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The natural extension of the Gauss map and the Hermite best approximations

par Nicolas CHEVALLIER

Résumé. À la suite de Humbert et Lagarias, étant donnée un réel θ, nous appelons vecteur meilleure approximation de Hermite de θ, tout vecteur non nul à coordonnées entières qui minimise une forme quadratique $f_\Delta(x,y) = (x-y\theta)^2 + \frac{y^2}{\Delta}$ pour au moins un réel $\Delta > 0$. Hermite a observé que si (p,q) est un tel minimum avec $q > 0$, alors la fraction p/q doit être une réduite du développement en fraction continue de θ. En utilisant les vecteurs minimaux dans les réseaux, nous donnons de nouvelles preuves de certains résultats de Humbert et Meignen et complétons leurs travaux. En particulier, nous montrons que la proportion des vecteurs meilleures approximations de Hermite parmi les réduites est presque sûrement de $\ln 3/\ln 4$. L'outil principal des preuves est l'extension naturelle de l'application de Gauss $x \in]0,1[\to \{1/x\}$.

Abstract. Following Humbert and Lagarias, given a real number θ, we call a nonzero vector $(p,q) \in \mathbb{Z}\times\mathbb{N}$ a Hermite best approximation vector of θ if it minimizes a quadratic form $f_\Delta(x,y) = (x-y\theta)^2 + \frac{y^2}{\Delta}$ for at least one real number $\Delta > 0$. Hermite observed that if (p,q) is such a minimum with $q > 0$, then the fraction p/q must be a convergent of the continued fraction expansion of θ. Using minimal vectors in lattices, we give new proofs of some results of Humbert and Meignen and complete their works. In particular, we show that the proportion of Hermite best approximation vectors among convergents is almost surely $\ln 3/\ln 4$. The main tool of the proofs is the natural extension of the Gauss map $x \in]0,1[\to \{1/x\}$.

1. Introduction

In 1850, Hermite observed that the fractions $\frac{p_\Delta}{q_\Delta}$ associated with the minima $(p_\Delta, q_\Delta) \in \mathbb{Z}\times\mathbb{N}\setminus\{0\}$ of the quadratic forms

$$f_\Delta(p,q) = (p-q\theta)^2 + \frac{q^2}{\Delta},\ \Delta > 0,$$

are all convergents of the continued fraction expansion of the real number θ (see [5, p. 295]). We shall say that a pair $(p,q) \in \mathbb{Z}\times\mathbb{N}$ is a Hermite best approximations vector of θ if it minimizes the quadratic forms f_Δ

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on $\mathbb{Z}^2 \setminus \{(0,0)\}$ for at least one $\Delta > 0$. In [6], Humbert spoke rather of "Hermite fraction" whereas "Hermite best approximation vector" was used instead by Lagarias in [8]. Humbert observed that if (p,q) is a Hermite best approximation vector of θ with $q > 0$, then

$$|q\theta - p| \leq \frac{1}{\sqrt{3}q}$$

(this follows easily from the Gauss reduction of binary positive quadratic forms). Therefore, some convergents of the continued fraction expansion of θ can be skipped. In [6], Humbert addressed three issues:

- Given two consecutive Hermite fractions, find the next one,
- Recognize if a given fraction, is a Hermite fraction,
- Study the connection between the ordinary convergents of θ and the Hermite fractions.

In [3], Grabiner and Lagarias continued and extended the work of Humbert on the third issue. They studied the deep relationships between the one-dimensional Minkowski geodesic continued fraction algorithm, the additive and multiplicative continued fraction algorithms, and the cutting sequences of the geodesic flow in the hyperbolic plane.

In [10], Meignen also continued and extended the work of Humbert. As Humbert, his main tools are the action of the isometry group of the hyperbolic plane and the geodesic flow. It can be noticed that Meignen used the whole isometry group, $\mathrm{PGL}(2,\mathbb{R})$, rather than the sub-group $\mathrm{PSL}(2,\mathbb{R})$ preserving the orientation.

In this note, we shall explain how to recover two striking results of Humbert and Meignen with a different starting point. We shall use minimal vectors in two-dimensional lattices rather than the action of the isometry group on the hyperbolic plane. We shall talk of best approximation vectors (p_n, q_n) rather than convergents $\frac{p_n}{q_n}$. We shall prove the following results.

Theorem 1 (Humbert 1916)**.** *Let θ be in $\mathbb{R}$ and let $(p_n, q_n)_{n\geq 0}$ be the sequence of best approximation vectors of θ. Then, for all $n \geq 0$, one at least of the best approximation vectors (p_n, q_n) and (p_{n+1}, q_{n+1}) is a Hermite best approximation vector.*

Theorem 2. *Let θ be in $\mathbb{R}$ and let $(p_n, q_n)_{n\geq 0}$ be the sequence of best approximation vectors of θ. Then, for almost all $\theta \in \mathbb{R}$,*

$$\lim_{n\to\infty} \frac{1}{n}\operatorname{card}\{0 \leq k < n : (p_k, q_k) \textit{ is a Hermite best approximation vector}\} = \frac{\ln 3}{\ln 4}.$$

Corollary 1. *Let θ be in $\mathbb{R}$ and let (g_n, h_n) be the n-th Hermite best approximation vector of θ. Then, for almost all $\theta \in \mathbb{R}$,*

$$\lim_{n\to\infty} \frac{1}{n} \ln h_n = \frac{\pi^2}{6\ln 3}.$$

Corollary 2 in Meignen's paper [10] is very close to the above corollary. Actually, Meignen proved that the same almost everywhere limit holds for a set of geodesics of the hyperbolic plane in which the set of vertical geodesics associated with the real numbers $\theta \in \mathbb{R}$, is negligible. If one add, as we did, the standard argument using the expanding direction of the geodesic flow, the above corollary becomes a corollary of Meignen's result. Theorem 2 was not stated by Meignen but can be easily deduced from the corollary and from Levy's theorem about the growth rate of the denominators $(q_n)_{n\geq 0}$.

Our main ingredient in the proof of Theorem 2 is the natural extension of the Gauss map $x \in \,]0,1[\, \to \{\frac{1}{x}\}$ and its ergodicity. Instead, Meignen uses the ergodicity of the geodesic flow in the modular surface together with a cross section of the flow. In fact, another objective of this note is to introduce the natural extension of the Gauss map starting from minimal vectors in lattices. There are many ways to introduce the natural extension of the Gauss map, see for instance [1, 11, 13] and, although the idea of minimal vectors goes back to Voronoï ([15]), it seems that their use for the natural extension of the Gauss map is not so well known, a use the author learned from Yitwah Cheung. Recently, Yi Han, a student of Cheung, did a senior thesis where the same approach is explained with emphasis on the role of the diagonal flow, see [4].

The note is organized as follows. We first define minimal vectors in lattices of $\mathbb{R}^2$ and pairs of consecutive minimal vectors in these lattices. Then, we describe the algorithm that computes the minimal vector that immediately follows a pair of consecutive minimal vectors, this leads to the definition of the natural extension of the Gauss map. Afterward, we state and prove all the results about the natural extension that are needed to prove Theorem 2, even those that are well known, including Theorem 1. Among these intermediate results, Proposition 15 (b). characterizes Hermite vectors with the natural extension. Then, we prove Theorem 2 and its corollary. In our work, we do not emphasis on the cross section associated with the minimal vectors, however in the last section we compare this latter cross section to cross sections obtained with the hyperbolic plane.

The author would like to thank Yann Bugeaud for bringing the work of Meignen to his attention.

2. Minimal vectors in lattices of $\mathbb{R}^2$

Notation. For a and $b \geq 0$, the box $B(a,b)$ is the set of vectors $(x,y) \in \mathbb{R}^2$ such that $|x| \leq a$ and $|y| \leq b$. When $u = (u_1, u_2)$ and $v = (v_1, v_2)$ are in

$\mathbb{R}^2$, the box $B(u)$ is defined by $B(u) = B(|u_1|, |u_2|)$ and the box $B(u,v)$ is defined by $B(u,v) = B(\max(|u_1|,|v_1|), \max(|u_2|,|v_2|))$.

Recall that a lattice $\Lambda \subset \mathbb{R}^2$ is a discrete additive subgroup of $\mathbb{R}^2$ that generates the vector space $\mathbb{R}^2$. Equivalently, it is the set of vectors with integral coordinates in a basis of the vector space $\mathbb{R}^2$.

Definition 2. Let Λ be a lattice in $\mathbb{R}^2$.

- A nonzero vector $u = (u_1, u_2) \in \Lambda$ is a *minimal* vector in Λ if for every nonzero $v \in \Lambda$, $v \in B(u) \Rightarrow |v_1| = |u_1|$ and $|v_2| = |u_2|$.
- Two minimal vectors $u = (u_1, u_2)$ and $v = (v_1, v_2)$ are *consecutive* if $|u_2| < |v_2|$ and there are no minimal vector $w = (w_1, w_2)$ such that $|u_2| < |w_2| < |v_2|$.
- A sequence $X_n = (x_n, y_n)$, $n \in I$, is a *complete* sequence of minimal vectors in Λ if
  - $I \subset \mathbb{Z}$ is an interval,
  - for all $n \in I$, X_n is a minimal vector in Λ,
  - for all $n \in I$ such that $n+1 \in I$, $|y_{n+1}| > |y_n|$,
  - for all minimal vectors $u = (x, y)$, there exists $n \in I$ such that $|y_n| = |y|$.

Example 1. When
$$\Lambda_\theta = \{(p - q\theta, q) : (p,q) \in \mathbb{Z}^2\}$$
where $\theta \in \mathbb{R}$, the vectors $(1,0)$ and $(-\theta', 1)$ with $\theta' \in [-\frac{1}{2}, \frac{1}{2}] \cap (\theta + \mathbb{Z})$, are always consecutive minimal vectors in the lattice Λ_θ.

Remark 1. If X_n and X_{n+1} are two elements of a complete sequence of minimal vectors in a lattice, they are consecutive minimal vectors.

Remark 2. Since lattices are discrete subsets, complete sequences of minimal vectors always exist. These sequences are not unique, and depending on whether a lattice contains non-zero vectors on the axes, they can be finite, infinite one sided, or infinite two sided.

Lemma 3. *Two minimal vectors $u = (u_1, u_2)$ and $v = (v_1, v_2)$ in a lattice $\Lambda \subset \mathbb{R}^2$ are consecutive iff $|u_2| < |v_2|$ and the only lattice point in the interior of $B(u,v)$ is zero.*

Consider the lexicographic preorder on $\mathbb{R}^2$ defined by $(x_1, x_2) \prec (y_1, y_2)$ iff $|x_2| < |y_2|$ or $|x_2| = |y_2|$ and $|x_1| \leq |y_1|$.

Proof. Let $u = (u_1, u_2)$ and $v = (v_1, v_2)$ be two minimal vectors with $|u_2| < |v_2|$. If the set $\overset{\circ}{B}(u,v) \cap \Lambda \setminus \{0\}$ is nonempty, then it is finite and there is a w minimal for the lexicographic preorder $\prec$ in this set. On the one hand, w is a minimal vector in Λ. On the other hand, $|w_1| < |u_1|$ and $|w_2| < |v_2|$

and since u is a minimal vector we have $|w_2| > |u_2|$. Hence u and v are not consecutive.

Conversely, if u and v are not consecutive there is a minimal vector w with $|u_2| < |w_2| < |v_2|$. Since w is minimal $|u_1| > |w_1|$, hence $w \in \overset{\circ}{B}(u,v) \cap \Lambda$. $\square$

Proposition 4. *Let $u = (u_1, u_2)$ and $v = (v_1, v_2)$ be two consecutive minimal vectors in a lattice $\Lambda \subset \mathbb{R}^2$. Then the pair (u, v) is primitive, i.e. $\Lambda = \mathbb{Z}u + \mathbb{Z}v$.*

Proof. We can suppose $v_2 > u_2 \geq 0$. Since v is minimal, $|u_1| > |v_1|$. Let $w = (w_1, w_2) = xu + yv$ be in Λ with $0 \leq x, y < 1$, we want to show that $x = y = 0$.

Suppose $x + y \leq 1$. If x and $y > 0$, then $w = (x+y)\frac{xu+yv}{x+y}$ is $(x+y)$ times a vector in the open line segment $]u, v[$, thus w in the interior of the box $B(u, v)$ which contradicts Lemma 3. If $x = 0$, then $w = yv$ and since v is minimal, $y = 0$. If $y = 0$, then $x = 0$ as well.

Suppose that $x + y > 1$. The vector $w' = u + v - w = x'u + y'v$ is in Λ. Since $x' = 1 - x$ and $y' = 1 - y$ are both in $]0, 1[$ and since $x' + y' = 2 - x - y < 1$, w' is in the interior of the box $B(u, v)$ which contradicts Lemma 3. $\square$

Remark 3. This proposition is still true when one considers minimal vectors in $\mathbb{R}^d = \mathbb{R}^{d-1} \times \mathbb{R}$ defined with the boxes $B((u_1, u_2)) = \{(x_1, x_2) \in \mathbb{R}^{d-1} \times \mathbb{R} : |x_1|_{\mathbb{R}^{d-1}} \leq |u_1|_{\mathbb{R}^{d-1}} \text{ and } |x_2| \leq |u_2|\}$ where $|\cdot|_{\mathbb{R}^{d-1}}$ is any norm on $\mathbb{R}^{d-1}$ (see [2]). However, the proposition no longer holds when one consider triples of consecutive minimal vectors rather than pairs in a lattice in $\mathbb{R}^3$. One can find examples with 3 consecutive minimal vectors that are not primitive. This was observed by Lagarias in terms of best approximation vectors (see [7]). This observation explains why a multidimensional continued fraction algorithm cannot share all the properties of the one dimensional continued fraction algorithm.

2.1. Minimal vectors and Diophantine approximations.

Definition 5. Let θ be a real number. A pair $(p, q) \in \mathbb{Z} \times \mathbb{N}^*$ is a best approximation vector of θ if for all $(a, b) \in \mathbb{Z}^2$,

$$\begin{cases} 0 < |b| < |q| \Rightarrow |p - q\theta| < |a - b\theta| \\ 0 < |b| \leq |q| \Rightarrow |p - q\theta| \leq |a - b\theta|. \end{cases}$$

Proposition 6. *Let θ be a real number and consider the lattice Λ_θ defined by*

$$\Lambda_\theta = \{(p - q\theta, q) : (p, q) \in \mathbb{Z}^2\}.$$

Then $X = (p - q\theta, q) \in \Lambda_\theta$ is a minimal vector with $q \neq 0$ iff (p, q) is a best approximation vector of θ.

Proof. Suppose that $X = (p - q\theta, q)$ is a minimal vector with $q \neq 0$. If a and b are integers with $0 < |b| < |q|$, then $Y = (a - b\theta, b) \notin B(X)$ which implies $|a - b\theta| > |p - q\theta|$. If $|b| = |q|$ and if $|a - b\theta| \leq |p - q\theta|$ then $Y \in B(X)$ which implies $|a - b\theta| = |p - q\theta|$.

Conversely, suppose that (p, q) is a best approximation vector of θ. For any $(a, b) \in \mathbb{Z}^2$, $Y = (a - b\theta, b) \in B(X)$ implies

$$\begin{cases} |a - b\theta| \leq |p - q\theta| \\ |b| \leq |q|. \end{cases}$$

If $b \neq 0$, this in turn, implies $|a - b\theta| = |p - q\theta|$ and $|b| = |q|$ by definition of best approximation vectors. If $b = 0$ and $a \neq 0$ then $|a| \geq 1 > \frac{1}{2} \geq |p - q\theta|$, hence $Y \notin B(X)$. □

3. Minimal vectors and the natural extension of the Gauss map

3.1. Definition of the natural extension. Let denote $\lfloor x \rfloor$ the lower integer part of the real number x and $\{x\} = x - \lfloor x \rfloor$ its fractional part. Set

$$U =]0, 1[^2 \cup (\{0\} \times [0, \tfrac{1}{2}]) \cup ([0, \tfrac{1}{2}] \times \{0\}).$$

Proposition 7. *Let Λ be a lattice in $\mathbb{R}^2$ and let $u = (u_1, u_2)$ and $v = (v_1, v_2)$ be a pair of consecutive minimal vectors in Λ with $v_2 > u_2 \geq 0$.*

(a) *If $u_2 > 0$, then $u_1 v_1 \leq 0$. So that we can suppose that $u_1 v_1 \leq 0$ by changing u to $-u$ when $u_2 = 0$ and $u_1 v_1 > 0$.*

(b) *The first coordinate u_1 cannot be zero and there exist $(x, y) \in U$ and $\varepsilon \in \{-1, 1\}$ unique such that*

$$u = (u_1, u_2) = (\varepsilon |u_1|, v_2 y)$$
$$v = (v_1, v_2) = (-\varepsilon |u_1| x, v_2).$$

(c) *If $v_1 \neq 0$, then, with $a = \lfloor \frac{1}{x} \rfloor$,*

$$w = u + av$$

is the minimal vector that follows immediately v. Furthermore,

$$v = (\varepsilon' |v_1|, w_2 y')$$
$$w = (-\varepsilon' |v_1| x', w_2)$$

where

$$\varepsilon' = -\varepsilon,\ w_2 = v_2(a + y),$$
$$x' = \{\tfrac{1}{x}\} \text{ and } y' = \frac{1}{a + y}.$$

Remark 4. When $u_2 = 0$, u is the first minimal vector.

Proof. (a). Since v is minimal and $v_2 > u_2 \geq 0$, we have $|u_1| > |v_1|$. Therefore, if $u_2 > 0$ and $u_1 v_1 > 0$, then $u - v$ is in the interior of the box $B(u, v)$ which contradicts Lemma 3.

(b). Since $|u_1| > |v_1|$, $v_2 > u_2 \geq 0$ and $u_1 v_1 \leq 0$, there exist $\varepsilon = \pm 1$ and $x, y \in [0, 1[$ unique such that $u_1 = \varepsilon |u_1|$, $v_1 = -\varepsilon |u_1| x$ and $u_2 = v_2 y$.

If $x = 0$, then $u - v = (u_1, v_2(y - 1))$. Therefore, $|y - 1| \geq y$ because u is minimal. It implies $y \leq 1/2$.

If $y = 0$, then $u + v = (u_1(1 - x), v_2)$. Therefore, $1 - x \geq x$ because v is minimal. It implies $x \leq 1/2$.

It follows that $(x, y) \in U$.

(c). If $v_1 \neq 0$, then $x > 0$ and the vector $u + av$ is in the strip $\{(x_1, x_2) : |x_1| < |v_1|\}$ for its first coordinate is $\varepsilon |u_1| x(\frac{1}{x} - a) = -v_1 \{\frac{1}{x}\}$. If $w = (w_1, w_2) \in \Lambda \setminus \{0\}$ with $w_2 \geq 0$ is minimal for the lexicographic preorder $\prec$ in this strip, then w is the minimal vector that immediately follows v. By Proposition 4, $\Lambda = \mathbb{Z}u + \mathbb{Z}v = \mathbb{Z}v + \mathbb{Z}w$, hence $\det_{(u,v)}(v, w) = \pm 1$ where $\det_{(u,v)}(v, w)$ is the determinant of the matrix of the coordinates of the vectors v and w in the basis (u, v). Therefore, $w = \pm u + nv$ where $n \in \mathbb{Z}$. Since $w_2 \geq 0$ and $v_2 > u_2 \geq 0$, the integer n is > 0. Now,

$$|v_1| > |w_1| = |nv_1 \pm u_1| = |v_1| |\pm \tfrac{1}{x} - n|,$$

hence $\left|\pm \frac{1}{x} - n\right| < 1$. Since $n \geq 1$, the sign $\pm$ must be $+$ and n must be equal to $a = \lfloor \frac{1}{x} \rfloor$ or to $a + 1$. Since $w_2 = u_2 + nv_2$ and since w is minimal for the lexicographic preorder $\prec$, $n = a$. Therefore, the minimal vector that immediately follows v is

$$w = u + av.$$

Finally, we obtain

$$\begin{aligned} v &= (-\varepsilon |u_1| x, v_2) = (\varepsilon' |v_1|, w_2 y'), \\ w &= (\varepsilon |u_1| x(\tfrac{1}{x} - a), v_2(y + a)) = (-\varepsilon' |v_1| x', w_2) \end{aligned}$$

where

$$\varepsilon' = -\varepsilon,\ x' = \{\tfrac{1}{x}\} \text{ and } y' = \frac{1}{a + y}. \qquad \square$$

Definition 8. Let Λ be a lattice in $\mathbb{R}^2$.

(1) Let $u = (u_1, u_2)$ and $v = (v_1, v_2)$ be two consecutive minimal vectors in Λ in *standard form*, i.e., such that $v_2 > u_2 \geq 0$ and $u_1 v_1 \leq 0$. The triple $(\varepsilon, x, y) \in \{-1, 1\} \times U$ associated with (u, v) by Proposition 7 is called the *intrinsic* coordinates of the pair (u, v).

(2) The map $T :]0,1[^2 \cup (]0,\frac{1}{2}] \times \{0\}) \to [0,1[^2$ defined by

$$T(x,y) = \left(\{\tfrac{1}{x}\}, \frac{1}{\lfloor \frac{1}{x} \rfloor + y}\right)$$

is "the natural extension" of the Gauss map.

Remark 5. Natural extensions of measure preserving maps were introduced by Rokhlin in 1961 (see [12]). Here we shall not prove that the map T is the natural extension of the Gauss map, i.e., the map T is a factor of any invertible extension of the Gauss map. We shall only prove that it is invertible and measure preserving.

3.2. Properties of the natural extension.

Lemma 9. *T is one to one and*

$$T(]0,1[^2 \cup (]0,\tfrac{1}{2}] \times \{0\})) =]0,1[^2 \cup (\{0\} \times]0,\tfrac{1}{2}]).$$

Furthermore, for $(x',y') \in U \setminus ([0,\frac{1}{2}] \times \{0\})$,

$$T^{-1}(x',y') = \left(\frac{1}{\lfloor \frac{1}{y'} \rfloor + x'}, \{\tfrac{1}{y'}\}\right).$$

Proof. If $T(x,y) = (x',y')$ then $0 < y' = \frac{1}{\lfloor \frac{1}{x} \rfloor + y} < 1$. With $b = \lfloor \frac{1}{y'} \rfloor$, we have, $b \leq \lfloor \frac{1}{x} \rfloor + y < b+1$, which implies $\lfloor \frac{1}{x} \rfloor = b$. In turn this implies $x' = \frac{1}{x} - b$ and $y = \frac{1}{y'} - b$, and then $(x,y) = (\frac{1}{b+x'}, \{\frac{1}{y'}\})$. Therefore, T is one to one. Moreover, if $x' = 0$, then $x = \frac{1}{b}$ and b cannot be 1 so that $y' \leq \frac{1}{2}$. It follows that $T(]0,1[^2 \cup (]0,\frac{1}{2}] \times \{0\})) \subset]0,1[^2 \cup (\{0\} \times]0,\frac{1}{2}])$. Conversely, it is easy to check that if $(x',y') \in]0,1[^2 \cup (\{0\} \times]0,\frac{1}{2}])$ then $T(\frac{1}{\lfloor \frac{1}{y'} \rfloor + x'}, \{\frac{1}{y'}\}) = (x',y')$. ☐

Lemma 10 (Contraction Lemma)**.** *Let $x \in]0,1[$ be such that $x' = \{\frac{1}{x}\} > 0$. Then for any $y,z \in]0,1[$, the four pairs $(x',y') = T(x,y)$, $(x'',y'') = T^2(x,y)$, $(x',z') = T(x,z)$ and $(x'',z'') = T^2(x,z)$ are defined and*

$$|z'-y'| \leq |z-y| \text{ and } |z''-y''| \leq \tfrac{1}{2}|z-y|.$$

Proof. With $a = \lfloor \frac{1}{x} \rfloor$ and $a' = \lfloor \frac{1}{x'} \rfloor$, we have

$$y' = \frac{1}{a+y},\ z' = \frac{1}{a+z},$$

$$|z'-y'| = \frac{|y-z|}{|a+z||a+y|} \leq |z-y|,$$

$$y'' = \frac{1}{a' + \frac{1}{a+y}} = \frac{a+y}{1+aa'+a'y},\ z'' = \frac{a+z}{1+aa'+a'z},$$

$$z'' - y'' = \frac{(1+aa'+a'y)(a+z) - (1+aa'+a'z)(a+y)}{(1+aa'+a'z)(1+aa'+a'y)},$$

$$|z'' - y''| \leq \frac{|z-y|}{(1+aa')^2} \leq \tfrac{1}{2}|z-y|. \qquad \square$$

Lemma 11. *The probability $\mu = \frac{1}{\ln 2(1+xy)^2}\mathrm{d}x\mathrm{d}y$ on U is T-invariant and ergodic.*

Proof. Since T is a diffeomorphism from $(]0,1[\setminus\{\frac{1}{n} : n \in \mathbb{N}^2\}) \times]0,1[$ onto $]0,1[\times (]0,1[\setminus\{\frac{1}{n} : n \in \mathbb{N}^2\})$, it suffices to show that, $\varphi \circ T \times |\mathrm{Jac}\, T| = \varphi$ where φ is the density of μ, which is easy to check.

Let us show that T is ergodic. By the contraction Lemma, for all $x \in]0,1[\setminus\mathbb{Q}$ and all $y,z \in]0,1[$,

$$\lim_{n\to\infty} \mathrm{d}(T^n(x,y), T^n(x,z)) = 0.$$

So that, if $f : [0,1]^2 \to \mathbb{R}$ is continuous and if for some $(x,y) \in]0,1[^2$, $\lim_{n\to+\infty} \frac{1}{n}\sum_{k=0}^{n-1} f\circ T^k(x,y) = l(x,y)$, then for all $z \in]0,1[$,

$$\lim_{n\to+\infty} \frac{1}{n}\sum_{k=0}^{n-1} f\circ T^k(x,z) = l(x,y).$$

Therefore, by Birkhoff Theorem, for almost all $(x,y) \in U$, the sequence $\frac{1}{n}\sum_{k=0}^{n-1} f\circ T^k(x,y)$ converges to a limit $l(x)$ which does not depend on y. Since $T^{-1} = s\circ T\circ s$ where $s(u,v) = (v,u)$, we also have that for almost all (x,y), the sequence

$$\frac{1}{n}\sum_{k=0}^{n-1} f\circ T^{-k}(x,y) = \frac{1}{n}\sum_{k=0}^{n-1} f\circ s\circ T^k(y,x)$$

almost everywhere converges to a limit $l'(y)$ which does not depend on x. Since the forward limit and the backward limit are almost surely equal, it follows that $l(x) = l'(y)$ for almost all (x,y). Therefore, the sequence $\frac{1}{n}\sum_{k=0}^{n-1} f\circ T^k(x,y)$ converges almost everywhere to a constant which must be the mean $M(f) = \int_U f\,\mathrm{d}\mu$. By Lebesgue Theorem, the convergence also holds in $L^1(\mu)$. It follows that the sequence of linear maps $A_n f = \frac{1}{n}\sum_{k=0}^{n-1} f\circ T^k$ converges in $L^1(\mu)$ on the everywhere dense set of continuous functions to $M(f)$. Since the sequence of linear maps $(A_n)_n$ is bounded for the operator norm in $L^1(\mu)$, it follows that for all $f \in L^1$, $A_n f \to M(f)$ in $L^1(\mu)$ which implies that T is ergodic. $\square$

4. Hermite best approximations vectors

Recall that a shortest vector in a lattice with respect to a norm $\|\cdot\|$ is a nonzero vector of the lattice whose norm is minimal.

Definition 12. A *Hermite vector* in a lattice $\Lambda \subset \mathbb{R}^2$ is a nonzero vector w in Λ that is a shortest vector in Λ for at least a Euclidean norm $|(x_1,x_2)|_t^2 = |tx_1|^2 + |\frac{1}{t}x_2|^2$ where t is a positive real number.

A *Hermite best approximation vector* of $\theta \in \mathbb{R}$ is a pair $(p,q) \in \mathbb{Z} \times \mathbb{N}$ such that $(p - q\theta, q)$ is a Hermite vector in the lattice $\Lambda_\theta = \{(p - q\theta, q) : (p,q) \in \mathbb{Z}^2\}$.

Remark 6. Observe that a vector $(p,q) \in \mathbb{Z} \times \mathbb{N}$ is a Hermite best approximation vector of θ if and only if it minimizes the quadratic form $f_\Delta(x,y) = (x - y\theta)^2 + \frac{y^2}{\Delta}$ for some $\Delta > 0$.

Lemma 13. *If $u = (u_1,u_2)$ is a Hermite vector in a lattice $\Lambda \subset \mathbb{R}^2$, then u is a minimal vector in Λ.*

Proof. By definition of a Hermite vector, there exists $t > 0$ such that $u = (u_1,u_2)$ is a shortest vector for the norm $|\cdot|_t$. Since

$$B(u) \subset \{v \in \mathbb{R}^2 : |v|_t \leq |u|_t\}$$

and

$$B(u) \setminus \{v \in \mathbb{R}^2 : |v|_t < |u|_t\} = \{(\pm u_1, \pm u_2)\},$$

u is a minimal vector in Λ. □

Lemma 14. *Let $u = (u_1,u_2)$ be a Hermite vector of a lattice $\Lambda \subset \mathbb{R}^2$. If $|u_1| > 0$, then*

(1) *there exists a Hermite vector $h = (h_1,h_2)$ with $|h_1| < |u_1|$,*
(2) *if h is such a Hermite vector with $|h_2|$ minimal, then there exists a positive real number t such that u and h are shortest vectors of Λ with respect to the same norm $|\cdot|_t$.*

Proof. (1). Since $|u_1| > 0$, there exists at least one nonzero vector $v = (v_1,v_2) \in \Lambda$ such that $|v_1| < |u_1|$. For $s > 0$, large enough, $|v|_s^2 = s^2|v_1|^2 + \frac{1}{s^2}|v_2|^2 < s^2|u_1|^2$. Let $h = (h_1,h_2)$ be a shortest vector in Λ for the norm $|\cdot|_s$. Then $s^2|h_1|^2 \leq |h|_s^2 \leq |v|_s^2 < s^2|u_1|^2$.

(2). Suppose now that $h = (h_1,h_2)$ is a Hermite vector with $|h_1| < |u_1|$ and $|h_2|$ minimal. Let

$$t = \sup\{s > 0 : u \text{ is a shortest vector with respect to the norm } |\cdot|_s\}.$$

By continuity, we see that u is still a shortest vector with respect to the norm $|\cdot|_t$. We want to show that $|u|_t = |h|_t$. We use the following short steps:

- If $v = (v_1,v_2)$ and $w = (w_1,w_2)$ are two Hermite vectors and if $|w_1| < |v_1|$ then $|w_2| > |v_2|$ because w is a minimal vector. Therefore the function $s \to |v|_s^2 - |w|_s^2$ is strictly increasing.

- There exists $r \geq t$ such that h is a shortest vector with respect to the norm $|\cdot|_r$. Otherwise, there exists $r < t$ such that h is a shortest vector with respect to $|\cdot|_r$ and we would have $|u|_t - |h|_t > |u|_r - |h|_r \geq 0$ and u would not be a shortest vector with respect to $|\cdot|_t$.
- If $v = (v_1, v_2)$ is a shortest vector of Λ with respect to $|\cdot|_s$ for some $s > t$ then $|v_1| \leq |u_1|$. Otherwise, $|v|_t - |u|_t < |v|_s - |u|_s \leq 0$ and u would not be a shortest vector with respect to $|\cdot|_t$.
- Since Λ is discrete, there exists a vector $v = (v_1, v_2)$ and a sequence (s_n) of real numbers decreasing to t such that v is a shortest vector with respect to the norm $|\cdot|_{s_n}$ for each n.
- We have $|v_1| \neq |u_1|$. Otherwise $|v_2| = |u_2|$ and u would be a shortest vector with respect to a norm $|\cdot|_{s_n}$ with $s_n > t$.
- We have $|v_1| < |u_1|$, otherwise $|v|_t - |u|_t < |v|_{s_n} - |u|_{s_n} < 0$.
- If $|h_1| = |v_1|$ we are done.
- If $|h_1| \neq |v_1|$ then by definition of h, we have $|h_2| < |v_2|$ and therefore $|v_1| < |h_1|$. It follows that $|h|_t - |v|_t \leq |h|_r - |v|_r \leq 0$ and therefore $|h|_t \leq |v|_t = |u|_t$. □

Theorem 1 is a particular case of (a) in the next proposition.

Proposition 15. *Let Λ be a lattice in $\mathbb{R}^2$ and let $u = (u_1, u_2)$ and $v = (v_1, v_2)$ be a pair of consecutive minimal vectors in standard form in Λ. Let $(\varepsilon, x, y) \in \{-1, 1\} \times U$ be the intrinsic coordinates of (u, v) (see Definition 8).*

(a) *One at least of the two vectors u and v is a Hermite vector (Theorem 1).*

(b) *u is a Hermite vector and v is not a Hermite vector iff*

$$x > \frac{2y+1}{y+2}.$$

Furthermore, if this inequality holds then v and $v+u$ are consecutive minimal vectors.

Proof. Step1. Let us show that if u is a Hermite vector and v is not a Hermite vector then $w = u + v$ is a Hermite vector and is the minimal vector that follows v. We proceed by contradiction and suppose that w is not a Hermite vector.

Call $h = (h_1, h_2)$ the Hermite vector with $|h_1| < |u_1|$ and h_2 non negative and minimal. By Lemma 14, the Hermite vector h exists and there exists a $t > 0$ such that u and h are shortest vectors of Λ for the same norm $|\cdot|_t$. Since v is not a Hermite vector, v is not a shortest vector of Λ for the norm $|\cdot|_t$, hence

$$t^2|u|_t^2 = t^4u_1^2 + u_2^2 = t^2|h|_t^2 = t^4h_1^2 + h_2^2 < t^2|v|_t^2 = t^4v_1^2 + v_2^2.$$

It follows that $t^4 = \frac{h_2^2-u_2^2}{u_1^2-h_1^2}$ and that $t^4(u_1^2 - v_1^2) + u_2^2 - v_2^2 < 0$, hence

$$\Delta = (h_2^2 - u_2^2)(u_1^2 - v_1^2) + (u_1^2 - h_1^2)(u_2^2 - v_2^2) < 0.$$

Let us show that $|v_1| \le \frac{1}{2}|u_1|$ is not possible. Set $a = \left\lfloor \frac{|u_1|}{|v_1|} \right\rfloor$. On the one hand, by Proposition 7, v and $u + av$ are the two minimal vectors that follows u, on the other hand, h is a minimal vector that follows u and $(|h_1|, |h_2|) \neq (|v_1|, |v_2|)$, hence $h = u + av$ or h is after $u + av$. In both cases, $h_2 \ge u_2 + av_2$. If $a \ge 2$, we have

$$\begin{aligned}\Delta &\ge (4v_2^2 + 4v_2u_2)\tfrac{3}{4}u_1^2 + u_1^2(u_2^2 - v_2^2)\\ &= 2u_1^2v_2^2 + 3u_1^2u_2v_2 + u_1^2u_2^2\\ &\ge u_1^2(2v_2^2 - \tfrac{3}{2}(v_2^2 + u_2^2) + u_2^2) > 0,\end{aligned}$$

a contradiction. It follows that $|v_1| > \frac{1}{2}|u_1|$ and $a = 1$.

Since $a = 1$, $w = u+v = (w_1, w_2)$ is a minimal vector and $h \neq w$ because we have assumed that w is not a Hermite vector. Now, $w = (w_1, w_2)$ where $|w_1| = |u_1| - |v_1|$ and $w_2 = u_2 + v_2$, hence

$$h_2 \ge w_2 + v_2 = 2v_2 + u_2.$$

We have $|w|_t > |u|_t$, hence

$$\Delta' = t^2(|u|_t^2 - |w|_t^2) < 0.$$

But

$$\begin{aligned}(u_1^2 - h_1^2)\Delta' &= (h_2^2 - u_2^2)(u_1^2 - w_1^2) + (u_1^2 - h_1^2)(u_2^2 - w_2^2)\\ &\ge (4v_2^2 + 4v_2u_2)(2|u_1||v_1| - v_1^2) + u_1^2(-2u_2v_2 - v_2^2)\\ &\ge (4v_2^2 + 4v_2u_2)|u_1||v_1| + u_1^2(-2u_2v_2 - v_2^2)\\ &\ge (4v_2^2 + 4v_2u_2)\tfrac{1}{2}u_1^2 + u_1^2(-2u_2v_2 - v_2^2)\\ &\ge 0\end{aligned}$$

a contradiction. Hence w is Hermite vector and by Proposition 7, w is the minimal vector that follows v.

Step 2. For every minimal vector $u = (u_1, u_2)$ with $u_2 > 0$, there exists a Hermite vector $z = (z_1, z_2)$ with $0 \le z_2 < u_2$. Indeed, if Λ contains a nonzero vector whose second coordinate vanishes, just take $z = (z_1, 0)$ with $0 \le z_1$ minimal. z is a Hermite vector with respect to $|\cdot|_t$ when $t > 0$ is small enough. Otherwise there is a vector $x = (x_1, x_2) \in \Lambda$ with $0 < x_2 < u_2$. One can find $s > 0$ such that

$$\frac{1}{s^2}u_2^2 > |x|_s^2.$$

A shortest vector $z = (z_1, z_2)$ associated with such a s, is by definition a Hermite vector and we have $0 < |z_2| < u_2$ because $\frac{1}{s^2}|z_2|^2 \leq |z|_s^2 \leq |x|_s^2 < \frac{1}{s^2}|u_2|^2$.

Step 3. Let us show (a). If $u_2 = 0$ then u is a Hermite vector. If $u_2 > 0$, by Step 2, there exists a Hermite vector $z = (z_1, z_2)$ with $0 \leq z_2 < u_2$. There exists a complete sequence of minimal vectors $(X_n = (a_n, b_n))_{n \in I}$ in Λ such that $(|z_1|, |z_2|) = (|a_{n_1}|, |b_{n_1}|)$ and $(|u_1|, |u_2|) = (|a_{n_2}|, |b_{n_2}|)$ with $n_1 < n_2 \in I$. Let us show by induction that for all $n \geq n_1$ such that $n+1 \in I$, X_n or X_{n+1} is a Hermite vector. If $n = n_1$ it holds because z is a Hermite vector. If X_n is not a Hermite vector, by induction hypothesis, X_{n-1} is a Hermite vector. By 1, $X_{n-1} + X_n$ is a Hermite vector and is the minimal vector that follows X_n. Therefore, X_{n+1} is a Hermite vector. It follows that X_{n_2} or X_{n_2+1} is a Hermite vector which proves (a).

Step 4. Let u and v be two consecutive minimal vectors. With $r = |u_1|, q = v_2 > 0$ by definition of the intrinsic coordinate, $u = (\varepsilon r, qy)$ and $v = (-\varepsilon rx, q)$. Suppose that u is a Hermite vector and v is not. Let $w = u + v$. By 1, there exists $t > 0$ such that

$$|u|_t = |w|_t < |v|_t.$$

As in Step 1, this implies

$$t^4 = \frac{q^2((1+y)^2 - y^2)}{r^2(1-(1-x)^2)},$$
$$t^4 r^2 (1 - x^2) + q^2(y^2 - 1) < 0,$$

thus

$$\frac{((1+y)^2 - y^2)}{(1-(1-x)^2)}(1 - x^2) + (y^2 - 1) < 0.$$

which is equivalent to

$$(2y + y^2)x^2 + 2(1 - y^2)x - (2y + 1) > 0$$

Solving in x, the discriminant is $(1-y^2)^2 + (2y+y^2)(2y+1) = (1+y+y^2)^2$, thus we obtain

$$x > \frac{2y+1}{y+2} \text{ or } x < -\frac{1}{y}$$

and since $x \geq 0$, $x > \frac{2y+1}{y+2}$.

Step 5. Conversely if the inequality $x > \frac{2y+1}{y+2}$ holds then with the value $t^4 = \frac{q^2((1+y)^2 - y^2)}{r^2(1-(1-x)^2)}$, we obtain

$$|u|_t = |w = u + v|_t < |v|_t$$

which implies that v is not a Hermite vector. Actually, if $s > t$ then $|w|_s < |v|_s$ and if $s < t$ then $|u|_s < |v|_s$.

Step 6. It remains to show that if

$$x > \frac{2y+1}{y+2},$$

then u is a Hermite vector. Since v is not a Hermite vector this follows from (a). $\square$

5. Proportion and growth rate of Hermite best approximations

Lemma 16. *Let $V = \{(x,y) \in U : x > \frac{2y+1}{y+2}\}$. Then*

$$\int\int_V \frac{1}{(1+xy)^2}\mathrm{d}x\mathrm{d}y = \ln 2 - \tfrac{1}{2}\ln 3$$

Proof. The lemma follows from the two standard calculations:

$$\begin{aligned}\int_{\frac{2y+1}{y+2}}^{1} \frac{1}{(1+xy)^2}\mathrm{d}x &= \left[-\frac{1}{y}\frac{1}{1+xy}\right]_{\frac{2y+1}{y+2}}^{1} \\ &= \frac{1-y}{2(1+y)(1+y+y^2)}\end{aligned}$$

and

$$\int \frac{1-y}{2(1+y)(1+y+y^2)}\mathrm{d}y = \ln(1+y) - \tfrac{1}{2}\ln(1+y+y^2). \qquad \square$$

Lemma 17. *Let Λ be a lattice in $\mathbb{R}^2$ and let $X_n = (r_n, q_n)$, $n \in I \subset \mathbb{Z}$, be a complete sequence of minimal vectors with $q_n \geq 0$ for all n. Suppose that $0, 1 \in I$ and let (ε, x, y) be the intrinsic coordinates of the pair (X_0, X_1). Then, for all $n \in I$ such that $n+1 \in I$, X_{n+1} is not a Hermite vector of Λ iff $T^n(x,y) \in V$.*

Proof. By Proposition 7, for all $n \in I$ such that $n+1 \in I$, the intrinsic coordinates of (X_n, X_{n+1}) are

$$\varepsilon_n = (-1)^n\varepsilon,\ (x_n, y_n) = T^n(x,y).$$

If X_{n+1} is not a Hermite vector, X_n is a Hermite vector by Proposition 15 (a), and therefore $(x_n, y_n) \in V$ by (b). Conversely, if $(x_n, y_n) \in V$ then X_{n+1} is not a Hermite vector again by Proposition 15 (b). $\square$

Proof of Theorem 2. Let θ be in $\mathbb{R}$. The first two minimal vectors of Λ_θ are $X_0 = \pm(1,0)$ and $X_1 = (-\theta', 1)$ where $\theta' = \theta - [\theta]$ and $[\theta]$ is the integer nearest to θ. The intrinsic coordinates of these two consecutive minimal vectors are $(\varepsilon, x, 0) = (\operatorname{sgn}\theta', |\theta'|, 0)$. So that, thanks to the previous Lemma, it is enough to prove that

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n} 1_V \circ T^k(x,0) = 1 - \frac{\ln 3}{2\ln 2}$$

for almost all $x \in [0, \frac{1}{2}]$. By Birkhoff Theorem applied to the natural extension of the Gauss map and to the indicator function 1_V, we know that for almost all $(x, y) \in U$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n} 1_V \circ T^k(x, y) = 1 - \frac{\ln 3}{2\ln 2}.$$

The problem is that the limit holds for almost all (x, y) and not for almost all x. The method to overcome this problem is standard, we have just to use that T does not increase the distance along the y direction (contraction Lemma).

Suppose on the contrary that there exist $a > 0$ and a measurable set $S \subset [0, \frac{1}{2}]$ of positive measure such that for all $x \in S$

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n} 1_V(T^k(x, 0)) \geq 1 - \frac{\ln 3}{2\ln 2} + a$$

or

$$\liminf_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n} 1_V(T^k(x, 0)) \leq 1 - \frac{\ln 3}{2\ln 2} - a.$$

We deal with the first case, the second is similar. Let t be positive real number and let

$$V_t = \{(x, y) \in U : \exists\, (x, y') \in V, |y - y'| \leq t\}.$$

We can choose t small enough so that $\mu(V_t) < \mu(V) + \frac{a}{2}$. By the contraction Lemma, for all $(x, y) \in S \times [0, t]$ and all integers $n \geq 0$,

$$T^n(x, 0) - T^n(x, y) = (0, z_n)$$

with $|z_n| \leq t$. It follows that for all n and all $(x, y) \in S \times [0, t]$,

$$\begin{aligned}\limsup_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n} 1_{V_t}(T^k(x, y)) &\geq \limsup_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n} 1_V(T^k(x, 0)) \\ &\geq \mu(V) + a \\ &\geq \mu(V_t) + \frac{a}{2},\end{aligned}$$

Which contradicts Birkhoff Theorem used with the function 1_{V_t}. □

Proof of Corollary 1. Let $(g_n - h_n\theta, h_n)_{n\geq 0}$ be the sequence of Hermite vectors in Λ_θ and $X_n(\theta) = (p_n - q_n\theta, q_n)$, $n \geq 0$, be the complete sequence of minimal vectors of Λ_θ. We can suppose that the q_n and h_n are ≥ 0. By Lemma 13, the sequence $(h_n)_{n\geq 0}$ is a sub-sequence of the sequence $(q_n)_{n\geq 0}$. Therefore, there exists an increasing sequence $(n_k)_{k\geq 0}$ such that for all $k \geq 0$, $h_k = q_{n_k}$. By definition,

$$\{0 \leq n < 1 + n_k : X_k(\theta) \text{ is a Hermite vector}\} = \{n_0, \ldots, n_k\},$$

and by Theorem 2, for almost all θ,

$$\lim_{k\to\infty}\frac{k+1}{1+n_k}=\frac{\ln 3}{2\ln 2},$$

so that by Levy's Theorem ([9]),

$$\begin{aligned}\frac{1}{k+1}\ln h_k &= \frac{1}{k+1}\ln q_{n_k}\\ &= \frac{n_k}{k+1}\times\frac{1}{n_k}\ln q_{n_k}\\ &\longrightarrow \frac{2\ln 2}{\ln 3}\times\frac{\pi^2}{12\ln 2}=\frac{\pi^2}{6\ln 3}\end{aligned}$$

when k goes to infinity. □

6. Minimal vectors, cross sections and the hyperbolic plane

Thanks to Proposition 7, there is an algorithm that associates to a pair of consecutive minimal vectors (u,v) in a lattice $\Lambda\subset\mathbb{R}^2$, the minimal vector w that follows v. With the intrinsic coordinates, the map $(u,v)\to(v,w)$ is given by the natural extension T of the Gauss map and a sign (see Proposition 7). We can think of this map another way. For each pair of consecutive minimal vectors $u=(u_1,u_2)$, $v=(v_1,v_2)$, since $|u_1|>|v_1|$ and $|v_2|>|u_2|$, the diagonal matrix

$$g_t=\begin{pmatrix}e^t & 0\\ 0 & e^{-t}\end{pmatrix},\ \text{with } t=\tfrac{1}{2}\ln\frac{|v_2|}{|u_1|},$$

is such that $|ug_t|_\infty=|vg_t|_\infty$ where $|(x,y)|_\infty=\max(|x|,|y|)$. This brings the lattice Λ in the cross section $\mathcal{S}$ of the diagonal flow in $\mathrm{SL}(2,\mathbb{Z})\backslash\mathrm{SL}(2,\mathbb{R})$ defined by $\Lambda\in\mathcal{S}$ iff there exist two vectors $u=(u_1,u_2)$ and $v=(v_1,v_2)$ in Λ such that

- $|u_2|$ and $|v_1|<r=|u_1|=|v_2|$ and
- $\pm u$ and $\pm v$ are the only nonzero vectors in Λ that are in the closed ball $B_\infty(0,r)$ associated with the sup norm.

Observe that the vectors u and v in the definition of $\mathcal{S}$ are consecutive minimal vectors. Now we can replace the previous map $(u,v)\to(v,w)$ by the first return map of the flow $(g_t)_{t\in\mathbb{R}}$ in the cross section $\mathcal{S}$. The ergodicity of the natural extension T of the Gauss map can be deduced from the ergodicity of the diagonal flow in the space of unimodular lattices.

Many authors used a first return map defined on the quotient of the unit tangent bundle of the hyperbolic plane $\mathrm{PSL}(2,\mathbb{Z})\backslash T_1\mathbb{H}$ (see [3, 14]). Identifying the space of unimodular lattices with the quotient $\mathrm{PSL}(2,\mathbb{Z})\backslash T_1\mathbb{H}$, the diagonal flow becomes the geodesic flow. Meignen used the action of the whole isometry group instead of just $\mathrm{PSL}(2,\mathbb{R})$. To be precise, on the one hand, $T_1\mathbb{H}$ can be identified with $\mathrm{SL}_\pm(2,\mathbb{R})/\{\mathrm{diag}(\pm1,\pm1)\}$ and on the other hand, the quotient of the space of unimodular lattices by the symmetries with respect to the two axes can be identified with $\mathrm{PGL}(2,\mathbb{Z})\backslash T_1\mathbb{H}$. Now the geodesic flow in the space of unimodular lattices can be view as a billiard in a well-chosen fundamental domain for the action of $\mathrm{PGL}(2,\mathbb{Z})$. Meignen used a hyperbolic triangle with one vertex at infinity as fundamental domain. The trajectories of the billiard are piecewise geodesics with the usual laws of reflection along the sides of the triangle. So that the geodesics in the modular surface become billiard trajectories. Then the ergodicity of the geodesic flow and a cross section allowed Meignen to prove his version of Corollary 1.

The common characteristic of the methods based on the hyperbolic plane is to use the geodesic flow on a quotient of $T_1\mathbb{H}$ and the first return map on a cross section whose projection in the corresponding quotient of $\mathbb{H}$, is a finite union of geodesic segments. For instance, the cross section used by Meignen projects on the bounded side of the hyperbolic triangle.

We want to point out that the projection in $\mathbb{H}$ of the cross section $\mathcal{S}$ defined with the minimal vectors, has non empty interior. Indeed, thanks to Proposition 7, it is easy to see that the projection of $\mathcal{S}$ contains all the complex numbers

$$z = \frac{i+\varepsilon y}{-\varepsilon x i + 1} \text{ with } (x,y) \in \,]0,1[^2, \varepsilon = \pm 1.$$

Thus, the cross section $\mathcal{S}$, when seen in the hyperbolic plane, is not of the usual type, i.e., the projection of $\mathcal{S}$ is not a finite union of geodesic segments.

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