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Badly approximable numbers, Kronecker's theorem, and diversity of Sturmian characteristic sequences

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Résumé. Nous donnons une version optimale du théorème classique des "trois distances" concernant les parties fractionnaires de $n\theta$, dans le cas où θ est un nombre irrationnel qui est mal approchable. Comme conséquence, nous obtenons une version du théorème d'approximation inhomogène de Kronecker, en une dimension, pour les nombres mal approchables. Nous appliquons ces résultats à l'obtention d'une mesure améliorée de la "diversité" des suites sturmiennes caractéristiques dont la pente est mal approchable.

Abstract. We give an optimal version of the classical "three-gap theorem" on the fractional parts of $n\theta$, in the case where θ is an irrational number that is badly approximable. As a consequence, we deduce a version of Kronecker's inhomogeneous approximation theorem in one dimension for badly approximable numbers. We apply these results to obtain an improved measure of sequence diversity for characteristic Sturmian sequences, where the slope is badly approximable.

1. Introduction

About twenty-five years ago, the second author [10] published an article in which several results about numbers with bounded partial quotients were proved. In this paper we improve those results—in some cases, optimally.

Let us recall what was proved previously. If θ is a real number with simple continued fraction $\theta = [a_0, a_1, a_2, \ldots]$, then we say that θ has *bounded partial quotients* or is *badly approximable* if there exists a positive integer B such that $|a_i| \leq B$ for all $i \geq 1$. The set of real numbers with partial quotients bounded by B is denoted by $\mathcal{S}_B$. For a survey about such numbers and their properties, see [9].

Kronecker's theorem is a celebrated theorem about inhomogeneous Diophantine approximation. In the one-dimensional version, it states that if θ

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is an irrational real number, β is a real number, and N and ϵ are positive real numbers, there exist integers n, p with $n > N$ such that

$$|n\theta - p - \beta| < \epsilon.$$

See, for example, [5, Chap. 23]. Note that Kronecker's theorem provides no estimate of the sizes of the numbers n, p, and indeed, no such estimate is possible, in general, since (for example) the ratio β/θ could be arbitrarily large.

However, if θ is badly approximable and β is bounded, then it is possible to bound n and p. We recall Theorem 17 from [10]:

Theorem 1.1. *Let θ be an irrational real number, $0 < \theta < 1$, with partial quotients bounded by B. Let $0 \le \beta < 1$ be a real number. Then for all $N \ge 1$ there exist integers n, p with $0 \le n, |p| \le (B+2)N^2$ such that $|n\theta - p - \beta| \le \frac{1}{N}$.*

In this paper we improve the quadratic upper bound in Theorem 1.1 to a linear one. Our main tool is an optimal and apparently new estimate, for badly approximable numbers, on the size of largest interval in the celebrated "three-gap theorem" (aka the Steinhaus conjecture) [11, 3, 12, 13, 14, 4, 1]. This is done in Section 2.

Finally, we apply these results to prove a new measure of sequence diversity for the so-called characteristic Sturmian sequences. Roughly speaking, this measure shows that linearly-indexed subsequences of Sturmian sequences cannot agree for "too long".

2. The three-gap theorem

Let us begin by recalling the three-gap theorem [7]. For a real irrational number θ, let $\{\theta\}$ denote its fractional part, which can also be written $\theta \bmod 1$. Let $\|\theta\|$ denote the distance from θ to the nearest integer, which is $\min\{\theta \bmod 1, (-\theta) \bmod 1\}$.

Let N be a positive integer, and sort the $N+2$ numbers

$$0, \{\theta\}, \{2\theta\}, \ldots, \{N\theta\}, 1$$

in ascending order, viz.,

$$s_0 = 0 < s_1 < s_2 < \cdots < s_N < 1 = s_{N+1}.$$

The *gaps* are the numbers $s_{i+1} - s_i$ for $0 \le i \le N$, and the *gap set* $G(\theta, N)$ is the set $\{s_{i+1} - s_i \ : \ 0 \le i \le N\}$. One version of the three-gap theorem is as follows [8]:

Theorem 2.1. *For integers $N \ge 1$ the set $G(\theta, N)$ is always of cardinality either two or three, and if it is of cardinality three, the larger of the three numbers is the sum of the smaller two.*

Suppose the continued fraction expansion of θ is $[a_0; a_1, \ldots]$, and the convergents to θ are p_i/q_i for $i \geq 0$. In addition, define the following notation for irrational θ:

$$\theta_k := [0; a_k, a_{k+1}, a_{k+2}, \ldots]; \quad \phi_k := [0; a_k, a_{k-1}, \ldots, a_1].$$

Let us recall some basic formulae about continued fractions, most of which can be found in [5, Chap. 10]:

$$\phi_k = \frac{q_{k-1}}{q_k}; \tag{2.1}$$

$$q_k\|q_{k-1}\theta\| + q_{k-1}\|q_k\theta\| = 1; \tag{2.2}$$

$$q_k\|q_k\theta\| = \frac{1}{a_{k+1} + \theta_{k+2} + \phi_k}; \tag{2.3}$$

$$q_k\|q_{k-1}\theta\| = \frac{1}{1 + \theta_{k+1}\phi_k}. \tag{2.4}$$

It turns out that the gaps are quantifiable in terms of the continued fraction for θ. More precisely, Van Ravenstein [7] proved the following result:

Theorem 2.2.

$$G(\theta, N) \subseteq \begin{cases} \{\|q_k\theta\|,\ \|q_{k-1}\theta\|,\ \|q_k\theta\| + \|q_{k-1}\theta\|\}, \\ \qquad \text{if } q_k \leq N < q_k + q_{k-1}; \\ \{\|q_k\theta\|,\ \|q_{k-1}\theta\| - (l-1)\|q_k\theta\|,\ \|q_{k-1}\theta\| - l\|q_k\theta\|\}, \\ \qquad \text{if } lq_k + q_{k-1} \leq N < (l+1)q_k + q_{k-1} \text{ for } 0 < l < a_{k+1}. \end{cases}$$

Moreover, for $q_k \leq N < q_k + q_{k-1} - 1$ *we have* $\|q_k\theta\| + \|q_{k-1}\theta\| \in G(\theta, N)$ *and for* $lq_k + q_{k-1} \leq N < (l+1)q_k + q_{k-1} - 1$ *we have* $\|q_{k-1}\theta\| - (l-1)\|q_k\theta\| \in G(\theta, N)$.

Let $H(\theta, N) = \max G(\theta, N)$, the largest gap in the gap set. For each integer $B \geq 1$ we set

$$f(B) := \sup_{\theta \in \mathcal{S}_B} \sup_{N \in \mathbb{N}} N \cdot H(\theta, N),$$

where for us $\mathbb{N} = \{1, 2, 3, \ldots\}$. As we will see soon, the values of $f(B)$ are always finite. Moreover it is easy to see that $f(B)$ is the smallest value that satisfies the following property: for all $\theta \in \mathcal{S}_B$, for all $N \in \mathbb{N}$ and for all $g \in G(\theta, N)$ we have $g \leq f(B)/N$. The core result of this paper is

Theorem 2.3. *Let* $B \geq 1$ *be an integer. Then*

$$f(B) = \begin{cases} 1 + \dfrac{(a+1)^2}{2\sqrt{a^2 + 2a}}, & \text{if } B = 2a,\ a \in \mathbb{N}; \\ 1 + \dfrac{a^2 + 3a + 2}{\sqrt{4a^2 + 12a + 5}}, & \text{if } B = 2a + 1,\ a \in \mathbb{N} \cup \{0\}. \end{cases} \tag{2.5}$$

In what follows we always assume that $\theta \in \mathcal{S}_B$. First, we prove the following easy lemma:

Lemma 2.4.

$$\min \mathcal{S}_B \setminus \mathbb{Q} = [0; \overline{B, 1}] = \frac{\sqrt{B^2+4B}-B}{2B};$$
$$\max \mathcal{S}_B \setminus \mathbb{Q} = [\overline{B; 1}] = \frac{\sqrt{B^2+4B}+B}{2}.$$

Here the vinculum denotes a repeating period of partial quotients.

Proof. The result is an immediate consequence of the following fact [5, Thm. 167]: as a real function of its partial quotients, the continued fraction

$$\theta = [a_0; a_1, a_2, \ldots]$$

is monotonically increasing in the even-numbered partial quotients a_0, $a_2, \ldots$, and is monotonically decreasing in the odd-numbered partial quotients $a_1, a_3, \ldots$. □

Proof of Theorem 2.3.

Case 1. Consider $q_k \le N < q_k + q_{k-1}$. Then we have

$$\sup_{q_k \le N < q_k+q_{k-1}} N \cdot H(\theta, N) \le (q_k + q_{k-1} - 1)(\|q_k\theta\| + \|q_{k-1}\theta\|).$$

Moreover, this upper bound is sharp. We first estimate the value of

$$(q_k + q_{k-1})(\|q_k\theta\| + \|q_{k-1}\theta\|). \tag{2.6}$$

Expanding and then simplifying (2.6), we get, using (2.2) and (2.3), that

$$\begin{aligned} &q_k\|q_k\theta\| + q_{k-1}\|q_{k-1}\theta\| + q_k\|q_{k-1}\theta\| + q_{k-1}\|q_k\theta\| \\ &\qquad = 1 + \frac{1}{a_{k+1}+\theta_{k+2}+\phi_k} + \frac{1}{a_k + \theta_{k+1} + \phi_{k-1}}. \end{aligned} \tag{2.7}$$

Introduce two variables $x = a_{k+1} + \theta_{k+2}$ and $y = \phi_k$. Then it is easy to check that $\theta_{k+1} = 1/x$ and $a_k + \phi_{k-1} = 1/y$. Finally the expression (2.6) simplifies to

$$1 + \frac{1}{x+y} + \frac{1}{x^{-1}+y^{-1}} = 1 + \frac{xy+1}{x+y}. \tag{2.8}$$

By looking at the partial derivatives of the right-hand side of (2.8), we see that it grows monotonically in x when $y > 1$ and decreases monotonically when $0 < y < 1$. By symmetry the same is true for y. Now, since $x > 1$ and $y < 1$, (2.8) is maximized when y is maximized and x is minimized. Since $\theta \in \mathcal{S}_B$, Lemma 2.4 says that the minimum possible x is

$1+\min\mathcal{S}_B = \frac{\sqrt{B^2+4B}+B}{2B}$. Among all values $y' \in \mathcal{S}_B$ the maximum possible is $y' = \max\mathcal{S}_B - B = \frac{\sqrt{B^2+4B}-B}{2}$. Then $xy' = 1$ and

$$x + y' = \frac{(B+1)\sqrt{B^2+4B}+B-B^2}{2B}.$$

Finally,

$$\begin{aligned} 1+\frac{xy'+1}{x+y'} &= 1+\frac{4B}{(B+1)\sqrt{B^2+4B}-B^2+B} \\ &= 1+\frac{(B+1)\sqrt{B^2+4B}+B^2-B}{2B^2+2B+1}. \end{aligned}$$

However, in our case, y is rational with partial quotients bounded by B. This value can be slightly bigger than y'. On the other hand, for all such y there exists $\xi \in \mathcal{S}_B$ such that y is a convergent of ξ. Since $y = \phi_k = \frac{q_{k-1}}{q_k}$, this implies that for the maximum possible y,

$$y - y' < \frac{1}{q_k^2}.$$

Then we have

$$\frac{xy+1}{x+y} - \frac{xy'+1}{x+y'} \leq \frac{\partial}{\partial y}\left(\frac{xy''+1}{x+y''}\right)(y-y') \leq \frac{x^2-1}{(q_k(x+y''))^2},$$

where y'' is some value between y and y'. Next, we have $x < 1+1/B$ and $x+y' > 2$. Therefore

$$\frac{xy+1}{x+y} - \frac{xy'+1}{x+y'} < \frac{2B+1}{4B^2q_k^2} < \|q_k\theta\| + \|q_{k-1}\theta\|.$$

Finally, we have that

$$\sup_{q_k \leq N < q_k+q_{k-1}} N\cdot H(\theta,N) \leq 1+\frac{(B+1)\sqrt{B^2+4B}+B^2-B}{2B^2+2B+1}. \tag{2.9}$$

Note that this estimate is sharp. Indeed, we can choose $\theta = [0; a_1, a_2, \ldots]$ and a positive integer k such that

$$x = [a_{k+1}; a_{k+2}, \ldots] = [1; B, 1, B, 1, \ldots]$$

is optimal and $y = [0; a_k, a_{k-1}, \ldots, a_1] = [0; 1, B, 1, \ldots, B]$ is as close to the optimal value as we wish. That is, for all $\epsilon > 0$, one can find $\theta \in \mathcal{S}_B$ and $k \in \mathbb{N}$ such that

$$\sup_{q_k \leq N < q_k+q_{k-1}} N\cdot H(\theta,N) > 1+\frac{(B+1)\sqrt{B^2+4B}+B^2-B}{2B^2+2B+1} - \epsilon.$$

Therefore we have

$$(2.10)\quad \sup_{\theta\in\mathcal{S}_B}\sup_{k\in\mathbb{N}}\max_{q_k\le N<q_k+q_{k-1}} N\cdot H(\theta,N) = 1+\frac{(B+1)\sqrt{B^2+4B}+B^2-B}{2B^2+2B+1}.$$

Case 2. Assume that $lq_k+q_{k-1}\le N<(l+1)q_k+q_{k-1}$ for some integer l between 1 and $a_{k+1}-1$. Then we have

$$\sup_{lq_k+q_{k-1}\le N<(l+1)q_k+q_{k-1}} N\cdot H(\theta,N) \le ((l+1)q_k+q_{k-1}-1)(\|q_{k-1}\theta\|-(l-1)\|q_k\theta\|).$$

We proceed as in Case 1. Define $x=a_{k+1}+\theta_{k+2}$ and $y=\phi_k$. We computed the values $q_k\|q_k\theta\|$ and $q_{k-1}\|q_{k-1}\theta\|$ in the previous case. Also from (2.4) we compute

$$q_k\|q_{k-1}\theta\| = \frac{1}{1+\theta_{k+1}\phi_k} = \frac{x}{x+y}.$$

Finally from (2.2), we have $q_{k-1}\|q_k\theta\| = 1-q_k\|q_{k-1}\theta\| = \frac{y}{x+y}$. Now we expand

$$\begin{aligned}((l+1)q_k+q_{k-1})(\|q_{k-1}\theta\|-(l-1)\|q_k\theta\|) &= \frac{1-l^2}{x+y}+\frac{(l+1)x}{x+y}-\frac{(l-1)y}{x+y}+\frac{xy}{x+y}\\ &= \frac{xy-(l-1)y+(l+1)x-l^2+1}{x+y}.\end{aligned}$$

Let the right-hand side be denoted by $F(x,y)$. We look for its maximum.

$$\frac{\partial}{\partial x}F(x,y) = \frac{y^2+2ly+l^2-1}{(x+y)^2} = \frac{(y+l)^2-1}{(x+y)^2}.$$

Since $0<y<1$ and $l\ge 1$, the quantity $F(x,y)$ is monotonically increasing in x. Therefore, by Lemma 2.4, the maximum is achieved at $x=\frac{\sqrt{B^2+4B}+B}{2}$. Next,

$$\frac{\partial}{\partial y}F(x,y) = \frac{x^2-2lx+l^2-1}{(x+y)^2} = \frac{(x-l)^2-1}{(x+y)^2}.$$

Notice that $x>B\ge l+1$. Therefore $F(x,y)$ monotonically increases in y. Among all values $y'\in\mathcal{S}_B$ the maximum possible is $y'=\frac{\sqrt{B^2+4B}-B}{2}$. By using the same argument as in case 1, the maximum rational value y satisfies $y-y'<q_k^{-2}$.

We have found that for each l between 1 and $B-1$, the maximum value of $F(x,y)$ is achieved at the same values of x and y' from $\mathcal{S}_B$. Now let us find l for which the value of $F(x,y')=F(x,y',l)$ is maximal. If we look at $F(x,y')$ as a function of l and look at its partial derivative we get

$$\frac{\partial}{\partial l}F(x,y',l)=\frac{-2l+x-y'}{x+y'}.$$

Notice that $x-y'=B$; therefore $F(x,y,l)$ is maximal for $l=B/2$ when B is even and $l=\frac{B\pm1}{2}$ if B is odd. One can easily check that, since $x-y'=B$ the values $F(x,y',(B+1)/2)$ and $F(x,y',(B-1)/2)$ coincide. So we can pick either of these two values of l; let us choose $l=(B-1)/2$. Finally, we notice that, as in Case 1, the upper bound for $F(x,y,l)$ is sharp because there exists $\theta\in\mathcal{S}_B$ and a positive integer k such that x takes the optimal value and y is as close to the optimal value as we wish.

Case 2.1. Let $B=2a$. Then $l=a$ and

$$F(x,y',l)=\frac{xy'+x+y'+1+l(x-y')-l^2}{x+y'}=1+\frac{(a+1)^2}{2\sqrt{a^2+2a}}.$$

Now we compute

$$F(x,y,l)-F(x,y',l)=\frac{(x-l)^2-1}{(x+y'')^2}(y-y')<\frac{(a+1)^2}{a^2q_k^2}.$$

The last is definitely smaller than

$$\|q_{k-1}\theta\|-(l-1)\|q_k\theta\|=\frac{x-(l-1)}{q_k(x+y)}>\frac{a}{2(a+1)q_k}.$$

Finally we compute

$$\sup_{\theta\in\mathcal{S}_B}\sup_{k\in\mathbb{N}}\max_{q_k+q_{k-1}\le N<q_{k+1}}N\cdot H(\theta,N)=1+\frac{(a+1)^2}{2\sqrt{a^2+2a}}. \tag{2.11}$$

Case 2.2. Let $B=2a+1$. Then $l=a$ and

$$F(x,y',l)=1+\frac{xy+1+a(2a+1)-a^2}{x+y}=1+\frac{a^2+3a+2}{\sqrt{4a^2+12a+5}}.$$

By computations similar to those in Case 2.1, we derive that

$$F(x,y,l)-F(x,y',l)<\|q_{k-1}\theta\|-(l-1)\|q_k\theta\|$$

and therefore

$$\sup_{\theta\in\mathcal{S}_B}\sup_{k\in\mathbb{N}}\max_{q_k+q_{k-1}\le N<q_{k+1}}N\cdot H(\theta,N)=1+\frac{a^2+3a+2}{\sqrt{4a^2+12a+5}}. \tag{2.12}$$

By careful computations one can notice that the right-hand side of (2.10) is always smaller (except the case $B=1$ when Case 2 is impossible) than the corresponding right-hand sides in (2.11) and (2.12). From this our result now follows. $\square$

Corollary 2.5. *We have* $\frac{B}{4} \le f(B) \le (1+\sqrt{4/5})B$.

Proof. This is an implication of the following estimates. For $B = 2a$, from (2.5) we have

$$\frac{B}{4} \le 1 + \frac{a+1}{2} \le 1 + \frac{(a+1)^2}{2\sqrt{(a+1)^2-1}} \le 1 + \frac{a+1}{2}\cdot\frac{2}{\sqrt{3}} < (1+\sqrt{4/5})B.$$

For $B = 2a+1$, we have

$$\begin{aligned} 1 + \frac{B/2+1}{2} \le f(B) &= 1 + \frac{a^2+3a+2}{\sqrt{4a^2+12a+5}} \\ &= 1 + \frac{(B/2+1)^2 - 1/4}{2\sqrt{(B/2+1)^2-1}} \le 1 + \frac{B/2+1}{2}\cdot\frac{8}{3\sqrt{5}}. \end{aligned}$$

Now it is easy to check that $f(B)$ is between $B/4$ and $(1+\sqrt{4/5})B$. $\square$

The proof of Theorem 2.3 also suggests a number $\theta \in \mathbb{R}$ and a sequence of positive integers N_i such that the largest gaps in $G(\theta, N_i)$ tend quickly to $f(B)/N_i$.

Corollary 2.6. *Let* $B \ge 1$ *be an integer and define* $\theta = [0, B, 1, B, 1, B, 1, \ldots]$. *Let* p_n/q_n *be the* n*-th convergent to* θ. *Then the largest gap* $H(\theta, N)$ *corresponding to* $N = q_{2n-1} + \lfloor\frac{B+2}{2}\rfloor q_{2n} - 2$ *equals* $\|q_{2n-1}\theta\| - \lfloor\frac{B-2}{2}\rfloor\|q_{2n}\theta\|$. *Further, as* $n \to \infty$ *the quantity* $N \cdot H(\theta, N)$ *tends to* $f(B)$.

Proof. Indeed, for this value of N we have

$$q_{2n-1} + lq_{2n} \le N < q_{2n-1} + (l+1)q_{2n} - 1,$$

and hence $l = \lfloor\frac{B}{2}\rfloor$ and Theorem 2.2 imply that

$$H(\theta, N) = \|q_{2n-1}\theta\| - \left\lfloor \frac{B-2}{2} \right\rfloor \|q_{2n-1}\theta\|.$$

For this value of N the values of x and y from the proof of Theorem 2.3 equal $x = [B; 1, B, 1, B, \ldots]$, which maximizes the value $F(x, y, l)$ and $y = [0; 1, b, 1, \ldots, 1]$, which tends to the optimal value of y as $n \to \infty$. Then Theorem 2.3 implies that $N \cdot H(\theta, N) \to f(B)$ as $n \to \infty$. $\square$

3. Kronecker's theorem for badly approximable numbers

Equipped with Theorem 2.3 we can improve Theorem 17 from [10] (see Theorem 1.1 above).

Theorem 3.1. *Let* θ *be an irrational real number with partial quotients bounded by* B, *and let* β *be an arbitrary real number. Suppose* $0 \le \theta, \beta < 1$. *Then there are integers* n, p *with* $n, |p| \le N$ *such that* $|n\theta - p - \beta| < \frac{f(B)}{2N}$.

Proof. Apply the three-gap theorem to θ and N. This involves sorting the $N+2$ points $0, \{\theta\}, \{2\theta\}, \ldots, \{N\theta\}, 1$ in ascending order and creating the $N+1$ intervals between the points, the union of which forms $[0,1)$.

Now consider the interval in which β lies, denote it by $[e_1, e_2]$ with $e_1 = \{n_1\theta\}$ and either $e_2 = \{n_2\theta\}$ or $e_2 = 1$. The distance from β to the closest of these two endpoints is at most $\frac{e_2 - e_1}{2}$, which by Theorem 2.3 is bounded above by $f(B)/2N$.

If the closest endpoint to β is of the form $\{n\theta\}$ then we notice that $\{n\theta\} = n\theta - p$ for some integer p with $0 \leq p < n$. Therefore $|n\theta - p - \beta| \leq \frac{f(B)}{2N}$ as required. If 1 is the closest endpoint to β then we take $n = 0, p = -1$ and verify $|0 \cdot \theta - 1 - \beta| \leq \frac{f(B)}{2N}$. $\square$

Remark 3.2. This improves the bounds $(B+2)N^2$ and $1/N$ given in [10].

Remark 3.3. The bound in the theorem is tight. We can choose θ and N from Corollary 2.6 and choose β arbitrarily close to the midpoint of the maximal gap. By choosing N large enough, we can ensure that the corresponding maximal gap will be as close as we like to $f(B)/N$.

4. Diversity

We now turn to the application of these results that concerned us in [10]. As usual, we use $f = O(g)$ (respectively, $f = \Omega(g)$) to mean that there exist constants $c > 0, N$ such that $f(n) \leq cg(n)$ (respectively, $f(n) \geq cg(n)$) for all $n > N$.

Let $\mathbf{s} = (s_n)_{n \geq 0}$ be a sequence. We say that $\mathbf{s}$ is *diverse* if, for all $k \geq 2$, the k subsequences $\{(s_{ki+a})_{i \geq 0} \ : \ 0 \leq a < k\}$ are all distinct.

If two sequences $\mathbf{t} = (t_i)_{i \geq 0}$ and $\mathbf{u} = (u_i)_{i \geq 0}$ are distinct, we define their *agreement* $\mathrm{ag}(\mathbf{t}, \mathbf{u})$ to be $\min\{i \ : \ t_i \neq u_i\}$. If a sequence $(s_n)_{n \geq 0}$ has the property that for all r, a, b with $0 \leq a < b < r$ we have

$$\mathrm{ag}((s_{ri+a})_{i \geq 0}, (s_{ri+b})_{i \geq 0}) \leq f(r),$$

then we say that the function f is a *diversity measure* for the sequence $\mathbf{s}$. In [10] it is shown that there exists a function $f \in O(\log r)$ that is a diversity measure for almost all binary sequences. However, no explicit example of a sequence with this diversity measure is known.

Thus, it is of interest to produce an explicitly-defined sequence with slowly-growing diversity measure. In [10] the second author looked at the case of Sturmian characteristic sequences (see, e.g., [2]); these are words of the form $\mathbf{s} = (s_i)_{i \geq 0}$ with

$$s_i = \lfloor (i+2)\theta \rfloor - \lfloor (i+1)\theta \rfloor. \tag{4.1}$$

(The indexing here is slightly atypical because we want to index $\mathbf{s}$ starting at 0 instead of the more conventional 1.)

Suppose θ has partial quotients bounded by B. Then the second author proved [10, Lem. 18] that $4(B+2)^3r^3$ is a diversity measure for $(s_i)_{i\geq 0}$. In this section we improve this result.

Theorem 4.1. *Suppose $0 < \theta < 1$ has partial quotients bounded by B. Consider the associated Sturmian characteristic sequence* $\mathbf{s}$ *as defined above. Then the function $2(B+2)^2r^2$ is a diversity measure for* $\mathbf{s}$.

Proof. We follow the proof of [10, Lem. 18] with some small changes.

We use the "circular representation" for subsets of $[0,1)$, identifying the endpoints 0 and 1 and considering each point modulo 1.

Then $s_{i-1} = 1$ iff $\{i\theta\} \in [1-\theta, 1)$. If we can find m such that $\{(rm+c)\theta\} \in [1-\theta, 1)$ while $\{(rm+d)\theta\} \in [0, 1-\theta)$, then for this m we have $s_{rm+c-1} \neq s_{rm+d-1}$.

But, using the circular representation of intervals,

$$\{(rm+c)\theta\} \in [1-\theta, 1) \text{ iff } \{rm\theta\} \in I_c := [-(c+1)\theta, -c\theta)$$

and

$$\{(rm+d)\theta\} \in [0, 1-\theta) \text{ iff } \{rm\theta\} \in I_d := [-d\theta, -(d+1)\theta).$$

Furthermore, $\mu(I_c) + \mu(I_d) = 1$ (where μ is Lebesgue measure) and so these two intervals have nontrivial intersection if $c \neq d$. The endpoints of these intervals are of the form $\{-i\theta\}$ for some i with $0 \leq i \leq r$. Then $\mu(I_c \cap I_d)$ is at least as big as the smallest gap g in the three-gap theorem corresponding to $N = r$, which by [10, Lem. 16], is at least $\frac{1}{(B+2)r}$.

Let m' be the midpoint of the interval $I_c \cap I_d$. If we could find integers m, t with

$$|rm\theta - m' - t| < \frac{1}{2(B+2)r},$$

then

$$\frac{1}{2}\mu(I_c \cap I_d) \geq \frac{1}{2}g \geq \frac{1}{2(B+2)r} > |rm\theta - m' - t|,$$

then $\{rm\theta\}$ would lie inside $I_c \cap I_d$.

Since θ has partial quotients bounded by B, we know that $r\theta$ has partial quotients bounded by $r(B+2)$; see, for example [6]. Hence by Theorem 3.1 applied to $r\theta$, we see that such an m exists with $m \leq r(B+2)f(r(B+2)) \leq 2(B+2)^2r^2$. $\square$

We now show that the bound in Theorem 4.1 is tight, up to a constant factor.

Let, as usual, the Fibonacci numbers F_n be defined by $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. Let the Lucas numbers L_n be defined by $L_0 = 2$, $L_1 = 1$, and $L_n = L_{n-1} + L_{n-2}$ for $n \geq 2$.

In what follows we let

$$\alpha = (1+\sqrt{5})/2 \doteq 1.61803$$
$$\beta = (1-\sqrt{5})/2 \doteq -0.61803$$

be the two zeros of $X^2 - X - 1$, and we let $\theta = 1/\alpha = -\beta = (\sqrt{5}-1)/2$. Recall the Binet formulas for the Fibonacci and Lucas numbers: $F_n = (\alpha^n - \beta^n)/\sqrt{5}$ and $L_n = \alpha^n + \beta^n$.

We start with three useful lemmas.

Lemma 4.2.
(i) $F_n\theta = F_{n-1} - \beta^n$ *for* $n \geq 1$*;*
(ii) $L_n\theta = L_{n-1} + \sqrt{5}\beta^n$ *for* $n \geq 1$*.*

Proof. Routine manipulation involving the Binet formulas. □

Define $\gamma(n) = \{n\theta\}$.

Lemma 4.3. *Let* $n \geq 2$*. Then*

$$\gamma(F_{4n}i + L_{2n}j + F_{2n-1}) = (\gamma(F_{4n}) - 1)i + \gamma(L_{2n})j + \gamma(F_{2n-1}) \tag{4.2}$$
$$\gamma(F_{4n}i + L_{2n}j + L_{2n}) = (\gamma(F_{4n}) - 1)i + \gamma(L_{2n})j + \gamma(L_{2n}). \tag{4.3}$$

for $0 \leq i \leq L_{2n+1} - 2$ *and* $0 \leq j \leq F_{2n} - 1$*.*

Proof. By Lemma 4.2 we get

$$\gamma(F_{2n-1}) = \theta^{2n-1}$$
$$\gamma(F_{4n}) = 1 - \theta^{4n}$$
$$\gamma(L_{2n}) = \sqrt{5}\theta^{2n},$$

which we use repeatedly in what follows. Then the desired relations (4.2) and (4.3) hold provided there is no "wrap-around" modulo 1 in those sums.

Lemma 4.4. *Let* $n \geq 2$*. Then*

$$\{F_{2n}i + j \,:\, 0 \leq i \leq L_{2n+1} - 2,\ 0 \leq j \leq F_{2n} - 1\}$$
$$= \{k \,:\, 0 \leq k \leq F_{4n+1} - F_{2n} - 2\}.$$

Proof. On the left-hand side, we have $(L_{2n+1} - 1)F_{2n}$ possibilities for the pairs (i,j); on the right-hand side we have a set of cardinality $F_{4n+1} - F_{2n} - 1$. Since $F_{2n}L_{2n+1} = F_{4n+1} - 1$, these two quantities are the same.

It now suffices to show that each element of the right-hand side set belongs to the left-hand side set. To do this for a particular k, use the division algorithm and define $i = \lfloor k/F_{2n} \rfloor$ and $j = k \bmod F_{2n}$, so that $k = F_{2n}i + j$. Then

$$i \leq k/F_{2n} \leq (F_{4n+1} - F_{2n} - 2)/F_{2n} = L_{2n+1} - 1 - 1/F_{2n};$$

since i is an integer, we see that $i \leq L_{2n+1} - 2$. □

We now define two rectangular arrays A and B, as follows:

$$\begin{aligned}A[i,j] &= (\gamma(F_{4n})-1)i+\gamma(L_{2n})j+\gamma(F_{2n-1})\\ &\qquad\text{for } 0\le i\le L_{2n+1}-2 \text{ and } 0\le j\le F_{2n}-1\\ B[i,j] &= (\gamma(F_{4n})-1)i+\gamma(L_{2n})j+\gamma(L_{2n})\\ &\qquad\text{for } 0\le i\le L_{2n+1}-2 \text{ and } 0\le j\le F_{2n}-1.\end{aligned}$$

Note that $B[i,j]-A[i,j]=\gamma(L_{2n})-\gamma(F_{2n-1})=\theta^{2n+1}$, which is independent of i and j.

We now claim that the entries of A (resp., B) can be read in ascending order as follows: start at the bottom left (i.e., at the entry $A[L_{2n+1}-2,0]$, resp., $B[L_{2n+1}-2,0]$). Proceed up each column to the top entry. When you reach the top of a column, continue at the bottom of the column to its right. This order is illustrated for $n=2$ and A in Figure 4.1. As shown in Figure 4.1, let d denote the difference between two entries in the same row, but adjacent columns, and let d' denote the difference between two entries in the same column, but adjacent rows. Finally, let d'' denote the difference between the entry at the top of a column, and the entry at the bottom of the column to its right. Then it is easy to verify from the definitions of A and B that

$$d=\gamma(L_{2n})=\sqrt{5}\theta^{2n}; \tag{4.4}$$

$$d'=1-\gamma(F_{4n})=\theta^{4n}; \tag{4.5}$$

$$d''=(L_{2n+1}-2)\gamma(F_{4n})+\gamma(L_{2n})=\theta^{2n+1}+2\theta^{4n}+\theta^{6n+1}. \tag{4.6}$$

Since all these quantities are positive, our claim about the ordering of the entries of A and B follows.

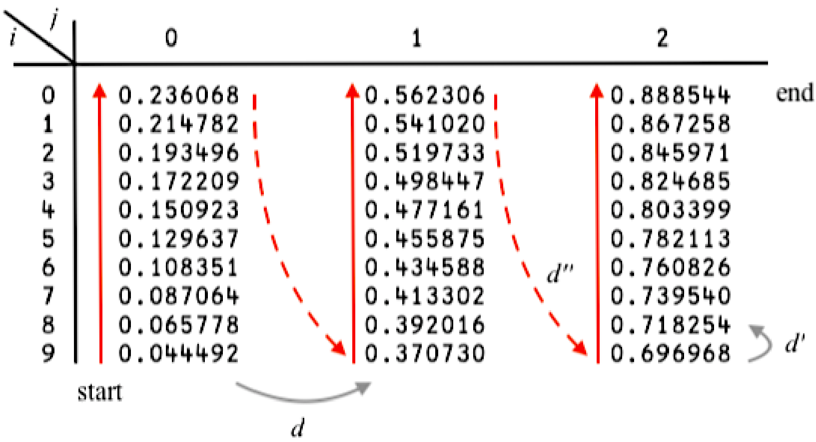

Figure 4.1. The array $A[i,j]$ for $n=2$

Finally, we observe that for both arrays, every entry is strictly between 0 and 1. By the increasing property of columns and rows, it suffices to verify this for the entries labeled "start" and "end". To see this, note that the entry labeled "start" in A is

$$A[L_{2n+1}-2,0] = (\gamma(F_{4n})-1)(L_{2n+1}-2)+\gamma(F_{2n-1}) = 2\theta^{4n}+\theta^{6n+1} > 0,$$

while the entry labeled "end" in B is

$$B[0,F_{2n}-1] = \gamma(L_{2n})(F_{2n}-1)+\gamma(L_{2n}) = 1-\theta^{4n} < 1.$$

Since $A[i,j] < B[i,j]$, the other two entries $A[0,F_{2n}-1]$ and $B[L_{2n+1}-2,0]$ are also between 0 and 1.

Putting all these facts together, we see there is no "wrap-around" modulo 1 in (4.2) and (4.3), so that

$$A[i,j] = \gamma(F_{4n}i + L_{2n}j + F_{2n-1})$$
$$B[i,j] = \gamma(F_{4n}i + L_{2n}j + L_{2n})$$

for $0 \le i \le L_{2n+1}-2$ and $0 \le j \le F_{2n}-1$. □

We can now prove our lower bound on diversity.

Theorem 4.5. *Let $n \ge 2$ be an integer, and let $(s_i)_{i\ge 0}$ be the Sturmian sequence, defined in* (4.1)*, and corresponding to $\theta = \frac{1}{2}(\sqrt{5}-1)$. Let $r = L_{2n}$, $a = F_{2n-1}-1$, and $b = L_{2n}-1$ for $n \ge 2$. Then*

$$s_{rk+a} = s_{rk+b} \text{ for } 0 \le k \le F_{4n+1}-F_{2n+1}-2, \text{ but}$$
$$0 = s_{rk+a} \ne s_{rk+b} = 1 \text{ for } k = F_{4n+1}-F_{2n+1}-1.$$

Since $F_{4n+1}/L_{2n}^2 \approx (\sqrt{5}+10)/10$, this implies that $\mathbf{s}$ *has diversity measure $\Omega(r^2)$.*

Proof. From the definition of s_n, we have that $s_n = 0$ iff $\gamma(n+1) \in (1-\theta, 1)$. Thus, for the choices of r, a, b above, it suffices to find the smallest $k \ge 0$ such that $\gamma(rk+a+1) \notin (1-\theta,1)$, but $\gamma(rk+b+1) \in (1-\theta,1)$ (or vice versa).

Let's look again at the arrays A and B. We claim that A contains each value $\gamma(rk+a+1)$ for $0 \le k \le F_{4n+1}-F_{2n}-2$ exactly once and similarly B contains each value $\gamma(rk+b+1)$ for $0 \le k \le F_{4n+1}-F_{2n}-2$ exactly once. To see this, it suffices to show that the numbers

$$F_{4n}i + L_{2n}j$$

for $0 \le i \le L_{2n+1}-2$, $0 \le j \le F_{2n}-1$, represent each multiple $k \cdot L_{2n}$, for $0 \le k \le F_{4n+1}-F_{2n}-2$, exactly once. This immediately follows from the observation that $F_{4n} = F_{2n}L_{2n}$, so (in effect) we are representing k in the mixed radix system with place values $(1, L_{2n}, F_{2n})$.

Now we have already observed above that $B[i,j]-A[i,j] > 0$, so if there exists k with $0 \le k \le F_{4n+1}-F_{2n}-2$ such that $s_{rk+a} \ne s_{rk+b}$, then it must be that $\gamma(rk+a+1) \in (0, 1-\theta)$ and $\gamma(rk+b+1) \in (1-\theta, 1)$. So we need to find i, j such that $A[i,j] < 1-\theta < B[i,j]$. We claim this occurs uniquely for $i = L_{2n+1} - 2$ and $j = F_{2n-2}$.

We can check now that, for these specific choices of i and j, we have

$$A[i,j] = \theta^2 + 2\theta^{4n} + \theta^{6n+1} - \theta^{4n-2} \tag{4.7}$$

$$B[i,j] = \theta^2 + \sqrt{5}\theta^{2n} + 2\theta^{4n} + \theta^{6n+1} - \theta^{2n-1} - \theta^{4n-2}, \tag{4.8}$$

and so we see

$$0 < \theta^2 - \theta^{4n-2} < A[i,j] < \theta^2 = 1-\theta$$

and

$$\begin{aligned} 1-\theta = \theta^2 < B[i,j] &= \theta^2 + (\sqrt{5} - 1/\theta)\theta^{2n} + (2 - 1/\theta^2 + \theta^{2n+1})\theta^{4n} \\ &< \theta^2 + \theta^{2n+1} < 1. \end{aligned}$$

It now follows that $s_{iF_{4n}+jL_{2n}+F_{2n-1}-1} = 0$, while $s_{iF_{4n}+jL_{2n}+L_{2n}-1} = 1$.

It remains to see this is the only possible choice (i,j) for which $A[i,j] < 1-\theta < B[i,j]$. To see this, note that by the ordering of the entries of A and B mentioned previously, it suffices to show that $A[i-1,j] > 1-\theta$ and $B[0, j-1] < 1-\theta$.

The first inequality follows from combining (4.7) and (4.5):

$$\begin{aligned} A[i-1,j] = A[i,j] + d' &= \theta^2 + 2\theta^{4n} + \theta^{6n+1} - \theta^{4n-2} + \theta^{4n} \\ &= \theta^2 + (3 - 1/\theta^2 + \theta^{2n+1})\theta^{4n} > \theta^2 = 1-\theta. \end{aligned}$$

The second inequality follows from combining (4.8) and (4.6):

$$\begin{aligned} B[0,j-1] &= B[i,j] - d'' \\ &= \theta^2 + \sqrt{5}\theta^{2n} + 2\theta^{4n} + \theta^{6n+1} - \theta^{2n-1} - \theta^{4n-2} \\ &\quad\; - (\theta^{2n+1} + 2\theta^{4n} + \theta^{6n+1}) \\ &= \theta^2 + (\sqrt{5} - 1/\theta - \theta)\theta^{2n} - \theta^{4n-2} = \theta^2 - \theta^{4n-2}. \end{aligned}$$

This completes the proof. □

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References

[1] P. Alessandri & V. Berthé, "Three distance theorems and combinatorics on words", *Enseign. Math.* **44** (1998), no. 1-2, p. 103-132.

[2] J. Berstel & P. Séébold, "Sturmian words", in *Algebraic Combinatorics on Words*, Encyclopedia of Mathematics and Its Applications, vol. 90, Cambridge University Press, 2002, p. 45-110.

[3] K. Florek, "Une remarque sur la répartition des nombres $n\xi$ (mod 1)", *Colloq. Math.* **2** (1951), p. 323-324.

[4] J. H. Halton, "The distribution of the sequence $\{n\xi\}$ $(n = 0, 1, 2, \ldots)$", *Proc. Camb. Philos. Soc.* **61** (1965), p. 665-670.

[5] G. H. Hardy & E. M. Wright, *An Introduction to the Theory of Numbers*, 5th ed., Oxford University Press, 1979.

[6] J. C. Lagarias & J. O. Shallit, "Linear fractional transformations of continued fractions with bounded partial quotients", *J. Théor. Nombres Bordeaux* **9** (1997), no. 2, p. 267-279, corrigendum in *ibid.* **15** (2003), no. 3, p. 741-743.

[7] T. van Ravenstein, "The three gap theorem (Steinhaus conjecture)", *J. Aust. Math. Soc.* **45** (1988), no. 3, p. 360-370.

[8] T. van Ravenstein, G. Winley & K. Tognetti, "Characteristics and the three gap theorem", *Fibonacci Q.* **28** (1990), no. 3, p. 204-214.

[9] J. O. Shallit, "Real numbers with bounded partial quotients", *Enseign. Math.* **38** (1992), no. 1-2, p. 151-187.

[10] ———, "Automaticity IV: Sequences, sets, and diversity", *J. Théor. Nombres Bordeaux* **8** (1996), no. 2, p. 347-367.

[11] N. B. Slater, "The distribution of the integers N for which $\{\Theta N\} < \Phi$", *Proc. Camb. Philos. Soc.* **46** (1950), p. 525-534.

[12] V. T. Sós, "On the theory of diophantine approximations. I", *Acta Math. Acad. Sci. Hung.* **8** (1957), p. 461-471.

[13] J. Suranyi, "Über die Anordnung der Vielfachen einer reellen Zahl mod 1", *Ann. Univ. Sci. Budap. Rolando Eötvös, Sect. Math.* **1** (1958), p. 107-111.

[14] S. Świerczkowski, "On successive settings of an arc on the circumference of a circle", *Fundam. Math.* **46** (1958), p. 187-189.

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