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Polynomial approximations in a generalized Nyman–Beurling criterion

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Résumé. Le critère de Nyman–Beurling, équivalent à l'hypothèse de Riemann (HR), est un problème d'approximation dans l'espace des fonctions de carré intégrable sur $(0,\infty)$, par des dilatations de facteurs $\theta_k \in (0,1)$, $k \geq 1$, de la fonction partie fractionnaire. Prendre les θ_k alétoires génère de nouvelles structures et de nouveaux critères. L'un d'eux est une condition suffisante pour HR qui revient à

(i) montrer que la fonction indicatrice peut être approximée par des convolutions de la partie fractionnaire, et

(ii) avoir un contrôle des coefficients de l'approximation.

Ce papier généralise les conditions (i) et (ii) afin d'obtenir un critère impliquant $\zeta(\sigma+it) \neq 0$ dans une bande $1/2 < \sigma \leq \sigma_0 < 1$. On identifie ensuite des fonctions pour lesquelles (i) est vérifiée inconditionnellement, grâce à des approximations polynomiales. Cela fournit, au passage, une courte preuve probabiliste d'une conséquence connue d'un théorème Taubérien. Dans ce contexte, la difficulté à prouver HR se reporte sur (ii), qui pourrait nécessiter une étude fine des matrices de Gram correspondantes. Nous obtenons deux structures remarquables de ces matrices. Nous montrons qu'un choix particulier des suites approximantes fournit une simplification remarquable de la matrice de Gram qui s'écrit alors sous la forme de matrices de Hankel par blocs.

Abstract. The Nyman–Beurling criterion, equivalent to the Riemann hypothesis (RH), is an approximation problem in the space of square integrable functions on $(0,\infty)$, involving dilations of the fractional part function by factors $\theta_k \in (0,1)$, $k \geq 1$. Randomizing the θ_k generates new structures and criteria. One of them is a sufficient condition for RH that splits into

(i) showing that the indicator function can be approximated by convolution with the fractional part,

(ii) a control on the coefficients of the approximation.

This self-contained paper generalizes conditions (i) and (ii) that involve a $\sigma_0 \in (1/2,1)$, and imply $\zeta(\sigma+it) \neq 0$ in the strip $1/2 < \sigma \leq \sigma_0 < 1$. We then identify functions for which (i) holds unconditionally, by means of polynomial approximations. This yields in passing a short probabilistic proof

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of a known consequence of Wiener's Tauberian theorem. In this context, the difficulty for proving RH is then reallocated in (ii), which heavily relies on the corresponding Gram matrices, for which two remarkable structures are obtained. We show that a particular tuning of the approximating sequence leads to a striking simplification of the second Gram matrix, then reading as a block Hankel form.

1. Introduction

1.1. Context and main definitions. The Riemann hypothesis (RH) is equivalent to the Nyman–Beurling (NB) criterion, which is an approximation problem of the indicator function χ of $(0,1)$ in the space of square integrable functions on $(0,\infty)$, involving dilations of the fractional part function $\{\cdot\}$ by factors $\theta_k \in (0,1]$:

Theorem 1.1 ([2]). *RH holds if and only if, given $\varepsilon > 0$, there exist $n \geq 1$, coefficients $c_1, \dots, c_n \in \mathbb{R}$, and $\theta_1, \dots, \theta_n \in (0,1]$ such that*

$$\int_0^\infty \left(\chi(t) - \sum_{k=1}^{n} c_k \left\{\frac{\theta_k}{t}\right\}\right)^2 \mathrm{d}t < \varepsilon. \tag{1.1}$$

The original version of the Nyman–Beurling criterion ([5, 17]) contains the linear constraint $\sum_{1\leq k\leq n} c_k\theta_k = 0$, which is removed by Báez-Duarte, Balazard, Landreau and Saias in [2].

Báez-Duarte [1] showed that it is possible to specify $\theta_k = 1/k$ in this criterion. The coefficients $(c_k)_{1\leq k\leq n}$ leading to the best approximation, then solve the linear system

$$Gx = b,$$

where $x = (c_1, \dots, c_n)^T$, $b = (b_1, \dots, b_n)^T$, with $b_k = \int_0^1 \{\frac{1}{kt}\}\mathrm{d}t$ and G is the Gram matrix with

$$G_{k,l} = \int_0^\infty \left\{\frac{1}{kt}\right\}\left\{\frac{1}{lt}\right\} \mathrm{d}t, \quad \text{for } 1 \leq k, l \leq n\,.$$

The computation of G is possible through a formula due to Vasyunin [21]. The evaluation of the distance in (1.1) may be then approximately evaluated by numerical means [2].

Over recent years, some research works have been devoted to the Nyman–Beurling criterion for the Riemann Hypothesis. For example, the articles [4] by Bettin and Conrey, or [13], [14] by Maier and Rassias, aim to study cotangent sums related to this problem and present a series of relevant results.

More recently, randomizing the θ_k has produced new characterizations and structures [10]. Among these, the following sufficient condition for RH is obtained. From now on, we write $\mathbb{E}Z$ for the expectation of a random variable (r.v.) Z, and we only consider positive r.v. Let us recall the result [10, Theorem 3.1 p. 53]:

Theorem 1.2. *Let $(Z_{k,n})_{1\leq k\leq n, n\geq 1}$ be square integrable r.v. satisfying for any $\alpha > 1$*

$$\sum_{k=1}^{n}\left(\mathbb{E}Z_{k,n}^2\right)^{\alpha/2} \ll_\alpha 1, \qquad n\to\infty. \tag{1.2}$$

If there exist real coefficients $(c_{k,n})_{1\leq k\leq n, n\geq 1}$ such that, for any $M_n\to\infty$,

$$D_n^2 = \int_0^\infty \left|\chi(t) - \sum_{k=1}^{n} c_{k,n}\mathbb{E}\left\{\frac{Z_{k,n}}{t}\right\}\right|^2 \mathrm{d}t \xrightarrow[n\to\infty]{} 0 \tag{1.3}$$

$$\sum_{k=1}^{n}|c_{k,n}|^2\mathbb{P}(Z_{k,n}\geq M_n) \xrightarrow[n\to\infty]{} 0, \tag{1.4}$$

then RH holds.

The relevance of the former theorem stems from the fact that the converse holds for some specific structures, as dilated or concentrated r.v., see [10] for more details. A particular case of dilated structure is given by exponentially distributed r.v. $Z_k\sim\mathcal{E}(k)$, which is related to a particular cotangent sum, studied in [9].

Notice that, for r.v. Z_k with densities ϕ_k, the involved functions

$$\mathbb{E}\left\{\frac{Z_k}{t}\right\} = \int_0^\infty \left\{\frac{x}{t}\right\}\phi_k(x)\mathrm{d}x, \tag{1.5}$$

can be written as a multiplicative convolution. Indeed, for $g:\mathbb{R}_+\to\mathbb{R}$, let $g^\times(t)$ denote the multiplicative convolution of g with the fractional part:

$$g^\times(t) = \int_0^\infty \left\{\frac{x}{t}\right\}g(x)\frac{\mathrm{d}x}{x} = \left(\left\{\frac{1}{\cdot}\right\}*g\right)(t). \tag{1.6}$$

If $g_k(x) = x\phi_k(x)$ then

$$g_k^\times(t) = \mathbb{E}\left\{\frac{Z_k}{t}\right\}.$$

We then observe that this latter expression allows for a generalization with functions g_k that possibly change sign.

From now on, we use two different notations: $g_k^\times$ when dealing with functions g_k, and $h_k^\times$ when dealing with r.v. Z_k:

$$g_k^\times(t) := \int_0^\infty \left\{\frac{x}{t}\right\}g_k(x)\frac{\mathrm{d}x}{x} \tag{1.7}$$

$$h_k^\times(t) := \mathbb{E}\left\{\frac{Z_k}{t}\right\}. \tag{1.8}$$

Definition 1. *We say that a sequence of real function* $(g_k)_k$*, resp. a sequence of r.v.* $(Z_k)_k$*, verifies* gNB *if there exist coefficients* $(c_{k,n})_{1\leq k\leq n, n\geq 1}$ *such that*

$$D_n^2 = \int_0^\infty \left|\chi(t) - \sum_{k=1}^n c_{k,n} g_k^\times(t)\right|^2 \mathrm{d}t \xrightarrow[n\to\infty]{} 0, \qquad \text{(gNB)} \tag{1.9}$$

resp. if

$$D_n^2 = \int_0^\infty \left|\chi(t) - \sum_{k=1}^n c_{k,n} h_k^\times(t)\right|^2 \mathrm{d}t \xrightarrow[n\to\infty]{} 0. \qquad \text{(gNB)} \tag{1.10}$$

The main purpose of this paper is to identify classes of functions g_k, or corresponding r.v. Z_k, that verify gNB, unconditionally (i.e. without assuming RH):

(1) In Section 3, we treat the case of r.v. $Z_k = Y/X_k$ where Y is a positive r.v. and X_k are $\Gamma(k,1)$-distributed r.v. independent of Y.
(2) In Section 4, we consider functions g_k, that are not necessarily non negative, defined by induction as, for all $k \geq 0$:

$$g_{k+1}(x) = -xg_k'(x) - r_k g_k(x), \quad x \geq 0,$$

where $(r_k)_{k\geq 0}$ is a real sequence and g_0 is a suitable initialization.

In each case, we also provide expressions of the scalar products $\langle h_k^\times, h_j^\times\rangle$, and $\langle g_k^\times, g_j^\times\rangle$, which may be necessary to tackle Condition (1.4) in the future. Two remarkable structures are obtained for the Gram matrices. Moreover, taking $r_k = 1/2$ in (2) yields to a striking simplification.

To handle Case (2), we need to generalize Theorem 1.2, which is done in Section 2.

1.2. Preliminaries.

We say that $g : \mathbb{R}^+ \to \mathbb{R}$ verifies Assumption (M) if

$$\text{(M)} \qquad \int_0^\infty |g(x)|\mathrm{d}x < \infty \quad \text{and} \quad \int_0^\infty |g(x)|\frac{\mathrm{d}x}{x} < \infty.$$

We write a complex number $s = \sigma + it$, $\sigma, t \in \mathbb{R}$. The previous assumption allows to define the Mellin transform $\widehat{g}$ of g in the critical strip $0 < \sigma < 1$:

$$\widehat{g}(s) = \int_0^\infty g(x)x^{s-1}\mathrm{d}x,$$

since $\int_0^\infty |g(x)|x^{\sigma-1}\mathrm{d}x \leq \left(\int_0^\infty |g(x)|\mathrm{d}x\right)^\sigma \left(\int_0^\infty |g(x)|\frac{\mathrm{d}x}{x}\right)^{1-\sigma}$, due to the Hölder inequality. Assumption (M) is sufficient but not necessary for the Mellin transform to be defined, and we recall that Mellin–Plancherel theory allows to define $\widehat{g}(s)$ whenever $g \in L^2(0,\infty)$. In this case, we have the isometry:

$$\int_0^\infty |g(x)|^2 \mathrm{d}x = \frac{1}{2\pi} \int_{-\infty}^\infty \left| \widehat{g}\left(\frac{1}{2} + it\right)\right|^2 \mathrm{d}t.$$

We finally recall the fundamental identity on which relies the NB criterion, see [20, (2.1.5)]:

$$\int_0^\infty \left\{\frac{1}{x}\right\} x^{s-1} \mathrm{d}x = -\frac{\zeta(s)}{s}, \quad 0 < \sigma < 1. \tag{1.11}$$

In the case where $\phi(x) = g(x)/x$ is the density of a r.v. $Z \geq 0$, Assumption (M) simply translates into $\mathbb{E}Z < \infty$. The following lemma is standard, but we give a proof in our framework for the sake of completeness.

Lemma 1.

(i) *Let $g : \mathbb{R}^+ \to \mathbb{R}$ satisfying* (M). *Then $g^\times \in L^2(\mathbb{R}_+)$ and its Mellin transform $\widehat{g^\times}(s)$ is well defined for $\sigma \in (0,1)$. Moreover,*

$$\widehat{g^\times}(s) = -\frac{\zeta(s)}{s}\widehat{g}(s), \quad 0 < \sigma < 1\,.$$

(ii) *If $Z \geq 0$ is an integrable r.v. then $t \longmapsto h^\times(t) := \mathbb{E}\{\frac{Z}{t}\}$ belongs to $L^2(\mathbb{R}_+)$ and*

$$\widehat{h^\times}(s) = -\frac{\zeta(s)}{s}\mathbb{E}Z^s, \quad 0 < \sigma < 1\,.$$

Proof. (i). Writing $|g(x)| = |g(x)|^{\frac{1}{2}}|g(x)|^{\frac{1}{2}}$ and using the Cauchy–Schwarz inequality, one obtains:

$$\begin{aligned}\int_0^\infty |g^\times(t)|^2 \mathrm{d}t &\leq \int_0^\infty \left(\int_0^\infty \left\{\frac{x}{t}\right\} |g(x)| \frac{\mathrm{d}x}{x}\right)^2 \mathrm{d}t \\ &\leq \int_0^\infty \left(\int_0^\infty |g(x)|\frac{\mathrm{d}x}{x}\right)\left(\int_0^\infty \left\{\frac{x}{t}\right\}^2 |g(x)|\frac{\mathrm{d}x}{x}\right) \mathrm{d}t \\ &= \int_0^\infty |g(x)|\frac{\mathrm{d}x}{x} \int_0^\infty |g(x)|\mathrm{d}x \int_0^\infty \left\{\frac{1}{t}\right\}^2 \mathrm{d}t < \infty,\end{aligned}$$

due to Assumption (M). We already noticed that for all $\sigma \in (0,1)$, $\int_0^\infty |g(x)|x^{\sigma-1}\mathrm{d}x < +\infty$, so

$$\int_0^\infty \int_0^\infty \left|\left\{\frac{x}{t}\right\} \frac{g(x)}{x} t^{s-1}\right| \mathrm{d}x\mathrm{d}t \leq \int_0^\infty \left\{\frac{1}{t}\right\} t^{\sigma-1}\mathrm{d}t \int_0^\infty |g(x)|x^{\sigma-1}\mathrm{d}x < +\infty\,.$$

Hence $\widehat{g^\times}$ is well defined and we can apply Fubini's theorem, which justifies:

$$\widehat{g^\times}(s) = \widehat{\left(\left\{\tfrac{1}{\cdot}\right\} * g\right)}(s) = \widehat{\left\{\frac{1}{\cdot}\right\}}(s)\widehat{g}(s) = -\frac{\zeta(s)}{s}\widehat{g}(s).$$

(ii). Now, let $Z \geq 0$ be a integrable r.v. and set $h^\times(t) = \mathbb{E}\{\frac{Z}{t}\}$. The situation reads even simpler:

$$\int_0^\infty |h^\times(t)|^2 \mathrm{d}t \leq \mathbb{E} \int_0^\infty \left\{\frac{Z}{t}\right\}^2 \mathrm{d}t = \mathbb{E} Z \int_0^\infty \left\{\frac{1}{t}\right\}^2 \mathrm{d}t < \infty\,.$$

Again, $\widehat{h^\times}$ is well defined by Fubini, and the result simply follows from a change of variable:

$$\begin{aligned}\widehat{h^\times}(s) = \int_0^\infty \mathbb{E}\left\{\frac{Z}{t}\right\} t^{s-1}\mathrm{d}t = \mathbb{E}\int_0^\infty \left\{\frac{Z}{t}\right\} t^{s-1}\mathrm{d}t &= \mathbb{E}\left[Z^s \int_0^\infty \left\{\frac{1}{u}\right\} u^{s-1}\mathrm{d}u\right] \\ &= -\frac{\zeta(s)}{s}\mathbb{E}Z^s,\end{aligned}$$

without even assuming that Z has a density. $\square$

Remark 1. *Notice that weaker assumptions, such as $\mathbb{E}Z^\sigma < \infty$, $0 < \sigma < 1$, are sufficient to show $h^\times \in L^2$, following the first lines of the proof with a different use of the Cauchy–Schwarz inequality.*

Since the methodology developed hereafter mainly uses the density of polynomials in a suitable weighted L^2 space, we recall the classical following result:

Lemma 2 ([16, (c) p. 76])**.** *Let ν be a positive measure on $\mathbb{R}$ with $\int_{\mathbb{R}} e^{a|t|}\mathrm{d}\nu(t) < \infty$ for some $a > 0$. Then the polynomials are dense into $L^2(\mathbb{R}, \nu)$.*

2. A sufficient condition for $\zeta(s) \neq 0$ in the strip $\frac{1}{2} < \sigma \leq \sigma_0 < 1$

We generalize Theorem 1.2 replacing $\mathbb{E}\{Z_k/t\}$ by general functions $g_k^\times$, which allows us to obtain some sharper estimates.

Theorem 2.1. *Let $\sigma_0 \in (\frac{1}{2}, 1)$. Let $(g_k)_{k\geq 1}$ be functions verifying Assumption* (M)*.*

Let $\phi : \mathbb{R}_+ \to \mathbb{R}$ be such that $\int_0^\infty (1+t)\,|\phi(t)|^2\,\mathrm{d}t < \infty$ and $\widehat{\phi}$ does not vanish in $\frac{1}{2} < \sigma \leq \sigma_0$.

If there exist real coefficients $(c_{k,n})_{1\leq k\leq n}$ and $M_n \to \infty$ such that

$$M_n^{2\sigma_0-1}\int_0^\infty \left(\phi(t) - \sum_{k=1}^n c_{k,n} g_k^\times(t)\right)^2 \mathrm{d}t \xrightarrow[n\to\infty]{} 0 \tag{2.1}$$

$$M_n^{\sigma_0-1}\int_{M_n}^\infty \left|\sum_{k=1}^n c_{k,n} g_k(t)\right| \mathrm{d}t \xrightarrow[n\to\infty]{} 0, \tag{2.2}$$

then ζ does not vanish in the strip $\frac{1}{2} < \sigma \leq \sigma_0$.

Proof. Let us assume that we are given $M_n \geq 1$ tending to infinity and real coefficients $(c_{k,n})_{1\leq k\leq n}$ such that (2.1) and (2.2) hold. Notice that $\widehat{\phi}$ exists in the strip $\frac{1}{2} < \sigma < 1$ since

$$\left(\int_0^1 |\phi(t)|\, t^{\sigma-1}\,\mathrm{d}t\right)^2 \leq \frac{1}{2\sigma-1}\int_0^1 |\phi(t)|^2\,\mathrm{d}t < \infty$$
$$\left(\int_1^\infty |\phi(t)|\, t^{\sigma-1}\,\mathrm{d}t\right)^2 \leq \int_1^\infty t\,|\phi(t)|^2\,\mathrm{d}t \int_1^\infty t^{2(\sigma-1-\frac{1}{2})}\,\mathrm{d}t < \infty,$$

due to $2\sigma - 1 > 0$ and $3 - 2\sigma > 1$. Set

$$\epsilon_n^2 = \int_{M_n}^\infty t\,|\phi(t)|^2\,\mathrm{d}t.$$

Notice that $\epsilon_n \xrightarrow[n\to\infty]{} 0$ and, again by the Cauchy–Schwarz inequality,

$$\int_{M_n}^\infty |\phi(t)|\, t^{\sigma-1}\,\mathrm{d}t = \int_{M_n}^\infty t^{\frac{1}{2}}\,|\phi(t)|\, t^{\sigma-3/2}\,\mathrm{d}t \leq \frac{\epsilon_n M_n^{\sigma-1}}{(2-2\sigma)^{\frac{1}{2}}}. \tag{2.3}$$

We write for simplicity $G_n = \sum_{k=1}^n c_{k,n} g_k$, so

$$G_n^\times = \left\{\frac{1}{\cdot}\right\} * G_n = \sum_{k=1}^n c_{k,n} g_k^\times.$$

The first lines of the proof follows the same ones as those of the Nyman–Beurling criterion.

Assume for contradiction that $\zeta(s) = 0$, for some (now fixed) $s = \sigma + i\tau \in \mathbb{C}$ with $1/2 < \sigma \leq \sigma_0$.

Using $\widehat{G_n^\times}(s) = -\frac{\zeta(s)}{s}\widehat{G_n}(s) = 0$, we then obtain for all integer n:

$$\widehat{\phi}(s) = \int_0^\infty \left(\phi(t) - G_n^\times(t)\right) t^{s-1}\,\mathrm{d}t.$$

We will show that the right hand side tends to 0 as $n \to \infty$, contradicting $\widehat{\phi}(s) \neq 0$.

Following the proof in [10], we write:

$$\left|\int_0^\infty \left(\phi(t) - G_n^\times(t)\right) t^{s-1}\mathrm{d}t\right|$$
$$\leq \left|\int_0^{M_n} \left(\phi(t) - G_n^\times(t)\right) t^{s-1}\mathrm{d}t\right| + \left|\int_{M_n}^\infty \left(\phi(t) - G_n^\times(t)\right) t^{s-1}\mathrm{d}t\right| = \mathrm{I} + \mathrm{II}\,.$$

We start to estimate I, by the Cauchy–Schwarz inequality:

$$\mathrm{I} \ll M_n^{\sigma-1/2}\left(\int_0^{M_n} \left(\phi(t) - G_n^\times(t)\right)^2\mathrm{d}t\right)^{\frac{1}{2}}.$$

We now turn to II. Using the triangular inequality and $G_n^{\times} = \{\frac{1}{\cdot}\} * G_n$,

$$\mathrm{II} \leq \int_{M_n}^{\infty} |\phi(t)|\, t^{\sigma-1} \mathrm{d}t + \int_{M_n}^{\infty} \left| \int_0^{M_n} \left\{\frac{x}{t}\right\} \frac{G_n(x)}{x}\, \mathrm{d}x \right| t^{\sigma-1} \mathrm{d}t \\ + \int_{M_n}^{\infty} \left| \int_{M_n}^{\infty} \left\{\frac{x}{t}\right\} \frac{G_n(x)}{x}\, \mathrm{d}x \right| t^{\sigma-1} \mathrm{d}t.$$

Noticing that $\{\frac{x}{t}\} = \frac{x}{t}$ if $x \leq M_n \leq t$, and $0 \leq \{\frac{x}{t}\} \frac{t}{x} \leq 1$, we obtain

$$\int_{M_n}^{\infty} \left| \int_0^{M_n} \left\{\frac{x}{t}\right\} \frac{G_n(x)}{x}\, \mathrm{d}x \right| t^{\sigma-1} \mathrm{d}t = \frac{M_n^{\sigma-1}}{1-\sigma} \left| \int_0^{M_n} G_n(x)\, \mathrm{d}x \right|$$

$$\int_{M_n}^{\infty} \left| \int_{M_n}^{\infty} \left\{\frac{x}{t}\right\} \frac{G_n(x)}{x}\, \mathrm{d}x \right| t^{\sigma-1} \mathrm{d}t \leq \frac{M_n^{\sigma-1}}{1-\sigma} \int_{M_n}^{+\infty} |G_n(x)|\, \mathrm{d}x.$$

Hence, with (2.3),

$$\mathrm{II} \leq \frac{\varepsilon_n M_n^{\sigma-1}}{(2-2\sigma)^{\frac{1}{2}}} + \frac{M_n^{\sigma-1}}{1-\sigma} \left| \int_0^{M_n} G_n(x)\, \mathrm{d}x \right| + \frac{M_n^{\sigma-1}}{1-\sigma} \int_{M_n}^{\infty} |G_n(x)|\, \mathrm{d}x. \tag{2.4}$$

The first and third term of the right hand side tend to zero as $n \to \infty$ by hypothesis. We can estimate $\int_0^{M_n} G_n(x)\, \mathrm{d}x$ as in [10], in the spirit of "Báez-Duarte–Balazard–Landreau–Saias' trick" (see [2, Lemme 1 & Proposition 1 p. 133]).

On one hand, we first notice that

$$\begin{aligned} \left(\int_0^{M_n} G_n(x)\, \mathrm{d}x \right)^2 &= M_n \int_{M_n}^{+\infty} \frac{\mathrm{d}t}{t^2} \left(\int_0^{M_n} G_n(x)\, \mathrm{d}x \right)^2 \\ &= M_n \int_{M_n}^{+\infty} \left| \int_0^{M_n} \frac{x}{t} \frac{G_n(x)}{x}\, \mathrm{d}x \right|^2 \mathrm{d}t \\ &= M_n \int_{M_n}^{+\infty} \left| \int_0^{M_n} \left\{\frac{x}{t}\right\} \frac{G_n(x)}{x}\, \mathrm{d}x \right|^2 \mathrm{d}t, \end{aligned} \tag{2.5}$$

still using $\{\frac{x}{t}\} = \frac{x}{t}$ if $x \leq t$.

On the other hand, writing

$$\int_0^{M_n} \left\{\frac{x}{t}\right\} \frac{G_n(x)}{x} \mathrm{d}x = \int_0^{\infty} \left\{\frac{x}{t}\right\} \frac{G_n(x)}{x}\, \mathrm{d}x - \phi(t) - \int_{M_n}^{\infty} \left\{\frac{x}{t}\right\} \frac{G_n(x)}{x}\, \mathrm{d}x + \phi(t),$$

using $(a+b+c)^2 \le 3(a^2+b^2+c^2)$ and again $0 \le \{\frac{x}{t}\}\frac{t}{x} \le 1$, we have

$$\begin{aligned}\int_{M_n}^{\infty}\left|\int_0^{M_n}\left\{\frac{x}{t}\right\}\frac{G_n(x)}{x}\,\mathrm{d}x\right|^2 \mathrm{d}t \\ &\le 3\int_{M_n}^{\infty}\left(\int_0^{\infty}\left\{\frac{x}{t}\right\}\frac{G_n(x)}{x}\,\mathrm{d}x-\phi(t)\right)^2 \mathrm{d}t \\ &\quad +3\int_{M_n}^{\infty}\left|\int_{M_n}^{\infty}\left\{\frac{x}{t}\right\}\frac{G_n(x)}{x}\,\mathrm{d}x\right|^2 \mathrm{d}t+3\int_{M_n}^{\infty}\phi(t)^2\mathrm{d}t \\ &\le 3\int_{M_n}^{\infty}\left(\phi(t)-G_n^{\times}(t)\right)^2 \mathrm{d}t \\ &\quad +3M_n^{-1}\left(\int_{M_n}^{\infty}|G_n(x)|\,\mathrm{d}x\right)^2+3\int_{M_n}^{\infty}\phi(t)^2\mathrm{d}t\,.\end{aligned}$$

Due to (2.5) and the definition of ε_n, this leads to

$$\begin{aligned}\left(\int_0^{M_n}G_n(x)\,\mathrm{d}x\right)^2 \\ &\le 3M_n\int_{M_n}^{\infty}\left(\phi(t)-G_n^{\times}(t)\right)^2 \mathrm{d}t+3\left(\int_{M_n}^{\infty}|G_n(x)|\,\mathrm{d}x\right)^2+3\,\varepsilon_n^2,\end{aligned}$$

which, put back in the bound (2.4) of II with $\sqrt{a+b+c} \le \sqrt{a}+\sqrt{b}+\sqrt{c}$, gives

$$\mathrm{II} \ll M_n^{\sigma-\frac{1}{2}}\left(\int_{M_n}^{\infty}\left(\phi(t)-G_n^{\times}(t)\right)^2 \mathrm{d}t\right)^{\frac{1}{2}}+M_n^{\sigma-1}\int_{M_n}^{\infty}|G_n(x)|\;\mathrm{d}x\;+\;M_n^{\sigma-1}\varepsilon_n.$$

Combining the bounds for I and II, we thus end up with

$$\begin{aligned}\left|\int_0^{\infty}\left(\phi(t)-G_n^{\times}(t)\right)t^{s-1}\,\mathrm{d}t\right| \\ &\ll M_n^{\sigma-\frac{1}{2}}\left\|\phi-G_n^{\times}\right\|_{L^2}+M_n^{\sigma-1}\int_{M_n}^{\infty}|G_n(x)|\;\mathrm{d}x+\;\varepsilon_n M_n^{\sigma-1}.\end{aligned}$$

We eventually notice that the three last terms of the right hand side tend to 0 as $n\to\infty$ due to $M_n^{\sigma} \le M_n^{\sigma_0}$ (since $M_n \ge 1$ and $\sigma \le \sigma_0 < 1$), (2.1) and (2.2). □

3. Class of Inverse Gamma distributions

In this section, we deal with the probabilistic framework of Lemma 1 (ii), and consider a sequence of r.v. $Z_k = Y/X_k$ where $Y \ge 0$ is an integrable r.v. and X_k is a $\Gamma(k,1)$-distributed r.v. independent of Y. Hence, in this probabilistic context, we set

$$h_k^{\times}(t) = \mathbb{E}\left\{\frac{Y}{X_k t}\right\}.$$

3.1. Inverse Gamma distributions verify gNB.

Theorem 3.1. *The sequence $(Z_k)_{k\geq 1}$ verifies gNB.*

Proof. We recall that the density of the $\Gamma(k,1)$ distribution is $f_k(x) = \frac{1}{\Gamma(k)} x^{k-1} e^{-x}$, $k \geq 1$. We have for all $k \geq 2$, $\mathbb{E}Z_k = \mathbb{E}Y\ \mathbb{E}(1/X_k) < \infty$. For $k=1$, we only have $\mathbb{E}(1/X_1)^\sigma < \infty$ for all $\sigma \in (0,1)$, which by Remark 1 is also sufficient to ensure that $g_1^{\times} \in L^2(\mathbb{R}_+)$. Therefore we can apply Lemma 1 (ii): for all $k \geq 1$,

$$\widehat{h_k^{\times}}(s) = -\frac{\zeta(s)}{s}\mathbb{E}Z_k^s, \quad 0 < \sigma < 1.$$

Since X_k and Y are independent,

$$\mathbb{E}Z_k^s = \mathbb{E}Y^s\ \mathbb{E}X_k^{-s}, \quad 0 < \sigma < 1.$$

Moreover, we can compute

$$\mathbb{E}X_k^{-s} = \frac{1}{\Gamma(k)}\int_0^\infty x^{k-s-1}e^{-x}\mathrm{d}x = \frac{\Gamma(k-s)}{\Gamma(k)}.$$

Notice that, for $k \geq 2$ and $0 < \sigma < 1$,

$$\begin{aligned}\Gamma(k-s) &= (k-1-s)\Gamma(k-1-s)\\ &= (k-1-s)\dots(1-s)\Gamma(1-s)\\ &= P_{k-1}(s)\Gamma(1-s),\end{aligned}$$

where $P_0 = 1$ and $P_k(s) = (k-s)(k-1-s)\dots(1-s)$ is a polynomial of degree k, known as a Pochhammer symbol.

We then write for some coefficients $c_{k,n}$, setting $s = 1/2 + it$,

$$\begin{aligned}D_n^2 &= \int_0^\infty \left|\chi(x) - \sum_{k=1}^n c_{k,n} h_k^{\times}(x)\right|^2 \mathrm{d}x\\ &= \frac{1}{2\pi}\int_{-\infty}^\infty \left|\widehat{\chi}(s) - \sum_{k=1}^n c_{k,n}\widehat{h_k^{\times}}(s)\right|^2 \mathrm{d}t\\ &= \frac{1}{2\pi}\int_{-\infty}^\infty \left|\frac{1}{s} + \sum_{k=1}^n c_{k,n}\frac{\zeta(s)}{s}\mathbb{E}Y^s\Gamma(1-s)\frac{P_{k-1}(s)}{\Gamma(k)}\right|^2 \mathrm{d}t\\ &= \frac{1}{2\pi}\int_{-\infty}^\infty \left|\frac{1}{s\varphi(s)} + \sum_{k=1}^n \frac{c_{k,n}}{\Gamma(k)}P_{k-1}(s)\right|^2 |\varphi(s)|^2\mathrm{d}t\,,\end{aligned}$$

where $\varphi(s) = \frac{\zeta(s)}{s}\mathbb{E}Y^s\Gamma(1-s)$.

We now notice that $\zeta(s) = O(t)$ (see e.g. [19, Corollary 3.7 p. 234]), $|\mathbb{E}Y^s| \leq \mathbb{E}\sqrt{Y}$, and by the complement formula

$$|\Gamma(1-s)|^2 = \Gamma\left(\frac{1}{2} - it\right)\Gamma\left(\frac{1}{2} + it\right) = \frac{\pi}{\sin\left(\frac{\pi}{2} + i\pi t\right)} \ll e^{-\pi t}.$$

Hence, we can apply Lemma 2 with the measure $\nu(\mathrm{d}t) = |\varphi(s)|^2\mathrm{d}t$ and the function $t \mapsto 1/(s\varphi(s))$ that belongs to $L^2(\mathbb{R}, \nu)$. Therefore, since the polynomials P_k are graduated, there exist coefficients $c_{k,n}$ such that $D_n \to 0$. In other words, the family $(Y/X_k)_{k\geq 1}$ verifies gNB. □

As a consequence of Wiener's Tauberian theorem (see e.g. [3, Theorem 2 p. 25]), the following result is known:

Theorem 3.2. *Given $\varepsilon > 0$, there exist $m \geq 1$, $c'_1, \ldots, c'_m \in \mathbb{R}$, and $\theta_1 > 0, \ldots, \theta_m > 0$ such that*

$$\int_0^\infty \left(\chi(t) - \sum_{l=1}^m c'_l \left\{\frac{\theta_l}{t}\right\}\right)^2 \mathrm{d}t < \varepsilon. \tag{3.1}$$

Interestingly, the existence of r.v. verifying gNB in Theorem 3.1 provides a short probabilistic proof of this fact. The basic idea is to approximate $\mathbb{E}\{Z_k/t\}$ in D_n^2 by $\frac{1}{N}\sum_{j=1}^N \{Z_{k,j}/t\}$ where $(Z_{k,j})_j$ are independent copies of Z_k defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Then we use Erdős probabilistic method through the following instance: if $\mathbb{E}V \leq \varepsilon$ for a r.v. $V \geq 0$, then there exists $\omega \in \Omega$, s.t. $V(\omega) \leq \varepsilon$. Notice that $Z_k = X/Y_k$ are fully supported on $\mathbb{R}_+$, and this is why the θ's do not lie in $(0,1)$.

Proof. Fix $\varepsilon > 0$. We know by Theorem 3.1 that there exist n and coefficients $c_1, \ldots, c_n \in \mathbb{R}$ (that are fixed now) such that

$$D_n^2 = \int_0^\infty \left(\chi(t) - \sum_{k=1}^n c_k \mathbb{E}\left\{\frac{Z_k}{t}\right\}\right)^2 \mathrm{d}t < \varepsilon\,.$$

Then set $d_{n,N}^2 = \int_0^\infty (\chi(t) - \sum_{k=1}^n c_k/N \sum_{j=1}^N \{Z_{k,j}/t\})^2 \mathrm{d}t$, and write, using $(a+b)^2 \leq 2a^2 + 2b^2$,

$$\mathbb{E}d_{n,N}^2 \leq 2D_n^2 + 2\mathbb{E}\int_0^\infty \left(\sum_{k=1}^n c_k \left(\mathbb{E}\{Z_k/t\} - \frac{1}{N}\sum_{j=1}^N \{Z_{k,j}/t\}\right)\right)^2 \mathrm{d}t$$
$$= 2D_n^2 + 2R_{n,N}^2\,.$$

Using now the Cauchy–Schwarz inequality,

$$\begin{aligned} R_{n,N}^2 &\leq \sum_{k=1}^n c_k^2 \sum_{k=1}^n \int_0^\infty \mathbb{E}\left(\mathbb{E}\{Z_k/t\} - \frac{1}{N}\sum_{j=1}^N \{Z_{k,j}/t\}\right)^2 \mathrm{d}t \\ &\leq \frac{1}{N}\sum_{k=1}^n c_k^2 \sum_{k=1}^n \int_0^\infty \mathrm{Var}(\{Z_k/t\})\mathrm{d}t. \end{aligned}$$

We can then choose N sufficiently large so that $R_{n,N}^2 \leq \varepsilon$. Hence $\mathbb{E}d_{n,N}^2 \leq 4\varepsilon$, so there exists $\omega \in \Omega$ such that $d_{n,N}^2(\omega) \leq 4\varepsilon$, which concludes the proof, the desired θ_l's being the $Z_{k,j}(\omega)$. □

3.2. Remark on a specific distribution tail. The preceding arguments generalize a remark due to Vincent Alouin who noticed that the distribution tail $x \mapsto (1+x)^{-k}$ has a Mellin transform which satisfies, by the change of variables $u = 1/(1+x)$, i.e. $x = 1/u - 1$, the identity

$$\begin{aligned} \int_0^\infty \frac{x^{s-1}}{(1+x)^k}\mathrm{d}x &= -\int_0^1 \left(\frac{1}{u} - 1\right)^{s-1} u^k \frac{\mathrm{d}u}{-u^2} \\ &= \int_0^1 (1-u)^{s-1}u^{k-s-1}\mathrm{d}u = \frac{\Gamma(s)\Gamma(k-s)}{\Gamma(k)}, \end{aligned} \tag{3.2}$$

which is a product of a polynomial in s by a fixed (independent of k) function. It turns out that the corresponding r.v. are of the form $Z_k \sim Y/X_k$ where $Y \sim \mathcal{E}(1)$.

3.3. Computation of the Gram matrix. We now want to compute the corresponding scalar products. We introduce the functions

$$\begin{aligned} \rho(t) &= \mathbb{E}\left\{\frac{Y}{t}\right\}, \quad t > 0, \\ A(u) &= \int_0^\infty \rho(ut)\rho((1-u)t)\mathrm{d}t, \quad 0 < u < 1. \end{aligned}$$

Two interesting particular cases are:

(i) when $Y \sim \delta_1$, we have $\rho(t) = \left\{\frac{1}{t}\right\}$,

(ii) when $Y \sim \mathcal{E}(\lambda)$, we have $\rho(t) = \frac{1}{e^{\lambda t}-1} - \frac{1}{\lambda t}$.

Recall that $h_k^\times(t) = \mathbb{E}\left\{\frac{Y}{X_k t}\right\}$ where Y is a r.v. satisfying (M) independent of $X_k \sim \Gamma(k,1)$.

Proposition 1. *For $m, n \geq 0$, we have*

$$\left\langle h_{n+1}^\times, h_{m+1}^\times \right\rangle = \int_0^1 B_n^{m+n}(u)A(u)\mathrm{d}u,$$

where $B_n^{m+n}(u) = \binom{n+m}{n}u^n(1-u)^m$ is an elementary Bernstein polynomial.

Proof. We first notice that, since X_n and Y are independent,

$$\begin{aligned} h_n^{\times}(t) = \mathbb{E}\left\{\frac{Y}{X_n t}\right\} &= \mathbb{E}\left[\rho(X_n t)\right] \\ &= \frac{1}{(n-1)!}\int_0^\infty \rho(xt)x^{n-1}e^{-x}\mathrm{d}x \\ &= \frac{1}{(n-1)!t^n}\int_0^\infty \rho(x)x^{n-1}e^{-\frac{x}{t}}\mathrm{d}x. \end{aligned}$$

We then have, by Fubini,

$$\begin{aligned} \left\langle h_{n+1}^{\times}, h_{m+1}^{\times}\right\rangle &= \frac{1}{m!n!}\int_0^\infty\int_0^\infty \rho(x)\rho(y)x^n y^m \int_0^\infty \frac{1}{t^{m+n+2}}e^{-\frac{x+y}{t}}\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}y \\ &= \frac{1}{m!n!}\int_0^\infty\int_0^\infty \rho(x)\rho(y)x^n y^m \frac{(m+n)!}{(x+y)^{n+m-1}}\mathrm{d}x\,\mathrm{d}y\,, \end{aligned}$$

where we used the elementary formula $\int_0^\infty \frac{1}{t^\alpha}e^{-\frac{\beta}{t}}\mathrm{d}t = \frac{\Gamma(\alpha-1)}{\beta^{\alpha-1}}$.

We now consider the change of variables $u = \frac{x}{x+y}$, $z = x+y$, which gives $x = uz$, $y = (1-u)z$. We thus have

$$\begin{aligned} &\left\langle h_{n+1}^{\times}, h_{m+1}^{\times}\right\rangle \\ &= \frac{(m+n)!}{m!n!}\int_0^\infty\int_0^1 \rho(uz)\rho((1-u)z)(uz)^n((1-u)z)^m\frac{1}{z^{m+n+1}}z\,\mathrm{d}z\,\mathrm{d}u \\ &= \frac{(m+n)!}{m!n!}\int_0^1 u^n(1-u)^m\int_0^\infty \rho(uz)\rho((1-u)z)\,\mathrm{d}z\,\mathrm{d}u\,, \end{aligned}$$

as desired. □

4. Sequence defined by induction and a remarkable Gram matrix

In this section, we consider a sequence of functions g_k defined by the induction

$$g_{k+1}(x) = -xg_k'(x) - r_k g_k(x), \tag{4.1}$$

where (r_k) is a real sequence and $g_0 \in C^\infty(\mathbb{R}^+)$ such that for all $k \geq 0$ and some $\alpha > 0$,

$$\lim_{x\to 0} x^{k-\alpha}g_0^{(k)}(x) = \lim_{x\to\infty} x^{k+1+\alpha}g_0^{(k)}(x) = 0.$$

4.1. Condition on g_0 ensuring gNB.

Theorem 4.1. *The Mellin transform* $\widehat{g_k^{\times}}$ *is well defined and we have:*

$$\widehat{g_k^{\times}}(s) = -\prod_{j=0}^{k}(s-r_j)\frac{\zeta(s)}{s}\widehat{g_0}(s), \quad 0 < \sigma < 1. \tag{4.2}$$

If $\widehat{g_0}\left(\frac{1}{2}+it\right) \ll e^{-\delta|t|}$, $\delta>0$, *then* $(g_k)_{k\geq 0}$ *verifies gNB.*

Proof. One can show by induction on k that there exists a family of numbers $(a_{l,k})_{k\geq 0, 0\leq l\leq k}$, with $a_{k,k}=(-1)^k$ such that for all $k\geq 0$,

$$g_k(x)=\sum_{l=0}^{k} a_{l,k}x^l g_0^{(l)}(x). \tag{4.3}$$

Let s with $0<\sigma<1$. Due to the assumption on g_0, g_k verifies Assumption (M) and $\widehat{g_k}$ is well defined for all $k\geq 0$. By integration by parts

$$\begin{aligned}
-\int_0^\infty x^{s-1}xg_k'(x)\mathrm{d}x &= -\int_0^\infty x^s g_k'(x)\mathrm{d}x \\
&= -\left[x^s g_k(x)\right]_0^\infty + \int_0^\infty s x^{s-1} g_k(x)\mathrm{d}x \\
&= s\widehat{g_k}(s),
\end{aligned}$$

since $\lim_{x\to 0,\infty} x^\sigma g_k(x)=0$, again by the assumption on g_0.

Hence $\widehat{g_{k+1}}(s)=(s-r_k)\widehat{g_k}(s)$, and then

$$\widehat{g_{k+1}}(s)=\prod_{j=0}^{k}(s-r_j)\widehat{g_0}(s).$$

We found again the polynomial structure of $\widehat{g_k^\times}$, as in the proof of Theorem 3.1. We can then follow these lines to conclude. □

4.2. Gram matrix.

Proposition 2. *For* $k\geq 0$ *and* $j\geq 1$, *we have*

$$\left\langle g_k^\times, g_j^\times\right\rangle + \left\langle g_{k+1}^\times, g_{j-1}^\times\right\rangle = (1-r_k-r_{j-1})\left\langle g_k^\times, g_{j-1}^\times\right\rangle.$$

Proof. We compute the scalar products as:

$$\begin{aligned}
&\int_0^\infty \int_0^\infty \left\{\frac{x}{t}\right\} g_k(x)\frac{\mathrm{d}x}{x}\int_0^\infty \left\{\frac{y}{t}\right\} g_j(y)\frac{\mathrm{d}y}{y}\ \mathrm{d}t \\
&\qquad = \int_0^\infty\int_0^\infty \{x\}\{y\}\int_0^\infty g_k(tx)g_j(ty)\mathrm{d}t\ \frac{\mathrm{d}x}{x}\frac{\mathrm{d}y}{y}.
\end{aligned}$$

Moreover, using $g_j(ty)=-tyg_{j-1}'(ty)-r_{j-1}g_{j-1}(ty)$, we compute

$$\begin{aligned}
I_{k,j}=I_{k,j}(x,y) &= \int_0^\infty g_k(tx)g_j(ty)\mathrm{d}t \\
&= \int_0^\infty g_k(tx)(-tyg_{j-1}'(ty)-r_{j-1}g_{j-1}(ty))\mathrm{d}t \\
&= -\int_0^\infty g_k(tx)tyg_{j-1}'(ty)\mathrm{d}t - r_{j-1}I_{k,j-1}.
\end{aligned}$$

By integrating by parts and using $g_{k+1}(tx) = -txg'_k(tx) - r_k g_k(tx)$, we have

$$\begin{aligned}
\int_0^\infty & g_k(tx)tyg'_{j-1}(ty)\mathrm{d}t \\
&= [tg_k(tx)g_{j-1}(ty)]_0^\infty - \int_0^\infty (txg'_k(tx) + g_k(tx))g_{j-1}(ty)\mathrm{d}t \\
&= \int_0^\infty (g_{k+1}(tx) + (r_k - 1)g_k(tx))g_{j-1}(ty)\mathrm{d}t \\
&= I_{k+1,j-1} + (r_k - 1)I_{k,j-1}.
\end{aligned}$$

Hence,

$$\begin{aligned}
I_{k,j} &= -I_{k+1,j-1} - (r_k - 1)I_{k,j-1} - r_{j-1}I_{k,j-1} \\
&= (1 - r_k - r_{j-1})I_{k,j-1} - I_{k+1,j-1},
\end{aligned}$$

and the result follows due to the linearity of the double integral. $\square$

We now set $G_{k,j} = \left\langle g_k^\times, g_j^\times \right\rangle$. Choosing $r_q = 1/2$ for all $q \geq 0$ yields in the previous proposition to a remarkable structure, i.e.

$$G_{k,j} + G_{k+1,j-1} = 0.$$

In particular, setting $j = k+1$, we obtain $G_{k,k+1} = 0$. This is not a surprise due to the following computation by means of Mellin transform and (4.2). We set $s = 1/2 + it$ and use Mellin–Plancherel isometry:

$$\begin{aligned}
G_{k,k+1} &= \int_{-\infty}^\infty \widehat{g_k^\times}(s)\overline{\widehat{g_{k+1}^\times}(s)}\mathrm{d}t \\
&= \int_{-\infty}^\infty (s - 1/2)^k\overline{(s - 1/2)^{k+1}} \left|\frac{\zeta(s)}{s}\widehat{g_0}(s)\right|^2 \mathrm{d}t \\
&= -i\int_{-\infty}^\infty t^{2k+1} \left|\frac{\zeta(s)}{s}\widehat{g_0}(s)\right|^2 \mathrm{d}t = 0.
\end{aligned}$$

The Gram matrix G, sort of "alternate" Hankel matrix, is then only determined through its diagonal entries:

$$G_{k,k} = \left\langle g_k^\times, g_k^\times \right\rangle = \int_{-\infty}^\infty t^{2k}\mathrm{d}\nu(t)\,,$$

where $\mathrm{d}\nu(t) = \left|\frac{\zeta(s)}{s}\widehat{g_0}(s)\right|^2 \mathrm{d}t\,$. The Gram matrix then reads:

$$G = \begin{pmatrix} G_{11} & 0 & -G_{22} & 0 & \dots \\ 0 & G_{22} & 0 & -G_{33} & \dots \\ -G_{22} & 0 & G_{33} & 0 & \dots \\ 0 & -G_{33} & 0 & G_{44} & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{4.4}$$

Renumbering the rows and the columns, making appear first the odd indices and then the even ones, leads to the equivalent matrix

$$\widetilde{G} = \begin{pmatrix} \widetilde{G}_1 & 0 \\ 0 & \widetilde{G}_2 \end{pmatrix} \tag{4.5}$$

where the two blocks $\widetilde{G}_1$ and $\widetilde{G}_2$ are Hankel:

$$\begin{aligned} \widetilde{G}_1 &= \begin{pmatrix} G_{11} & -G_{22} & G_{33} & \dots \\ -G_{22} & G_{33} & \dots & \\ G_{33} & \vdots & & \\ \vdots & & & \end{pmatrix}, \\ \widetilde{G}_2 &= \begin{pmatrix} G_{22} & -G_{33} & G_{44} & \dots \\ -G_{33} & G_{44} & \dots & \\ G_{44} & \vdots & & \\ \vdots & & & \end{pmatrix}. \end{aligned} \tag{4.6}$$

Notice that if one wants to evaluate the squared distance (1.3) by computing the determinant of G, we can obtain the determinant of a moment matrix $H_{k,j} = \int_{-\infty}^{\infty} t^{k+j} \mathrm{d}\nu(t)$ by multiplying the rows of $\widetilde{G}$ by $(-1)^k$ and the columns by $(-1)^j$.

The study of such determinant falls into the theory of Hankel determinants with symbols that possess power-like singularities, a specific case of more general Fisher–Hartwig singularities. This theory is well established with a finite number of power-like singularities, see e.g. [8, 12]. However, the infinite number of zeros of $\zeta(1/2 + it)$, which appear in our symbol, seems to be a challenging issue for this theory to apply.

It is also possible to provide simpler expressions for the scalar products $b_k = \langle \chi, g_k^{\times} \rangle$. Indeed:

$$b_k = \int_0^{\infty} \chi(t) \int_0^{\infty} \left\{ \frac{x}{t} \right\} g_k(x) \frac{\mathrm{d}x}{x} \, \mathrm{d}t = \int_0^{\infty} \{x\} \int_0^{\infty} \chi(t) g_k(tx) \mathrm{d}t \frac{\mathrm{d}x}{x}.$$

But

$$\begin{aligned} \int_0^1 g_{k+1}(tx)\mathrm{d}t &= -\int_0^1 txg_k'(tx)\mathrm{d}t - \frac{1}{2}\int_0^1 g_k(tx)\mathrm{d}t \\ &= -[tg_k(tx)]_0^1 + \int_0^1 g_k(tx)\mathrm{d}t - \frac{1}{2}\int_0^1 g_k(tx)\mathrm{d}t \\ &= -g_k(x) + \frac{1}{2}\int_0^1 g_k(tx)\mathrm{d}t. \end{aligned}$$

Therefore

$$b_{k+1} = \frac{1}{2} b_k - \int_0^\infty \{x\}\, g_k(x) \frac{\mathrm{d}x}{x}.$$

4.3. Examples for g_0 and comments.

4.3.1. ***The Ξ-function.*** As an answer to a suggestion by P. Biane and C. Delaunay, it is possible to use the induction (4.1) to find a g_0 that produces within ν the Ξ-function:

$$\Xi(t) = \xi(s) = \frac{1}{2} s(s-1)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s), \quad s = 1/2 + it.$$

Indeed, to construct g_0 such that $\widehat{g_0}(s) = \frac{1}{2}(s-1)s^2\Gamma\left(\frac{s}{2}\right)$, we define first $h_0(x) = e^{-x^2}$, so that

$$\widehat{h_0}(s) = \int_0^\infty x^{s-1} e^{-x^2} \mathrm{d}x = \int_0^\infty u^{s/2-1/2} \frac{e^{-u}}{2\sqrt{u}} \mathrm{d}u = \frac{1}{2}\Gamma\left(\frac{s}{2}\right).$$

Then, in order to have $\widehat{g_0}(s) = (s^3 - s^2)\widehat{h_0}(s)$, we compute

$$\begin{aligned} g_0(x) &= \left(\left(-x\frac{\mathrm{d}}{\mathrm{d}x}\right)^3 - \left(-x\frac{\mathrm{d}}{\mathrm{d}x}\right)^2\right) h_0(x) \\ &= (8x^6 - 28x^4 + 12x^2)e^{-x^2}. \end{aligned}$$

Hence, again in the case $r_k = 1/2$, we can obtain, taking $\pi^{-1/4} g_0$,

$$G_{k,k} = \int_{-\infty}^\infty t^{2k} \Xi(t)^2 \mathrm{d}t.$$

As an historical nod, notice that quantities as $\int_{-\infty}^\infty t^{2k}\Xi(t)\mathrm{d}t$ or related to, have been used by Pólya and Hardy to study the zeros of ζ on the critical line, see e.g. [20, 10.2-10.4, p. 256-260].

4.3.2. ***Seed with compact support.*** If the seed g_0 has a compact support, say $(0, M)$, then the g_k's are also supported on $(0, M)$. This removes the control condition (2.2) on the coefficient $c_{k,n}$ as soon as $M_n \geq M$. If one wants to prove RH, one then only needs a density result.

Amazingly, in this compact support case, we then lose the density of the polynomials in the whole space $L^2(\nu)$. Indeed, Ingham [11] remarked (originally for Fourier transform) that if g_0 has compact support $[a, b] \subset (0, \infty)$, we cannot have $\widehat{g_0}(s) \ll e^{-\delta|t|}$, $\delta > 0$, where again $s = 1/2 + it$. We transfered here the result for Fourier transform to Mellin transform noticing that

$$\widehat{g_0}(s) = \int_{-\infty}^\infty e^{us} g_0(e^u)\mathrm{d}u = \int_{-\infty}^\infty e^{u/2} g_0(e^u) e^{iut} \mathrm{d}u$$

and that $u \mapsto e^{u/2} g_0(e^u)$ has compact support $[\log(a), \log(b)]$. More precisely, for any decreasing function $\varepsilon(t) = o(1)$, there exists a compactly supported function g_0 such that $\widehat{g_0}(s) \ll e^{-\varepsilon(|t|)|t|}$ if and only if $\int_1^\infty \frac{\varepsilon(t)}{t}\,\mathrm{d}t < \infty$. See [18, Annexe] for a nice account on these results. But this is precisely incompatible with the condition that ensures the density of the polynomials in weighted $L^2(\mathbb{R})$-spaces, namely $\int_{-\infty}^{\infty} \frac{\log w(t)}{1+t^2}\mathrm{d}t = \infty$. The link between the weight w and the function ε is $\log w(t) = \varepsilon(t)t$ here. See [16, 4.8.3 p. 77] for many aspects regarding such theorems.

Let us stress that Mellin isometry involves an integration on the whole real line. On the half line, a density result is obtained by Mergelyan [15] with the condition $\int_1^\infty \frac{\log w(t)}{t^{3/2}}\mathrm{d}t = \infty$. In our framework, Borichev [7] proved that for all $\varepsilon > 0$ there exist $Q \in \mathbb{C}[X]$ such that

$$\int_0^\infty \left| \frac{1}{s\varphi(s)} - Q(s) \right|^2 |\varphi(s)|^2 \mathrm{d}t < \varepsilon,$$

where $\varphi(s) = \frac{\zeta(s)}{s}\widehat{g_0}(s)$, $s = 1/2 + it$, g_0 has compact support and verifies $\widehat{g_0}(s) \ll e^{-|t|/\log^2|t|}$.

Although we do not have the density on the whole line $(-\infty, +\infty)$, there exist results regarding the closure of the polynomials, see e.g. [6]. So, we ask for the following question:

What is the closure of the space generated by the polynomials in $L^2\left(\mathbb{R}, |\varphi(s)|^2\mathrm{d}t\right)$ when g_0 has compact support?

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References

[1] L. BÁEZ-DUARTE, "A strengthening of the Nyman-Beurling criterion for the Riemann hypothesis", *Atti Accad. Naz. Lincei, Cl. Sci. Fis. Mat. Nat., IX. Ser., Rend. Lincei, Mat. Appl.* **14** (2003), no. 1, p. 5-11.

[2] L. BÁEZ-DUARTE, M. BALAZARD, B. LANDREAU & E. SAIAS, "Étude de l'autocorrélation multiplicative de la fonction "partie fractionnaire"", *Ramanujan J.* **9** (2005), no. 1, p. 5-11.

[3] M. BALAZARD, "Completeness problems and the Riemann hypothesis: an annotated bibliography", in *Number theory for the millennium I (Urbana, IL, 2000)*, A K Peters, 2000, p. 21-48.

[4] S. BETTIN & B. CONREY, "Period functions and cotangent sums", *Algebra Number Theory* **7** (2013), no. 1, p. 215-242.

[5] A. Beurling, “A closure problem related to the Riemann Zeta-function”, *Proc. Natl. Acad. Sci. USA* **41** (1955), p. 312-314.

[6] A. Borichev, “On the closure of polynomials in weighted spaces of functions on the real line”, *Indiana Univ. Math. J.* **50** (2001), no. 2, p. 829-846.

[7] ———, “Personal communication to Sébastien Darses”, 2020.

[8] C. Charlier, “Asymptotics of Hankel determinants with a one-cut regular potential and Fisher-Hartwig singularities”, *Int. Math. Res. Not.* **2019** (2019), no. 24, p. 7515-7576.

[9] S. Darses & E. Hillion, “An exponentially-averaged Vasyunin formula”, *Proc. Am. Math. Soc.* **149** (2021), no. 7, p. 2969-2982.

[10] ———, “On probabilistic generalizations of the Nyman-Beurling criterion for the zeta function”, *Confluentes Math.* **13** (2021), no. 1, p. 43-59.

[11] A. E. Ingham, “A note on Fourier transforms”, *J. Lond. Math. Soc.* **9** (1934), p. 29-32.

[12] I. V. Krasovsky, “Correlations of the characteristic polynomials in the Gaussian unitary ensemble or a singular Hankel determinant”, *Duke Math. J.* **139** (2007), no. 3, p. 581-619.

[13] H. Maier & M. T. Rassias, “The order of magnitude for moments for certain cotangent sums”, *J. Math. Anal. Appl.* **429** (2015), no. 1, p. 576-590.

[14] ———, “Generalizations of a cotangent sum associated to the Estermann zeta function”, *Commun. Contemp. Math.* **18** (2016), no. 1, article no. 1550078 (89 pages).

[15] S. N. Mergelyan, “Weighted approximation by polynomials”, *Usp. Mat. Nauk* **11** (1956), p. 107-152.

[16] N. K. Nikolski, *Operators, functions, and systems: an easy reading. Volume 2: Model operators and systems*, Mathematical Surveys and Monographs, vol. 93, American Mathematical Society, 2002.

[17] B. Nyman, “On the one-dimensional translation group and semi-group in certain function spaces”, PhD Thesis, University of Uppsala (Sweden), 1950.

[18] C. Swaenepoel, “Digits of prime numbers and other remarkable sequences”, PhD Thesis, Université d’Aix-Marseille (France), 2019.

[19] G. Tenenbaum, *Introduction à la théorie analytique et probabiliste des nombres*, 2nd ed., Contributions in Mathematical and Computational Sciences, vol. 1, Société Mathématique de France, 1995.

[20] E. C. Titchmarsh, *The theory of the Riemann zeta-function*, 2nd ed., Oxford Science Publications, Clarendon Press, 1986.

[21] V. I. Vasyunin, “On a biorthogonal system related with the Riemann hypothesis”, *Algebra Anal.* **7** (1995), no. 3, p. 118-135.

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