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Résumé. Nous prouvons l'existence de systèmes d'Euler pour les représentations galoisiennes modulaires adjointes p-ordinaires en utilisant les déformations de représentations galoisiennes provenant de formes modulaires p-ordinaires de Hilbert et nous leurs associons des fonctions L p-adiques via une formule conjecturale pour l'idéal de Fitting d'un module de congruences équivariant pour le changement de base abélien.

Abstract. We prove the existence of Euler systems for p-ordinary adjoint modular Galois representations using deformations of Galois representations coming from p-ordinary Hilbert modular forms, and relate them to p-adic L-functions under a conjectural formula for the Fitting ideals of some equivariant congruence modules for abelian base change.

1. Introduction

After the introduction of the notion of Euler systems by Kolyvagin [8] as a powerful tool to understand the structure of Selmer groups, a systematic study of them by Perrin-Riou [9] and Rubin [11] has led naturally to the notion of Euler systems of rank d for some positive integer d. In [10], Rubin shows that Stark type conjectures give some evidence of the existence of such systems but so far there was no construction when $d > 1$ without using the validity of the corresponding Main Iwasawa–Greenberg conjecture.

This work is one of a series of papers (see for example [13]) in which the author is investigating the construction of Euler systems via the study of congruences between automorphic forms of various levels and weights. The main purpose of this note is to give an illustration of this principle for adjoint modular Galois representations and must be seen as an example of a very general construction. In each situation, some technical difficulties arise which can or cannot be overcome depending of what is known about the structure of certain modules over the Hecke algebras involved. Nevertheless, it is the author's conviction that these principles shed light on the Iwasawa theory of the Galois representation at play and will eventually lead to the

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proof of new Iwasawa Main Conjectures by a successful study of these congruences. In the situation of this paper, we will also show that Euler systems of rank $d > 1$ can be constructed by this technique. Note that despite there being other works giving evidence of the existence of Euler systems of higher ranks based on the knowledge of the Main Conjectures, there is not yet any geometric constructions of such.

To describe the construction done in this paper, let us introduce some notations. Let p be an odd prime and F be a totally real number field of degree d over the rationals $\mathbb{Q}$. Let f be a nearly p-ordinary Hilbert cuspidal eigenform. Let us denote by ρ_f the Galois representation attached to f:

$$\rho_f\colon G_F \longrightarrow \mathrm{GL}_O(T_f)$$

where T_f is a O-free module of rank 2 for some finite extension O of the ring of p-adic integers $\mathbb{Z}_p$ and G_F is the absolute Galois group of F. We will assume throughout the paper that the residual representation $\bar{\rho}_f$ is absolutely irreducible. This representation is nearly ordinary at each place v dividing p, which means for such v there exists a O-direct factor $\mathrm{Fil}_v^+ T_f$ of rank 1 which is stable under the action of the decomposition subgroup $D_v \subset G_F$ at v. We will also assume that ρ_f is v-distinguished[1] at each place v dividing p. We write F_v for the completion of F at v and $d_v := [F_v : \mathbb{Q}_p]$.

We denote by $\mathrm{ad}(\rho_f) \subset \mathrm{End}_O(T_f)$ the adjoint representation on the endomorphisms of T_f having trace 0. The filtrations $\mathrm{Fil}_v^+ T_f$ on T_f induce for each v a three steps filtration on $\mathrm{ad}(\rho_f)$:

$$\mathcal{F}_v^+ \subset \mathcal{F}_v^0 \subset \mathcal{F}_v^- = \mathrm{ad}(\rho_f)$$

with rank 1 graded pieces. We denote by $\mathrm{Gr}_v^0 := \mathcal{F}_v^0/\mathcal{F}_v^+$ and fix an isomorphism of D_v-module $\mathrm{Gr}_v^0 \cong O$. We consider the restriction map at p:

$$\mathrm{res}_p\colon H^1(F, \mathrm{ad}(\rho_f)) \longrightarrow \bigoplus_{v|p} H^1(F_v, \mathrm{ad}(\rho_f))$$

Finally recall that for any Galois representation V of G_F and S a finite set of finite places, we denote by $L^S(V, s)$ the corresponding L-function defined as the Euler product:

$$L^S(V, s) := \prod_{v \notin S} P_v(q_v^{-s}; V)^{-1}$$

where $P_v(X, V) := \det(1 - X\,\mathrm{Frob}_v; V^{I_v})$ with $\mathrm{Frob}_v \in D_v$ a geometric Frobenius, $I_v \subset D_v$ the inertia subgroup at v and q_v the cardinality of the residue field at v. We denote by $\Gamma(V, s)$ the corresponding Γ-factor.

[1] The trace of the residual representation restricted to D_v is the sum of two characters which are distinct modulo the uniformizer ϖ of O.

Let us denote by $\bigotimes_F^{\mathbb{Q}} \bar{\rho}_f$ the tensor induction from G_F to $G_{\mathbb{Q}}$ of the residual representation $\bar{\rho}_F$. The beginning of this work starts by observing that Hida theory for Hilbert modular forms can provide a proof of the following theorem.

Theorem 1.1. *Let us assume that* $\bigotimes_F^{\mathbb{Q}} \bar{\rho}_f$ *is absolutely irreducible. Then, there exists a canonical element* $z_f \in \bigwedge_O^d H^1(F, \mathrm{ad}(\rho_f))$*, defined up to an element in* $O^\times$ *such that*

(i) $\wedge^d \mathrm{res}_p(z_f)$ *belongs to* $\bigotimes_{v|p} \bigwedge_O^{d_v} H^1(F_v, \mathcal{F}_v^0)$*.*

(ii) *The image of* $\wedge^d \mathrm{res}_p(z_f)$ *by the map*

$$\bigotimes_{v|p} \bigwedge_O^{d_v} H^1(F_v, \mathcal{F}_v^0) \longrightarrow \bigotimes_{v|p} \bigwedge_O^{d_v} H^1(I_v, \mathrm{Gr}_v^0)^{D_v/I_v} \cong O$$

is equal to

$$\xi_f := \frac{\Gamma(\mathrm{ad}(\rho_f), 1) L^{S_f}(\mathrm{ad}(\rho_f), 1)}{\Omega_f^{\Sigma} \Omega_f^{\Sigma_F \setminus \Sigma}}$$

where S_f *is the set of finite places containing those where* ρ_f *is ramified and* $(\Omega_f^{\Sigma})_{\Sigma \subset \Sigma_F}$ *are the canonical complex periods attached to the Hilbert modular form* f *in* [1]*.*

The proof of this result follows from examining the first fundamental exact sequence of Kähler differentials of the universal ordinary Hecke algebra and the interpretation of the latter in terms of Galois cohomology classes and congruence modules. In particular, it uses the fact due to the works of Hida and Wiles [4, 15] in the case $F = \mathbb{Q}$ and Dimitrov [1] in general that ξ_f measures the size of the congruence module $\wp_f/\wp_f^2$ where $\wp_f = \mathrm{Ker}(\lambda_f)$ with λ_f the homomorphism of the cuspidal Hecke algebra of the same weight and level as f that gives the Hecke eigenvalues associated to f.

Before stating the main result of this work which is a generalization of Theorem 1.1, let us introduce some more notations. For any number field or p-adic field K, we denote by K^{cyc} the cyclotomic $\mathbb{Z}_p$-extension of K. For any Galois module M over $G_K = \mathrm{Gal}(\overline{K}/K)$, let $H^1(K, M)$ (resp. $H^1_{Iw}(K, M)$) the Galois cohomology of M (resp. the Iwasawa Galois cohomology of K which is defined as the projective limit under the norm maps of $H^1(K', M)$ for finite extensions K'/K with $K' \subset K^{\mathrm{cyc}}$). We also write $\Lambda := O[\![\mathrm{Gal}(F^{\mathrm{cyc}}/F)]\!]$.

Let S be a finite set of places containing those dividing the level of f. Recall that the un-primitive Selmer group $\mathrm{Sel}^S(F^{\mathrm{cyc}}, \mathrm{ad}(\rho_f))$ attached to

$\mathrm{ad}(\rho_f)$ is defined as the kernel of the restriction map:

$$H^1(\mathrm{Gal}(F_S/F^{\mathrm{cyc}}), \mathrm{ad}(\rho_f) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p) \longrightarrow \bigoplus_{v|p} H^1(F_v^{\mathrm{cyc}}, \mathrm{ad}(\rho_f)/\mathcal{F}_v^+ \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p)$$

where F_S is the maximal extension of F unramified away from S and p. It is known thanks to the work of Wiles, Taylor–Wiles, Fujiwara and others on the modularity of Galois representations (often known as "R=T" Theorems), that this Selmer group is of co-torsion over the Iwasawa algebra Λ. We denote by $X^S(F^{\mathrm{cyc}}, \mathrm{ad}(\rho_f))$ its Pontrjagin dual. Thanks to the work of R. Greenberg [2, Prop. 4.1.1], it is known that this module does not contain any non trivial finite submodule and therefore its Fitting ideal is equal to its characteristic ideal and is therefore principal. We fix $\mathcal{L}_f^{S,\mathrm{alg}} \in \Lambda$ one of its generators.

By extending the arguments of Theorem 1.1 for the base change of f to the totally real abelian extensions E of F, we obtain the following Theorem[2]

Theorem 1.2. *There exists a (non trivial) Iwasawa–Euler system of rank d for $\mathrm{ad}(\rho_f)$. In other words, for totally real fields E running in an S-admissible set of abelian extensions of F which are unramified above S (see Definition 3.11), there exists an element $Z_{f,E} \in \bigcap^d_{\Lambda[\mathrm{Gal}(E/F)]} H^1_{Iw}(E, \mathrm{ad}(\rho_f))$ such that for any extensions E and E' such that $E' \supset E \supset F$, we have*

$$Cores_E^{E'}(Z_{f,E'}) = \prod_{v \in S(E'/E)} P_v(q_v^{-1}\,\mathrm{Frob}_v, \mathrm{ad}(\rho_f)).Z_{f,E}$$

where $S(E'/E)$ is the set of finite places of F that ramify in E' but not in E. Moreover,

(i) *$\bigcap^d \mathrm{res}_p(Z_{f,F})$ belongs to $\bigotimes_{v|p} \bigcap^{d_v}_\Lambda H^1_{Iw}(F_v, \mathcal{F}_v^0)$.*

(ii) *The image of $\bigcap^d \mathrm{res}_p(Z_{f,F})$ by the map*

$$\bigotimes_{v|p} \bigcap_{\Lambda}^{d_v} H^1_{Iw}(F_v, \mathcal{F}_v^0) \longrightarrow \bigotimes_{v|p} \bigcap_{\Lambda}^{d_v} H^1_{Iw}(I_v, \mathrm{Gr}_v^0)^{D_v/I_v} \cong \Lambda$$

is equal to $\mathcal{L}_f^{S,\mathrm{alg}}$.

The Iwasawa–Zeta elements $Z_{f,E}$ are defined as the projective limit over n of elements $z_{f,E_n} \in \bigwedge^d_{O[\Delta_{E_n}]} H^1(E_n, \mathrm{ad}(\rho_f))$; here E_n denotes the field of degree p^n over E inside the $\mathbb{Z}_p$-cyclotomic extension E^{cyc} of E. In order to construct the elements $z_{f,E} \in \bigwedge^d_{O[\Delta_E]} H^1(E, \mathrm{ad}(\rho_f))$ when E runs in a certain set of totally real abelian extensions of F with a version of (ii) similar to Theorem 1.1, we need to understand the structure of $\wp_{f_E}/\wp_{f_E}^2$

[2]See Section 3.4 for the definition of $\bigcap^r_A$ for any commutative ring A and positive integer r.

as a module over $O[\Delta_E]$. Here $\Delta_E := \mathrm{Gal}(E/F)$, f_E is the base change to E of the Hilbert modular form f and $\wp_{f_E}$ is determined similarly as $\wp_f$. The link between the Iwasawa theory of the Selmer groups attached to $\mathrm{ad}(\rho_f)$, the congruence modules $\wp_{f_E}/\wp_{f_E}^2$ and deformation theory of the Galois representation $\bar{\rho}_f$ has been systematically studied by Hida in [6]. In particular, it can be seen from deformation theory that we have a canonical surjection

$$X^S(E^{\mathrm{cyc}}, \mathrm{ad}(\rho_f)) \twoheadrightarrow \wp_{f_E}/\wp_{f_E}^2$$

and in the proof of Theorem 1.2, we use that the Fitting ideal of the dual Selmer group $X^S(E^{\mathrm{cyc}}, \mathrm{ad}(\rho_f))$ which therefore annihilates $\wp_{f_E}/\wp_{f_E}^2$ behaves well when E varies (see Proposition 3.18).

B. Perrin-Riou [9, App. B] and K. Rubin [11] have developed some arguments to extract Euler systems of rank ones from one of higher rank. However, our method allows us to obtain directly rank one Euler systems with prescribed local conditions at places dividing p. More precisely, we have the following result.

Theorem 1.3. *For each place $v|p$, there exists a system of classes $c_E^v \in H^1_{\mathrm{Iw}}(E, \mathrm{ad}(\rho_f))$ with E running in an S-admissible set of abelian extensions of F such that*

$$\mathrm{Cores}_E^{E'}(c_{E'}^v) = \prod_{w \in S(E'/E)} P_v(q_v^{-1}\,\mathrm{Frob}_v, \mathrm{ad}(\rho_f)).c_E^v$$

Moreover

(i) *For all place w of E dividing p, $\mathrm{res}_w(c_E^v)$ belongs to $H^1_{\mathrm{Iw}}(E_w, \mathcal{F}_w^0)$ and furthermore belongs to $H^1_{\mathrm{Iw}}(E_w, \mathcal{F}_w^+)$ if $w \nmid v$.*

(ii) *The image of $\mathrm{res}_v(c_F^v)$ by the map*

$$H^1_{\mathrm{Iw}}(F_v, \mathcal{F}_v^0) \longrightarrow H^1_{\mathrm{Iw}}(I_v, \mathrm{Gr}_v^0)^{D_v/I_v} \cong \Lambda$$

is equal to $\mathcal{L}_f^{S,\mathrm{alg}}$.

To obtain an Euler system with Iwasawa–Zeta elements that we can relate to p-adic L-functions, we would like to know that

$$\mathcal{L}_{f,E}^S \in \mathrm{Fitt}_{O[\Delta_E]}(\wp_{f_E}/\wp_{f_E}^2)$$

where $\mathcal{L}_{f,E}^S \in K[\Delta_E]$ is the unique element satisfying for every $\chi \in \mathrm{Hom}(\Delta_E, \mathbb{C}^\times)$

$$\chi(\mathcal{L}_{f,E}^S) = \frac{G(\chi)^2\Gamma(\mathrm{ad}(\rho_f)\otimes\chi, 1)L^{S_E}(\mathrm{ad}(\rho_f)\otimes\chi, 1)}{\Omega_f^{\Sigma}\Omega_f^{\Sigma_F\setminus\Sigma}} \tag{1.1}$$

where S_E is the union of S and the set of finite places that ramify in E/F and $G(\chi)$ is the Gauss sum attached to χ. We make the following conjecture.

Conjecture 1.4. For each totally real abelian extension E/F that ramifies away from S or at p, we have $\mathcal{L}^S_{f,E} \in \mathrm{Fitt}_{O[\Delta_E]}(\wp_{f_E}/\wp^2_{f_E})$.

Then we have:

Theorem 1.5. *If we assume Conjecture 1.4, the conclusion of Theorem 1.2 holds with $\mathcal{L}^{S,\mathrm{alg}}_f$ replaced by the un-primitive Coates–Schmidt p-adic L-function $\mathcal{L}^{S,\mathrm{an}}_f$.*

A weaker form of the conjecture is the following.

Conjecture 1.6. For each totally real abelian extension E/F that ramifies away from S or at p, $\mathcal{L}^S_{f,E}$ annihilates $\wp_{f_E}/\wp^2_{f_E}$.

Then we have:

Theorem 1.7. *If we assume Conjecture 1.6, the conclusion of Theorem 1.3 holds with $\mathcal{L}^{S,\mathrm{alg}}_f$ replaced by the un-primitive Coates–Schmidt p-adic L-function $\mathcal{L}^{S,\mathrm{an}}_f$.*

Note that either of these conjectures implies that the elements $\mathcal{L}^S_{f,E}$ are integral, in otherwords that they belong to $O[\Delta_E]$. Note that this fact could be proven using the techniques of [5]. Assuming this integrality statement for all extensions E_n/E, we define the equivariant Coates–Schmidt p-adic L-function

$$\mathcal{L}^{S,\mathrm{an}}_{f,E} := \varprojlim_n \mathcal{L}^S_{f,E_n} \in \Lambda = \varprojlim_n O[\mathrm{Gal}(E_n/E)].$$

where we write E_n for the extension of E of degree p^n inside E^{cyc}. For $E=F$, we just write $\mathcal{L}^{S,\mathrm{an}}_f \in \Lambda$. Up to a unit in Λ, $\mathcal{L}^{S,\mathrm{an}}_f$ is the un-primitive Coates–Schmidt p-adic L-function.

Using the Euler system machinery, we can then deduce, assuming conjecture 1.6, that the following divisibility holds in Λ:

$$\mathcal{L}^{S,\mathrm{alg}}_f \mid \mathcal{L}^{S,\mathrm{an}}_f \tag{1.2}$$

On the other hand, by refining the techniquess of Iwasawa theory and Euler systems to the equivariant setting, it is very likely that one can show that the existence of an Euler system with the bottom class satisfying the correct property is implied by the divisibility above for all the twist $\mathrm{ad}(\rho_f)\otimes\chi$. It would then follow that the existence of the Euler system satisfying the correct bottom class condition is equivalent to Conjecture 1.4. We leave the verification of these expectations to the interested reader.

This note is organized as follows. Section 2 is devoted to recalling facts about Hida families for Hilbert cuspidal eigenforms, their Galois representations and congruence modules. We also state the conjecture about their annihilators. Section 3 contains the construction and the properties of the cocycles and of the compatible systems of cohomology classes and zeta elements.

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Notations. Throughout this paper p is a fixed odd rational prime. Let $\overline{\mathbb{Q}}$ and $\overline{\mathbb{Q}}_p$ be respectively the algebraic closures of $\mathbb{Q}$ and $\mathbb{Q}_p$ and let $\mathbb{C}$ be the field of complex numbers. We fix embeddings $\iota_\infty : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ and $\iota_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$. Throughout this note, we implicitly view $\overline{\mathbb{Q}}$ as a subfield of $\mathbb{C}$ and $\overline{\mathbb{Q}}_p$ via the embeddings ι_∞ and ι_p and we fix an identification $\overline{\mathbb{Q}}_p \cong \mathbb{C}$ compatible with these embeddings. We denote by ϵ_{cyc} the p-adic cyclotomic character.

2. Congruences modules for Hilbert modular forms

2.1. Universal ordinary Hecke algebras. Let F be a totally real field and O_F its ring of integers. We denote by Σ_F the set of embeddings of F into $\overline{\mathbb{Q}}$. Let $T = T_F$ be the torus defined over $\mathbb{Z}$ by $T_F(A) = (O_F \otimes_{\mathbb{Z}} A)^\times$. Let p be an odd prime. We write $T(\mathbb{Z}_p)_{\mathrm{tor}}$ for the torsion subgroup of $T(\mathbb{Z}_p)$ and Γ_F for the subgroup of $T(\mathbb{Z}_p)$ such that $T(\mathbb{Z}_p) \cong T(\mathbb{Z}_p)_{\mathrm{tor}} \times \Gamma_F$. We have $\Gamma_F \cong \mathbb{Z}_p^d$ with $d = [F : \mathbb{Q}]$ is the degree of F over $\mathbb{Q}$. We fix O a finite extension of $\mathbb{Z}_p$, we denote $\Lambda_F := O[\![\Gamma_F]\!]$.

Let $\lambda = (\sum_\sigma k_\sigma.\sigma, \sum_\sigma l_\sigma.\sigma,) \in \mathbb{Z}[\Sigma_F]^2$ be an arithmetic weight (which means that $k_\sigma \geq 2$ for all $\sigma \in \Sigma_F$ and $w = k_\sigma + 2l_\sigma$ is independent of σ). We fix $\mathfrak{n}$ a non zero integral ideal of O_F prime to p and let $K^p(\mathfrak{n}) \subset \mathrm{GL}_2(\widehat{\mathbb{Z}}^p \otimes O_F)$ be the subgroup of matrices which are congruent to the identity matrix modulo $\mathfrak{n}$ and where we have written $\widehat{\mathbb{Z}}^p$ for the projective limit of $\mathbb{Z}/M\mathbb{Z}$ for the integers M prime to p. We also fix ω be an idèle class character of conductor dividing $\mathfrak{n}p^\infty$ and infinity type $|\cdot|^w$.

For each positive integer n, we denote by $K_0(p^n)$ the subgroup of $\mathrm{GL}_2(O_F \otimes \mathbb{Z}_p)$ of matrices which are upper triangular modulo p^n and by $K_1(p^n)$ its subgroup of those such that the diagonal entries are congruent modulo p^n. We will identify $K_0(p^n)/K_1(p^n)$ with $(O_F/p^nO_F)^\times$ via the map $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mapsto a^{-1}d$. Let $h^{\mathrm{ord}}_{F,\lambda}(\mathfrak{n}, p^n, \omega)$ be the ordinary Hecke algebra of lever $K^p(\mathfrak{n})K_1(p^n)$ and weight λ. We then consider the universal nearly ordinary Hecke algebra of weight λ and tame level K^p and action of the center given by ω.

$$\mathbf{h}^{\mathrm{ord}}_F = \mathbf{h}^{\mathrm{ord}}_{F,\lambda}(\mathfrak{n}) := \varprojlim_n h^{\mathrm{ord}}_{F,\lambda}(\mathfrak{n}, p^n)$$

This is the Hecke algebra denote $h(U^W_\infty, O)$ by Hida in [6, p. 403] with $W = \mathrm{Cl}_F(p^\infty)$. From the action of $K_0(p^n)/K_1(p^n)$ on the space of ordinary forms of weight λ and level $K^p(\mathfrak{n})K_1(p^n)$, we inherit an action of Γ_F on $\mathbf{h}_F^{\rm ord}$ which therefore has the structure of a Λ_F-algebra. For any finite order character ψ of Γ_F, we denote by P_ψ the kernel of the map from Λ_F into $\overline{\mathbb{Q}}_p$ induced by ψ.

Theorem 2.1 (Hida)**.** $\mathbf{h}^{\rm ord}(\mathfrak{n})$ *is free of finite rank over* Λ_F*. Moreover, for any arithmetic character* ψ_λ*, there is a canonical isomorphism*

$$\mathbf{h}_{F,\lambda}^{\rm ord}(\mathfrak{n})\otimes\Lambda_F/P_\psi\cong h_{F,\lambda}^{\rm ord}(\mathfrak{n},p^r,\psi)$$

where $h_{F,\lambda}^{\rm ord}(\mathfrak{n},p^r,\psi)$ *denotes the Hecke algebra over* $O(\psi)$ *generated by the Hecke operators acting on the space of ordinary Hilbert modular forms of weight* λ*, level* $K^p(\mathfrak{n})K_1(p^r)$ *and with nebentypus restricted to the image of* Γ_F *into* $(O_F/p^rO_F)^\times$ *given by* ψ *where* r *is the smallest integer such that* ψ *factorizes through that image.*

Proof. This follows from Hida's works. See for example Corollary 5.3 of [6] for $W^? = W^\phi$ or [7]. □

Let now f be a p-nearly ordinary Hilbert modular form of tame level $K^p(\mathfrak{n})$, unramified central character at p and weight λ with Hecke eigenvalues contained in O via the embedding of $\overline{\mathbb{Q}}$ into $\overline{\mathbb{Q}}_p$. It gives rise to a homomorphism: $\lambda_f : h_{F,\lambda}^{\rm ord}(\mathfrak{n},p^r,\psi)\to O$ for some character ψ of level p^r. We will denote $\widehat{\lambda}_f$ the composite of λ_f with the canonical surjective map $\mathbf{h}_{F,\lambda}^{\rm ord}(\mathfrak{n})\to h_\lambda^{\rm ord}(\mathfrak{n},p^r,\psi)$. We therefore have a map:

$$\widehat{\lambda}_f : \mathbf{h}_{F,\lambda}^{\rm ord}(\mathfrak{n})\longrightarrow O$$

Following H. Hida, we can define two congruence modules. Let B be the unique quotient of $h_\lambda^{\rm ord}(\mathfrak{n},p^r,\psi)$ such that

$$h_{F,\lambda}^{\rm ord}(\mathfrak{n},p^r,\psi)\hookrightarrow O\times B$$

where the first projection map is λ_f and the second λ_B is the canonical map induced by the fact B is a quotient of $h_\lambda^{\rm ord}(\mathfrak{n},p^r,\psi)$. Let $\eta_f=\ker(\lambda_B)$ which naturally imbeds into O via λ_f and let $\wp_f=\mathrm{Ker}(\lambda_f)$ which imbeds in B via λ_B. The we can define the congruence modules

$$C_0(f):=O/\eta_f\quad\text{and}\quad C_1(f):=\wp_f/\wp_f^2\cong\Omega_{h_\lambda^{\rm ord}(\mathfrak{n},p^r,\psi)/O}\otimes_{\lambda_f}O.$$

Via the canonical isomorphism $O/\eta_f\cong B/\wp_f$, we see that η_f annihilates $\wp_f/\wp_f^2$ and $\wp_f/\wp_f^2$ is therefore finite since it is finitely generated over O and $\eta_f\neq 0$.

Let $\widehat{\wp}_f:=\mathrm{Ker}(\widehat{\lambda}_f)$. The following exact sequence will play a fundamental role in this paper.

Lemma 2.2. With the notation as before, we have a canonical exact sequence

$$0\longrightarrow\Omega_{\Lambda_F/O}\otimes_{\Lambda_F}O\longrightarrow\widehat{\wp}_f/\widehat{\wp}_f^2\longrightarrow\wp_f/\wp_f^2\longrightarrow 0$$

Proof. We write the first fundamental exact sequence of Kahler differentials attached to the maps $O \to \Lambda_F \to \mathbf{h}^{\mathrm{ord}}_{F,\lambda}(\mathfrak{n})$ which we tensor by O through $\widehat{\lambda}_f$:

$$\Omega_{\Lambda_F/O} \otimes_{\Lambda_F} \mathbf{h}^{\mathrm{ord}}(\mathfrak{n}) \otimes_{\hat{\lambda}_f} O \xrightarrow{(1)} \Omega_{\mathbf{h}^{\mathrm{ord}}_{F,\lambda}(\mathfrak{n})/O} \otimes_{\hat{\lambda}_f} O \longrightarrow \Omega_{h^{\mathrm{ord}}_{F,\lambda}(\mathfrak{n})/\Lambda_F} \otimes_{\hat{\lambda}_f} O \longrightarrow 0$$

Since $\mathbf{h}^{\mathrm{ord}}_{F,\lambda}(\mathfrak{n}) \otimes \Lambda/P_{\psi,\lambda} \cong h^{\mathrm{ord}}_{F,\lambda}(\mathfrak{n}, p^r, \psi)$, we easily see that we have

$$\begin{aligned}\Omega_{\mathbf{h}^{\mathrm{ord}}(\mathfrak{n})/\Lambda_F} \otimes_{\hat{\lambda}_f} O &\cong \Omega_{\mathbf{h}^{\mathrm{ord}}_{F,\lambda}(\mathfrak{n})/\Lambda_F} \otimes_{\mathbf{h}^{\mathrm{ord}}(\mathfrak{n})} h^{\mathrm{ord}}_{F,\lambda}(\mathfrak{n}, p^r, \psi) \otimes_{\lambda_f} O \\ &\cong \Omega_{h^{\mathrm{ord}}_{F,\lambda}(\mathfrak{n},p^r,\psi)/O} \otimes_{\lambda_f} O = \wp_f/\wp_f^2\end{aligned}$$

and is therefore finite. To deduce that the map (1) is injective, since $\Omega_{\mathbf{h}^{\mathrm{ord}}(\mathfrak{n})/\Lambda_F} \otimes_{\hat{\lambda}_f} O = \Omega_{\Lambda_F/O} \otimes_{\Lambda_F} O \cong O^d$ is torsion free. it is sufficient to prove the injectivity after inverting p. Since after inverting p, the map (1) is surjective, it is therefore sufficient to prove that $\Omega_{\mathbf{h}^{\mathrm{ord}}_{F,\lambda}(\mathfrak{n})/O} \otimes_{\hat{\lambda}_f} O[1/p]$ is of rank at least d. This follows from the fact that $h^{\mathrm{ord}}_{F,\lambda}(\mathfrak{n})[1/p]$ is of Krull dimension d, by [6, p. 40 property ff], or because it is free of finite rank over $\Lambda_F[1/p]$ thanks to Theorem 2.1. Note that the fact that $\wp_f/\wp_f^2$ is finite says exactly that the map $\operatorname{Spec} h^{\mathrm{ord}}_{F,\lambda}(\mathfrak{n}) \to \operatorname{Spec} \Lambda_F$ is étale at $\widehat{\lambda}_f$ at the level of generic fibers. □

2.2. Base change. Let now E be a totally real abelian extension of F. The norm map from $O_E^\times$ to $O_F^\times$ induces a map from Γ_E to Γ_F and therefore from Λ_E to Λ_F inducing the base change transfer map for characters of p-power level. It induces an isomorphsim

$$\Lambda_E/I_E\Lambda_E \cong \Lambda_F$$

where I_E is the augmentation ideal of the group ring $A_E := O[\mathrm{Gal}(E/F)]$ acting on Λ_E via the Galois action of $\mathrm{Gal}(E/F)$ on $O_E^\times$. On the other hand, the imbedding $O_F^\times \hookrightarrow O_E^\times$ gives Λ_E the structure of a Λ_F-algebra.

Let us identify Σ_E with the set of embeddings of E into $\overline{\mathbb{Q}}_p$ via our fix embedding $\iota_p \colon \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$. Then we have a non-canonical isomorphism

$$\Lambda_E \cong O[\![T_\sigma, \sigma \in \Sigma_E]\!]$$

such that the map $\Lambda_E \to O$ induced by the evaluation at the weight of f_E is induced by $T_\sigma \mapsto 0$ for all $\sigma \in \Sigma_E$. We may choose and fix such isomorphisms so that, for any intermediate extension $E' \subset E$, the canonical surjective map $\Lambda_E \to \Lambda_{E'}$ is induced by the canonical surjection $\Sigma_E \twoheadrightarrow \Sigma_{E'}$. This clearly induces the isomorphism

$$\Omega_{\Lambda_E/O} \otimes O \cong \bigoplus_{\sigma \in I_E} O.dT_\sigma$$

where $\sum_\sigma f_\sigma.dT_\sigma$ is mapped to $\sum_\sigma f_\sigma(0,\dots,0).dT_\sigma$. Moreover the free action of $\mathrm{Gal}(E/F)$ on Σ_E shows that $\Omega_{\Lambda_{E/O}} \otimes O$ is free of rank d over A_E.

We now define the base change homomorphism for the ordinary universal Hecke algebras. Let $\mathfrak{n}_E := \mathfrak{n}O_E$. By the existence of solvable base change, there is a canonical algebra homomorphism:

$$\pi_F^E(\mathfrak{n}, p^n) \colon h_E^{\mathrm{ord}}(\mathfrak{n}_E, p^n)[1/p] \longrightarrow h_F^{\mathrm{ord}}(\mathfrak{n}, p^n)[1/p]$$

Let $\mathcal{H}_E^S$ be the restricted tensor product over O of the spherical Hecke algebras over O for $\mathrm{GL}_2(E_w)$ for all finite place w above a place of F not in S. The local base change for unramified representations implies we have a canonical algebra homomorphism $\Pi_F^E(S) \colon \mathcal{H}_E^S \to \mathcal{H}_F^S$ such that the following diagram commutes

$$\begin{array}{ccc} \mathcal{H}_E^S & \xrightarrow{\Pi_F^E(S)} & \mathcal{H}_F^S \\ \downarrow & & \downarrow \\ h_E^{\mathrm{ord}}(\mathfrak{n}_E, p^n)[1/p] & \longrightarrow & h_F^{\mathrm{ord}}(\mathfrak{n}, p^n)[1/p] \end{array}$$

Since the Hecke algebras $h_E^{\mathrm{ord}}(\mathfrak{n}_E, p^n)$ are torsion free, it follows from the above diagram that the base change homomorphism preserve the integrality of Hecke operators and defines a homomorphism $h_E^{\mathrm{ord}}(\mathfrak{n}_E, p^n) \to h_F^{\mathrm{ord}}(\mathfrak{n}, p^n)$. After passing to the projective limit over n, we deduce the base change map for the ordinary universal Hecke algebra

$$\pi_F^E : \mathbf{h}_E^{\mathrm{ord}}(\mathfrak{n}_E) \longrightarrow \mathbf{h}_F^{\mathrm{ord}}(\mathfrak{n})$$

becomes a homomorphism of Λ_E-algebras for the Λ_E-algebra structure of $\mathbf{h}_F^{\mathrm{ord}}(\mathfrak{n})$ induced by the base change transfer map for the character $\Lambda_E \to \Lambda_F$ described above.

Let us denote by f_E the ordinary Hilbert modular form for E defined by $\widehat{\lambda}_{f_E} = \widehat{\lambda}_f \circ \pi_{E,F}$. Recall that there is a natural action of the Galois group $\mathrm{Gal}(E/F)$ over $h_E^{\mathrm{ord}}(\mathfrak{n}_E)$ defined by $U_\mathfrak{p} \to U_{\mathfrak{p}^\sigma}$ for $\mathfrak{p}|p$ and $T_\mathfrak{q} \mapsto T_{\mathfrak{q}^\sigma}$ for any prime ideal $\mathfrak{q}$ of E and $\sigma \in \mathrm{Gal}(E/F)$. It induces an action on the characters of this algebra leaving λ_{f_E} invariant. It induces therefore an action of $\mathrm{Gal}(E/F)$ over $C_1(f_E) = \wp_{f_E}/\wp_{f_E}^2$.

Let us record the following lemma

Lemma 2.3. *With the notation as before, $\wp_{f_E}/\wp_{f_E}^2$ is O-torsion and we have a canonical exact sequence of A_E-modules*

$$0 \longrightarrow \Omega_{\Lambda_E/O} \otimes_{\Lambda_E} O \longrightarrow \widehat{\wp}_{f_E}/\widehat{\wp}_{f_E}^2 \longrightarrow \wp_{f_E}/\wp_{f_E}^2 \longrightarrow 0$$

Proof. The proof is similar to Lemma 2.2. The fact that the maps are equivariant for the action of A_E is clear. □

2.3. Galois representations. Let $G_F = \mathrm{Gal}(\overline{\mathbb{Q}}/F)$ be the absolute Galois group of F. For each finite place v, we denote respectively G_v, I_v and $\mathrm{Frob}_v \in G_v$ a decomposition subgroup at v, its inertia subgroup and a Frobenius lift at v. To each ordinary Hilbert modular form f for F of weight λ and tame level $\mathfrak{n}p^r$ and nebentypus at p given by ψ as in the section above, one can associate a nearly ordinary p-adic continuous Galois representation thanks to the work of many authors (for example see [14] and [6, p. 406–407]).

$$\rho_f : G_F \longrightarrow \mathrm{GL}_2(O)$$

such that

(i) ρ_f is unramified at finite places of F not dividing $\mathfrak{n}p$.
(ii) For each place v not dividing $\mathfrak{n}p$ and geometric Frobenius $\mathrm{Frob}_v \in G_F$ at v, $\mathrm{Trace}(\rho_f(\mathrm{Frob}_v)) = \lambda_f(T_v)$
(iii) For each place $v|p$, there exists $\alpha_{f,v} \in \mathrm{GL}_2(O)$ and characters ψ_v and ψ'_v of G_v, such that

$$\rho_f(g) = \alpha_{f,v} \begin{pmatrix} \psi'_v(g) & * \\ 0 & \psi_v(g) \end{pmatrix} \alpha_{f,v}^{-1} \quad \forall\, g \in G_v$$

such that $\psi_v|_{I_v}$ is the character of I_v corresponding to $\psi|_{O_{F_v}}$ via local Class Field Theory that identifies Frob_v with an uniformizer of O_{F_v}.
(iv) $\det \rho_f = \omega\epsilon_{\mathrm{cyc}}$.

Let us assume the following irreducibility assumption:

(Irred) The reduction $\bar{\rho}_f$ of ρ_f modulo the maximal ideal of O is absolutely irreducible.

Let $\mathrm{sl}_2(O)$ be the set of 2×2 matrices A with entries in O such that $\mathrm{tr}(A) = 0$. We consider the adjoint action $\mathrm{ad}(\rho_f)$ of G_F on $\mathrm{sl}_2(O)$ via ρ_f. It is defined by

$$\mathrm{ad}(\rho_f)(g).A = \rho_f(g)A\rho_f(g)^{-1}$$

Note that under the condition (Irred), we have

$$H^0(E, \mathrm{ad}(\bar{\rho}_f)) = 0 \tag{2.1}$$

for any totally real field extension E/F. Throughout this paper, we will further assume the conditions:

(Dist) For each $v|p$, the characters ψ_v and ψ'_v are distinct modulo p.
(Indec) For each $v|p$, $\rho_f|_{G_v}$ is indecomposable.

2.4. Twisted adjoint L-values. We are interested in the nature of the Fitting ideal of $\wp_{f_E}/\wp_{f_E}^2$ over A_E. We formulate a conjecture about it below. We first introduce some notations. For any finite order character χ of G_F and finite set of primes S, we consider the S-primitive twisted L-function $L^S(\mathrm{ad}(\rho_f) \otimes \chi, s)$. It is known to be holomorphic on the whole complexe plane except maybe if ρ_f is dihedral, a case we exclude by our hypothesis. For any subset $\Sigma \subset \Sigma_F$, M. Dimitrov has introduced in [1] some canonical periods $\Omega_f^{\Sigma} \in \mathbb{C}^\times/(O \cap \overline{\mathbb{Q}})^\times$ so that

$$\mathcal{L}^S(\mathrm{ad}(\rho_f)\otimes\chi,1) := G(\chi)^2\frac{\Gamma(\mathrm{ad}(\rho_f)\otimes\chi,1)L^S(\mathrm{ad}(\rho_f)\otimes\chi,1)}{\Omega_f^{\Sigma}\Omega_f^{\Sigma_F\setminus\Sigma}} \in K(\chi)$$

where $K(\chi) \subset \overline{\mathbb{Q}}_p$ is the extension of K generated by the values of χ and $G(\chi)$ is the Gauss sum attached to χ.

Let S_E be the set of places of F dividing $\mathfrak{n}p$ and those ramifying in the extension E/F. For each character $\chi\colon \mathrm{Gal}(E/F) \to \overline{\mathbb{Q}}^\times \subset \overline{\mathbb{Q}}_p^\times$, let $e_\chi \in K(\chi)\otimes_O A_E$ be the idempotent projecting any $K[\mathrm{Gal}(E/F)]$-modules on its χ-isotypical component. We then define

$$\mathcal{L}_E^{an}(\mathrm{ad}(\rho_f)) := \sum_\chi \mathcal{L}^{S_E}(\mathrm{ad}(\rho_f)\otimes\chi,1).e_\chi \ \in \overline{\mathbb{Q}}_p\otimes_O A_E$$

where the sum runs over all the characters of $\mathrm{Gal}(E/F)$. The part (iv) of the following conjecture is an equivariant formulation of the congruence number formula established by Hida in the eighties.

Conjecture 2.4. For any totally real abelian extension E/F that ramifies away from S or at p, we have:

(i) $\mathcal{L}_E^{an}(\mathrm{ad}(\rho_f)) \in A_E$,
(ii) $\mathcal{L}_E^{an}(\mathrm{ad}(\rho_f))$ annihilates $\wp_{f_E}/\wp_{f_E}^2$,
(iii) $\mathcal{L}_E^{an}(\mathrm{ad}(\rho_f))$ belongs to the Fitting ideal of $\wp_{f_E}/\wp_{f_E}^2$ over A_E.
(iv) $\mathcal{L}_E^{an}(\mathrm{ad}(\rho_f))$ generates the Fitting ideal of $\wp_{f_E}/\wp_{f_E}^2$ over A_E

The part (i) of the above conjecture should not be difficult to prove using the integral representation of these twisted L-values using Eisenstein series and Theta series of half integral weight as was done by Hida in [5] for $F = \mathbb{Q}$. We can think of (ii) as an automorphic Stickelberger Theorem and we will see that if we assume (ii) for many fields E, we can construct Euler system of rank one for $\mathrm{ad}(\rho_f)$. However part (iii) which implies part (ii) is stronger and allows one to construct Euler systems for $\mathrm{ad}(\rho_f)$ of rank d. When E/F is quadratic, J. Tilouine and the author have proved this conjecture in [12] which was formulated in the quadratic case by H. Hida in a slightly different way.

3. Congruences and Euler systems

3.1. Big Hecke rings and Galois representations. Let E/F like in Section 2.2. Let $\mathbf{T}_E$ be the local component of $\mathbf{h}_E^{\mathrm{ord}}(\mathfrak{n})$ associated to the maximal ideal containing $\mathrm{Ker}(\widehat{\lambda}_{f_E})$. It is flat over Λ_E and it is equipped with a natural action of $\mathrm{Gal}(E/F)$ such that the natural map $\Lambda_E \to \mathbf{T}_E$ is $\mathrm{Gal}(E/F)$-equivariant.

Then, there exists a Galois representation

$$\rho_{\mathbf{T}_E} : G_E \longrightarrow \mathrm{GL}_2(\mathbf{T}_E)$$

such that

(1) $\rho_{\mathbf{T}_E}$ is unramified at finite places of F not dividing $\mathfrak{n}p$.
(2) For each place v not dividing $\mathfrak{n}p$ and arithmetic Frobenius $\mathrm{Frob}_v \in G_E$ at v, $\mathrm{Trace}(\rho_{\mathbf{T}_E}(\mathrm{Frob}_v)) = T_v$
(3) For each place $v|p$ of F and each place $w|v$ of E, there exists $\alpha_w \in \mathrm{GL}_2(\mathbf{T}_E)$ and characters Ψ_w and Ψ'_w of G_v, such that

$$\rho_f(g) = \alpha_w \begin{pmatrix} \Psi'_w(g) & * \\ & \Psi_w(g) \end{pmatrix} \alpha_w^{-1} \quad \forall\, g \in G_w$$

and $\Psi_w|_{I_w} = \psi_v|_{I_w}\kappa_w$, where κ_w is the character of $I_w \overset{\mathrm{Art}}{\to} O_w^\times \hookrightarrow (O_E \otimes \mathbb{Z}_p)^\times \twoheadrightarrow \Gamma_E \subset \Lambda_E^\times$ where Art is the restriction to I_w of the isomorphsim of local Class Field Theory $G_w^{\mathrm{ab}} \cong E_w^\times$ that identifies Frob_w with an uniformizer of O_{E_w}.
(4) $\det \rho_{\mathbf{T}_E} = \omega\epsilon_{\mathrm{cyc}}^{-1}$.

3.2. Construction of cocycles. Let $\widehat{\wp}_{f_E} = \mathrm{Ker}(\widehat{\lambda}_{f_E} : \mathbf{T} \to O)$. We denote $\mathfrak{q}_E := \wp_{f_E}/\wp_{f_E}^2$ and $\widehat{\mathfrak{q}}_E := \widehat{\wp}_{f_E}/\widehat{\wp}_{f_E}^2$, and we see them as Galois modules with action of G_F given via the natural action of $\mathrm{Gal}(E/F)$ on $\mathbf{T}_E$. Note that the exact sequence of Lemma 2.3 induces an exact sequence of $O[\mathrm{Gal}(E/F)]$-modules:

$$0 \longrightarrow \Omega_{\Lambda_E/O} \otimes O \longrightarrow \widehat{\mathfrak{q}}_E \longrightarrow \mathfrak{q}_E \longrightarrow 0 \tag{3.1}$$

where $\Omega_{\Lambda_E/O} \otimes O$ is free of rank d over $A_E = O[\mathrm{Gal}(E/F)]$.

Let us denote by $\mathrm{sl}_2(O)$ the Lie algebra over O of 2×2 matrices of trace 0. We will most of the time use the notation $\mathrm{ad}(\rho_f)$ instead of $\mathrm{sl}_2(O)$ to emphasis that it is a Galois module for the adjoint action of ρ_f given by

$$g.X := \rho_f(g) X \rho_f(g)^{-1} \qquad \forall\, g \in G_F, \quad \forall\, X \in \mathrm{sl}_2(O)$$

For $g \in G_E$, we define an element $x_{f,E}(g) := x(g) \in \mathrm{sl}_2(O) \otimes_O \widehat{\wp}_{f_E}/\widehat{\wp}_{f_E}^2$ by the equality

$$x(g) := \rho_{\mathbf{T}}(g)\rho_f(g)^{-1} - \mathbb{1}_2 \pmod{\widehat{\wp}_{f_E}^2}$$

Lemma 3.1. The map $g \mapsto x_{f,E}(g)$ defines a cocycle in $Z^1(G_E, \mathrm{ad}(\rho_f) \otimes \widehat{\mathfrak{q}}_E)$. Moreover, for any $\tau \in \mathrm{Gal}(E/F)$, we have

$$x^\tau(g) = \tau^{-1}.x(g)\ , \quad \forall\, g \in G_E$$

where on the left hand side the action of τ is the natural one on the set of cocycles and the right hand side action is the one on $\widehat{\mathfrak{q}}_E$.

Proof. From the definition of $x(g)$ we have

$$\rho_{\mathbf{T}}(g) = (\mathbb{1}_2 + x(g))\rho_f(g) \pmod{\widehat{\wp}_{f_E}^2}$$

Since $\rho_{\mathbf{T}}$ and ρ_f have the same determinant, we have $\det(1 + x(g)) = 1 \pmod{\widehat{\wp}_{f_E}^2}$ and therefore $x(g)$ has trace zero modulo $\widehat{\wp}_{f_E}^2$. Now, an easy computation provides that

$$\begin{aligned} x(gg') &= x(g) + \rho_f(g)x(g')\rho_f(g)^{-1} + x(g)\rho_f(g)x(g')\rho_f(g)^{-1} \\ &= x(g) + \rho_f(g)x(g')\rho_f(g)^{-1} \pmod{\widehat{\wp}_{f_E}^2} \end{aligned}$$

which implies the first claim.

Let now $\tau \in \mathrm{Gal}(E/F)$. By definition of the action of τ on $\mathbf{T}$, we have the matrix identity

$$\tau^{-1}(\rho_{\mathbf{T}}(g)) = A_\tau \rho_{\mathbf{T}}(\tau^{-1} g \tau) A_\tau^{-1} \tag{3.2}$$

for some matrix $A_\tau \in \mathrm{GL}_2(\mathbf{T})$. Since $\tau.\lambda_{f_E} = \lambda_{f_E}$, from the equality $\rho_f(\tau g \tau^{-1}) = \rho_f(\tau)\rho_f(g)\rho_f(\tau)^{-1}$ and the fact ρ_f is residually irreducible, we can easily see that we may choose A_τ so that

$$A_\tau \equiv \rho_f(\tau) \pmod{\widehat{\wp}_{f_E}}$$

Therefore, since $\tau^{-1}.\rho_f(g) = \rho_f(g)$ for all $g \in G_E$, we have the following identities modulo $\widehat{\wp}_{f_E}^2$

$$\begin{aligned} (\mathbb{1}_2 + \tau^{-1}.x(g))\rho_f(g)\tau^{-1} &\equiv (\mathbb{1}_2 + A_\tau x(\tau^{-1} g \tau) A_\tau^{-1}) A_\tau \rho_f(\tau^{-1} g \tau) A_\tau^{-1} \\ &\equiv (\mathbb{1}_2 + \rho_f(\tau) x(\tau^{-1} g \tau) \rho_f(\tau)^{-1}) \rho_f(g) \end{aligned}$$

Therefore we deduce:

$$\tau^{-1}.x(g) \equiv \rho_f(\tau) x(\tau^{-1} g \tau) \rho_f(\tau)^{-1} \pmod{\widehat{\wp}_{f_E}^2}$$

which proves the second claim. □

Corollary 3.2. The cocycle x_{f_E} descends canonically to a cohomology class

$$\widetilde{x}_E \in H^1(F, \mathrm{ad}(\rho_f) \otimes_O \widehat{\mathfrak{q}}_E).$$

Proof. It is clear that x_{f_E} defines a cohomology class in $H^1(E, \mathrm{ad}(\rho_f) \otimes_O \widehat{\mathfrak{q}}_E)$. This class is invariant by $\mathrm{Gal}(E/F)$ by the previous Lemma. The inflation restriction exact sequence gives

$$\begin{aligned} 0 \longrightarrow & H^1(\mathrm{Gal}(E/F), H^0(E, \mathrm{ad}(\rho_f)) \otimes_O \widehat{\mathfrak{q}}_E) \\ & \longrightarrow H^1(F, \mathrm{ad}(\rho_f) \otimes_O \widehat{\mathfrak{q}}_E) \longrightarrow H^1(E, \mathrm{ad}(\rho_f) \otimes_O \widehat{\mathfrak{q}}_E)^{\mathrm{Gal}(E/F)} \\ & \qquad \longrightarrow H^2(\mathrm{Gal}(E/F), H^0(E, \mathrm{ad}(\rho_f) \otimes_O \widehat{\mathfrak{q}}_E) \end{aligned}$$

This implies that $H^1(F, \mathrm{ad}(\rho_f) \otimes_O \widehat{\mathfrak{q}}_E) \cong H^1(E, \mathrm{ad}(\rho_f) \otimes_O \widehat{\mathfrak{q}}_E)^{\mathrm{Gal}(E/F)}$ since it is easily seen that $H^0(E, \mathrm{ad}(\rho_f) \otimes_O \widehat{\mathfrak{q}}_E) = 0$ from the irreducibility of the residual representation $\bar{\rho}_f|_{G_E}$. Our result follows from this fact and the previous Lemma. □

In what follows, we will consider the map

$$\mathbf{c}_E : \widehat{\mathfrak{q}}_E^* = \mathrm{Hom}_{A_E}(\widehat{\mathfrak{q}}_E, A_E) \longrightarrow H^1(F, \mathrm{ad}(\rho_f) \otimes \Lambda_E) = H^1(E, \mathrm{ad}(\rho_f))$$

sending $\phi \in \mathrm{Hom}_{A_E}(\widehat{\mathfrak{q}}_E, A_E)$ to the cohomology class of the cocycle

$$g \mapsto (\mathrm{id}_{\mathrm{sl}_2(O)} \otimes \phi) \circ \widetilde{x}_E(g).$$

3.3. Local non-triviality. Let L be a finite extension of $\mathbb{Q}_p$ with absolute Galois group G_L. We denote by $I_L \subset G_L$ the inertia subgroup of G_L and by $\mathrm{Frob}_L \in G_L$ a Frobenius element. Let S_L be the universal deformation ring representing the deformation functor $\mathcal{F}_1$ of the trivial character of G_L defined on the category of local noetherian O-algebra. Let κ_L be the corresponding universal deformation $\kappa_L : G_L \to S_L$. We also denote by Γ_L the p-torsion free part of $O_L^\times$. The following lemma is well known.

Lemma 3.3. With the previous notations, the following holds.

(i) For any O-deformation $\chi \in \mathcal{F}_1(O)$, we have a canonical isomorphism $S_L \cong O[\![T]\!][\![\Gamma_L]\!]$ with $1 + T = \kappa_L(\mathrm{Frob}_L)$ such that the map induced by reducing modulo the augmentation ideal of Γ_L and T to 0 corresponds to the character χ.

(ii) $\kappa_L|_{I_L}$ takes values in $O[\![\Gamma_L]\!]^\times$ and is induced by the Artin reciprocity map $\mathrm{Art}_L : G_L^{\mathrm{ab}} \cong L^\times$.

(iii) For any O-deformation $\chi \in \mathcal{F}_1(O)$, we have canonical isomorphisms:

$$\mathcal{F}_1(O[\epsilon](\epsilon^2)) = \mathrm{Hom}_{S_L}(\Omega_{S_L/O}, O) \cong H^1(L, O)$$

where O is seen as a S_L-module via the map $S_L \to O$ induced by χ. In particular, it is free of rank $[L : \mathbb{Q}_p] + 1$. Moreover, we have the following commutative square

$$\begin{array}{ccc} \mathrm{Hom}_{S_L}(\Omega_{S_L/O}, O) & \xrightarrow{\sim} & H^1(L, O) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{O[\![\Gamma_L]\!]}(\Omega_{O[\![\Gamma_L]\!]/O}, O) & \xrightarrow[\delta_L]{\sim} & H^1(I_L, O)^{G_L/I_L} \end{array}$$

where the horizontal arrows are isomorphism and where the left arrow is induced by the inclusion $O[\![\Gamma_L]\!] \subset S_L$ and O is an S_L-module by the character χ.

(iv) These isomorphisms are norm compatible for any extension L'/L.

Proof. This is an elementary exercise in deformation theory using the reciprocity law of Local Class Field theory. □

Let v be a place of F dividing p and let F_v be the corresponding completion. Let $\mathrm{ad}(\rho_f) = \mathcal{F}_v^- \supset \mathcal{F}_v^0 \supset \mathcal{F}_v^+$ be the G_{F_v} stable filtration with graded pieces $\mathrm{Gr}_v^-, \mathrm{Gr}_v^0$ and Gr_v^+ of rank 1 over O where

$$\mathcal{F}_v^0 = \left\{ A \in \mathrm{sl}_2(O) \,\middle|\, A = \alpha_{f,v} \begin{pmatrix} a & b \\ 0 & -a \end{pmatrix} \alpha_{f,v}^{-1} \text{ for some } a, b \in O \right\}.$$

and

$$\mathcal{F}_v^+(\mathrm{ad}(\rho_v)) = \left\{ A \in \mathrm{sl}_2(O) \,\middle|\, A = \alpha_{f,v} \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \alpha_{f,v}^{-1}, \text{ for some } b \in O \right\}.$$

On the other hand, we have an isomorphism

$$\Omega_{\Lambda_E/O} \otimes O \cong \bigoplus_{w|p} \Omega_{O[\![\Gamma_{E_w}]\!]/O} \otimes O \tag{3.3}$$

For any $\phi \in \mathrm{Hom}_{A_E}(\widehat{\mathfrak{q}}_E, A_E)$, we denote by ϕ_w, the restriction of ϕ to $\Omega_{O[\![\Gamma_{E_w}]\!]/O} \otimes O \subset \Omega_{\Lambda_E/O} \otimes O \hookrightarrow \widehat{\mathfrak{q}}_E$.

Lemma 3.4. For each $\phi \in \mathrm{Hom}_{A_E}(\widehat{\mathfrak{q}}_E, A_E)$ and each place w of E above v, we have

(i) $\mathbf{c}_w(\phi) := \mathbf{c}_E(\phi)|_{G_{E_w}} \in H^1(E_w, \mathcal{F}_v^0)$.

(ii) Let $\mathbf{c}_w^0(\phi)$ be the image of $\mathbf{c}_w(\phi)$ in $H^1(I_w, \mathrm{Gr}_v^0) \cong H^1(I_w, O)$. Then

$$\mathbf{c}_w^0(\phi) = \delta_{E_w}(\phi_w)$$

where δ_{E_w} is the isomorphism of Lemma 3.3 (iii).

Proof. Since $\rho_{\mathbf{T}_E}|_{G_w}$ is nearly ordinary, for $g \in G_w$, we have

$$\rho_{\mathbf{T}_E}(g) = \alpha_w \begin{pmatrix} * & * \\ & \Psi_w(g) \end{pmatrix} \alpha_w^{-1}$$

where thanks to (Indec), we may choose α_w so that $\alpha_{f,v} = \lambda_{f_E}(\alpha_w)$ with

$$\rho_f(g) = \alpha_{f,v} \begin{pmatrix} * & * \\ 0 & \psi_v(g) \end{pmatrix} \alpha_{f,v}^{-1} \quad \forall\, g \in I_v$$

Let $X_w \in \mathrm{sl}_2(O) \otimes_O \widehat{\wp}_{f_E}$ such that $X_w \equiv \alpha_w^{-1}\alpha_{f,v} - \mathbb{1}_2 \pmod{\widehat{\wp}_{f_E}^2}$ For all $g \in G_w$, then an elementary computation gives the equality in $\mathrm{sl}_2(O) \otimes \widehat{\wp}_{f_E}/\widehat{\wp}_{f_E}^2$:

$$\alpha_w^{-1} x(g) \alpha_w \equiv \begin{pmatrix} * & * \\ 0 & \Psi_w \psi_v^{-1}(g) \end{pmatrix} - \mathbb{1}_2 + \alpha_{f,v}^{-1}[\mathrm{ad}(\rho_f(g))(X_v) - X_v]\alpha_{f,v} \pmod{\widehat{\wp}_{f_E}^2}$$

The cocycle defined by the second term in the above equation being a co-boundary, we may assume that $X_v = 0$ in the computation. We see that from this expression and the definition of the cocycle that (i) follows. Now we look at the restriction to the inertia subgroup $I_w \subset G_w$. Recall that $\Psi_w|_{I_w}$ is the universal deformation of the character $\psi|_{I_w}$, where we see ψ as a character of I_v for v the place of F below w associated to ψ via the Reciprocity Law of Local Class Field Theory. In other words, $\Psi_w|_{I_w} = \psi_v|_{I_w}\kappa_{E_w}$. Now, for $\phi_w \in \mathrm{Hom}_O(\Omega_{O[\![\Gamma_{E_w}]\!]}, O)$, we have

$$\delta_{E_w}(\phi_w)(g) = \phi_w(\kappa_{E_w}(g) - 1)$$

where we see ϕ_w as a linear form on the augmentation ideal $I_{\Gamma_{E_w}}$ of $O[\![\Gamma_{E_w}]\!]$ via $I_{\Gamma_{E_w}} \twoheadrightarrow I_{\Gamma_{E_w}}/I^2_{\Gamma_{E_w}} \cong \Omega_{O[\![\Gamma_{E_w}]\!]/O}$. Our claim now follows easily from interpreting the last three equations in the definition of $\mathbf{c}^0_w(\phi)$ after identifying Gr^0_v with the trivial G_{F_v}-module via the map

$$\alpha_{f,v}\begin{pmatrix} -a & * \\ 0 & a \end{pmatrix}\alpha_{f,v}^{-1} \mapsto a. \qquad \square$$

The following corollary is immediate.

Corollary 3.5. The map $\bigoplus_{w|p}\mathrm{Res}_w \circ \mathbf{c}_E$ induces an isomorphism of A_E-modules:

$$\bigoplus_{w|p}\mathrm{Hom}_O(\Omega_{O[\![\Gamma_{E_w}]\!]/O}, O) \cong \bigoplus_{w|p} H^1(I_{E_w}, O)^{G_{E_w}/I_{E_w}}$$

In particular, the map $\mathbf{c}_E$ is injective.

3.4. Zeta elements. We construct zeta elements in Rubin's lattice attached to the Galois cohomology of $\mathrm{ad}(\rho_f)$. We first recall some definitions. Let A be a finite free O-algebra. Let n be a positive integer. If L is a A-module, we put following Rubin:

$$\bigcap_A^n L := \left(\bigwedge_A^n L^*\right)^*$$

where for any A-module M, we set $M^* := \mathrm{Hom}_A(M, A)$. $L \mapsto \bigcap_A^n L$ defines clearly a covariant functor from the category of A-modules to itself. In what follows we take $A = A_E = O[\mathrm{Gal}(E/F)]$. Now we consider the map we have defined in the end of Section 3.2:

$$\mathbf{c}_E : \widehat{\mathfrak{q}}_E^* = \mathrm{Hom}_{A_E}(\widehat{\mathfrak{q}}_E, A_E) \longrightarrow H^1(F, \mathrm{ad}(\rho_f) \otimes \Lambda_E) = H^1(E, \mathrm{ad}(\rho_f))$$

and apply the co-variant functor $\bigcap^d$:

$$\bigcap^d \mathbf{c}_E : \bigcap_{A_E}^d \widehat{\mathfrak{q}}_E^* \longrightarrow \bigcap_{A_E}^d H^1(E, \mathrm{ad}(\rho_f)) \tag{3.4}$$

Since $\Omega_{\Lambda_E/O} \otimes O$ is free of rank d over A_E, we can fix an isomorphism

$$\bigcap_{A_E}^d (\Omega_{\Lambda_E/O} \otimes O)^* \cong A_E \tag{3.5}$$

From the previous corollary, we get easily the following

Corollary 3.6. We have a commutative diagram

$$\begin{array}{ccc} \bigcap^d_{A_E} \mathfrak{q}_E^* & \longrightarrow & \bigcap^d_{A_E} (\Omega_{\Lambda_E/O} \otimes O)^* \cong A_E \\ {\scriptstyle \bigcap^q \mathbf{c}_E}\downarrow & & \downarrow{\scriptstyle \cong} \\ \bigcap^d_{A_E} H^1(E, \mathrm{ad}(\rho_f)) & \xrightarrow{\bigcap^q \mathrm{Res}_p} & \bigcap^d_{A_E} \bigoplus_{w|p} H^1(I_{E_w}, O)^{G_{E_w}/I_{E_w}} \end{array}$$

Proof. We just apply the covariant functor $\bigcap^q_{A_E}$ to the isomorphism of Corollary 3.5. □

We define the local Zeta elements $z_{E,\mathrm{loc}} \in \bigcap^d_{A_E} \bigoplus_{w|p} H^1(I_w, \mathrm{Gr}{*}0))^{G_{E_w}/I_{E_w}}$ the element corresponding to 1 via the isomorphism of Corollary 3.6 and (3.5). The following lemma will be useful to construct elements in $\bigcap^d_{A_E} H^1(E, \mathrm{ad}(\rho_f))$.

Lemma 3.7. Let $\widetilde{\mathfrak{q}}_E$ be the cokernel of the map $\bigcap^d_{A_E} \widehat{\mathfrak{q}}_E^* \to A_E$ induced by the exact sequence (3.1) and the isomorphism (3.5). Then we have

$$\mathrm{Fitt}_{A_E}(\widetilde{\mathfrak{q}}_E) \supset \mathrm{Fitt}_{A_E}(\wp_{f_E}/\wp_{f_E}^2).$$

Proof. Notice that for any A_E-module M, we have a canonical isomorphism $M^* = \mathrm{Hom}_{A_E}(M, A_E) \cong \mathrm{Hom}_O(M, O) =: M'$. Moreover if we have an injective homomorphism $f\colon X \to Y$ of A_E–modules which are O-free of same rank over O, inducing by duality an injective homomorphism $f'\colon Y' \to X'$. Then $\mathrm{Coker}\, f'$ and $\mathrm{Coker}\, f$ are Pontryagin dual to each other. Moreover if both X and Y are equipped with a O-linear action of a group G and if f is itself G-equivariant, then the corresponding duality isomorphism is G-equivariant. This implies that $\mathrm{Fitt}_{A_E}(\mathrm{Coker}\, f') = \mathrm{Fitt}_{A_E}(\mathrm{Coker}\, f)$. We consider now the map $f_E\colon \Omega_{\Lambda_E/O} \otimes O \to \widehat{\mathfrak{q}}_E$. We have $f_E^{**}\colon \Omega_{\Lambda_E/O} \otimes O \to (\widehat{\mathfrak{q}}_E)_f$ where $(\widehat{\mathfrak{q}}_E)_f$ is the maximal O-free quotient of $\widehat{\mathfrak{q}}_E$, and therefore

$$\mathrm{Fitt}_{A_E}(\mathrm{Coker}\, f_E^{**}) \supset \mathrm{Fitt}_{A_E}(\mathrm{Coker}\, f_E) = \mathrm{Fitt}_{A_E}(\mathfrak{q}_E) = \mathrm{Fitt}_{A_E}(\wp_{f_E}/\wp_{f_E}^2)$$

Now by the previous discussion for $f = \bigwedge^d f_E^{**}$, we have

$$\mathrm{Fitt}_{A_E}(\widetilde{\mathfrak{q}}_E) = \mathrm{Fitt}_{A_E}\left(\mathrm{Coker}\left(A_E \longrightarrow \bigwedge^d \mathfrak{q}_E^{**}\right)\right)$$

On the other hand,

$$\mathrm{Fitt}_{A_E}\left(\mathrm{Coker}\left(A_E \longrightarrow \bigwedge^d \mathfrak{q}_E^{**}\right)\right) = \mathrm{Fitt}_{A_E}(\mathrm{Coker}\, f_E^{**}).$$

The equalities and inclusion above imply our claim. □

We can now give the definition of global Zeta elements.

Definition 3.8. For any element ξ annihilating $\widetilde{\mathfrak{q}}_E$, we consider $z_{E,\xi} \in \bigcap_{A_E}^d H^1(E, \mathrm{ad}(\rho_f))$ to be the image of $\xi.1_{A_E}$ viewed as an element of $\bigcap_{A_E}^d \widehat{\mathfrak{q}}_E^*$ by the map $\bigcap^q \mathbf{c}_E$ of (3.4).

It follows immeditely from Corollary 3.6, that

$$\bigcap^d \mathrm{Res}_p(z_{E,\xi}) = \xi.z_{E,\mathrm{loc}} \tag{3.6}$$

Proof of Theorem 1.1. Notice first that when $E = F$, we have $A_E = O$ and $\bigcap_{A_E}^d = \bigwedge_O^d$. Let η_f be the O-Fitting ideal of the congruence module attached to f. Under our hypothesis, it is known that $\mathrm{Fitt}_O(\wp_f/\wp_f^2) = \eta_f$ since in that case that the Hecke algebra is complete intersection thanks to the works of Taylor–Wiles–Fujiwara–Dimitrov (see [1] for the most complete statement). Since the cohomology of the Hilbert–Blumenthal variety is free over the Hecke algebra again thanks to the work of Dimitrov [1], we have η_f is the ideal of O generated by

$$\xi_f := \frac{\Gamma(\mathrm{ad}(\rho_f), 1) L^{S_f}(\mathrm{ad}(\rho_f), 1)}{\Omega_f^{\Sigma} \Omega_f^{\Sigma_F \setminus \Sigma}} \in O$$

with the notation of Theorem 1.1.

Now, we just apply Lemma 3.7 to the case $E = F$ and the construction of Definition 3.8 with $\xi := \xi_f$. This gives the construction of the element $z_f \in \wedge_O^d H^1(F, \mathrm{ad}(\rho_f))$. The point (i) now follows from Lemma 3.4 (i), and the point (ii) is a direct consequence of the construction and of the relation (3.6). $\square$

3.5. Compatible systems for the norm map.

3.5.1. ***Construction.*** We start by the following lemma

Lemma 3.9. Let E'/F be an extension of F contained in E. Then the base change map $\mathbf{T}_E \to \mathbf{T}_{E'}$ is surjective. In particular, the induced map $\widehat{\mathfrak{q}}_E \to \widehat{\mathfrak{q}}_{E'}$ is surjective and for any $\phi \in \mathfrak{q}_E^*$, there exists a unique $\phi' \in \mathfrak{q}_{E'}^*$ such that the following diagram of A_E-module commutes

$$\begin{array}{ccccccc}
\Omega_{\Lambda_{E/O}} \otimes O & \longrightarrow & \Omega_{\mathbf{T}_E/O} \otimes_{\lambda_{f_E}} O & = & \widehat{\mathfrak{q}}_E & \xrightarrow{\phi} & A_E \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \pi_{E'}^E \\
\Omega_{\Lambda_{E'/O}} \otimes O & \longrightarrow & \Omega_{\mathbf{T}_{E'}/O} \otimes_{\lambda_{f_{E'}}} O & = & \widehat{\mathfrak{q}}_{E'} & \overset{\phi'}{\dashrightarrow} & A_{E'}
\end{array}$$

where $\pi_{E'}^E$ is induced by $\mathrm{Gal}(E/F) \to \mathrm{Gal}(E'/F)$ and the left vertical arrows are induced by the base change maps $\Lambda_E \to \Lambda_{E'}$ and $\mathbf{T}_E \to \mathbf{T}_{E'}$.

Proof. Let R_E be the universal nearly ordinary (at places dividing p) deformation ring with fixed determinant equal to $\omega\epsilon_{\mathrm{cyc}}$ of G_E unramified at places not dividing $\mathfrak{n}p$ (see Section 3.7 for the precise definition). It follows from [6, Prop. 3.1] that $R_E \to R_{E'}$ is surjective. Since for every extension E, the canonical map $R_E^{\phi} \to \mathbf{T}_E$ is surjective, we deduce that $\mathbf{T}_E \to \mathbf{T}_{E'}$ is surjective. This implies immediately that $\widehat{\mathfrak{q}}_E \to \widehat{\mathfrak{q}}_{E'}$ is surjective. On the other hand, we see easily from the exact sequence (3.1) for E and E', that the kernel of the surjective map $\widehat{\mathfrak{q}}_E \otimes_{A_E} A_{E'} \to \widehat{\mathfrak{q}}_E$ is O-torsion. We deduce easily the rest of the Lemma from the surjectivity and that last observation. $\square$

By the previous lemma, $\phi \mapsto \phi'$ defines a A_E-linear map

$$\widehat{\pi}^E_{E'} : \widehat{\mathfrak{q}}^*_E \longrightarrow \widehat{\mathfrak{q}}^*_{E'}$$

where the action of A_E onto $\mathfrak{q}_{E'}$ is given via the ring homomorphism $\pi^E_{E'}$. Similarly, we have a map $(\Omega_{\Lambda_{E/O}} \otimes O)^* \to (\Omega_{\Lambda_{E'/O}} \otimes O)^*$ that we denote $\widehat{\pi}^E_{E'}$ too. We deduce that we have a canonical diagram

$$\begin{array}{ccccccc} (\Omega_{\Lambda_{E/O}} \otimes O)^* & \longleftarrow & \widehat{\mathfrak{q}}^*_E & \xrightarrow{\mathbf{c}_E} & H^1(F, \mathrm{ad}(\rho_f) \otimes A_E) & = & H^1(E, \mathrm{ad}(\rho_f)) \\ \downarrow{\scriptstyle \hat{\pi}^E_{E'}} & & \downarrow{\scriptstyle \hat{\pi}^E_{E'}} & & \downarrow{\scriptstyle \mathrm{id}_{\mathrm{ad}(\rho_f)} \otimes \pi^E_{E'}} & & \downarrow{\scriptstyle N^E_{E'}} \\ (\Omega_{\Lambda_{E'/O}} \otimes O)^* & \longleftarrow & \widehat{\mathfrak{q}}^*_{E'} & \xrightarrow{\mathbf{c}_{E'}} & H^1(F, \mathrm{ad}(\rho_f) \otimes A_{E'}) & = & H^1(E', \mathrm{ad}(\rho_f)) \end{array} \tag{3.7}$$

where $N^E_{E'}$ stands for the natural corestriction or norm map. We can now choose the isomorphisms (3.5) so that for any extension E/E' as before, the following diagram commutes.

$$\begin{array}{ccc} \bigcap^d_{A_E} \Omega_{\Lambda_{E/O}} \otimes O & \xrightarrow{\sim} & A_E \\ \downarrow & & \downarrow{\scriptstyle \pi^E_{E'}} \\ \bigcap^d_{A_E} \Omega_{\Lambda_{E/O}} \otimes O & \xrightarrow{\sim} & A_{E'} \end{array}$$

The following lemma results from the above discussion.

Lemma 3.10. With the previous notations, we have canonical commutative diagram:

$$\begin{array}{ccccc} A_E & \longleftarrow & \bigcap^d_{A_E} \widehat{\mathfrak{q}}^*_E & \xrightarrow{\bigcap^q \mathbf{c}_E} & \bigcap^d_{A_E} H^1(E, \mathrm{ad}(\rho_f)) \\ \downarrow{\scriptstyle \pi^E_{E'}} & & \downarrow & & \downarrow{\scriptstyle \bigcap^q N^E_{E'}} \\ A_{E'} & \longleftarrow & \bigcap^d_{A_{E'}} \widehat{\mathfrak{q}}^*_{E'} & \xrightarrow{\bigcap^q \mathbf{c}_{E'}} & \bigcap^d_{A_{E'}} H^1(E', \mathrm{ad}(\rho_f)) \end{array}$$

3.5.2. *Compatible systems.* The goal of this section is to now explain the definition and construction of compatible systems under the norm map of zeta elements and Galois cohomology classes. We start by defining a class of sets of abelian extensions of F.

Definition 3.11. Let S be a finite set of places of F. A set $\mathfrak{F}$ of abelian extensions of F is said S-admissible if

(i) For all $E \in \mathfrak{F}$, E/F is unramified at places in S,

(ii) For any $E, E' \in \mathfrak{F}$ with $E \subset E'$, if the index of ramification in E' of a place w of E is divisible by p, then E' contains the maximal p-abelian extension of F unramified away from the places of F dividing w.

For E'/E as in (ii), we denote by $\mathrm{Ram}(E'/E)$ the set of place of F which ramify in E' but not in E.

In the next sections, we will discuss how one can choose elements ξ satisfying the condition of Definition 3.8 in an optimal way when E varies in an S-admissible set of abelian extension of F. We will explain how this is connected to the existence of an Euler system of rank d whose definition we now recall.

Definition 3.12. Let S be a finite set of primes of F and (ρ_V, V) be a p-adic representation of G_F which is unramified away from finitely many primes S and p:

$$\rho_V : G_F \longrightarrow \mathrm{GL}_L(V)$$

Let $\mathfrak{F}$ an S-admissible set of abelian extensions of F. An Euler system of rank d for a G_F-stable O-lattice $T \subset V$ is a collection of elements $(z_E)_{E\in\mathfrak{F}}$ with $z_E \in \bigcap_{A_E} H^1(E,T)$ for each $E \in \mathfrak{F}$ such that for any $E', E \in \mathfrak{F}$ with $E' \supset E$, we have:

$$\bigcap_{A_E}^{q} \mathrm{Cores}_E^{E'}(z_{E'}) = \prod_{v\in\mathrm{Ram}(E'/E)} P_v(q_v^{-1}\sigma_{v,E}; V) \cdot z_E$$

where

- $\bigcap_{A_E}^{q} \mathrm{Cores}_E^{E'}$ is the norm map induced by the corestriction map in Galois cohomology
$$\bigcap_{A_E}^{q} \mathrm{Cores}_E^{E'} : \bigcap_{A_{E'}}^{q} H^1(E', T) \longrightarrow \bigcap_{A_E}^{q} H^1(E, T),$$
- $P_v(X; V) = \det_V(1 - X\rho_V(\mathrm{Frob}_v))$,
- $\sigma_{v,E}$ is the geometric Frobenius in $\mathrm{Gal}(E/F)$ at v.

Definition 3.13. Let $\mathfrak{F}$ as above. A compatible system of equivariant numbers for V, S and $\mathfrak{F}$ is the data of elements $\xi_E \in A_E$ for each $E \in \mathfrak{F}$ such that for any $E', E \in \mathfrak{F}$ with $E' \supset E$, we have:

$$N_{E'/E}(\xi_{E'}) = \prod_{v \in \mathrm{Ram}(E'/E)} P_v(q_v^{-1}\sigma_{v,E}; V) \cdot \xi_E$$

where $N_{E'/E}$ is the natural projection map $A_{E'} \to A_E$,

For $V = \mathrm{ad}(\rho_f)$, we say that such a system is a compatible system of congruence numbers (resp. of congruence anihilators) if $\xi_E \in \mathrm{Fitt}_{A_E}(\wp_E/\wp_E^2)$ (resp. if ξ_E annihilates $\wp_E/\wp_E^2$) for each $E \in \mathfrak{F}$.

An typical example of compatible system of elements in A_E as in the definition above is given by the equivariant L-values as defined in Section 2.4.

Corollary 3.14. If $(\xi_E)_{E\in\mathfrak{F}}$ is a compatible system of congruence numbers, then $(z_{E,\xi_E})_{E\in\mathfrak{F}}$ is an Euler system of rank d.

Proof. It is a straightforward consequence from the definitions, the construction and Lemma 3.10. □

For any $h_E = \sum_{w|p} h_w \in \bigoplus_{w|p} H^1(I_{E_w}, O)^{G_{F_w}/I_{F_w}}$, let us denote by ℓ_{h_E} the O-linear form on $\Omega_{\Lambda_E/O} \otimes O \to O$ induced by the isomorphisms of Lemma 3.3 (iii). The following corollary gives many Euler systems with prescribed conditions at each place dividing p.

Corollary 3.15. If $(\xi_E)_{E\in\mathfrak{F}}$ is a compatible system of congruence annihilators, then for any $h := (h_E)_E \in \varprojlim_E \bigoplus_{w|p} H^1(I_{E_w}, O)^{G_{F_w}/I_{F_w}}$, the system of classes $(c_E^h)_{E\in\mathfrak{F}}$ defined by

$$c_E^h := \mathbf{c}_E(\xi_E \cdot \ell_{h_E}) \qquad \forall\, E \in \mathfrak{F}$$

defines an Euler system of rank 1. Moreover, for each place w of E dividing p above the place v of F, we have $\mathrm{res}_w(c_E^h) \in H^1(I_{E_w}, \mathcal{F}_v^0)$ and its image $res_w^0(c_E^h)$ by the map $H^1(E_w, \mathcal{F}_v^0) \to H^1(I_w, \mathrm{Gr}_v^0) = H^1(I_w, O)$ is given by

$$res_w^0(c_E^h) = \xi_E \cdot h_w.$$

Proof. We just need to ensure the definition is meaningful. The norm relations on the classes will follow from the definitions and the diagram (3.7). For each $E \in \mathfrak{F}$, let t_E be the image in $\mathfrak{q}_E$ of the O-torsion submodule of $\widehat{\mathfrak{q}}_E$. From the arguments of the beginning of the proof of Lemma 3.7, we have an exact sequence

$$o \longrightarrow \widehat{\mathfrak{q}}_E^* \longrightarrow (\Omega_{\Lambda_E/O} \otimes O)^* \longrightarrow ((\mathfrak{q}_E)/t_E)^\vee \longrightarrow 0$$

where $*$ means O-dual and $\vee$ means Pontrjagin dual. Since ξ_E annihilates $\mathfrak{q}_E$, we deduce that $\xi_E \cdot \ell_{h_E} \in \widehat{\mathfrak{q}}_E^*$ and therefore we can evaluate $\mathbf{c}_E$ on this elements. The local property follows from the discussions of Corollary 3.4 and the discussion preceding it. □

3.6. Variants.

3.6.1. ***Iwasawa theory variant.*** We now give a variant of the constructions made in the previous section. For any field totally real field E as in the previous section, we denote by E_{cyc} the cyclotomic $\mathbb{Z}_p$-extension of E, and by $E_n \subset E_{\mathrm{cyc}}$ the intermediate extension of degree p^n over E. We define the objects A_E^{Iw}, $\widehat{\mathfrak{q}}_E^{\mathrm{Iw}}$, $\mathfrak{q}_E^{\mathrm{Iw}}$, $\mathbf{c}_E^{\mathrm{Iw}}$, $H^1_{\mathrm{Iw}}(E, \cdot\,)$ as the projective limit over n for the norm or corestriction maps of the objects A_{E_n}, $\widehat{\mathfrak{q}}_{E_n}$, $\mathfrak{q}_{E_n}$, c_{E_n}, $H^1_{\mathrm{Iw}}(E_n, \cdot\,)$. It is straightforward to prove by taking projective limits to see that the results and constructions of the previous section go *mutatis mutandis* with the obvious modifications.

3.6.2. ***Hida family variant.*** Similarly we can define the objects and constructions of the previous sections and its Iwasawa theory variant by replacing f by families of modular forms. We leave this to the interested reader.

3.7. Deformations rings and Selmer groups. The goal of this section is to recall the relation between the modules $\mathfrak{q}_E$'s and Selmer groups in orders to exhibit a system of annihilators of congruence modules.

Let LCN_O be the category of local noetherian complete O-algebras with residue field $\kappa = O/\varpi_O$. Let E/F as in the previous sections. We denote by $\mathrm{Ram}(E/F)$ the places of F which are ramified in the extension E/F. Recall that S is a finite set of places of F containing those dividing $\mathfrak{n}p$ and $\mathrm{Ram}(E/F)$. We consider the deformation functor:

$$\mathcal{F}_E^S\colon LCN_O \longrightarrow \mathrm{Set}$$

where $\mathcal{F}_E^S(A)$ is the set of strict equivalence classes of deformations

$$\rho\colon G_E \longrightarrow \mathrm{GL}_2(A)$$

such that

- $\rho \pmod{\mathfrak{m}_A} \cong \rho_f \pmod{\varpi_O}$,
- ρ is unramified at places not dividing those in $S \cup \{p\}$,
- ρ is nearly ordinary at each place w above p
- $\det \rho = \omega\epsilon_{\mathrm{cyc}}^{-1}$

It is well known by now classical arguments originally due Mazur, that this functor is representable by a local complete noetherian O-algebra R_E. See for example [6]. Let $\lambda_E\colon R_E \to O$ be the homomorphism corresponding to the Galois representation ρ_f, and $\wp_{R_E} := \mathrm{Ker}(\lambda_E)$. It is proved in loc. cit. using again arguments due to Mazur that there is an isomorphism

$$\mathrm{Hom}_O(\wp_{R_E}/\wp_{R_E}^2, K/O) \cong \mathrm{Sel}_S(E, \mathrm{ad}(\rho_f))$$

where $\mathrm{Sel}_S(E, \mathrm{ad}(\rho_f)$ is the Selmer group defined as the kernel of the following restriction map:

$$H^1(\mathrm{Gal}(\overline{\mathbb{Q}}/E), \mathrm{ad}(\rho_f) \otimes K/O)$$
$$\longrightarrow \prod_{v \notin S_E} H^1(I_v, \mathrm{ad}(\rho_f) \otimes K/O) \oplus \bigoplus_{v|p} H^1(I_v, \mathrm{ad}(\rho_v)/\mathcal{F}_v^+(\mathrm{ad}(\rho_v)) \otimes K/O)$$

For any extension E/F which is unramified above places outside S and those dividing p, we extend the definition by defining $\mathrm{Sel}_S(E, \mathrm{ad}(\rho_f))$ to be the inductive limit of $\mathrm{Sel}_S(E', \mathrm{ad}(\rho_f))$ when E runs in the set of finite extension of F contained in E. In particular, we can define

$$\mathrm{Sel}_S(E^{\mathrm{cyc}}, \rho_f) := \varinjlim_n \mathrm{Sel}_S(E_n, \rho_f)$$

It follows from result of Taylor–Wiles–Fujiwara and Hida [6], that this Selmer group is co-torsion over A_W^{cyc}. We denote by $\mathcal{L}_{E,f}^{S,\mathrm{alg}}$ the A_E^{cyc}-Fitting ideal of the Pontryagin dual of $\mathrm{Sel}_S(E^{\mathrm{cyc}}, \rho_f)$.

Lemma 3.16. Let E and S as above, then we have $\mathcal{L}_{E,f}^{S,\mathrm{alg}} \subset \mathrm{Fitt}_{A_E^{\mathrm{cyc}}}(\mathfrak{q}_E^{\mathrm{Iw}})$.

Proof. For each integer n, by the universal property of R_{E_n}, we have a surjective[3] homomorphism:

$$\psi_n \colon R_{E_n} \twoheadrightarrow \mathbf{T}_{E_n}$$

such that $\lambda_{E_n} := \lambda_{f_{E_n}} \circ \psi_n$. By surjectivity of ψ_n, we have a surjection of A_{E_n}-torsion modules:

$$\wp_{R_{E_n}}/\wp_{R_{E_n}}^2 \twoheadrightarrow \wp_{f_{E_n}}/\wp_{f_{E_n}}^2 = \mathfrak{q}_{E_n}$$

By passing to the projective limit over n, we deduce a surjective homomorphism of A_E^{cyc}-torsion modules:

$$\mathrm{Sel}_S(E^{\mathrm{cyc}}, \mathrm{ad}(\rho_f))^\vee \twoheadrightarrow \mathfrak{q}_E^{\mathrm{Iw}}$$

which implies our claim. □

3.8. Proof of the main Theorems. Using the Iwasawa version of Corollary 3.14 and 3.15, it just suffices to construct compatible system of congruence number in A_E^{Iw}. We will do so using Lemma 3.16. We start by the following elementary lemma.

Lemma 3.17. For each $E \in \mathfrak{F}$ like in Definition 3.13, let $I_E \subset A_E^{\mathrm{Iw}}$ be a non zero ideal such that for any $E', E \in \mathfrak{F}$ such that $E \subset E'$, we have $\pi_{E',E}(I_{E'}) = Q_{E',E} I_E$ for some $Q_{E',E} \in A_E^{\mathrm{Iw}}$ satisfying

- $Q_{E',E}$ is not a zero divisor in A_E^{Iw}

[3]The surjectivity is a classical fact obtained using the local properties of the Galois representation $\rho_{\mathbf{T}_{E_n}}$.

- For any $E, E', E'' \in \mathfrak{F}$ with $E \subset E' \subset E''$, we have

$$Q_{E'',E} = \pi_{E'',E'}(Q_{E'',E'})Q_{E',E}.$$

Then there exists a non trivial compatible system of elements $(\xi_E)_{E\in\mathfrak{F}}$ such that $\xi_E \in I_E$ for each $E \in \mathfrak{F}$ satisfying the relation $\pi_{E',E}(\xi_{E'}) = Q_{E',E}.\xi_E$ for any $E', E \in \mathfrak{F}$ such that $E \subset E'$, for any choice of $\xi_F \in I_F$.

Proof. For each E', E as above let $\phi_{E',E} \colon I_E \to I_{E'}$ such that $\pi_{E',E}(x) = Q_{E',E}\phi_{E',E}(x)$ for any $x \in I_E$. This is well defined since we assumed that the $Q_{E',E}$'s are not zero divisors. Moreover $\phi_{E',E}$ are surjective by hypothesis, and they define a projective system for $(I_E)_{E\in\mathfrak{F}}$ with surjective transition maps. Therefore we have a surjection

$$\varprojlim_{E\in\mathfrak{F},\phi.} I_E \twoheadrightarrow I_F$$

which implies our claim. □

Proposition 3.18. Let $\mathfrak{F}$ be a S-admissible set of abelian extension of F. Let $E', E \in \mathfrak{F}$ with $E \subset E'$ as above. Then we have

$$\pi_{E',E}(\mathcal{L}^{S,\mathrm{alg}}_{E',f}) = \prod_{v\in\mathrm{Ram}(E'/E)} P_v(q_v^{-1}\sigma^{\mathrm{Iw}}_{v,E}) \cdot \mathcal{L}^{S,\mathrm{alg}}_{E,f}$$

where

- $P_v(X, \mathrm{ad}(\rho_f)) := \det(1 - X.\,\mathrm{ad}(\rho_f)(\mathrm{Frob}_v))$,
- $\sigma^{\mathrm{Iw}}_{v,E}$ is the image of a geometric Frobenius automorphism at v in $\mathrm{Gal}(E_{\mathrm{cyc}}/F) \subset (A^{\mathrm{Iw}}_E)^\times$.

Proof. By induction on the cardinality of the set $\mathrm{Ram}(E'/E)$, it is sufficient to prove the result in the following two cases:

(a) $\mathrm{Ram}(E'/E) = \emptyset$,
(b) $\mathrm{Ram}(E'/E) = \{v_0\}$,

The case (a) follows from the isomorphism below which is valid for any finite set S containing the places dividing the level of f

$$\mathrm{Sel}_S(E'_{\mathrm{cyc}}, \mathrm{ad}(\rho_f)) = \mathrm{Sel}_S(E_{\mathrm{cyc}}, \mathrm{ad}(\rho_f))^{\mathrm{Gal}(E'/E)}$$

which follows easily from the inflation-restriction exact sequence since $\mathrm{ad}(\bar\rho_f)$ has no invariants by $G_{E'}$, and because the ramification conditions are the same on both Selmer groups since E'/F is unramified at all places where E/F is (see [6] for example). Moreover a class in $H^1_{\mathrm{Iw}}(E, \rho_f)$ which is unramified at a place w' after restriction to E' is already unramified at the place w of E below w' since the ramification index is prime to p, by the hypothesis on the set $\mathfrak{F}$.

For the case (b), we may assume that $E' = EF_0$ where F_0 is the p-maximal abelian extension of F unramified away from v_0. In that case, we have

$$\mathrm{Sel}_{S\cup\{v_0\}}(E_{\mathrm{cyc}}, \mathrm{ad}(\rho_f)) = \mathrm{Sel}_S(E'_{\mathrm{cyc}}, \mathrm{ad}(\rho_f))^{\mathrm{Gal}(E'/E)}$$

We argue as in the isomorphism used for (a). We only need to show that the local condition of a class in $H^1_{\mathrm{Iw}}(E, \mathrm{ad}(\rho_f))$ that is ramified possibly at places dividing v_0 becomes unramified after restriction to E'. This is obvious since E' contains the maximal p-abelian extension of F that is unramified away from v_0. From the previous isomorphism, we deduce that

$$\pi_{E',E}(\mathcal{L}^{S,\mathrm{alg}}_{E,f}) = \mathcal{L}^{S\cup\{v_0\},\mathrm{alg}}_{E',f} \tag{3.8}$$

Now consider the exact sequence

$$\begin{aligned} 0 \longrightarrow \mathrm{Sel}_S(E_{\mathrm{cyc}}, \mathrm{ad}(\rho_f)) \longrightarrow \mathrm{Sel}_{S\cup\{v_0\}}(E_{\mathrm{cyc}}, \mathrm{ad}(\rho_f)) \\ \longrightarrow H^1(I_{v_0}, \mathrm{ad}(\rho_f) \otimes (A_E^{\mathrm{Iw}})^\vee)^{G_{v_0}/I_{v_0}} \longrightarrow 0 \end{aligned}$$

The left exactness follows from the definition of the ramification conditions of the Selmer group on the left and the fact due to Shapiro's lemma that

$$\varinjlim_n H^1(E_n, \mathrm{ad}(\rho_f) \otimes K/O) = H^1(F, \mathrm{ad}(\rho_f) \otimes_O (A_E^{\mathrm{Iw}})^\vee)$$

and

$$\begin{aligned} &\bigoplus_{w|v_0} (H^1(I_w, \mathrm{ad}(\rho_f) \otimes \Lambda^\vee)^{D_v/I_v})^\vee \\ &\qquad = H^1(I_{v_0}\, \mathrm{ad}(\rho_f) \otimes O[\mathrm{Gal}(F_{\mathfrak{m}}/F)] \otimes \Lambda^\vee)^{D_{\mathfrak{q}}/I_{\mathfrak{q}}})^\vee \end{aligned}$$

where w runs through the set of places of E above v_0. The exactness on the right is a Theorem of Greenberg–Vatsal [3] see also [2, §4.2], from which we get

$$\mathcal{L}^{S\cup\{v_0\},\mathrm{alg}}_{E,f} = P_v(q_{v_0}^{-1}\sigma^{\mathrm{Iw}}_{v_0,E}) \cdot \mathcal{L}^{S,\mathrm{alg}}_{E,f}. \tag{3.9}$$

Combining (3.8) and (3.9), we get (b). □

Proof of Theorem 1.2 and 1.3. Note that Lemma 3.17 applies with $I_E = \mathcal{L}^{S,\mathrm{alg}}_{E,f}$ as it is straightforward to see that $P_v(q_v^{-1}\sigma_{v,E})$ is not a zero divisor in A_E^{Iw} since its image on every irreducible component of A_E^{Iw} is clearly non-zero. By Proposition 3.18 and Lemma 3.17, there exists therefore a compatible system of congruence numbers $(\xi_E)_{E\in\mathfrak{F}}$ such that $\xi_F = \mathcal{L}^{S,\mathrm{alg}}_f \subset \Lambda = A_F^{\mathrm{Iw}} = O[\![\mathrm{Gal}(F_{\mathrm{cyc}}/F)]\!]$. It now suffices to apply the Iwasawa variant of Corollaries 3.14 and 3.15 to this system, and we are done. □

Proof of Theorem 1.5 and 1.7. The Conjectures 1.4 and 1.6 state that $\mathcal{L}^{S,\mathrm{an}}_{E,f}$ is respectively a compatible system of congruence numbers and of congruence annihilators. It therefore suffices to apply the Iwasawa variant of Corollaries 3.14 and 3.15 to this system. □

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