frac{1}{N^{d+1}}\sum_{m\ge 1} h^0(X,\mathscr{L}^N(-mD)) \\
&\qquad\le \frac{1}{N^{d+1}}\sum_{m\ge 1} h^0(X,\mathscr{L}^{lk}(-mD)) \\
&\qquad\le \frac{1}{N^{d+1}}\sum_{m\ge 0} h^0\left(X,\mathscr{L}^{lk}\left(-\left\lfloor\frac{m}{l}\right\rfloor lD\right)\right) \\
&\qquad= \frac{1}{N^{d+1}}\sum_{m'=0}^{\infty} l h^0(X,\mathscr{L}^{lk}(-m'lD)) \\
&\qquad< \left(1+\frac{1}{l_0}\right)^{d+1} \frac{1}{l^d k^{d+1}}\sum_{m=0}^{\infty} h^0(X,\mathscr{L}^{lk}(-mlD)) \\
&\qquad\le \left(1+\frac{1}{l_0}\right)^{d+1} \left(\frac{1}{k}\sum_{m=0}^{\infty} f\left(\frac{m}{k}\right) + \epsilon_1\right) \\
&\qquad\le \left(1+\frac{1}{l_0}\right)^{d+1} \left(I + \frac{f(0)}{k} + \epsilon_1\right) \\
&\qquad\le I + \epsilon .
\end{aligned}$$

A similar argument gives

$$\frac{1}{N^{d+1}}\sum_{m\ge 1} h^0(X,\mathscr{L}^N(-mD)) > I - \epsilon ,$$

and this implies (7.2.3), concluding the proof of the theorem.

Corollary 7.3. *Let X be a complete variety over a field of characteristic 0, let $\mathscr{L}$ be a big line sheaf on X, and let $\mathbf{D}$ be a nonzero effective Cartier b-divisor on X. Then, recalling Definition 6.3, the limit*

$$\lim_{N\to\infty} \frac{\sum_{m=1}^{\infty} h^0_{\mathrm{bir}}(X, \mathscr{L}^N(-m\mathbf{D}))}{N h^0_{\mathrm{bir}}(X, \mathscr{L}^N)}$$

converges. Thus, the $\liminf$ *in* (6.5.1) *is actually a limit.*

Proof. Let $\pi\colon W \to X$ be a normal model of X on which $\mathbf{D}$ is represented by an effective Cartier divisor D. By Lemma 6.2 (a), we then have

$$h^0_{\mathrm{bir}}(X, \mathscr{L}^N(-m\mathbf{D})) = h^0(W, \pi^*\mathscr{L}^N(-mD))$$

for all $m, N \in \mathbb{N}$ (notably including $m = 0$). Thus

$$\frac{\sum_{m=1}^{\infty} h^0_{\mathrm{bir}}(X, \mathscr{L}^N(-m\mathbf{D}))}{N h^0_{\mathrm{bir}}(X, \mathscr{L}^N)} = \frac{\sum_{m=1}^{\infty} h^0(W, \pi^*\mathscr{L}^N(-mD))}{N h^0(W, \pi^*\mathscr{L}^N)}$$

for all $N \in \mathbb{Z}_{>0}$, and the corollary then follows from Theorem 7.2. $\square$

Corollary 7.4. *If* $\operatorname{char} k = 0$*, then the limits infima in* (1.4.1) *and* (1.5.1) *converge as limits.*

Proof. This is immediate from Corollaries 7.3 and 6.9 (b). $\square$

8. An Inequality of B-divisors

This section continues with the proof of Theorem 1.15, by applying the method of Autissier [1, §4] as adapted in Ru and the author [9], leading up to an inequality of $\mathbb{R}$-Cartier b-divisors (Lemma 8.9). This closely follows the proof in [9, §6], but we simplify it here by eliminating the sets Σ and $\triangle_\sigma$ (see Remark 8.10).

We start with some notation. Let X be a complete variety over a field k of characteristic zero, and let $Y_1, \dots, Y_q$ be proper closed subschemes of X that have the Autissier property. Let $\beta_1, \dots, \beta_q \in \mathbb{R}_{>0}$. Let b and N be large positive integers, to be chosen later (Corollary 9.12 and Theorem 10.4).

Let

$$\triangle = \{\mathbf{t} \in \mathbb{R}^q_{\ge 0} : t_1 + \dots + t_q = 1\} .$$

Recalling that $b \in \mathbb{Z}_{>0}$, let

$$\triangle_b = \left\{ \mathbf{a} \in \prod_{i=1}^q \beta_i^{-1}\mathbb{N} : \sum \beta_i a_i = b \right\} ,$$

so that $b^{-1}\triangle_b$ is a finite discrete subset of $\triangle$.

Recall from Sections 3 and 5 that $\square = \mathbb{R}^q_{\ge 0} \setminus \{\mathbf{0}\}$ and that

$$N(\mathbf{t}, x) = \{\mathbf{b} \in \mathbb{N}^q : \sum t_i b_i \ge x\} , \qquad \mathbf{t} \in \square ,\ x \in \mathbb{R}_{\ge 0} .$$

Let $F = F_{\mathscr{L}^N} : \square \to \mathbb{R}$ be the function of Definition 5.1. Write $b^{-1}\mathbf{a} = \mathbf{a}/b$ for all $\mathbf{a} \in \triangle_b$, and recall that

$$F(\mathbf{a}/b) = \int_0^\infty \frac{\dim \mathscr{F}_{\mathscr{L}^N}(\mathbf{a}/b)_x}{h^0(X, \mathscr{L}^N)}\, \mathrm{d}x\ ,$$

where $(\mathscr{F}(\mathbf{a}/b))_x = (\mathscr{F}_{\mathscr{L}^N}(\mathbf{a}/b))_x$ is the filtration of $H^0(X, \mathscr{L}^N)$ given by

$$\mathscr{F}(\mathbf{a}/b)_x = H^0(X, \mathscr{L}^N \otimes \mathscr{J}_X(\mathbf{a}/b, x))$$

and

$$\mathscr{J}_X(\mathbf{a}/b, x) = \sum_{\mathbf{b}\in N(\mathbf{a}/b,x)} \mathscr{I}_1^{b_1} \cdots \mathscr{I}_q^{b_q}\ .$$

For all $\mathbf{a} \in \triangle_b$ and $x \in \mathbb{R}_{\ge 0}$ let $K = K(\mathbf{a}/b, x)$ be the set of minimal elements in $N(\mathbf{a}/b, x)$. Then

$$\mathscr{J}_X(\mathbf{a}/b, x) = \sum_{\mathbf{b}\in K} \mathscr{I}_1^{b_1} \cdots \mathscr{I}_q^{b_q}\ , \tag{8.1}$$

and this is a finite sum since K is a finite set.

Following Ru and the author [9, §6], for all $\mathbf{a} \in \triangle_b$ and all $s \in H^0(X, \mathscr{L}^N) \setminus \{0\}$ we define

$$\mu_{\mathbf{a}/b}(s) = \sup\{x : \mathscr{F}(\mathbf{a}/b)_x \ni s\}\ . \tag{8.2}$$

Lemma 8.3. *Let $\mathbf{a} \in \triangle_b$ and $s \in H^0(X, \mathscr{L}^N) \setminus \{0\}$. Let $\mu = \mu_{\mathbf{a}/b}(s)$. Then*

$$s \in H^0\left(X, \sum_{\mathbf{b}\in K(\mathbf{a}/b,\mu)} \mathscr{L}^N \otimes \mathscr{I}_1^{b_1} \cdots \mathscr{I}_q^{b_q}\right)\ . \tag{8.3.1}$$

Proof. The union $\bigcup_{x\in[0,\mu]} K(\mathbf{a}/b, x)$ is finite, and each $\mathbf{b}$ in this union occurs in the sum (8.1) for a closed set of x. Therefore the supremum in (8.2) is actually a maximum. In particular, $s \in \mathscr{F}(\mathbf{a}/b)_\mu$, and this gives (8.3.1). □

Remark 8.4. Since the injection (6.8.3) is not necessarily bijective, it is important in this section to carefully distinguish between objects defined on X (non-birational objects) and the birational objects defined in Section 6. So far in this Section 8, everything has been non-birational. This will now change.

Corollary 8.5. *Let $\mathbf{a}$, s, and μ be as in Lemma 8.3, and let $K = K(\mathbf{a}/b, s)$. Let $\mathbf{Y}_1, \dots, \mathbf{Y}_q$ be the b-divisors on X corresponding to $Y_1, \dots, Y_q$, respectively. Then*

$$(s) \ge \bigwedge_{\mathbf{b}\in K} \sum_{i=1}^q b_i \mathbf{Y}_i\ . \tag{8.5.1}$$

At this point, for the convenience of the reader, we briefly recall the notation in (8.5.1). For b-divisors $\mathbf{D}_1$ and $\mathbf{D}_2$ on X, we define a relation $\mathbf{D}_1 \ge \mathbf{D}_2$ to hold if there is a model $\pi\colon W \to X$ on which the pull-back $\pi^*(\mathbf{D}_1 - \mathbf{D}_2)$ is represented by an effective Cartier divisor. This gives a partial ordering on the set of all b-Cartier divisors on X. Moreover, this partial ordering is a *lattice*; i.e., every nonempty finite set has a least upper bound and a greatest lower bound (see [9, §4] for details). The symbols $\bigwedge$ in (8.5.1) and $\bigvee$ below denote the greatest lower bound and the least upper bound of the given b-divisors, respectively (also known as their *meet* and *join,* respectively).

Proof of Corollary 8.5. Let $\pi\colon W \to X$ be a model of X on which all $\mathbf{Y}_i$ are represented by Cartier divisors D_i. Then, by (8.3.1), π^*s is a global section of the subsheaf of $\pi^*\mathscr{L}^N$ generated by the set $\{\pi^*\mathscr{L}^N(-b_1D_1-\dots-b_qD_q) : \mathbf{b} \in K\}$. By [9, Prop. 4.18], since this set is finite, we have

$$(\pi^*s) \ge \bigwedge_{\mathbf{b}\in K} (b_1D_1 + \dots + b_qD_q)\ .$$

This gives (8.5.1). ☐

Definition 8.6. Let $\mathscr{F} = (\mathscr{F}_x)_{x\in\mathbb{R}_{\ge 0}}$ be a filtration of a finite dimensional vector space V, and let $\mathscr{B}$ be a basis of V. Then $\mathscr{B}$ is *adapted* to $\mathscr{F}$ if $\mathscr{B}\cap\mathscr{F}_x$ is a basis of $\mathscr{F}_x$ for all x.

Definition 8.7. Let $\mathscr{B}$ be a basis of $H^0(X,\mathscr{L}^N)$. Then

$$\operatorname{div}(\mathscr{B}) = \sum_{s\in\mathscr{B}} (s)\ .$$

Remark 8.8. At this point we start using $\mathbb{R}$-Cartier b-divisors. These are basically finite formal linear combinations of Cartier b-divisors with real coefficients. An $\mathbb{R}$-Cartier b-divisor is said to be *effective* if it is a finite linear combination of effective Cartier b-divisors with positive (real) coefficients. For more details on $\mathbb{R}$-Cartier divisors and $\mathbb{R}$-Cartier b-divisors, see [10, §2].

Lemma 8.9. *Assume that* $\operatorname{char} k = 0$*. For each* $\mathbf{a} \in \triangle_b$ *let* $\mathscr{B}_{\mathbf{a}}$ *be a basis of* $H^0(X,\mathscr{L}^N)$ *adapted to the filtration* $\mathscr{F}(\mathbf{a}/b)$*. Then*

$$\bigvee_{\mathbf{a}\in\triangle_b} \operatorname{div}(\mathscr{B}_{\mathbf{a}}) \ge \frac{b}{b+q}\left(\min_{1\le i\le q}\sum_{m=1}^{\infty}\frac{h^0(X,\mathscr{L}^N\otimes\mathscr{I}_i^m)}{\beta_i}\right)\sum_{i=1}^{q}\beta_i\mathbf{Y}_i\ . \tag{8.9.1}$$

Proof. Let $\mathbf{D}'$ be the left-hand side of (8.9.1), and let $\pi\colon W\to X$ be a model of X on which $\mathbf{D}'$ and $\mathbf{Y}_1,\dots,\mathbf{Y}_q$ are represented by Cartier divisors D' and $D_1,\dots,D_q$, respectively. We also assume that W is nonsingular.

Let E be a prime divisor on W. Let ν', $\nu_{\mathbf{a}}$ for all $\mathbf{a}\in\triangle_b$, and $\nu_1,\dots,\nu_q$ be the multiplicities of E in D', $\operatorname{div}(\mathscr{B}_{\mathbf{a}})$ for all $\mathbf{a}$, and $D_1,\dots,D_q$, respectively. Let $\nu = \sum\beta_i\nu_i$.

We claim that there is an $\mathbf{a} \in \triangle_b$ (depending on E) such that

$$\nu_{\mathbf{a}} \geq \frac{b}{b+q} h^0(X, \mathscr{L}^N) F(\mathbf{a}/b) \nu \;. \tag{8.9.2}$$

Since $\operatorname{div}(\mathscr{B}_{\mathbf{a}})$ is effective for all $\mathbf{a}$ (and $\triangle_b$ is nonempty), the claim is trivial if $\nu = 0$, so we assume that $\nu > 0$.

Let

$$t_i = \frac{\nu_i}{\nu} \;, \qquad i = 1, \dots, q \;. \tag{8.9.3}$$

Since $\sum \beta_i \nu_i = \nu$, we have $\sum \beta_i t_i = 1$ and therefore $b \leq \sum \lfloor (b+q)\beta_i t_i \rfloor \leq b+q$. Therefore we may choose $\mathbf{a} \in \triangle_b$ such that

$$a_i \leq (b+q) t_i \;, \qquad i = 1, \dots, q \;. \tag{8.9.4}$$

Let $s \in \mathscr{B}_{\mathbf{a}}$, and let ν_s be the multiplicity of E in the divisor $(\pi^* s)$. Let $K = K(\mathbf{a}/b, \mu_{\mathbf{a}/b}(s))$. By (8.5.1), (8.9.3), (8.9.4), and the fact that $\sum a_i b_i \geq b \mu_{\mathbf{a}/b}(s)$ for all $\mathbf{b} \in K \subseteq N(\mathbf{a}/b, \mu_{\mathbf{a}/b}(s))$,

$$\frac{\nu_s}{\nu} \geq \frac{1}{\nu} \min_{\mathbf{b} \in K} \sum_{i=1}^q b_i \nu_i = \min_{\mathbf{b} \in K} \sum_{i=1}^q b_i t_i \geq \min_{\mathbf{b} \in K} \sum_{i=1}^q \frac{a_i b_i}{b+q} \geq \frac{b}{b+q} \mu_{\mathbf{a}/b}(s) \;. \tag{8.9.5}$$

Since $\mathscr{B}_{\mathbf{a}}$ is adapted to the filtration $\mathscr{F}(\mathbf{a}/b)$, we have

$$h^0(X, \mathscr{L}^N) F(\mathbf{a}/b) = \int_0^\infty \dim \mathscr{F}(\mathbf{a}/b)_x \, \mathrm{d}x = \sum_{s \in \mathscr{B}_{\mathbf{a}}} \mu_{\mathbf{a}/b}(s)$$

(see [9, Rem. 6.6]). Combining this with (8.9.5) and the fact that $\nu_{\mathbf{a}} = \sum_{s \in \mathscr{B}_{\mathbf{a}}} \nu_s$ then gives (8.9.2).

Since $Y_1, \dots, Y_n$ have the Autissier property, Theorem 5.4 gives

$$h^0(X, \mathscr{L}^N) F(\mathbf{a}/b) \geq \min_{1 \leq i \leq q} \sum_{m=1}^\infty \frac{h^0(X, \mathscr{L} \otimes \mathscr{I}_i^m)}{\beta_i} \;.$$

Therefore, by (8.9.2) and the definition of ν, we have

$$\nu' \geq \frac{b}{b+q} \left(\min_{1 \leq i \leq q} \sum_{m=1}^\infty \frac{h^0(X, \mathscr{L}^N \otimes \mathscr{I}_i^m)}{\beta_i} \right) \sum_{i=1}^q \beta_i \nu_i \;.$$

We conclude that the difference of the two sides of (8.9.1) is represented on W by a finite sum of effective Cartier divisors with nonnegative real coefficients (these divisors are the finitely many prime divisors E occurring in $\operatorname{Supp} D'$, and they are Cartier because W is nonsingular). This proves (8.9.1). $\square$

Remark 8.10. As noted in the introductory paragraph of this section, we have simplified the argument somewhat by eliminating the dependence on subsets $\sigma \subseteq \{1, \dots, q\}$. It would be easy to put this dependence back (by Remark 3.10 it is still true that at most $\dim X$ of the Y_i can pass through any point of X). With this change, the fraction $b/(b+q)$ in Lemma 8.9 can be replaced by $b/(b+\dim X)$, as in [9].

9. A Birational Growth Condition for B-divisors

This section formulates a definition of μ-b-growth. This is the same as [10, Def. 3.1], except that the divisor is an $\mathbb{R}$-Cartier b-divisor instead of an $\mathbb{R}$-Cartier divisor. See also [9, Def. 4.12].

Most of this section discusses in more detail the structure of the group of $\mathbb{R}$-Cartier b-divisors. We then restate Lemma 8.9 in terms of μ-b-growth.

Definition 9.1. Let X be a complete variety, let $\mathbf{D}$ be an effective $\mathbb{R}$-Cartier b-divisor on X, let $\mathscr{L}$ be a line sheaf on X, let V be a linear subspace of $H^0(X,\mathscr{L})$ with $\dim V > 1$, and let $\mu > 0$ be a real number. We say that $\mathbf{D}$ has *μ-b-growth with respect to V and $\mathscr{L}$* if there is a model $\phi\colon W \to X$ of X such that $\mathbf{D}$ is represented by an $\mathbb{R}$-Cartier divisor D on W and such that for all $Q \in W$ there is a basis $\mathscr{B}$ of V such that

$$\phi^* \operatorname{div}(\mathscr{B}) \ge \mu D \tag{9.1.1}$$

in a Zariski-open neighborhood U of Q, relative to the cone of effective $\mathbb{R}$-Cartier divisors on U.

At this point, we will take the opportunity to provide a cleaner treatment of $\mathbb{R}$-Cartier (b-)divisors. As was noted in [10, §2.2], the group of $\mathbb{R}$-Cartier b-divisors on a variety X is usually not a lattice-ordered group. This difficulty was handled by working with b-Weil functions at times (for example, in [10, Prop. 2.2]).

Instead, we provide here an explicit embedding of this group of divisors into a lattice-ordered group.

Let X be a variety over a field k. We recall some definitions and notation from [10]. Let $\mathfrak{X}$ denote the Zariski–Riemann space

$$\mathfrak{X} = \varprojlim_{\pi} X_\pi$$

of X, where the projective limit is over all models $\pi\colon X_\pi \to X$ of X. Also, $\operatorname{CDiv}(X)$ and

$$\operatorname{CDiv}(\mathfrak{X}) = \varinjlim_{\pi} \operatorname{CDiv}(X_\pi)$$

denote the groups of Cartier divisors on X and on $\mathfrak{X}$, respectively, and $\operatorname{CDiv}_{\mathbb{R}}(X) = \operatorname{CDiv}(X) \otimes \mathbb{R}$ denotes the group of $\mathbb{R}$-Cartier divisors on X. Then an *$\mathbb{R}$-Cartier b-divisor* on X is an element of the group

$$\begin{aligned} \operatorname{CDiv}_{\mathbb{R}}(\mathfrak{X}) &:= \operatorname{CDiv}(\mathfrak{X}) \otimes \mathbb{R} \\ &\cong \varinjlim_{\pi} \operatorname{CDiv}_{\mathbb{R}}(X_\pi) \ . \end{aligned}$$

An $\mathbb{R}$-Cartier b-divisor on X is *effective* if it comes from an effective $\mathbb{R}$-Cartier divisor on some model X_π.

Lemma 9.2. *Let X be a variety over a field k of characteristic zero.*

(a) *For any given $\mathbb{R}$-Cartier b-divisors $\mathbf{D}_1, \dots, \mathbf{D}_n$ on X, there is a model $\pi\colon X_\pi \to X$ of X such that, for all i, $\mathbf{D}_i$ is represented by an $\mathbb{R}$-Cartier divisor D_i on X_π, and all irreducible components of* $\operatorname{Supp} D_i$ *are Cartier divisors.*

(b) *Let $\mathbf{D}$ be an $\mathbb{R}$-Cartier b-divisor on X, and let $\pi\colon X_\pi \to X$ be a model for which the conclusion of part* (a) *is true. Write $\pi^*\mathbf{D} = \sum_i c_i D_i$, where the D_i are distinct Cartier divisors on X_π, and are prime as Weil divisors. Then $\mathbf{D}$ is effective if and only if $c_i \ge 0$ for all i.*

Proof. (a). For each i let X_i be a model of X on which $\mathbf{D}_i$ is represented by an $\mathbb{R}$-Cartier divisor, and let $\pi\colon X_\pi \to X$ be a model that dominates all of the X_i We may assume that X_π is nonsingular (since $\operatorname{char} k = 0$). Then X_π satisfies the conditions.

(b). Let $\pi\colon X_\pi \to X$, (c_i), and (D_i) be as above, and assume that $\mathbf{D}$ is effective. By Remark 8.8 and [9, Def. 4.1d], the latter means that there are (integral) Cartier b-divisors $\mathbf{E}_1, \dots, \mathbf{E}_m$ on X and positive real numbers $e_1, \dots, e_m$ such that $\mathbf{D} = \sum e_j \mathbf{E}_j$, and moreover that for each $j = 1, \dots, m$ there is a model $\pi_j\colon X_j \to X$ such that $\pi_j^* E_j$ is an effective Cartier divisor on X_j.

Let $\sigma\colon X_\sigma \to X$ be a model that dominates $X_1, \dots, X_m$ and X_π, and such that X_σ is nonsingular. Then $\sigma^* E_j$ is an effective Cartier divisor on X_σ for all j, and also, letting $\tau\colon X_\sigma \to X_\pi$ be the (unique) morphism over X, $\sigma^*\mathbf{D} = \sum_i c_i \tau^* D_i$. For each i let $\widetilde{D}_i$ be the strict transform of D_i in X_τ; this also has multiplicity c_i in $\sigma^*\mathbf{D}$. But also $\sigma^*\mathbf{D} = \sum_j e_j \sigma^*\mathbf{E}_j$, and $\sigma^*\mathbf{E}_j$ is an effective Cartier divisor on X_τ. Therefore, for all i, the multiplicity of $\widetilde{D}_i$ in $\sigma^*\mathbf{E}_j$ is nonnegative for all j, so $c_i \ge 0$.

Conversely, if $c_i \ge 0$ for all i, then $\mathbf{D}$ is effective because each prime divisor D_i (as above) defines an effective Cartier b-divisor $\mathbf{D}_i$ on X, and we then have that $\mathbf{D} = \sum c_i \mathbf{D}_i$ satisfies the condition in Remark 8.8. ☐

Now we are ready to define the embedding of $\operatorname{CDiv}_{\mathbb{R}}(\mathfrak{X})$.

Definition 9.3. Let X be a variety over a field k of characteristic zero, and let $\mathfrak{X}$ be its Zariski–Riemann space.

(a) Let $\mathbb{R}^{\mathfrak{X}}$ be the set of all functions $\mathfrak{X} \to \mathbb{R}$ (as is standard). This is a group under pointwise addition of functions, and is partially ordered by the relation $f \ge g$ if $f(x) \ge g(x)$ for all $x \in \mathfrak{X}$. This is then a lattice-ordered group (in which, for example, the join of two functions is their pointwise maximum).

(b) We define a map $\Phi\colon \operatorname{CDiv}_{\mathbb{R}}(\mathfrak{X}) \to \mathbb{R}^{\mathfrak{X}}$ as follows. Let $\mathbf{D}$ be an $\mathbb{R}$-Cartier b-divisor on X, and let $\pi\colon X_\pi \to X$ be a model on which

$\mathbf{D}$ is represented by an $\mathbb{R}$-Cartier divisor D. Let $x \in \mathfrak{X}$. This point is represented by a prime Weil divisor on some model of X, and we choose some model X_x that dominates both X_π and this model. Then D pulls back to an $\mathbb{R}$-Cartier divisor D_x on X_x, and x is represented by a prime divisor E_x on X_x. We then define $\operatorname{ord}_x \mathbf{D}$ to be the (real) multiplicity of E_x in D_x. This is independent of the choices of the various models. We then define $\Phi(\mathbf{D})$ to be the function $x \mapsto \operatorname{ord}_x \mathbf{D}$. This is clearly a group homomorphism.

Proposition 9.4. *Let X, k, $\mathfrak{X}$, and $\Phi\colon \operatorname{CDiv}_{\mathbb{R}}(\mathfrak{X}) \to \mathbb{R}^{\mathfrak{X}}$ be as in Definition 9.3. Then Φ preserves the orderings; in other words, $\Phi(\mathbf{D}_1) \ge \Phi(\mathbf{D}_2)$ if and only if $\mathbf{D}_1 \ge \mathbf{D}_2$. In particular, Φ is injective.*

Proof. For the first assertion, we may assume that $\mathbf{D}_2 = 0$, so it suffices to show that an $\mathbb{R}$-Cartier b-divisor $\mathbf{D}$ is effective if and only if $\operatorname{ord}_x \mathbf{D} \ge 0$ for all $x \in \mathfrak{X}$. The forward implication is immediate. The converse holds by Lemma 9.2 (b).

The second assertion then follows because if $\Phi(\mathbf{D}) = 0$ then both $\mathbf{D}$ and $-\mathbf{D}$ are effective, so $\mathbf{D} = 0$ by Lemma 9.2 (b) (where one assumes that X_π is nonsingular). $\square$

From now on, we regard $\operatorname{CDiv}_{\mathbb{R}}(\mathfrak{X})$ as a subgroup of $\mathbb{R}^{\mathfrak{X}}$ (via Φ).

Lemma 9.5. *Let $\mathbf{D}$ be an effective $\mathbb{R}$-Cartier b-divisor on X, and let $\pi\colon X_\pi \to X$ be a model of X that satisfies the condition of Lemma 9.2*(a) *for $\mathbf{D}$. For all $f \in \mathbb{R}^{\mathfrak{X}}$, let $\operatorname{Supp} f$ denote $\{x \in \mathfrak{X} : f(x) \ne 0\}$ (as is standard in analysis). Finally, let $\phi\colon \mathfrak{X} \to X_\pi$ be the canonical map. Then $\operatorname{Supp} \Phi(\mathbf{D}) = \phi^{-1}(\operatorname{Supp} \pi^*\mathbf{D})$.*

Proof. Write $\pi^*\mathbf{D} = \sum c_i D_i$ as in Lemma 9.2 (b). We may assume that $c_i > 0$ for all i.

Since $\mathbf{D}$ is effective, $\operatorname{ord}_x \mathbf{D} \ge 0$ for all $x \in \mathfrak{X}$, and likewise $\operatorname{ord}_x D_i \ge 0$ for all $x \in \mathfrak{X}$ and all i. Therefore $\operatorname{Supp} \Phi(\mathbf{D}) = \bigcup_i \operatorname{Supp} \Phi(D_i)$. Since we also have $\operatorname{Supp}(\sum c_i D_i) = \bigcup \operatorname{Supp} D_i$, it suffices to show that $\operatorname{Supp} \Phi(D_i) = \phi^{-1}(\operatorname{Supp} D_i)$; in other words, that $\operatorname{ord}_x D_i > 0$ if and only if $\phi(x) \in \operatorname{Supp} D_i$. This is trivial, because if U is an open neighborhood of $\phi(x)$ in X_π on which D_i is represented by a principal divisor (f), then both conditions are equivalent to $f(\phi(x)) = 0$. $\square$

We also will need to know that Φ preserves the meet and join operations (when they are defined). (This will also provide a more complete proof of the assertion in [10, §2.2] that the group of $\mathbb{R}$-Cartier b-divisors is generally not lattice-ordered. See Remark 9.8.)

Definition 9.6. Let $p > 0$ and let $\mathbf{D}_1, \dots, \mathbf{D}_p$ be $\mathbb{R}$-Cartier b-divisors on X. Then we define

$$\bigwedge_{i=1}^{p} \mathbf{D}_i = \operatorname{glb}\{\mathbf{D}_1, \dots, \mathbf{D}_p\} \tag{9.6.1}$$

if the indicated greatest lower bound exists. Otherwise it is undefined. Note that, for example, $\mathbf{D}_1 \wedge \mathbf{D}_2 \wedge \mathbf{D}_3$ may exist even if $\mathbf{D}_1 \wedge \mathbf{D}_2$ does not. We also define $\bigvee_{i=1}^{p} \mathbf{D}_i$ similarly (as a least upper bound).

Lemma 9.7. *Let X be a variety over a field k of characteristic zero.*

(a) *Let $\mathbf{E}_1, \dots, \mathbf{E}_p$ be effective $\mathbb{R}$-Cartier b-divisors on X such that*

$$\bigwedge_{i=1}^{p} \mathbf{E}_i = 0\ .$$

Then $\bigcap \operatorname{Supp} \Phi(\mathbf{E}_i) = \emptyset$ and $\Phi(\mathbf{E}_1) \wedge \dots \wedge \Phi(\mathbf{E}_p) = 0$.

(b) *Let $\mathbf{D}_1, \dots, \mathbf{D}_p$ be $\mathbb{R}$-Cartier b-divisors on X, and assume that*

$$\bigvee_{i=1}^{p} \mathbf{D}_i = \mathbf{b} \in \operatorname{CDiv}_{\mathbb{R}}(\mathfrak{X})\ .$$

Then

$$\Phi(\mathbf{b}) = \bigvee_{i=1}^{p} \Phi(\mathbf{D}_i)\ . \tag{9.7.1}$$

(c) *If $\mathbf{D}_1, \dots, \mathbf{D}_p$ are $\mathbb{Q}$-Cartier b-divisors on X, then*

$$\Phi\left(\bigvee_{i=1}^{p} \mathbf{D}_i\right) = \bigvee_{i=1}^{p} \Phi(\mathbf{D}_i)\ . \tag{9.7.2}$$

Proof. (a). Let $\mathbf{E}_1, \dots, \mathbf{E}_p$ be as given, and assume by way of contradiction that $\bigcap \operatorname{Supp} \Phi(\mathbf{E}_i)$ is nonempty. Pick $x \in \bigcap \operatorname{Supp} \Phi(\mathbf{E}_i)$, and let $\pi \colon W \to X$ be a model of X satisfying the conclusion of Lemma 9.2 (a). By passing to a larger model, we may assume that x corresponds to the support of a prime divisor G on W. (This can be done, for example, by starting with a model W_0 as in Lemma 9.2 (a), blowing up the closure of the image of x in W_0, and desingularizing the blowup.) For all i let F_i be an $\mathbb{R}$-Cartier divisor on W corresponding to $\pi^*\mathbf{E}_i$. Since $x \in \operatorname{Supp} \Phi(\mathbf{E}_i)$ for all i, we have $\operatorname{ord}_x(\mathbf{E}_i) > 0$ for all i. Let $c > 0$ be the minimum such order. Then $F_i - cG$ is effective for all i, so the $\mathbb{R}$-Cartier b-divisor $\mathbf{D}$ on X corresponding to cG is effective, is nonzero, and satisfies $\mathbf{E}_i - \mathbf{D} \ge 0$ for all i. This contradicts the assumption that $\operatorname{glb}\{\mathbf{E}_1, \dots, \mathbf{E}_p\} = 0$; thus $\bigcap \operatorname{Supp} \Phi(\mathbf{E}_i) = \emptyset$.

For the second assertion, the functions $\Phi(\mathbf{E}_i)$ are nonnegative real-valued functions whose supports have no point in common; therefore their pointwise minimum is zero.

(b). By part (a), $\bigwedge \Phi(\mathbf{b} - \mathbf{D}_i) = 0$, and this implies (9.7.1).

(c). This is immediate from (b), since the group of $\mathbb{Q}$-Cartier b-divisors on X is lattice-ordered. □

Remark 9.8. As noted earlier, Lemma 9.7 (b) supports a more complete proof of the assertion at the end of [10, §2.2] that the group of $\mathbb{R}$-Cartier b-divisors is not always lattice-ordered. Indeed, following [10], let L_1 and L_2 be two distinct lines on $X := \mathbb{P}^2$, let $\alpha, \beta \in \mathbb{R}_{>0}$ be real numbers with α/β irrational, and let $D_1 = \alpha L_1$ and $D_2 = \beta L_2$. Assume that $D_1 \vee D_2$ exists as an $\mathbb{R}$-Cartier b-divisor $\mathbf{b}$. Then $\mathbf{b} - D_1$ and $\mathbf{b} - D_2$ are effective $\mathbb{R}$-Cartier b-divisors on X such that $(\mathbf{b} - D_1) \wedge (\mathbf{b} - D_2) = 0$.

Let $\pi \colon W \to X$ be a nonsingular model that satisfies the conclusion of Lemma 9.2 (a) for $\mathbf{b}$, $\mathbf{b} - D_1$, and $\mathbf{b} - D_2$. Then $\mathbf{b}$ is represented by a Cartier divisor B on W, and by Lemmas 9.2 (b), 9.7 (a), and 9.5, $B - \pi^* D_1$ and $B - \pi^* D_2$ are effective $\mathbb{R}$-Cartier divisors on W with disjoint supports. One can then construct, using Hartshorne [2, V 5.3], arbitrarily long sequences of monoidal transformations over X such that π factors through their composition, contradicting [2, V 5.4]. The details are left to the reader.

Remark 9.9. The following facts are also not hard to see. They are not specifically needed for any proofs in this paper, hence are not proved here, but can be helpful for intuition.

(a) An $\mathbb{R}$-Cartier b-divisor $\mathbf{D}$ is a $\mathbb{Q}$-Cartier b-divisor (resp. a Cartier b-divisor) if and only if $\Phi(\mathbf{D}) \in \mathbb{Q}^{\mathfrak{X}}$ (resp. $\Phi(\mathbf{D}) \in \mathbb{Z}^{\mathfrak{X}}$).

(b) Let $\mathbf{D}_1, \dots, \mathbf{D}_p, \mathbf{b}$ be $\mathbb{R}$-Cartier b-divisors. If $\bigvee \Phi(\mathbf{D}_i) = \Phi(\mathbf{b})$, then $\bigvee \mathbf{D}_i = \mathbf{b}$.

Lemma 9.10. *Let X be a variety over a field k of characteristic zero, let D_i $(i \in I)$ be $\mathbb{R}$-Cartier divisors on X, and let $\mathbf{D}$ be an $\mathbb{R}$-Cartier b-divisor on X. Assume that $\bigvee_{i \in J} D_i$ exists (as an $\mathbb{R}$-Cartier b-divisor on X) for all nonempty finite $J \subseteq I$. Then the following conditions are equivalent.*

(i) *There is a model $\pi \colon W \to X$ of X such that for all $Q \in W$ there exist an element $i \in I$ and a Zariski-open neighborhood U of Q in W such that*

$$(\pi^* D_i)\big|_U \ge \mathbf{D}\big|_U , \tag{9.10.1}$$

relative to the cone of effective $\mathbb{R}$-Cartier b-divisors on U.

(ii) *There is a finite list $i_1, \dots, i_n$ of indices in I such that*

$$\bigvee_{j=1}^{n} D_{i_j} \ge \mathbf{D} . \tag{9.10.2}$$

Proof. By replacing X with a suitable model of X, we may assume that $\mathbf{D}$ is an $\mathbb{R}$-Cartier divisor D on X. Then, by replacing D_i with $D_i - D$ for all i, we may assume that $D = 0$.

(i) $\Rightarrow$ (ii). Let $\pi\colon W \to X$ be as in (i). As Q varies over all points of W, the corresponding open sets U_Q as in (i) cover W, so by quasi-compactness there are finitely many points $Q_1, \dots, Q_n$ such that $U_{Q_1}, \dots, U_{Q_n}$ cover W. For each $j = 1, \dots, n$, let $i_j \in I$ be an index such that the restriction of $\pi^* D_{i_j}$ to U_{Q_j} is effective. Therefore $\operatorname{ord}_x D_{i_j} \ge 0$ for all $x \in \mathfrak{X}$ lying over U_{Q_j}. Since the join operation corresponds to taking the pointwise maximum of elements of $\mathbb{R}^{\mathfrak{X}}$ (by Lemma 9.7 (b)), and since the U_{Q_j} cover W, it follows that $\operatorname{ord}_x \bigvee_{j=1}^n D_{i_j} \ge 0$ for all $x \in \mathfrak{X}$. Therefore $\bigvee_{j=1}^n D_{i_j}$ is effective.

(ii) $\Rightarrow$ (i). Assume that $i_1, \dots, i_n \in I$ satisfy condition (ii). For simplicity of notation, write D_j in place of D_{i_j} for all $j = 1, \dots, n$.

Let $\mathbf{E} = D_1 \vee \cdots \vee D_n$. Then $\mathbf{E} - D_j$ is an effective $\mathbb{R}$-Cartier b-divisor for all j, and $\bigwedge_{j=1}^n (\mathbf{E} - D_j) = 0$. By Lemma 9.7 (a), $\bigcap_j \operatorname{Supp} \Phi(\mathbf{E} - D_j) = \emptyset$. Also $\mathbf{E}$ is effective, because $\mathbf{E} \ge \mathbf{D} = 0$.

Now let $\pi\colon W \to X$ be a model of X that satisfies the condition of Lemma 9.7 (a) for $\mathbf{E} - D_j$ for all j. Then $\pi^*(\mathbf{E} - D_j)$ is an effective $\mathbb{R}$-Cartier divisor on W for all j, and by Lemma 9.5, $\bigcap_j \operatorname{Supp} \pi^*(\mathbf{E} - D_j) = \emptyset$.

For all j, let $U_j = W \setminus \operatorname{Supp} \pi^*(\mathbf{E} - D_j)$; then $\bigcup U_j = W$ and $(\pi^*(\mathbf{E} - D_j))\big|_{U_j} = 0$. Therefore condition (i) is satisfied. Indeed, given any $Q \in W$, pick j such that $Q \in U_j$ and let $U = U_j$. Then $(\pi^* D_j)\big|_U = (\pi^* \mathbf{E})\big|_U$ is effective because $\mathbf{E}$ is effective. $\square$

Everything so far in this section has been leading up to the following result.

Proposition 9.11. *Let X, $\mathbf{D}$, $\mathscr{L}$, V, and μ be as in Definition 9.1. Then $\mathbf{D}$ has μ-b-growth with respect to V and $\mathscr{L}$ if and only if there is a finite list $\mathscr{B}_1, \dots, \mathscr{B}_\ell$ of bases of V such that*

$$\bigvee_{i=1}^{\ell} \operatorname{div}(\mathscr{B}_i) \ge \mu \mathbf{D}\,. \tag{9.11.1}$$

Proof. This is immediate from Lemma 9.10 (applied to $\operatorname{div}(\mathscr{B})$ for all bases $\mathscr{B}$ of V). $\square$

Corollary 9.12. *Let k be a field of characteristic zero, let X be a complete variety over k, let $\mathscr{L}$ be a big line sheaf on X, and let $Y_1, \dots, Y_q$ be proper closed subschemes of X that have the Autissier property. For each $i = 1, \dots, q$ let $\mathbf{Y}_i$ be the Cartier b-divisor on X corresponding to Y_i, and let $\mathscr{I}_i$ be the ideal sheaf corresponding to Y_i. Let $\beta_1, \dots, \beta_q \in \mathbb{R}_{>0}$, and let $\mathbf{D}$ be the effective $\mathbb{R}$-Cartier b-divisor $\beta_1 \mathbf{Y}_1 + \cdots + \beta_q \mathbf{Y}_q$. Let $b \in \mathbb{Z}_{>0}$ and*

$N \in \mathbb{Z}_{>0}$, and assume that N satisfies $H^0(X, \mathscr{L}^N \otimes \mathscr{I}_i) \neq 0$ for all i and $h^0(X, \mathscr{L}^N) > 1$. Let

$$\mu = \left(\frac{b}{b+q}\right) \min_{1 \le i \le q} \sum_{m=1}^{\infty} \frac{h^0(X, \mathscr{L}^N \otimes \mathscr{I}_i^m)}{N\beta_i} . \tag{9.12.1}$$

Then $N\mathbf{D}$ has μ-b-growth with respect to $H^0(X, \mathscr{L}^N)$ and $\mathscr{L}^N$.

Furthermore, if $\beta_i = \beta(\mathscr{L}, Y_i)$ for all i, then for all $\epsilon > 0$ there exist b_0 and N_0 such that μ satisfies

$$\frac{h^0(X, \mathscr{L}^N)}{\mu} < 1 + \epsilon \tag{9.12.2}$$

whenever $b \ge b_0$ and $N \ge N_0$.

Proof. By Lemma 8.9, there exists a finite collection of bases of $H^0(X, \mathscr{L}^N)$ that satisfy (9.11.1) (with μ as in (9.12.1) and $\mathbf{D}$ replaced by $N\mathbf{D}$). This gives the first assertion.

Next, by (9.12.1),

$$\frac{h^0(X, \mathscr{L}^N)}{\mu} = \frac{b+q}{b} \left(\min_{1 \le i \le q} \frac{1}{\beta_i} \sum_{m=1}^{\infty} \frac{h^0(X, \mathscr{L}^N \otimes \mathscr{I}_i^m)}{N h^0(X, \mathscr{L}^N)} \right)^{-1} . \tag{9.12.3}$$

By the assumption on the β_i, (1.4.1), Corollary 7.4, and the fact that $H^0(X, \mathscr{L}^N) \neq 0$ for all $N \gg 0$ (see Lazarsfeld [6, II, Ex. 11.4.7]), the right-hand side of (9.12.3) is arbitrarily close to 1 for all sufficiently large b and N. This gives the second assertion. □

10. A Birational Multidivisor Nevanlinna Constant for B-divisors

In this section we introduce the birational Nevanlinna constant of Ru and the author [9], modified (i) to use $\mathbb{R}$-Cartier b-divisors, and (ii) to allow for multiple divisors (as in [10, §4.1]).

Throughout this section, X is a complete variety over a field of characteristic zero.

The following definition is essentially [10, Def. 4.1a], except that $\mathbf{D}_1, \dots, \mathbf{D}_p$ are allowed to be $\mathbb{R}$-Cartier b-divisors instead of $\mathbb{R}$-Cartier divisors.

Definition 10.1. Let $\mathbf{D}_1, \dots, \mathbf{D}_p$ ($p > 0$) be effective $\mathbb{R}$-Cartier b-divisors on X, and let $\mathscr{L}$ be a line sheaf on X. Then we define the *birational multidivisor Nevanlinna constant* for $\mathscr{L}$ and $\mathbf{D}_1, \dots, \mathbf{D}_p$ as

$$\operatorname{Nev}_{\mathrm{bir}}(\mathscr{L}, \mathbf{D}_1, \dots, \mathbf{D}_p) = \inf_{N,V,\mu} \frac{\dim V}{\mu} , \tag{10.1.1}$$

where the infimum passes over all triples (N, V, μ) such that $N \in \mathbb{Z}_{>0}$, V is a linear subspace of $H^0(X, \mathscr{L}^N)$ with $\dim V > 1$, and $\mu \in \mathbb{R}_{>0}$, such that

$N\mathbf{D}_i$ has μ-b-growth with respect to V and $\mathscr{L}^N$ for all i. Here, as usual, we use the convention that the infimum of the empty set is $+\infty$.

Note that one can use the either the condition of Definition 9.1 or the condition of Proposition 9.11 to check μ-b-growth.

Remark 10.2. The above definition requires that V be independent of i. This is because the proof of Theorem 12.1 ultimately relies on a map from X to projective space defined using V, and this map cannot depend on i.

Remark 10.3. One could make Definition 10.1 "fully birational" by allowing $\mathscr{L}$ to be a b-line-sheaf, where a *b-line-sheaf* could be defined as an element of $\varinjlim \operatorname{Pic} W$, taking the direct limit over all models W of X. However, this would basically amount to replacing X with some model W on which the hypothetical b-line-sheaf lies in $\operatorname{Pic} W$, so nothing new would be added.

Theorem 10.4. *Let $\mathscr{L}$ be a big line sheaf on X. Let $p > 0$, and for each $i = 1, \dots, p$ let $Y_{i,1}, \dots, Y_{i,q_i}$ be proper closed subschemes of X that have the Autissier property; let $\beta_{ij} = \beta(\mathscr{L}, Y_{ij})$ for all $j = 1, \dots, q_i$; let $\mathbf{Y}_{ij}$ be the Cartier b-divisor on X corresponding to Y_{ij} for all j; and let $\mathbf{D}_i = \beta_{i1}\mathbf{Y}_{i1} + \dots + \beta_{i,q_i}\mathbf{Y}_{i,q_i}$. Then*

$$\operatorname{Nev}_{\mathrm{bir}}(\mathscr{L}, \mathbf{D}_1, \dots, \mathbf{D}_p) \le 1 . \tag{10.4.1}$$

Proof. Let $\epsilon > 0$. For each $i = 1, \dots, p$, each $b > 0$, and each $N > 0$ satisfying the hypotheses of Corollary 9.12, let $\mu_{i,b,N}$ be as defined by (9.12.1) with Y_j equal to $Y_{i,j}$ and $\beta_j = \beta_{ij}$ for all $j = 1, \dots, q_i$. Then $N\mathbf{D}_i$ has $\mu_{i,b,N}$-b-growth with respect to $H^0(X, \mathscr{L}^N)$ and $\mathscr{L}^N$. Moreover, under the conditions of (9.12.2), for each i there are integers b_i and N_i such that (9.12.2) holds with μ replaced by $\mu_{i,b,N}$ for all $b \ge b_i$ and all $N \ge N_i$. Fix b and N such that $b \ge b_i$ and $N \ge N_i$ for all i, and let $\mu = \min_i \mu_{i,b,N}$. Then $N\mathbf{D}_i$ has μ-b-growth with respect to $H^0(X, \mathscr{L}^N)$ and $\mathscr{L}^N$ for all i, and

$$\frac{h^0(X, \mathscr{L}^N)}{\mu} < 1 + \epsilon .$$

Then $\operatorname{Nev}_{\mathrm{bir}}(\mathscr{L}, \mathbf{D}_1, \dots, \mathbf{D}_p) \le 1 + \epsilon$. Letting $\epsilon \to 0$ then gives (10.4.1). □

The following result will be used in Section 12.

Proposition 10.5. *Let $\mathbf{D}_1, \dots, \mathbf{D}_p$ and $\mathscr{L}$ be as in Definition 10.1. Then*

(a) *The equation* (10.1.1) *remains true if the infimum instead passes over all triples (N, V, μ) such that $N \in \mathbb{Z}_{>0}$, V is a linear subspace of $H^0(X, \mathscr{L}^N)$ with $\dim V > 1$, $\mu \in \mathbb{R}_{>0}$, and there is a finite list $\mathscr{B}_1, \dots, \mathscr{B}_\ell$ of bases of V such that*

$$\bigvee_{j=1}^{\ell} \operatorname{div}(\mathscr{B}_j) \ge \mu N \mathbf{D}_i \qquad \textit{for all } i . \tag{10.5.1}$$

(b) *Let* $\Phi\colon \operatorname{CDiv}_{\mathbb{R}}(\mathfrak{X}) \to \mathbb{R}^{\mathfrak{X}}$ *be as in Definition 9.3. Then part* (a) *remains true when* (10.5.1) *is replaced by*

$$\bigvee_{j=1}^{\ell} \Phi(\operatorname{div}(\mathscr{B}_j)) \ge \mu N \bigvee_{i=1}^{p} \Phi(\mathbf{D}_i) \;. \tag{10.5.2}$$

(c) *If* $\mathbf{D}_1, \dots, \mathbf{D}_p$ *are* $\mathbb{Q}$*-Cartier b-divisors, then* $\bigvee_{i=1}^p \mathbf{D}_i$ *exists as a* $\mathbb{Q}$*-Cartier b-divisor, and*

$$\operatorname{Nev}_{\text{bir}}(\mathscr{L}, \mathbf{D}_1, \dots, \mathbf{D}_p) = \operatorname{Nev}_{\text{bir}}\left(\mathscr{L}, \bigvee_{i=1}^{p} \mathbf{D}_i\right) . \tag{10.5.3}$$

Proof. (a). Let $N \in \mathbb{Z}_{>0}$, let V be a linear subspace of $H^0(X, \mathscr{L}^N)$ with $\dim V > 1$, and let $\mu \in \mathbb{R}_{>0}$. By Proposition 9.11, the condition on (N, V, μ) in Definition 10.1 is equivalent to the condition that, for each i, there is a finite list $\mathscr{B}_{i,1}, \dots, \mathscr{B}_{i,\ell_i}$ of bases of V such that

$$\bigvee_{j=1}^{\ell_i} \operatorname{div}(\mathscr{B}_{i,j}) \ge \mu N \mathbf{D}_i \;.$$

Without loss of generality, we may assume that the list of bases is independent of i, say $\mathscr{B}_1, \dots, \mathscr{B}_\ell$. This gives (10.5.1).

(b). By Proposition 9.4, $\mathbb{R}$-linearity of Φ, (9.7.2), and the definition of least upper bound, (10.5.2) is equivalent to (10.5.1).

(c). Since the group of $\mathbb{Q}$-Cartier b-divisors on X is lattice-ordered, (9.7.2) applies to both sides of (10.5.2), and then by Proposition 9.4, (10.5.2) is equivalent to

$$\bigvee_{i=1}^{\ell} \operatorname{div}(\mathscr{B}_i) \ge \mu N \bigvee_{i=1}^{p} \mathbf{D}_i \;.$$

This is the condition on (N, V, μ) for the $p = 1$ case of Definition 10.1 (with the divisor equal to $\bigvee_{i=1}^p \mathbf{D}_i$). This gives (10.5.3). □

Remark 10.6. It is easy to check that, if $\mathbf{D}_1, \dots, \mathbf{D}_p$ and $\mathscr{L}$ are as in Definition 10.1, and if $c \in \mathbb{R}_{>0}$, then

$$\operatorname{Nev}_{\text{bir}}(\mathscr{L}, c\mathbf{D}_1, \dots, c\mathbf{D}_p) = c \operatorname{Nev}_{\text{bir}}(\mathscr{L}, \mathbf{D}_1, \dots, \mathbf{D}_p) \;.$$

(This extends [9, Rem. 1.8].)

11. Multidivisor Proximity Functions for B-divisors

As was done in the previous section for Nevanlinna constants, we need to extend the definition of multidivisor proximity functions to b-divisors.

Throughout this section, k is either a number field or the field $\mathbb{C}$ of complex numbers, and X is a complete variety over k.

We start by introducing Weil functions, and the resulting proximity and counting functions, for $\mathbb{R}$-Cartier b-divisors $\mathbf{D}$.

Recall that, classically, a Weil function for a Cartier divisor D on X is a continuous function $\lambda\colon U(M) \to \mathbb{R}$ that satisfies certain growth conditions near $\operatorname{Supp} D$. Here $U(M)$ means $U(\mathbb{C})$ if $k = \mathbb{C}$ and $\coprod_{v \in M_k} U(\mathbb{C}_v)$ if k is a number field, where M_k is the set of places of k and $\mathbb{C}_v$ is the completion of the algebraic closure of the completion k_v of k at v. Also, U equals $X \setminus \operatorname{Supp} D$. For simplicity of exposition, if $k = \mathbb{C}$ then we let M_k be the set consisting of the single place v of $\mathbb{C}$ corresponding to the classical absolute value on $\mathbb{C}$, and $\mathbb{C}_v = \mathbb{C}$.

For more information on the growth conditions mentioned above, as well as basic properties of Weil functions, see [9, §2.3], Lang [5, Ch. 10], or [16, §8].

Weil functions are extended to the group of $\mathbb{R}$-Cartier divisors on X by $\mathbb{R}$-linearity. (When adding Weil functions, we take the sum over the intersection of their domains, then extend uniquely to a larger domain by continuity if necessary.)

We then define Weil functions for $\mathbb{R}$-Cartier b-divisors as follows.

Definition 11.1. Let $\mathbf{D}$ be an $\mathbb{R}$-Cartier b-divisor on X. We define $\operatorname{Supp} \mathbf{D}$ to be the closure of the image of $\operatorname{Supp} \Phi(\mathbf{D})$ under the map $\mathfrak{X} \to X$ (see Definition 9.3 (b) and Lemma 9.5 for definitions). Let $U = X \setminus \operatorname{Supp} \mathbf{D}$. Then a *Weil function for* $\mathbf{D}$ is a continuous function $\lambda\colon U(M) \to \mathbb{R}$ such that there is a model $\pi\colon W \to X$ that satisfies the following property. The pull-back $\pi^*\mathbf{D}$ is an $\mathbb{R}$-Cartier divisor D on W, and there is a Weil function $\lambda'\colon V(M) \to \mathbb{R}$ for D, where $V = W \setminus \operatorname{Supp} D$, such that there is a nonempty open $V' \subseteq V$ such that $\lambda'(w) = \lambda(\pi(w))$ for all $w \in V'(M)$.

Let Y be a proper closed subscheme of X and let $\mathbf{Y}$ be the corresponding b-divisor on X, represented by the exceptional divisor E on the blowing-up $\pi\colon W \to X$ of X along Y (as in Definition 6.7). Then we can use this W as the model in Definition 11.1, and can use $U = X \setminus Y$. Therefore the resulting Weil function coincides with the Weil function as defined by Silverman [13] or Yamanoi [18] (up to an M-bounded function, as usual).

If $\mathbf{D}$ is an $\mathbb{R}$-Cartier b-divisor on X, then we can define a proximity function for $\mathbf{D}$ in the obvious way. We leave out the details here because this is the special case $p = 1$ of the following definition of multidivisor proximity functions.

Definition 11.2. Let $\mathbf{D}_1, \dots, \mathbf{D}_p$ $(p > 0)$ be effective $\mathbb{R}$-Cartier b-divisors on X, and let $\lambda_{\mathbf{D}_1}, \dots, \lambda_{\mathbf{D}_p}$ be Weil functions for $\mathbf{D}_1, \dots, \mathbf{D}_p$, respectively. Let U be a nonempty open subset of X which is disjoint from $\operatorname{Supp} \mathbf{D}_i$ for all i, and such that for each i there is a model $\pi\colon W \to X$ for X such that $\pi^{-1}(\mathbf{D}_i)$ is an $\mathbb{R}$-Cartier divisor on W and π is an isomorphism over U.

(a) Assume that k is a number field, and let S be a finite set of places of k. Then the *multidivisor proximity function* for $\mathbf{D}_1, \dots, \mathbf{D}_p$ is

$$m_S(\mathbf{D}_1, \dots, \mathbf{D}_p, x) = \frac{1}{[k:\mathbb{Q}]} \sum_{v \in S} \max_{1 \le i \le p} \lambda_{\mathbf{D}_i, v}(x) \tag{11.2.1}$$

for all $x \in U(k)$. This proximity function depends on the choices of the Weil functions, but the dependence is only up to $O(1)$. (One can also extend this definition to handle algebraic points $x \in U(\overline{k})$.)

(b) Assume that $k = \mathbb{C}$, and let $f \colon \mathbb{C} \to X$ be a holomorphic function whose image meets U. Then the *multidivisor proximity function* for $\mathbf{D}_1, \dots, \mathbf{D}_p$ and f is

$$m_f(\mathbf{D}_1, \dots, \mathbf{D}_p, r) = \int_0^{2\pi} \max_{1 \le i \le p} \lambda_{\mathbf{D}_i}(f(re^{\sqrt{-1}\theta})) \frac{\mathrm{d}\theta}{2\pi} \tag{11.2.2}$$

for all $r > 0$.

Remark 11.3. Let $\mathscr{L}$ and $Y_{i,j}$ $(i = 1, \dots, p;\ j = 1, \dots, q_i)$ be as in the statement of Theorem 1.9. For each i and j let $\mathbf{Y}_{i,j}$ be the Cartier b-divisor corresponding to $Y_{i,j}$, and for each i let $\mathbf{D}_i = \sum_{j=1}^{q_i} \beta(\mathscr{L}, Y_{i,j}) \mathbf{Y}_{i,j}$. Then the left-hand sides of (1.9.1) and (1.9.2) equal the multidivisor proximity functions $m_S(\mathbf{D}_1, \dots, \mathbf{D}_p, x)$ and $m_f(\mathbf{D}_1, \dots, \mathbf{D}_p, r)$, respectively.

The following proposition is analogous to Proposition 10.5, and will also be used in Section 12.

Proposition 11.4. *Let $\mathbf{D}_1, \dots, \mathbf{D}_p$ be effective $\mathbb{Q}$-Cartier divisors on X.*

(a) *Let S be as in Definition 11.2(a). Then*

$$m_S(\mathbf{D}_1, \dots, \mathbf{D}_p, x) = m_S\left(\bigvee_{i=1}^p \mathbf{D}_i, x\right) + O(1)$$

for all x as in Definition 11.2(a).

(b) *Let f be as in Definition 11.2(b). Then*

$$m_f(\mathbf{D}_1, \dots, \mathbf{D}_p, r) = m_f\left(\bigvee_{i=1}^p \mathbf{D}_i, r\right) + O(1)$$

for all $r > 0$.

Proof. In both cases this follows from the fact that if $\lambda_{\mathbf{D}_i}$ is a Weil function for $\mathbf{D}_i$ for all i, then $\max_{1 \le i \le p} \lambda_{\mathbf{D}_i}$ is a Weil function for $\bigvee_{i=1}^p \mathbf{D}_i$ [9, Prop. 4.10c]. □

Remark 11.5. Although it may be possible to extend this result to $\mathbb{R}$-Cartier b-divisors, that would likely require extending the definition of Weil function to $\mathbb{R}^{\mathfrak{X}}$ (or at least a sufficiently large subgroup of it). This does not seem to be easy.

12. Conclusion of the Main Proof

This section gives the last step of the proof of Theorems 1.15 and 1.9. This relies on the following theorem, which generalizes [9, Thms. 1.4 and 1.5] (the case in which $p = 1$ and $\mathbf{D}_1$ is an effective Cartier divisor), as well as [10, Thms. 4.4 and 4.7] (in which $\mathbf{D}_1, \dots, \mathbf{D}_p$ are effective $\mathbb{R}$-Cartier divisors). It corresponds to the penultimate step in the proof of the Main Theorem of [9].

(This theorem may be regarded as the Second Main Theorem for birational multidivisor Nevanlinna constants.)

Theorem 12.1. *Let X be a complete variety over a field k, let $\mathscr{L}$ be a line sheaf on X with $h^0(X, \mathscr{L}^N) > 1$ for some $N > 0$, and let $\mathbf{D}_1, \dots, \mathbf{D}_p$ ($p > 0$) be effective $\mathbb{R}$-Cartier b-divisors on X.*

(a) (Arithmetic part) *Assume that k is a number field, and let S be a finite set of places of k. Then, for all $\epsilon > 0$ and all $C \in \mathbb{R}$, there is a proper Zariski-closed subset Z of X such that the inequality*

$$m_S(\mathbf{D}_1, \dots, \mathbf{D}_p, x) \le (\operatorname{Nev}_{\text{bir}}(\mathscr{L}, \mathbf{D}_1, \dots, \mathbf{D}_p) + \epsilon) h_{\mathscr{L}}(x) + C \tag{12.1.1}$$

holds for all points $x \in X(k) \setminus Z$.

(b) (Analytic part) *Assume that $k = \mathbb{C}$. Then, for all $\epsilon > 0$, there is a proper Zariski-closed subset Z of X such that the inequality*

$$m_f(\mathbf{D}_1, \dots, \mathbf{D}_p, r) \le_{\text{exc}} (\operatorname{Nev}_{\text{bir}}(\mathscr{L}, \mathbf{D}_1, \dots, \mathbf{D}_p) + \epsilon) T_{f,\mathscr{L}}(r) \tag{12.1.2}$$

holds for all holomorphic mappings $f \colon \mathbb{C} \to X$ whose image is not contained in Z.

Proof. We will prove only the arithmetic case. The analytic case is similar.

This proof closely follows the proof of [10, Thm. 4.4], which in turn follows Ru [8], as summarized at the end of [9, §5].

Fix $\epsilon > 0$ and $\epsilon' \in (0, \epsilon)$.

By Definition 10.1 and Proposition 10.5 (a), there exist a triple (N, V, μ) and a finite list $\mathscr{B}_1, \dots, \mathscr{B}_\ell$ such that $N \in \mathbb{Z}_{>0}$, V is a linear subspace of $H^0(X, \mathscr{L}^N)$ with $\dim V > 1$, $\mu \in \mathbb{R}_{>0}$, $\mathscr{B}_j$ is a basis of V for all j,

$$\bigvee_{j=1}^{\ell} \operatorname{div}(\mathscr{B}_j) \ge \mu N \mathbf{D}_i \qquad \text{for all } i\ , \tag{12.1.3}$$

and

$$\frac{\dim V}{\mu} \le \operatorname{Nev}_{\text{bir}}(\mathscr{L}, \mathbf{D}_1, \dots, \mathbf{D}_p) + \epsilon'\ . \tag{12.1.4}$$

Choose a sufficiently large model $\pi \colon W \to X$ such that $D := \pi^*\bigl(\bigvee_{j=1}^{\ell} \operatorname{div}(\mathscr{B}_j)\bigr)$ is a Cartier divisor on W and $D_i := \pi^*(\mathbf{D}_i)$ is an $\mathbb{R}$-Cartier divisor on W for all i. Choose Weil functions $\lambda_{\mathscr{B}_j}$ for $\pi^* \operatorname{div}(\mathscr{B}_j)$ for all j and λ_{D_i} for D_i for all i. By (12.1.3) and standard properties of Weil functions, for all $v \in S$ and all $i = 1, \dots, p$ there is a constant $c_{v,i}$ such that

$$\max_{1 \le j \le \ell} \lambda_{\mathscr{B}_j,v}(w) \ge \mu \lambda_{D_i,v}(w) - c_{v,i} \tag{12.1.5}$$

for all $w \in W(k)$ outside of a proper Zariski-closed subset. After adjusting the λ_{D_i} for all i, we may assume that $c_{v,i} = 0$ for all v and i.

As is the case for Weil functions (see Definition 11.1), we have

$$m_S(\mathbf{D}_1, \dots, \mathbf{D}_p, x) = m_S(D_1, \dots, D_p, \pi^{-1}(x)) \tag{12.1.6}$$

where we consider only $x \in U$ with U as in Definition 11.1. We also assume that the Weil functions used for computing the two proximity functions are related as in Definition 11.1.

We then claim that, if $\epsilon'' > 0$ is sufficiently small, then

$$\begin{aligned} \mu \cdot m_S(\mathbf{D}_1, \dots, \mathbf{D}_q, \pi(w)) &= \mu \cdot m_S(D_1, \dots, D_q, w) \\ &\le \frac{1}{[k:\mathbb{Q}]} \sum_{v \in S} \max_{1 \le j \le \ell} \lambda_{\mathscr{B}_j,v}(w) \\ &\le (\dim V + \epsilon'') h_{\pi^*\mathscr{L}}(w) + \mu C \\ &= (\dim V + \epsilon'') h_{\mathscr{L}}(\pi(w)) + \mu C \\ &\le \mu(\mathrm{Nev}_{\mathrm{bir}}(\mathscr{L}, \mathbf{D}_1, \dots, \mathbf{D}_p) + \epsilon) h_{\mathscr{L}}(\pi(w)) + \mu C \end{aligned}$$

for all $w \in W(k)$ outside of a proper Zariski-closed subset. Indeed, the first step is (12.1.6), the second is (12.1.5) (noting that $c_{v,i} = 0$ for all v and i), the third step follows from Schmidt's Subspace Theorem in the form of [9, Thm. 2.10], the fourth step holds by functoriality of heights, and the last step holds by (12.1.4) and the choices of ϵ' and ϵ''.

Dividing this by μ then gives (12.1.1). □

Proof of Theorems 1.15 and 1.9. Theorem 1.15 is immediate from Theorems 12.1 and 10.4. Combining Theorem 1.15 with Theorem 1.14 (Proposition 4.3) then gives Theorem 1.9. □

Remark 12.2. When $\mathbf{D}_1, \dots, \mathbf{D}_p$ are $\mathbb{Q}$-Cartier b-divisors, Theorem 12.1 for general p reduces to the special case $p = 1$, with the divisor $\bigvee_{i=1}^p \mathbf{D}_i$. This follows immediately from Propositions 10.5 (c) and 11.4. However, this is not true in the more general case of $\mathbb{R}$-Cartier b-divisors, because $\bigvee \mathbf{D}_i$ may not exist.

13. An Example: Linear Subspaces of $\mathbb{P}^n_k$

This section gives an example involving linear subspaces of $\mathbb{P}^n_k$.

Let $Y_1, \dots, Y_q$ be linear subvarieties of $\mathbb{P}^n_k$ that intersect properly. In this case, Definition 4.1 reduces to the condition that they intersect properly in the sense of intersection theory; i.e.,

$$\operatorname{codim} \bigcap_{i\in I} Y_i = \sum_{i\in I} \operatorname{codim} Y_i \tag{13.1}$$

for all nonempty $I \subseteq \{1,\dots,q\}$ such that $\bigcap_{i\in I} Y_i \ne \emptyset$.

We now compute $\beta(\mathscr{O}(1), Y_i)$ for these subschemes.

Proposition 13.2. *Let k be a field, let $X = \mathbb{P}^n_k$ with $n > 0$, and let Y be an integral linear subscheme of X of codimension $r > 0$. Then*

$$\beta(\mathscr{O}(1), Y) = \frac{r}{n+1}\ . \tag{13.2.1}$$

Proof. Let $x_0,\dots,x_n$ be homogeneous coordinates on X. We may assume that Y is the subscheme $x_1 = \dots = x_r = 0$. Let $\mathscr{I}$ be the ideal sheaf corresponding to Y.

First, for all $N \in \mathbb{N}$, $H^0(X, \mathscr{O}(N))$ has a basis over k consisting of all homogeneous monomials of degree N in $x_0,\dots,x_n$. For all $m \in \mathbb{N}$ the subspace $H^0(X, \mathscr{O}(N)\otimes\mathscr{I}^m)$ of $H^0(X,\mathscr{O}(N))$ is generated by

$$\{x_0^{j_0}\cdots x_n^{j_n} : j_0+\dots+j_n = N \text{ and } j_1+\dots+j_r \ge m\}\ .$$

Therefore

$$\begin{aligned}
&\sum_{m=1}^{\infty} h^0(X, \mathscr{O}(N)\otimes\mathscr{I}^m)\\
&\qquad = \sum_{m=1}^{\infty} \left|\left\{(j_0,\dots,j_n)\in\mathbb{N}^{n+1} : \sum j_i = N \text{ and } j_1+\dots+j_r \ge m\right\}\right|\\
&\qquad = \sum_{\substack{(j_0,\dots,j_n)\in\mathbb{N}^{n+1}\\ \sum j_i = N}} (j_1+\dots+j_r)\ ,
\end{aligned}$$

where the second equality holds because each tuple $(j_0,\dots,j_n)$ occurs in the set in the second sum for $j_1+\dots+j_r$ values of m.

Let $J = \{(j_0,\dots,j_n)\in\mathbb{N}^{n+1} : \sum j_i = N\}$. As noted in Ru [7, Example 1.2], $\sum_J j_l$ is independent of l (by symmetry), so

$$\sum_J (j_1+\dots+j_r) = \frac{r}{n+1}\sum_J (j_0+\dots+j_n) = \frac{rN}{n+1}|J|\ .$$

Therefore,

$$\lim_{N\to\infty} \frac{\sum_{m=1}^{\infty} h^0(X,\mathscr{O}(N)\otimes\mathscr{I}^m)}{N h^0(X,\mathscr{O}(N))} = \lim_{N\to\infty}\frac{(rN/(n+1))|J|}{N|J|} = \frac{r}{n+1}\ ,$$

which implies (13.2.1). ∎

As a corollary of Theorem 1.9, we then obtain:

Theorem 13.3. *Let k be a number field, let S be a finite set of places of k, let $X = \mathbb{P}^n_k$, and let $Y_1, \dots, Y_q$ be linear subvarieties of X in general position (according to* (13.1)*). Then, for all $\epsilon > 0$ and all $C \in \mathbb{R}$, there is a proper Zariski-closed subset Z of X such that the inequality*

$$\sum_{i=1}^{q} (\operatorname{codim} Y_i) m_S(Y_i, x) \le (n+1+\epsilon) h_k(x) + C$$

holds for all $x \in X(k)$ outside of Z.

This is a consequence of [15, (3.9)], in which $\mathscr{D}$ is a finite collection of hyperplanes containing, for each i, a subset whose intersection is Y_i.

It is also a special case of the Main Theorem of Heier and Levin [3].

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