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Résumé. Soit $\{d_i n + b_i : \ n \in \mathbb{Z}\}_{i \in I}$ une famille de suites arithmétiques qui est une couverture disjointe de l'ensemble des nombres entiers. Barát and Varjú [1] ont prouvé que si $d_i = p_1^{\alpha_1} p_2^{\alpha_2}$ pour deux nombres premiers p_1, p_2 et des entiers $\alpha_1, \alpha_2 \geq 0$, alors il existe i et j tels que $j \neq i$ et $d_i | d_j$. Nous montrons que ce résultat reste vrai si $d_i = p_1^{\alpha_1} \dots p_n^{\alpha_n}$ pour un ensemble fixé $\{p_1, \dots, p_n\}$ de n nombres premiers.

Abstract. Let $\{d_i n + b_i : \ n \in \mathbb{Z}\}_{i \in I}$ be a family of disjoint arithmetic progressions covering the integers. Barát and Varjú [1] have proved that if $d_i = p_1^{\alpha_1} p_2^{\alpha_2}$ for two prime numbers p_1, p_2 and integers des $\alpha_1, \alpha_2 \geq 0$, then there exist $j \neq i$ such that $d_i | d_j$. We show that this result remains true if $d_i = p_1^{\alpha_1} \dots p_n^{\alpha_n}$ for a fixed set $\{p_1, \dots, p_n\}$ of n prime numbers.

Covering systems were introduced by Paul Erdős [3] in 1950. In particular, a *disjoint covering system* (DCS) is an infinite collection of pairwise disjoint infinite arithmetic progressions $A(d_i, b_i) = \{nd_i + b_i : \ n \in \mathbb{Z}\}$ whose union is $\mathbb{Z}$. One of the early results about disjoint covering systems, proved by Mirsky, Newman, Davenport, Rado and Stein (see [4]) is that in a finite disjoint covering system the two largest moduli are equal. For an infinite DCS, the case $n = 2$ of the following result was proved by Barát and Varjú [1].

Theorem 1. *If $\{A(d_i, b_i) : \ i \in I\}$ is an infinite DCS such that the prime factors of each modulus d_i belong to the same finite set of primes, then $d_i | d_j$ for some $i \neq j$.*

Our aim in this note is to point out that the considerably stronger Theorem 2 below, which has nothing to do with covering systems, is an old (and simple) result in combinatorics.

Theorem 2. *Let $\{d_i : \ i \in I\}$ be an infinite set of natural numbers such that the prime factors of each d_i belong to a set $P = \{p_1, \dots, p_n\}$ of n primes. Then $d_i | d_j$ for some $i \neq j$.*

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To restate this assertion, let $(\mathbb{N}^n, \prec)$ be the partially ordered set with the partial order $x = (x_i)_1^n \prec y = (y_i)_1^n \in \mathbb{N}^n$ if $x_i \le y_i$ for every i and $x \neq y$. As in every poset, a set $A \subset \mathbb{N}^n$ is an *antichain* if $a \neq b \in A$ implies that $a \not\prec b$, i.e. $b_j < a_j$ for some j. Identifying $n_i = p_1^{\alpha_1} \dots p_n^{\alpha_n}$ with the n-tuple $(\alpha_1 + 1, \dots, \alpha_n + 1) \in \mathbb{N}^n$, Theorem 2 has the following simple reformulation.

Theorem 3. *In the poset* $(\mathbb{N}^n, \prec)$ *every antichain is finite.*

Although this is Dickson's Lemma [2] from 1913, let us prove it by induction on n, starting with the trivial case $n = 1$. Suppose that $n \ge 2$ and the assertion holds for smaller values of n. For $1 \le j \le n$ and $k \ge 1$, let $M_j(k) = \{x = (x_i)_1^n \in \mathbb{N}^n : \ x_j = k\}$, and note that $(M_j(k), \prec)$ is isomorphic to $(\mathbb{N}^{n-1}, \prec)$.

Let $A \subset \mathbb{N}^n$ be an antichain and let $a = (a_i)_1^n \in A$. If $b \in A$ and $b \neq a$ then $b \in M_j(k)$ for some j and k with $1 \le j \le n$ and $1 \le k \le a_j - 1$. Since $A \cap M_j(k)$ is an antichain in $(M_j(k), \prec) \cong (\mathbb{N}^{n-1}, \prec)$, it is a finite set by the induction hypothesis. Consequently, A is also finite.

This result is the starting point of the rich and deep theory of *well-quasi-ordered sets*: for a review of the early results, see Kruskal [5], and for highlights of an important series of papers on graph minors, see Robertson and Seymour [6, 7].

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