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par Marc-Hubert NICOLE et Giovanni ROSSO

Résumé. Dans la première partie, nous revenons sur les courbes modulaires de Drinfeld associée à GL(2) en adoptant le point de vue perfectoïde, et nous montrons comment récupérer une portion (perfectisée) de la théorie des formes modulaires de Drinfeld π-adiques surconvergentes. Dans la seconde partie, nous présentons quelques problèmes ouverts portant sur les familles de formes modulaires de Drinfeld pour GL(n).

Abstract. In the first part, we revisit Drinfeld modular curves associated to GL(2) from the perfectoid point of view, and we show how to recover (a perfectized) part of the theory of overconvergent π-adic Drinfeld modular forms. In the second part, we review open problems for families of Drinfeld modular forms for GL(n).

1. Introduction

Let $\mathcal{C}$ be a projective smooth geometrically irreducible curve over $\mathbb{F}_q$, $F = \mathbb{F}_q(\mathcal{C})$ its function field, ∞ an $\mathbb{F}_q$-rational point and A the ring of regular functions outside ∞. Fix an A-ideal $\mathfrak{N}$ that we can assume to be principal and a prime ideal $\mathfrak{p}$ of norm q^d, coprime with $\mathfrak{N}$. Let π be a generator of $\mathfrak{p}$ in $A_{\mathfrak{p}}$.

In our previous paper [28], we explained how to adapt the eigenvariety machinery to the non-noetherian context of Drinfeld modular varieties associated to GL(n) for $n \in \mathbb{N}$, including Hida theory in the form of an analogue of the Vertical Control Theorem, a continuous analogue of Coleman's finite slope families and a classicality theorem of overconvergent Drinfeld modular forms. This led to a variety of open questions, some intrinsic to the set-up of Drinfeld modular forms.

In the first part of this paper, we illustrate in detail that the perfectoid approach to Shimura varieties pioneered by Scholze [33] also works well for Drinfeld modular curves associated to GL(2) (where there are no technical difficulties at the boundary, exactly as for classical modular curves). This is groundwork towards an alternative treatment of overconvergent Drinfeld modular forms following Chojecki–Hansen–Johansson [10] and its generalisation to the Hilbert setting [6]. At this point, we are only able to recover

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their defining sheaves over the perfection. Further, treating higher dimensions presents a non-trivial technical challenge due to a more complicated boundary and the absence of Tate traces.

After reviewing the theory of Drinfeld modules and the Hodge–Tate–Taguchi map, we first show that there exists an infinite level Drinfeld modular curve which is a perfectoid space; we follow closely the construction of Scholze, first constructing an anti-canonical tower of a strict neighbourhood of the ordinary locus and then using the Hodge–Tate–Taguchi map to extend it to the whole Drinfeld modular curve, thus proving

Theorem (Theorem 2.18)**.** *Let $\mathcal{X}(\pi^m)$ be the Drinfeld modular curve of full level $\mathfrak{p}^m$. There exists a preperfectoid space*

$$\mathcal{X}_\infty \sim \varprojlim_m \mathcal{X}(\pi^m).$$

This space is equipped with a $\mathrm{GL}_2(F_\mathfrak{p})$-equivariant Hodge–Tate–Taguchi map of adic spaces

$$\Pi_{\mathrm{HTT}} : \mathcal{X}_\infty \to \mathbb{P}^1.$$

Here we use the definition of preperfectoid of [24, Definition 3.7.1 (a)] meaning that the inverse limit in the theorem becomes perfectoid after extension of scalars to $A_\mathfrak{p}[\![\pi^{1/p^\infty}]\!][1/\pi]$.

Then, given an analytic weight $s \in \mathbb{Z}_p$ we would like, following [10], to define overconvergent Drinfeld modular forms in a concrete fashion as functions on a subset of $\mathcal{X}_\infty$ which satisfy the transformation formula

$$\gamma^* f = (b\mathfrak{z} + d)^{-s} f, \tag{1.1}$$

for all $\gamma \in \Gamma_0(\mathfrak{p}) \subset \mathrm{GL}_2(A_\mathfrak{p})$ and where $\mathfrak{z}$ is a so-called fake Hasse invariant i.e., it is the pullback via the Hodge–Tate–Taguchi map of the coordinate z on $\mathbb{P}^1$. More precisely, let $\mathcal{X}_0(\pi)(v)$ be a strict neighbourhood of the ordinary locus of Drinfeld modules for which the Hasse invariant is bounded by v. We construct a line bundle ω^s on the counterimage of $\mathcal{X}_0(\pi)(v)$ in $\mathcal{X}_\infty$, consisting of functions satisfying the above formula (1.1) and we can only show that this sheaf is the pullback from $\mathcal{X}_0(\pi)(v)$ of *the perfection of* the sheaf of overconvergent Drinfeld modular forms of weight s defined in [28]. In the second part of this note, we treat a variety of open problems of widely varying level of difficulty in some detail. In brief: a conjectural $r = t$ theorem; asking for a better definition of the Fredholm determinant in the non-noetherian context; asking about families of generalized modular forms for Anderson motives; the study of slopes à la Gouvêa–Mazur in higher rank; classicity in infinite slope (an example of a problem arising only for Drinfeld modular forms).

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2. Perfectoid Drinfeld modular curves

Let $\mathcal{X} = \mathcal{X}(\mathfrak{N})$ be the compactified Drinfeld modular curve of full level $\mathfrak{N}$ seen as an adic space over $\mathrm{Spa}(F_{\mathfrak{p}}, A_{\mathfrak{p}})$.

The main theorem of the section is the following:

Theorem 2.1. *There exists a preperfectoid space*

$$\mathcal{X}_\infty \sim \varprojlim_n \mathcal{X}(\pi^m)$$

equipped with a natural map $\mathrm{GL}_2(F_{\mathfrak{p}})$*-equivariant Hodge–Tate–Taguchi period map to* $\mathbb{P}^1$*.*

As in [10] we shall use this map to define overconvergent π-adic modular forms of p-adic weight s as functions on $\mathbb{P}^1$ satisfying the usual transformation property

$$f|_s\gamma(z) = j(\gamma, z)^s f(z),$$

for a π-adic cocycle $j(\,\cdot\,, z)$. The proof of the theorem follows the lines of [33]: we first construct a perfectoid anti-canonical tower over a strict neighbourhood of the ordinary locus using the fact that we have a map

$$\mathcal{X}(q^{-dm}v) \longrightarrow \mathcal{X}_0(\pi^m).$$

On points, the map sends a rank two Drinfeld module φ to $(\varphi/C_m, \varphi[\pi^m]/C_m)$, for C_m the canonical subgroup of level m. The remarkable feature of this map is that the Hasse invariant of φ/C_m is the Hasse invariant of φ multiplied by q^d. Hence, the overconvergence radius on the image is constant independent of m.

This allows us to construct an intermediate perfectoid object $\mathcal{X}_{0,\infty}(v)$ over $\mathcal{X}(v)$. Then we use the purity theorem to go from level Γ_0 to full level without much ado, as we are working over Drinfeld modular curves. The map from level Γ_1 to level Γ_0 is generally not étale on the boundary in higher dimension. We follow Scholze's strategy for modular curves.

Remark 2.2. The same construction should work for general rank $r-1$, but studying the boundary becomes trickier. The Hodge–Tate–Taguchi map will take values in the flag variety parameterizing flags with blocks of size $r-1$ and 1, which is isomorphic to $\mathbb{P}^{r-1}$. One can then define overconvergent Drinfeld modular forms as functions on preimages of neighbourhoods of $\mathbb{P}^{r-1}(F_{\mathfrak{p}})$ inside $\mathbb{P}^{r-1}$ satisfying suitable transformation properties, exactly as in the analytic case [4].

2.1. Reminder on canonical subgroups and the Hodge–Tate–Taguchi decomposition. Fix a lift Ha of the Hasse invariant as in [28, Section 4] and let Ha be the truncated valuation of the Hasse invariant. For $v \in \mathbb{Q} \cap [0,1]$ let $\mathcal{X}(v)$ be a strict neighbourhood of the ordinary locus of Drinfeld modules for which $\mathrm{Ha} \leq v$. Fix a formal model $\mathfrak{X}(v)$ for $\mathcal{X}(v)$ obtained as an open of in admissible blow-up of the formal model $\mathfrak{X}$ of $\mathcal{X}$. Given a (formal) Drinfeld module φ, we can take its Taguchi dual $\varphi^D := \mathrm{Hom}(\varphi, \mathrm{CH})$, where CH denotes the Carlitz–Hayes module, the unique (formal) Drinfeld module of rank 1 and good reduction at $\mathfrak{p}$.

There is a Hodge–Tate–Taguchi map

$$\mathrm{HTT}^D_{\varphi,m} : \varphi^D[\mathfrak{p}^n](\overline{K}) \to \omega_\varphi/\pi^n\mathcal{O}_{\bar{K}}$$

sending a torsion point $x_m \in \mathrm{Hom}(\varphi, \mathrm{CH})$ to $x_m^* \mathrm{d}z$, for $\mathrm{d}z$ the canonical differential on CH. We have a so-called dual version of it (without using a base of ω_{CH}):

$$\mathrm{HTT}_\varphi : T_{\mathfrak{p}}(\varphi) \to \mathrm{Lie}(\varphi^D)^\vee.$$

Indeed, by definition of the Tate module, any $x \in T_{\mathfrak{p}}(\varphi)$ can be seen as a map $F_{\mathfrak{p}}/A_{\mathfrak{p}} \to \varphi$. There is a dual map $x^D : \varphi^D \to \mathrm{CH}$, which defines a map

$$\mathrm{HTT}_\varphi(x) = \mathrm{Lie}(x^D) \in \mathrm{Lie}(\varphi^D)^\vee.$$

We recall the following theorem [28, Théorème 4.4].

Theorem 2.3. *Let $m \geq 1$ be a positive integer. Let $v \in \mathbb{Q} \cap [0,1]$ such that $v < \frac{1}{2q^{d(m-1)}}$.*

(i) *The $\mathfrak{p}^m$-torsion of the universal family of generalised Drinfeld module $(\overline{\mathcal{E}}, \bar{\varphi})$ over $\mathfrak{X}(v)$ (see [29, Theorem 4.2]) has a canonical subgroup $C_{\bar{\mathcal{E}},m}$ of 1, dimension 1 and level m;*

(ii) *For all formal open $\mathrm{Spf}(R)$ of $\mathfrak{X}(v)$, the linearisation of the Hodge–Tate–Taguchi map*

$$\mathrm{HTT} : C^D_{\bar{\mathcal{E}},m}(\overline{R}) \otimes \overline{R}/\pi^m\overline{R} \to \omega_{C_{\bar{\mathcal{E}},m}} \otimes \overline{R}/\pi^m\overline{R}$$

has cokernel killed by π^ϑ, for all $\vartheta \in \mathbb{Q}_{>0}$ such that $\vartheta \geq \frac{v}{q^d-1}$. Here $\overline{R}$ is the integral closure of R in an algebraic closure of its fraction field.

We now prove a useful lemma:

Lemma 2.4. Let $x = (\mathcal{E}_x, \varphi_x)$ be a Drinfeld module and suppose that $\mathrm{ha}(x) = v \leq \frac{1}{q^d+1}$. Then $\mathrm{ha}((\mathcal{E}_x/C_{\bar{\mathcal{E}},1}, \varphi_x/C_{\bar{\mathcal{E}},1})) = q^d v$.

Proof. As $\frac{1}{q^d+1} \leq \frac{1}{2}$ we have a canonical subgroup and the proof is, exactly as in the case of elliptic curves, a study of the Newton polygon, see [23, Theorem 3.10.7]. □

Let $(\mathcal{E}, \varphi)$ be a Drinfeld module over an algebraically closed and complete field $K/F_{\mathfrak{p}}$. We will define the Hodge–Tate–Taguchi decomposition of $T_{\mathfrak{p}}(\varphi)$.

Theorem 2.5. *Let φ be a Drinfeld module over K of rank two, and let φ^D its Taguchi dual. We have a surjective map*

$$T_{\mathfrak{p}}(\varphi^D) \otimes_{A_{\mathfrak{p}}} \overline{K} \to \mathrm{Lie}(\varphi)^{\vee} \otimes_K \overline{K}$$

which induces a line $\mathrm{Lie}(\varphi_{\bar{K}})$ *inside* $T_{\mathfrak{p}}(\varphi) \otimes_{A_{\mathfrak{p}}} \overline{K}$.

Proof. Recall that $\mathcal{X}$ admits a proper model $\mathfrak{X}$ over $\mathrm{Spa}(A_{\mathfrak{p}}, A_{\mathfrak{p}})$ equipped with the universal family of generalised Drinfeld modules [28, Théorème 2.6]. Given that φ corresponds to a K-point of $\mathcal{X}$, by properness we can lift this to an $\mathcal{O}_K$-point of $\mathfrak{X}$ and it will correspond to a generalised Drinfeld module $\widetilde{\varphi}$ over $\mathcal{O}_K$ whose generic fiber is φ.

If $\widetilde{\varphi}$ corresponds to a point in $\mathfrak{X}(v)$, then let x be a lift to $T_{\mathfrak{p}}(\varphi^D)$ of a generator of its canonical subgroup $C^D_{\widetilde{\varphi},m}(\overline{K})$. By Theorem 2.3 we know that, up to π^{wn}, $\mathrm{HTT}^D_{\widetilde{\varphi}}(x)$ generates $\omega_{C_{\tilde{\varphi},m}}$, and hence over K the Hodge–Tate–Taguchi map is surjective. Note that $T_{\mathfrak{p}}(\varphi^D)$ is the dual representation: given $y \in T_{\mathfrak{p}}(\varphi^D) = \mathrm{Hom}(F_{\mathfrak{p}}/A_{\mathfrak{p}}, \mathrm{Hom}(\varphi, \mathrm{CH}))$ and $x \in T_{\mathfrak{p}}(\varphi) = \mathrm{Hom}(F_{\mathfrak{p}}/A_{\mathfrak{p}}, \varphi)$, we get $\langle x, y\rangle \in \mathrm{Hom}(F_{\mathfrak{p}}/A_{\mathfrak{p}}, \mathrm{CH})$ defined by

$$\langle x, y\rangle(z) := y(z)(x(z)).$$

By duality, we get a line in $T_{\mathfrak{p}}(\varphi) \otimes_{A_{\mathfrak{p}}} \overline{K}$. Note that this line is the kernel of $\mathrm{HTT}_{\varphi} \otimes \mathrm{id}$.

If the point corresponding to $\widetilde{\varphi}$ is not in $\mathfrak{X}(v)$ then it is not on the boundary as Tate–Drinfeld modules are ordinary, see e.g., [18, Lemma 4.1]. We apply [28, Lemme 5.2 and Proposition 5.4] to show that quotienting $\widetilde{\varphi}$ by a suitable subgroup of the $\mathfrak{p}^m$-torsion will move the point into $\mathfrak{X}(v)$; this is enough as the Tate modules of two isogenous Drinfeld modules are simply different lattices in $T_{\mathfrak{p}}(\varphi) \otimes_{A_{\mathfrak{p}}} F_{\mathfrak{p}}$. Note that if v' is the truncated valuation of the Hasse invariant of $\widetilde{\varphi}$, then one can use the explicit description of the canonical subgroup given in the proof of [28, Théorème 4.4] to see that its degree à la Fargues is $1 - v'$. Then Proposition 5.4 of loc. cit. tells us that iterations of the correspondence U_{π} pushes $\widetilde{\varphi}$ to points of Fargues degree $1 - v''$, with $1 - v' < 1 - v'' < 1$ and v'' chosen as close to 0 as we want. Hence, the Hasse invariant of $\widetilde{\varphi}$ quotiented by suitable subgroups of the $\mathfrak{p}^m$-torsion will have Hasse invariant v'', and choosing $v'' < v$ allows us to conclude. ☐

Remark 2.6. If $r > 2$, we can only show that the Hodge–Tate–Taguchi map is surjective over the locus of good reduction. Indeed, if the reduction of the Drinfeld modules φ is not good, the corresponding point in $\mathcal{X}(\mathcal{O}_C)$ will fall in the boundary which for $r > 2$ is not necessarily ordinary.

Definition 2.7 (Hodge–Tate–Taguchi period map)**.** Given a trivialization $\eta : T_{\mathfrak{p}}(\varphi) \cong A_{\mathfrak{p}}^2$, we can define an element $\Pi_{\mathrm{HTT}}(\varphi, \eta) \in \mathbb{P}^1(\overline{K})$ by picking the line in $\overline{K}^2$ given by $\mathrm{Lie}(\varphi_{\bar{K}})$.

Before constructing the perfectoid tower, we prove an important property of the Hodge–Tate–Taguchi period map:

Lemma 2.8. Let $(\mathcal{E}, \varphi)$ be a rank 2 Drinfeld module over an algebraically closed field K, then $\Pi_{\mathrm{HTT}}(\varphi, \eta) \in \mathbb{P}^1(F_{\mathfrak{p}})$ if and only if $(\mathcal{E}, \varphi)$ is $\mathfrak{p}$-ordinary.

Remark 2.9. This is the exact analogue of what happens for classical (perfectoid) modular curves mapping the supersingular locus to the Drinfeld upper-half plane via the Hodge–Tate map.

Proof. If $(\mathcal{E}, \varphi)$ is ordinary, then we can identify $\mathrm{Lie}(\varphi)$ with the Lie algebra of the canonical subgroup (which is identified with the Carlitz–Hayes module), and then its π-torsion is an $A_{\mathfrak{p}}$-line in $T_{\mathfrak{p}}(\varphi)$. Conversely, let $(\mathcal{E}, \varphi)$ be a Drinfeld module over K and suppose that there is a trivialization η such that $\Pi_{\mathrm{HTT}}(\varphi, \eta) \in \mathbb{P}^1(F_{\mathfrak{p}})$. Using matrices in $\mathrm{GL}_2(F_{\mathfrak{p}})$, we can suppose that the rational line lies in $T_{\mathfrak{p}}(\varphi)$. We proceed as in [33, Remark III.3.7] and we first show that the kernel of

$$\mathrm{HTT}_{\varphi} : T_{\mathfrak{p}}(\varphi) \to \mathrm{Lie}(\varphi^D)^{\vee}$$

is given by $T_{\mathfrak{p}}(\varphi^{\mathrm{CH}})$, where φ^{CH} is the sub-module of φ isomorphic to a power of CH (so either 0 or CH, by dimension count). This means that we have to show that if $\varphi^{\mathrm{CH}} = 0$, then HTT is injective. Suppose that $\varphi^{\mathrm{CH}} = 0$, then φ^D is a formal group, as the étale π-divisible group $F_{\mathfrak{p}}/A_{\mathfrak{p}}$ is CH^D. Given $x \in T_{\mathfrak{p}}(\varphi)$, denote again $x^D : \varphi^D \to \mathrm{CH}$, and

$$\mathrm{HTT}_{\varphi}(x) = \mathrm{Lie}(x^D) \in \mathrm{Lie}(\varphi^D).$$

As φ^D is a formal Drinfeld module, if $\mathrm{Lie}(x^D) = 0$, then $x^D = 0$, hence $x = 0$. So if $\mathrm{Lie}(\varphi) \subset T_{\mathfrak{p}}(\varphi)$ then $\mathrm{Lie}(\varphi)$ is in the kernel of HTT_{φ} and hence $\varphi^{\mathrm{CH}} \neq 0$, proving that φ is ordinary. □

Remark 2.10. As in the classical Hodge–Tate decomposition of an abelian variety, the decomposition of $T_{\mathfrak{p}}(\varphi) \otimes_{A_{\mathfrak{p}}} \overline{K}$ is not the one induced by the Hodge decomposition of $\mathrm{H}^1_{\mathrm{dR}}(\varphi)$ (see [14, (3.11)] or [20, Lemma 2.21]) and the comparison isomorphism of [16, Theorem 4.12].

2.2. The perfectoid tower. For each adic space $h : \mathcal{Y} \to \mathcal{X}$ we define $\mathcal{Y}(v) := h^{-1}(\mathcal{X}(v))$. We follow the definition of perfection of [33, Definition III.2.18]. Given an $A_{\mathfrak{p}}$-algebra R which is uniform, we define R^{perf} as the π-adic completion of $\varinjlim R$, where the transition morphisms are given by the relative Frobenius. For an adic space $\mathcal{Y} = \mathrm{Spa}(R[1/\pi], R)$, we denote by $\mathcal{Y}^{\mathrm{perf}}$ the adic space associated with $\mathrm{Spa}(R^{\mathrm{perf}}[1/\pi], R^{\mathrm{perf}})$. From now on until the end of section, we base change all adic spaces to

$\mathrm{Spa}(A_{\mathfrak{p}}[\![\pi^{1/p^\infty}]\!][1/\pi], A_{\mathfrak{p}}[\![\pi^{1/p^\infty}]\!])$ but, by a slight abuse of notation, we shall denote all these base-changed adic spaces by the same symbol as before the base change. Our first goal is to prove the following theorem:

Theorem 2.11. *Let $v \leq \frac{1}{q^d+1}$. Then there is a perfectoid space*

$$\mathcal{X}_\infty(v) \sim \varprojlim_m \mathcal{X}(\pi^n)(v),$$

where $\sim$ means that we have an isomorphism of topological spaces

$$|\mathcal{X}_\infty(v)| \cong \varprojlim_m |\mathcal{X}(\pi^n)(v)|,$$

and further that the direct limit of functions on the right hand side are dense in the left hand side, notation as in [34].

The first step to its proof is as follows:

Theorem 2.12. *Let $v \leq \frac{1}{q^d+1}$. There is an affinoid perfectoid space*

$$\mathcal{X}_{0,\infty}(v)_a \sim \varprojlim_m \mathcal{X}_0(\pi^m)(v)_a,$$

where $\mathcal{X}_0(\pi^m)(v)_a$ denotes the anticanonical neighbourhood in $\mathcal{X}_0(\pi^m)$.

Proof. We choose as before a formal model $\mathfrak{X}(q^{-dm}v)$ for $\mathcal{X}(q^{-dm}v)$ as before; we choose a formal model $\mathfrak{X}_0(\pi^m)$ via the normalisation of the formal model of $\mathfrak{X}$. Let

$$h : \mathcal{X}(q^{-dm}v) \to \mathcal{X}_0(\pi^m).$$

be the map that, on points, sends a Drinfeld module φ to $(\varphi/C_{\varphi,m}, \varphi[\pi^m]/C_{\varphi,m})$, where $C_{\varphi,m}$ is the canonical subgroup of level m. By Lemma 2.4 the image of h is contained in $\mathcal{X}_0(\pi^m)(v)$. We denote the image of h by $\mathcal{X}_0(\pi^m)(v)_a$ where "a" stands for anticanonical (as it parametrises Drinfeld modules with a subgroup of the π^m-torsion that does not intersect the canonical subgroup). We show that h is an open immersion, using the same proof as [33, Theorem III.2.15]. First note that composing h with the map

$$\mathcal{X}_0(\pi^m) \to \mathcal{X}$$

sending (φ, H) to φ/H is the open immersion of $\mathcal{X}(q^{-dm}v)$ in $\mathcal{X}$. (This is the Fricke involution composed with the forgetful projection.) On the open part of $\mathcal{X}$ the second map is étale hence the first map is étale on the open part. At the cusps, this second map is étale when restricted to the image of h as the cusps of $\mathcal{X}(q^{-dm}v)$ are sent to the ramified cusps, but the Fricke involution swaps the ramified and unramified cusps.

We take as an integral model of $\mathcal{X}_0(\pi^m)(v)_a$ the integral model $\mathfrak{X}(q^{-dm}v)$ of $\mathcal{X}(q^{-dm}v)$. As the canonical subgroup is a lift of the kernel of the Frobenius modulo π (see the first formula in the proof of [28, Théorème 4.4 (i)]) and as the image of $C_{\varphi,m+1}$ in $\varphi/C_{\varphi,m}$ is the canonical subgroup, the projection

$$\mathcal{X}_0(\pi^{m+1})(v)_a \to \mathcal{X}_0(\pi^m)(v)_a$$

coincides with the relative Frobenius map relative to $A_{\mathfrak{p}}/\pi$, relative Frobenius that we denote by Fr. This is a map of degree q^d, purely inseparable modulo π. Let $\mathfrak{X}(v) = \mathrm{Spf}(R)$ and $\mathfrak{X}(q^{-dm}v) = \mathrm{Spf}(R_m)$. If $\mathrm{Spf}(A)$ is an open of $\mathfrak{X}(0)$ and if $v = b/a$ then we can write locally on $\mathfrak{X}(0)$ after trivializing the sheaf to which the Hasse invariant belongs,

$$R_m = A\langle t_m\rangle/(\mathrm{Ha}^{aq^m}\, t_m - \pi^b),$$

where by a small abuse of notation we consider Ha as function and not as a section of the sheaf. We have a diagram

$$\begin{array}{ccccccc} R = R_0 & \longrightarrow & R_1 & \longrightarrow \cdots \longrightarrow & R_\infty = \varinjlim_m R_m \\ \downarrow & & \downarrow & & \downarrow \\ R/\pi & \longrightarrow & R_1/\pi & \longrightarrow \cdots \longrightarrow & (R/\pi)^{\mathrm{perf}} \end{array}$$

Let $f_m : R_m \to R_{m+1}$.

Given the explicit formula for the canonical subgroup in the proof of [28, Théorème 4.4 (ii)], we see that, if we further extend scalars to include π^v, then as expected the canonical subgroup is the kernel of the relative Frobenius modulo π^{1-v}, and so $f_m \equiv \mathrm{Fr} \bmod \pi^{1-v}$.

Consider now the isomorphism

$$R_\infty/\pi^{(1-v)/q^d} = \varinjlim_m R_m/\pi^{(1-v)/q^d} \cong \varinjlim_m R_m/\pi^{(1-v)} = R_\infty/\pi^{(1-v)}$$

which is induced by the absolute Frobenius of $A_{\mathfrak{p}}$, which raises to the q^d power.

This means that the completion of R_∞ is an perfectoid algebra over $A_{\mathfrak{p}}[\![\pi^{1/p^\infty}]\!]^a$ (in the sense of almost mathematics, see [30, Definition 5.1 (ii)]). By Theorem 5.2 of loc. cit. this implies that $R_\infty[1/\pi]$ is a perfectoid algebra over $A_{\mathfrak{p}}[\![\pi^{1/p^\infty}]\!][1/\pi]$, which is what we wanted. □

Remark 2.13. Consider the diagram

$$R_0 \longrightarrow R_1 \longrightarrow \cdots \longrightarrow R_\infty.$$

As perfection commutes with direct limits, we get

$$R_0^{\mathrm{perf}} \longrightarrow R_1^{\mathrm{perf}} \longrightarrow \cdots \longrightarrow R_\infty^{\mathrm{perf}}.$$

It follows that we also have

$$\mathcal{X}_{0,\infty}(v)_a \sim \varprojlim_m \mathcal{X}_0(\pi^m)(v)_a^{\mathrm{perf}}.$$

Note the limit of underlying topological spaces is the same.

Lemma 2.14. The map

$$\mathcal{X}(\pi^m)_a \longrightarrow \mathcal{X}_0(\pi^m)_a$$

is étale.

Proof. We just have to check this at the cusps, as the result is known on the open part. From level $\Gamma_0(\pi^n)$ to $\Gamma_1(\pi^n)$ it is a direct calculation on the Tate–Drinfeld module of rank 2 [22, Lemma 6.5], that we denote by TD. This is a rank two Drinfeld module over $A_{\mathfrak{p}}[\![x]\!]$ which reduces modulo x to the Carlitz–Hayes Drinfeld module. Anyway, we have that $\mathrm{TD}[\pi]/\mathrm{CH}[\pi]$ is generated by an element of positive x-adic valuation and it is étale as π-divisible module (cf. the explicit calculation of [22, Lemma 6.5] or [18, Lemma 4.4]). Hence the passage from $\Gamma_0(\pi^n)$ to $\Gamma_1(\pi^n)$ is done choosing a generator of this étale group, which is unramified. To pass to level $\Gamma(\pi^n)$ we proceed as in [33, Lemma III.2.35]. $\square$

Note that Lemma 2.14 holds also after perfection as the Frobenius is a universal homeomorphism.

Remark 2.15. For higher rank r, the transition maps from level Γ_1 to Γ_0 will not necessarily be étale on the boundary, and Scholze uses Tate traces in [33, III.2.4] to deal with this issue. Note that normalized Tate traces are not available in positive characteristic, so it is far from clear to us how to adapt Scholze's strategy for higher ranks.

Using this lemma, [5, Lemma 3.4 (xi)], and Scholze's almost purity result [30, Theorem 7.9 (iii)], we obtain

Theorem 2.16. *Let $v \leq \frac{1}{q^d+1}$. Then there is an affinoid perfectoid space*

$$\mathcal{X}_\infty(v)_a \sim \varprojlim_m \mathcal{X}(\pi^m)(v)_a.$$

We now use the Hodge–Tate–Taguchi map to extend the construction to the whole Drinfeld modular curve. We consider the topological space

$$|\mathcal{X}_\infty| := \varprojlim_m |\mathcal{X}(\pi^m)|.$$

For any complete extension $\widetilde{K}$ of $A_{\mathfrak{p}}[\![\pi^{1/p^\infty}]\!][1/\pi]$ with valuation ring $\widetilde{K}^+$, we can define

$$\mathcal{X}_\infty(\widetilde{K},\widetilde{K}^+) := \varprojlim_m \mathcal{X}(\pi^m)((\widetilde{K},\widetilde{K}^+))$$

and clearly we have a map

$$\mathcal{X}_\infty(\widetilde{K},\widetilde{K}^+) \longrightarrow |\mathcal{X}_\infty|.$$

We can describe then

$$|\mathcal{X}_\infty| = \varinjlim_{\widetilde{K}} \mathcal{X}_\infty(\widetilde{K},\widetilde{K}^+),$$

where the transition maps in the direct limit are field extensions $\widetilde{K} \to \widetilde{K}'$.

Given that $\mathcal{X}(\pi^m)$ has a moduli interpretation as (Tate–)Drinfeld modules with a full trivialization of the π^m-torsion, the points of $|\mathcal{X}_\infty|$ are in bijection with isomorphism classes of (Tate–)Drinfeld modules φ equipped with an isomorphism $A_\mathfrak{p}^2 \to T_\mathfrak{p}(\varphi)$. There is a natural action of $\mathrm{GL}_2(A_\mathfrak{p})$ by pre-composition but thanks to the inverse limit we can even extend this action to an action of $\mathrm{GL}_2(F_\mathfrak{p})$: given a matrix $\gamma \in \mathrm{GL}_2(F_\mathfrak{p})$ with determinant in $A_\mathfrak{p}$ and a Drinfeld module φ with a trivialization η, we can define a submodule $L := \eta \circ \gamma(A_\mathfrak{p}^2) \subset T_\mathfrak{p}(\varphi)$, and this corresponds to a subgroup L_{coker} of $\varphi[\mathfrak{p}^m]$, then L is the Tate module of $\varphi/L_{\mathrm{coker}}$ and $\eta \circ \gamma$ defines an isomorphism of L with $A_\mathfrak{p}^2$. We proceed similarly if the determinant has negative valuation. Note that the action of $\mathrm{GL}_2(F_\mathfrak{p})$ is continuous as, at all finite levels, it can be interpreted via the moduli problem defining $\mathcal{X}(\pi^m)$, so it comes from maps of adic spaces.

Let $[x : y]$ be a point in $\mathbb{P}^1$, we let $\mathrm{GL}_2(F_\mathfrak{p})$ act on $\mathbb{P}^1$ via

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} .[x : y] = [dx - by : -cx + ay]. \tag{2.1}$$

This is simply $\det(\gamma)\gamma^{-1}$ applied to the vector $\binom{x}{y}$. If $z = -\frac{y}{x}$ we have the neat formula

$$\gamma(z) = \frac{az + c}{bz + d}.$$

Lemma 2.17. We have a continuous and $\mathrm{GL}_2(F_\mathfrak{p})$-equivariant map

$$|\Pi_{\mathrm{HTT}}| : |\mathcal{X}_\infty| \to |\mathbb{P}^1|$$

that is induced by a map of adic spaces

$$\Pi_{\mathrm{HTT}} : \mathcal{X}_\infty(v)_a \to \mathbb{P}^1.$$

Proof. Pointwise the map is defined using the Hodge–Tate–Taguchi period map of Definition 2.7. As $\mathrm{GL}_2(F_\mathfrak{p})$ acts on the trivialization of $T_\mathfrak{p}(\varphi)$ in the same way as it acts on $\mathbb{P}^1$ and the map is equivariant by the very definition of the action on $\mathbb{P}^1$. To prove continuity we first show that it comes from a map of adic spaces

$$\Pi_{\mathrm{HTT}} : \mathcal{X}_\infty(v)_a \to \mathbb{P}^1.$$

We consider a smaller perfectoid subspace $S = \mathrm{Spa}(A, A^+)$ of $\mathcal{X}_\infty(v)_a$, which is the pullback of the generic fibre of $\mathrm{Spf}(R)$ as in Theorem 2.3, so that the argument of Theorem 2.5 works over the whole S and we get a map

$$T_\mathfrak{p}(\overline{\mathcal{E}}^D) \to \mathrm{Lie}(\overline{\mathcal{E}})^\vee \cong \mathcal{O}_S,$$

where $\overline{\mathcal{E}}$ is the generalised Drinfeld module over $\mathcal{X}_\infty(v)_a$. After trivializing the π^∞-torsion of the universal Drinfeld module $\gamma : \mathcal{O}_S^2 \cong T_\mathfrak{p}(\overline{\mathcal{E}})$ and of its dual $\gamma^D : \mathcal{O}_S^2 \cong T_{\mathfrak{p}}(\bar{\mathcal{E}}^D)$, we get a map

$$\mathcal{O}_S^2 \cong T_{\mathfrak{p}}(\bar{\mathcal{E}}^D) \to \mathrm{Lie}(\bar{\mathcal{E}})^\vee \cong \mathcal{O}_S$$

which gives the desired adic map to $\mathbb{P}^1$. This shows that the map is continuous on $|\mathcal{X}_\infty(v)_a|$. We now show that we can move every point of $|\mathcal{X}_\infty|$ to $|\mathcal{X}_\infty(v)_a|$ with the $\mathrm{GL}_2(F_{\mathfrak{p}})$-action. Indeed, if the underlying (Tate-)Drinfeld module φ lies in $\mathcal{X}(v)$, we can change the level structure using a matrix in $\mathrm{GL}_2(A_{\mathfrak{p}})$ that will move the first coordinate of the trivialization $T_{\mathfrak{p}}(\varphi) \cong A_{\mathfrak{p}}^2$ to something not intersecting the canonical subgroup at any level. If φ is not in $\mathcal{X}(v)$, as in Theorem 2.5, we can use isogenies (hence the importance of the full $\mathrm{GL}_2(F_{\mathfrak{p}})$-action!) to move the point to $\mathcal{X}(v)$. The pointwise-defined Hodge–Tate–Taguchi map is indeed the extension of $|\Pi_{\mathrm{HTT}}|$ on the whole $|\mathcal{X}_\infty|$, requiring the equivariance of the action of $\mathrm{GL}_2(F_{\mathfrak{p}})$. Given that the action of $\mathrm{GL}_2(F_{\mathfrak{p}})$ is continuous on both spaces, continuity follows.

We conclude the lemma giving the explicitly description of what happens at a point $(\varphi, \gamma) \in S$. We let $\bar{\mathcal{E}}_m$ be the universal Drinfeld module over $\mathcal{X}(q^{-dm}v)$, and we have $\bar{\mathcal{E}} = \bar{\mathcal{E}}_m / C_{\bar{\mathcal{E}},m}$. At the level of Tate modules we have

$$T_{\mathfrak{p}}(\bar{\mathcal{E}}_m) \cong (\pi^m \mathcal{O}_S e_1 \oplus \mathcal{O}_S e_2) \to \mathcal{O}_S e_1 \oplus \mathcal{O}_S e_2 \cong T_{\mathfrak{p}}(\bar{\mathcal{E}})$$

(after possibly choosing another basis e_1 and e_2.) From the moduli interpretation of the transition maps $\mathcal{X}_0(\pi^m)(v)_a$, we know that $\gamma^D(A_{\mathfrak{p}} \oplus 0)$ will project non-trivially on the e_2-line (as the first point which determines the $\Gamma_1(\pi^m)$-structure is a generator of $\bar{\mathcal{E}}_m[\pi^m]/C_{\bar{\mathcal{E}},m}$). This is telling us that the image of $\mathcal{X}_\infty(0)_a$ will be a point $[x : y] \in \mathbb{P}^1(F_{\mathfrak{p}})$ with $|x| \leq |y| \neq 0$.

Moreover the image of the canonical locus over the ordinary locus will be the point $[1 : 0]$. ☐

We have the following theorem:

Theorem 2.18. *There is a perfectoid space*

$$\mathcal{X}_\infty \sim \varprojlim_m \mathcal{X}(\pi^m).$$

There is a $\mathrm{GL}_2(F_{\mathfrak{p}})$*-equivariant map of adic spaces*

$$\Pi_{\mathrm{HTT}} : \mathcal{X}_\infty \to \mathbb{P}^1.$$

Proof. In the proof of Lemma 2.17 we have shown that every point x can be moved to $|\mathcal{X}_\infty(v)_a|$ via the $\mathrm{GL}_2(F_{\mathfrak{p}})$-action; as the action is continuous, for every x in $|\mathcal{X}_\infty|$ we can find a small open around x that is moved inside $\mathcal{X}_\infty(v)_a$ and this gives the structure of perfectoid space to the small open. Using again the action of $\mathrm{GL}_2(F_{\mathfrak{p}})$ we can extend the map of adic spaces Π_{HTT} on $\mathcal{X}_\infty(v)_a$ of Lemma 2.17 to the desired map on $\mathcal{X}_\infty$ that we label by the same name. ☐

As in [33], one can show that the whole construction is compatible with changing the tame level, and hence all objects at infinite level admit an action of the prime-to-$\mathfrak{p}$ Hecke operators, which act trivially on the flag variety.

We obtain the following corollary:

Corollary 2.19. The space $\mathcal{X}_\infty$ can be described also as

$$\mathcal{X}_\infty \sim \varprojlim_m \mathcal{X}(\pi^m)^{\mathrm{perf}}.$$

2.3. Overconvergent Drinfeld modular forms. In this section we give a definition of (perfectoid) overconvergent Drinfeld modular forms of weight $s \in \mathbb{Z}_p$ à la Chojecki–Hansen–Johansson [10]. In order to proceed, we need the following lemma:

Lemma 2.20. Let $\omega = \mathrm{Lie}(\bar{\mathcal{E}})^\vee$ be the sheaf of weight one Drinfeld modular forms on $\mathcal{X}$ and ω_∞ the pullback to $\mathcal{X}_\infty$. Over $\mathcal{X}_\infty(v)$, we have

$$\omega_\infty = \Pi^*_{\mathrm{HTT}}\mathcal{O}(1).$$

Proof. As in the proof of Lemma 2.17, the map Π_{HTT} is defined via the rank one quotient of $\mathcal{O}_S^2 \to \mathrm{Lie}(\bar{\mathcal{E}})^\vee$. If one interprets a map to $\mathbb{P}^1$ as a giving a rank one quotient of a locally free rank two sheaf, then the pullback of $\mathcal{O}(1)$ on $\mathbb{P}^1$ by the map is exactly the rank one quotient sheaf. In our setting, this says $\mathrm{Lie}(\bar{\mathcal{E}})^\vee = \Pi^*_{\mathrm{HTT}}\mathcal{O}(1)$. □

For w an integer we define $\mathbb{P}^1_w = \left\{z \in \mathbb{P}^1 \middle| \exists\ z_0 \in \pi A_{\mathfrak{p}} \text{ s.t. } |z - z_0| \leq q^{-dw} < 1\right\}$. This set is stable for the action of

$$\Gamma_0(\mathfrak{p}) := \left\{\gamma \in \mathrm{GL}_2(A_{\mathfrak{p}}) \middle| \gamma \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \bmod \mathfrak{p}\right\}.$$

We also define $\mathcal{U}_{\infty,w}$ to be $\Pi^{-1}_{\mathrm{HTT}}(\mathbb{P}^1_w)$.

Lemma 2.21. Let $q_\infty : \mathcal{X}_\infty \to \mathcal{X}_0(\pi)$ be the natural projection and $q_\infty^{\mathrm{perf}} : \mathcal{X}_\infty \to \mathcal{X}_0^{\mathrm{perf}}(\pi)$. Then the $\mathcal{X}_w := q_\infty(\mathcal{U}_{\infty,w})$, with w tending to 0, are a set of strict affinoid neighbourhoods of the ordinary multiplicative locus in $\mathcal{X}_0(\pi)$, and similarly for $\mathcal{X}_w^{\mathrm{perf}} := q_\infty^{\mathrm{perf}}(\mathcal{U}_{\infty,w})$.

Proof. First note that $\mathbb{P}^1_w$ is a rational subset of the standard affinoid of $\mathbb{P}^1$ defined by $|y| \leq |x|$ (denoted by $\mathbb{P}^1_1$ in [33, Section III.3]). (Recall that $z = -\frac{y}{x}$, for $[x : y]$ the standard coordinates on $\mathbb{P}^1$.) The same argument of [33, Theorem III.3.18 (i)] tells us that the inverse image by Π_{HTT} of the standard affinoid is a rational subset of $\mathcal{X}_\infty$, and by [32, Proposition 2.2] we see that $\mathcal{U}_{\infty,w}$ are affinoid too, as preimages of rational subsets.

Reasoning as in the proof of that proposition we also see that rational subsets in $\mathcal{X}_\infty$ always come from a finite level i.e., there exists $\mathcal{X}(\pi^m)_w \subset$

$\mathcal{X}(\pi^m)$ such that $\mathcal{U}_{\infty,w} = q_{\infty,m}^{-1}(\mathcal{X}(\pi^m)_w)$ with

$$q_{\infty,m} : \mathcal{X}_\infty \to \mathcal{X}(\pi^m)$$

the natural projection. One takes invariants with [10, Corollary 6.26] and obtain that $\mathcal{X}_w$ are affinoid and open.

We have to show that a point $(\varphi, \eta) \in \mathcal{U}_{\infty,w}$ defines a Drinfeld module in $\mathcal{X}(v)$ for some $0 \leq v < 1$, and that η restricted to the first coordinate trivializes the canonical subgroup of φ. By Lemma 2.8 we know that the image of the ordinary locus is contained in $\mathbb{P}^1(F_\mathfrak{p})$. Excluding the case of the Tate–Drinfeld module (which is ordinary), we can suppose that φ has good reduction and hence φ belongs to $\mathcal{X}_0(\pi)(v)^{\mathrm{mult}}$, the counterimage in $\mathcal{X}_0(\pi)$ of $\mathcal{X}(v)$, for a suitable v. The triangular inequality tells us that if $|z| \leq q^{-d}$ then clearly $|y| \leq q^{-d}|x|$, so we can assume that the point $[x : y]$ has $x = 1$ and y is an element divisible by π, which means that, by the same reasoning in the proof of Lemma 2.17, η restricted to the first coordinate e_1 (same notation as that proof, we suppose that e_1 corresponds to the canonical subgroup on the ordinary locus) is not-zero modulo π, while the second coordinate is. This means that the projection to level $\Gamma_0(\mathfrak{p})$ lies in the multiplicative neighbourhood of the ordinary locus.

To conclude, we show that $\mathcal{X}_w$ and $\mathcal{X}(v)$ are cofinal. First note that $\bigcap_w \mathbb{P}^1_w \subset \mathbb{P}^1(A_\mathfrak{p})$ and that $\Pi_{\mathrm{HTT}}^{-1}(P^1(F_\mathfrak{p})) = \mathcal{X}_\infty(0)$ so $\bigcap_w \mathcal{X}_w \subset \mathcal{X}_0(\pi)(0)$ (we already know that it is contained in the multiplicative part).

We cover $\mathcal{X}_w$ with $\mathcal{X}_w \cap \mathcal{X}_0(\pi)(v)^{\mathrm{mult}}$ and by quasi-compactness we find a single largest v that works. Conversely, if φ is in $\mathcal{X}(v)$ we can find, using the canonical subgroup of order m which being a subgroup of $\varphi[\mathfrak{p}^m]$ has rational coordinates modulo $\pi^{m-\vartheta}$ (we need to divide by π^ϑ to go from the image of HHT to a basis of ω_φ), a trivialization whose z is $z_0 + z_1$ with $z_0 \in A_\mathfrak{p}$ and z_1 divisible by $\pi^{m-\vartheta}$, so $z \in \mathbb{P}^1_{m-\vartheta}$. To conclude note that $m - \vartheta$ diverges as v tends to 0, see Theorem 2.5. $\square$

Let $\mathfrak{z}$ be the pullback via the Hodge–Tate–Taguchi map of the coordinate z on $\mathbb{P}^1$. In [33] this is called a fake Hasse invariant, as it commutes with prime-to-$\mathfrak{p}$ Hecke operators. Note that $\omega_{\infty|_{\mathbb{P}^1_w}} = \mathcal{O}_{\mathcal{U}_{\infty,w}}$; indeed we can define an element s which trivializes $\mathcal{O}(1)$ as in [10, Section 2.4]. Identify $\mathcal{O}(1)$ with the contracted product $(\mathrm{GL}_2 \times \mathbb{A}^1)/B$ where B is the Borel subgroup of lower triangular matrices and γ in B acts on $a \in \mathbb{A}^1$ via multiplication by d^{-1} and on GL_2 by right multiplication. A global section is hence a map $f : \mathrm{GL}_2 \to \mathbb{A}^1$ such that $f(gh) = d_h f(\gamma)$, for g in GL_2, h in B and d_h is the right lower entry of h. The function s sending g to $-b_g$ i.e., minus the upper right entry of g, satisfies this condition. Moreover $s = 0$ if and only if g is in B. Hence s is a non-vanishing section of $\mathcal{O}(1)$ on $\mathbb{P}^1 \setminus \{\infty\}$. We want to see how γ in $\mathrm{GL}_2(F_\mathfrak{p})$ acts on s; recall the action on $\mathbb{P}^1$ given in (2.1). If g corresponds to $[x : y]$, then $\gamma.[x : y]$ is the image of $\det(\gamma)\gamma^{-1}g$

in $(\mathrm{GL}_2/B)(\mathrm{F}_{\mathfrak{p}})$. Let $g = \left(\begin{smallmatrix} 0 & -1 \\ 1 & z \end{smallmatrix}\right)$, then

$$\gamma^* s(g) = s(\det(\gamma)\gamma^{-1} g) = (bz+d)s(g).$$

Then, let $\mathfrak{s}$ be the pullback via the HTT map of s, which trivializes $\omega_{\infty|_{\mathbb{P}^1_w}}$. We then have a cocycle

$$j(\gamma, \mathfrak{z}) = \frac{\gamma^* \mathfrak{s}}{\mathfrak{s}} = (b\mathfrak{z} + d).$$

Definition 2.22. Let $v \leq \frac{1}{q^d+1}$. The space of perfectoid overconvergent Drinfeld modular forms of weight s and radius of overconvergence v is

$$M_s(w) := \left\{ f : \mathcal{U}_{\infty,w} \to C \,\middle|\, \gamma^* f = (b\mathfrak{z}+d)^{-s} f, \forall\, \gamma \in \Gamma_0(\mathfrak{p}) \right\}.$$

We want to compare this definition to that in [28, Définition 4.11]. We define a sheaf ω_w^s on $q_\infty^{\mathrm{perf}}(\mathcal{U}_{\infty,w})$ as

$$\omega_w^s(\mathcal{U}) = \left\{ f : q_\infty^{\mathrm{perf}\,-1}(\mathcal{U}) \to C \,\middle|\, f(\gamma\mathfrak{z}) = (b\mathfrak{z}+d)^{-s} f(\mathfrak{z}), \forall\, \gamma \in \Gamma_0(\mathfrak{p}) \right\}.$$

We shall show that this sheaf is a line bundle on $q_\infty^{\mathrm{perf}}(\mathcal{U}_{\infty,w})$. This amounts to finding locally a generator for it.

As in [31, Definition 4.1] we give the following definition:

Definition 2.23. Let $\mathcal{X}^{\mathrm{perf}}$ be the perfection a rigid analytic variety over $F_{\mathfrak{p}}$. We let $\mathcal{X}^{\mathrm{perf}}_{\mathrm{proet}}$ be the pro-étale site of [34, Definition 8.2.6].

Lemma 2.24. For every level $\Gamma \subset \mathrm{GL}_2(A_{\mathfrak{p}})$ we define $\mathcal{X}(\Gamma)$ the corresponding rigid space. Let

$$q_{\infty,\Gamma}^{\mathrm{perf}} : \mathcal{X}_\infty \to \mathcal{X}(\Gamma)^{\mathrm{perf}}$$

be the projection induced by Corollary 2.19. Then we have

$$\mathcal{O}_{\mathcal{X}_\infty}^{\Gamma} = \mathcal{O}_{\mathcal{X}(\Gamma)^{\mathrm{perf}}},$$

which means $\mathcal{O}_{\mathcal{X}_\infty}(q_\infty^{-1}(\mathcal{U}))^\Gamma = \mathcal{O}_{\mathcal{X}(\Gamma)^{\mathrm{perf}}}(\mathcal{U})$ for every admissible open $\mathcal{U}$ in $\mathcal{X}(\Gamma)^{\mathrm{perf}}$.

Proof. We can check it locally. First suppose that $\mathcal{U}$ does not intersect the boundary. In this case $q_{\infty,\Gamma}^{\mathrm{perf},-1}(\mathcal{U})$ belongs to the pro-étale site of $\mathcal{U}$. Applying [10, Lemma 2.24] to $\mathcal{O}^+_{\mathcal{X}_\infty}/\pi^m(^{\mathrm{perf},-1}_{\infty,\Gamma}(\mathcal{U}))$ we get

$$\left(\mathcal{O}^+_{\mathcal{X}_\infty}/\pi^m(q_{\infty,\Gamma}^{\mathrm{perf},-1}(\mathcal{U}))\right)^\Gamma = \mathcal{O}^+_{\mathcal{X}(\Gamma)^{\mathrm{perf}}}/\pi^m(\mathcal{U}).$$

As by definition we have $\mathcal{O}^+_{\mathcal{X}^{\mathrm{perf}}} = \varprojlim \mathcal{O}^+_{\mathcal{X}^{\mathrm{perf}}}/\pi^n$ we get that

$$\mathcal{O}^+_{\mathcal{X}_\infty}(q_{\infty,\Gamma}^{\mathrm{perf},-1}(\mathcal{U}))^\Gamma = \mathcal{O}^+_{\mathcal{X}(\Gamma)}(\mathcal{U}),$$

and to conclude we just have to invert π.

If $\mathcal{U}$ intersects the boundary, any section f of $\mathcal{O}_{\mathcal{X}_\infty}(q_{\infty,\Gamma}^{\mathrm{perf},-1}(\mathcal{U}))^\Gamma$ defines a bounded section of $\mathcal{O}_{\mathcal{X}_\infty}(q_{\infty,\Gamma}^{\mathrm{perf},-1}(\mathcal{U})\setminus\{\mathrm{cusps}\})^\Gamma$ and hence, by the previous case, a bounded element of the sheaf $\mathcal{O}_{\mathcal{X}(\Gamma)^{\mathrm{perf}}}(\mathcal{U}\setminus\{\mathrm{cusps}\})$. Write f as the π-adic limit $\varinjlim f_n$, where every f_n comes from pullback from a finite level in the limit $\mathcal{X}(\Gamma)^{\mathrm{perf}} = \varprojlim \mathcal{X}(\Gamma)$. Each f_n can be extended uniquely to an element $\widetilde{f}_n$ of $\mathcal{O}_{\mathcal{X}(\Gamma)}(\mathcal{U})$ by [26, Theorem 1.6I)]. We want to show that $\widetilde{f} = \varinjlim \widetilde{f}_n$ is well-defined. It is enough to check that they form a Cauchy sequence for the π-adic topology. If $\widetilde{f}_n$ and $\widetilde{f}_m$ are defined on the same $\mathcal{X}(\Gamma)$ and $\sup|f_n - f_m|_\pi < \epsilon$ on $\mathcal{U}\setminus\{\mathrm{cusps}\}$, the extension $\widetilde{f}_n - \widetilde{f}_m$ has sup-norm $\leq \epsilon$ by the maximum modulus principle, hence $\widetilde{f}$ is a well-defined element of $\mathcal{O}_{\mathcal{X}(\Gamma)^{\mathrm{perf}}}(\mathcal{U})$. $\square$

Let $\mathcal{U}$ be an admissible open of $q_\infty^{\mathrm{perf}}(\mathcal{U}_{\infty,w})$ such that ω^{perf} is trivialized by a basis $\xi_{\mathcal{U}}$ and let $\mathfrak{s} = t_{\mathcal{U}} q_\infty^{\mathrm{perf}^{-1}}(\xi_{\mathcal{U}})$, for $t_{\mathcal{U}}$ a unit in $\mathcal{O}_{\mathcal{X}_\infty}(q_\infty^{\mathrm{perf}^{-1}}(\mathcal{U}))$. We want to make sense of $t_{\mathcal{U}}^s$ for $s \in \mathbb{Z}_p$. As in [10, Proposition 2.27] we have:

Lemma 2.25. We can write

$$t_{\mathcal{U}} = t'_{\mathcal{U}} s'_{\mathcal{U}},$$

with $t'_{\mathcal{U}} \in 1+\pi\mathcal{O}_{\mathcal{X}_\infty}((q_\infty^{\mathrm{perf}^{-1}}(\mathcal{U})))$ and $s'_{\mathcal{U}} \in \mathcal{O}_{\mathcal{X}_0(\pi)^{\mathrm{perf}}}(\mathcal{U})$. Moreover $\gamma^* t_{\mathcal{U}} = (b\mathfrak{z}+d)t_{\mathcal{U}}$.

Let f in $\omega_w^s(\mathcal{U})$ and note that

$$\gamma^*(f{t'_{\mathcal{U}}}^s) = f{t'_{\mathcal{U}}}^s \left(\frac{s'_{\mathcal{U}}}{\gamma^* s'_{\mathcal{U}}}\right)^s.$$

As $s'_{\mathcal{U}}$ is invariant by $\Gamma_0(\mathfrak{p})$, applying Lemma 2.24 to $f{t'_{\mathcal{U}}}^s$ we can then embed

$$\omega_w^s(\mathcal{U}) \to \mathcal{O}_{\mathcal{X}_0(\pi)^{\mathrm{perf}}}(\mathcal{U})$$

via $f \mapsto f{t'_{\mathcal{U}}}^s$.

Theorem 2.26. *The sheaf ω_w^s is coherent and locally free of rank one.*

Proof. For $s = 0$ this is consequence of Lemma 2.24. Note that we can make $\Gamma_0(\mathfrak{p})$ acts on $\mathcal{O}_{\mathcal{X}_0(\pi)^{\mathrm{perf}}}(\mathcal{U})$ via

$$\gamma.f = \left(\frac{\gamma^* s'_{\mathcal{U}}}{s'_{\mathcal{U}}}\right)^s \gamma^* f.$$

Then $\omega_w^s(\mathcal{U})$ falls in the invariant part for this action.

Note that if $\mathcal{U}$ does not intersect the boundary, the cover is étale Galois and by Galois descent the invariant part is locally free on $\mathcal{U}$ of rank one (the rank of $\mathcal{O}_{\mathcal{X}_0(\pi)}$). So ω_w^s is a subsheaf of a rank one sheaf. It is indeed coherent: take a section f and suppose it is not vanishing (shrinking $\mathcal{U}$ if necessary). Division by f induces an isomorphism between ω_w^s and ω_w^0, which is coherent. We can extend everything to the boundary reasoning as in the last part of the proof of Lemma 2.24. □

We are nearly ready to compare this notion of overconvergent Drinfeld modular forms with our previous notion. First, to give some context, we recall the relationship between line bundles on $\mathcal{X}_0(\pi)^{\mathrm{perf}}$ and $\mathcal{X}_0(\pi)$. If X is a scheme, it is known that

$$\mathrm{Pic}(X)\left[\frac{1}{p}\right] = \mathrm{Pic}(X^{\mathrm{perf}}),$$

i.e., every line bundle comes from finite level by pullback of line bundles on the Frobenius twist $X^{(p^n)}$ [5, Lemma 3.5]. The same holds for affinoid rigid spaces, thanks to a result of Kedlaya and Liu:

Proposition 2.27. Let $\mathcal{Y}$ be a affinoid rigid analytic space over $F_{\mathfrak{p}}$ and let $\mathcal{Y}^{\mathrm{perf}}$ be its perfection. Then

$$\mathrm{Pic}(\mathcal{Y})\left[\frac{1}{p}\right] = \mathrm{Pic}(\mathcal{Y}^{\mathrm{perf}}).$$

Proof. Let us write $\mathcal{Y} = \mathrm{Spa}(R, R^+)$ and $\varprojlim \mathcal{Y}_i = \mathcal{Y}^{\mathrm{perf}} = \mathrm{Spa}(R, R^+)^{\mathrm{perf}}$. A line bundle $\mathcal{L}_\infty$ over $\mathcal{Y}^{\mathrm{perf}}$ is nothing else but a free module of rank one over R^{perf}; by [25, Lemma 5.6.8] we know that every finite projective module on the perfection comes via extension of scalars from a finite projective module at finite level, hence from a line bundle on some $\mathcal{Y}_i$. □

We can now compare our two definitions of sheaves of Drinfeld modular forms. Recall the rigid torsor $\mathcal{F}$ of [28, Définition 4.5]: it parametrises generators of the image of the Hodge–Tate–Taguchi map in ω. It is a torsor for $G := A_{\mathfrak{p}}^\times(1 + \pi^v \mathcal{O}_{\mathcal{X}(v)})$.

We pull it back to a sheaf $\mathcal{F}^{\mathrm{perf}}$ on $q_\infty^{\mathrm{perf}}(\mathcal{U}) \to \mathcal{X}_0(\pi)^{\mathrm{perf}}$ and it is now a torsor for $G^{\mathrm{perf}} = (A_{\mathfrak{p}}^\times)^{1/p^\infty}(1 + \pi^v \mathcal{O}_{\mathcal{X}(v)^{\mathrm{perf}}})$. Here $(A_{\mathfrak{p}}^\times)^{1/p^\infty}$ is the group of all p^n-roots of elements of $A_{\mathfrak{p}}^\times$ which is isomorphic to a finite product of infinitely many copies of $\mathbb{Q}_p$. Using the arguments of the proof of Lemma 2.21 on the fact that X_v is a neighbourhood of the ordinary locus we get:

Lemma 2.28. The section $s'_{\mathcal{U}} {q_\infty^{\mathrm{perf}}}^{-1}(\xi_{\mathcal{U}})$ of Lemma 2.25 is a generator of $\mathcal{F}^{\mathrm{perf}}$.

By abuse of notation, we denote by the same symbol the corresponding element of G^{perf} (which can be seen as an inverse limit of generators of the images of the Hodge–Tate–Taguchi map in ω along the Frobenius tower).

We can go from the new sheaf ω_w^s to the perfection of the older sheaf $\omega^{s,\mathrm{NR}}$ as follows. First note that there is no action of G on the sections of

ω_w^s. So given f in $\omega^{s,\mathrm{NR}}(\mathcal{U})$ the function $(s'_{\mathcal{U}} {q_\infty^{\mathrm{perf}}}^{-1}(\xi_{\mathcal{U}}))^s f$ is an element of $\omega_w^s(\mathcal{U})$ which transforms correctly for the action of $\Gamma_0(\pi)$ by the cocycle relation and it is invariant by the action of G as f is homogenous of weight $-s$. This map is bijective, hence:

Theorem 2.29. *The sheaf ω_w^s is the pullback of $\omega^{s,\mathrm{NR}}$ on the perfection $\mathcal{X}_w^{\mathrm{perf}}$.*

3. Some open problems on families of Drinfeld modular forms

3.1. Modularity theorem. Let f be a double cuspidal Drinfeld modular form of level $\mathfrak{n}$ and rank 2, and let $\bar{\chi}_f$ be the reduction modulo $\mathfrak{p}$ of the associated Galois character defined by [7]. We know that $\bar{\chi}_f$ factors through the Galois extension corresponding, by class field theory, to $(A/\mathfrak{n}')^\times$, for $\mathfrak{n}'$ a squarefree ideal dividing the product of $\mathfrak{n}\mathfrak{p}$ and $\mathfrak{m}$, for $\mathfrak{m}$ the set of places of good supersingular reduction of the Drinfeld modular curve of level $\mathfrak{p}\mathfrak{n}$ [8, Theorem 12.22]. Let $\Lambda_\infty := A_{\mathfrak{p}}[\![A_{\mathfrak{p}}^\times]\!]$; we define a representation $\chi^{\mathrm{univ}} : G_F \to r^\times \cong \Lambda_\infty^\times$ as the composition of the tautological character of class field theory $G_F \to (A_{\mathfrak{p}}[\![(A/\mathfrak{n}')^\times \times (1+\pi A_{\mathfrak{p}})]\!])^\times$ with the projection to the $\bar{\chi}_f$-isotypic component.

Every character of G_F with coefficients into a local $A_{\mathfrak{p}}$-algebra R factoring through $(A/\mathfrak{n}')^\times \times (1+\pi A_{\mathfrak{p}})$ and congruent to $\bar{\chi}_f$, can be obtained by composition of a unique morphism $\alpha : r \to R$ with χ^{univ}.

Suppose that f is ordinary and let t be the localisation at a maximal ideal containing f of the ordinary Hecke algebra $\mathbb{T}$ of [28, Proposition 3.17]. One can construct a t-adic Galois representation interpolating the Galois representations of classical forms in the family.

The first question is: is there a surjective $r \to t$ morphism? If so, is it an isomorphism?

The answer to the first question is affirmative if $F = \mathbb{F}_q(T)$ and $\mathfrak{n} = \mathfrak{p} = T$; this restriction on working only with level T is imposed by the fact that the ramification of the Galois representation associated with a Drinfeld modular forms is not known. More precisely, given a $\mathfrak{p}$-adic Galois representation, we do not know if the ramification at a different prime $\mathfrak{q}$ is finite or not. If we could develop a theory of vanishing cycles for Böckle–Pink τ-sheaves [8], it is likely that we could show that the ramification at $\mathfrak{q}$ is finite, and thus get the desired $r \to t$ morphism. Once one has this, can one prove that $r = t$, as is the case for most elliptic modular forms?

We present a conjectural application which has been suggested to us by C. Popescu. Let χ be an element of $\mathrm{Pic}^0(A)$ and φ_χ the corresponding Carlitz–Hayes module. It is known that the Galois representation ρ_{φ_χ} is unramified outside $\mathfrak{p}$ and ∞ (see [36, Theorem 5] and [21, Theorem 3.2]). An $r = t$ theorem would show that ρ_{φ_χ} is "modular" of type II [15, Definition 6], i.e., the Galois representation on the $\mathfrak{p}$-torsion of φ_χ arises from a

π-adic Drinfeld modular form. This is only known for the Carlitz module of $\mathbb{F}_q[T]$ [15, Example 10] which is associated to the Drinfeld modular form Δ, which is not ordinary for any prime.

3.2. Non noetherian eigenvarieties. In his eigenvariety paper, Buzzard points out that Lemma A1.6 of [11] is not complete: roughly speaking, Coleman claims that given a completely continuous operator U on an orthonormalisable module M over a Banach algebra A and a finitely generated submodule M', a finite number of coordinates are enough to determine if an element of M' is 0 or not. In the noetherian case, this is handled by [9, Lemma 2.3]. It would interesting to investigate the following:

Question 3.1. Given a completely continuous operator U on an orthonormalisable module M over a non-noetherian Banach algebra A, can one define a Fredholm determinant $F_{\mathrm{U}}(X)$? If not, are there extra conditions on A under which this holds true?

Once we have a good definition of the Fredholm determinant $F_{\mathrm{U}}(X)$, the next step is the construction of the spectral variety, which is defined as the closed subset

$$\mathscr{Z} := V(F_{\mathrm{U}}(X)) \subset \mathbb{A}^1_{\mathrm{Spa}(A,A^\circ)}.$$

A difficult point is the generalisation of [9, Lemma 4.1] which proves that $\mathscr{Z}$ is flat, as for the moment we lack a flatness criterion for non-noetherian rings. For example, we believe it would be enough to generalise [35, Lemma 10.127.4], replacing "essentially of finite presentation" by "topologically of finite type".

3.3. Families of modular forms for *t*-motives. An obstacle to generalising Drinfeld modular forms in higher dimensions is our lack of understanding of algebraic families of Anderson A-motives. Still, the local theory as developed in [17] gives a nice description of the duality between local Anderson modules and local shtukas. The properties of the Hodge–Tate–Taguchi map in higher dimension are not as tractable as in dimension one: indeed, the proof of the almost surjectivity in Theorem 2.3, in our one-dimensional case of arbitrary rank, was done by hand. For abelian varieties (or more generally, p-divisible groups), the proof of almost surjectivity is done using p-adic Hodge theory (see [13, Appendix C] or [33, Section III.2.1]), whose analogue is lacking in our context.

3.4. The maximal slope. Given an elliptic modular form over $\mathbb{Q}$ of weight k, we know that if p does not divide the level, the possible U_p-eigenvalues have slopes between 0 and $k-1$. This is because the constant term of the Hecke polynomial is, up to a root of unit, p^{k-1}.

For Drinfeld modular forms, the Hecke polynomial has constant term 0, which a priori allows any possible slope.

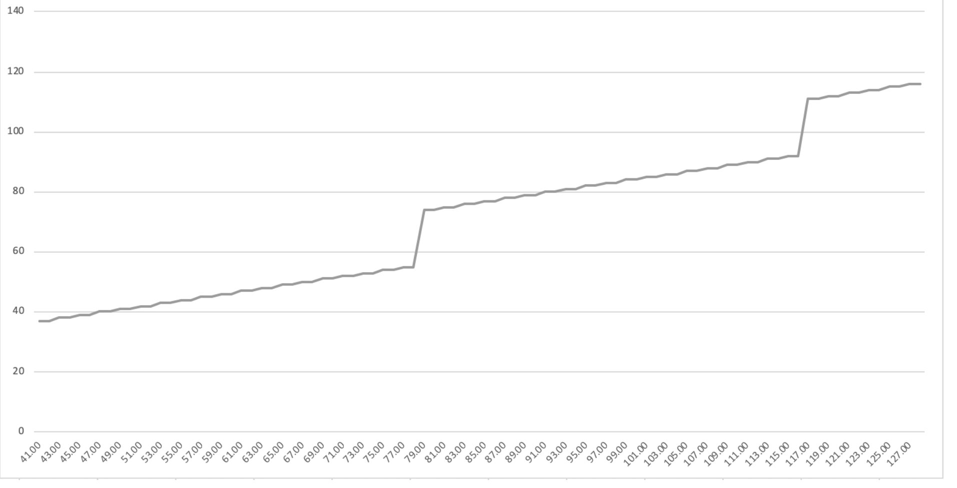

FIGURE 3.1. $q = 5$

In level $\Gamma_0(T)$, Bandini and Valentino conjecture that the maximal slope is always $(k-2)/2$, and this maximal slope arises exclusively from (suitably defined) newforms. Recently, they proved a quadratic bound for the maximal slope [3, Theorem 6.4].

In level $\Gamma_1(T)$, explicit computations for $A = \mathbb{F}_q[T]$ hint to the fact that the maximal slope is at least always bounded by $k-1$, if not better estimates. The following Figures 1–4 are calculated using Hattori's tables [19] relying on the formulae of Bandini and Valentino [1, 2]. We use the x-axis to indicate the weight and the y-axis to indicate the maximal slope appearing in that fixed weight. Note that the patterns of the maximal slope distribution vary widely with q. At the time of writing, we have no clue towards a conceptual explanation of this variation.

3.5. Infinite slope Drinfeld modular forms. Suppose that $\mathfrak{p}$ is principal and let f be an eigenform of T_π of prime-to-π level and eigenvalue λ. In rank two, the explicit formula for the π-stabilisation of f [2, Section 3.2] (with $m = 0$ and a slightly different normalisation of U_π, as ours is theirs divided by π) tells us that

$$\mathrm{U}_\pi\left(f(z) - \frac{\pi^{k-1}}{\lambda}f(\pi z)\right) = \lambda\left(f(z) - \frac{\pi^{k-1}}{\lambda}f(\pi z)\right), \quad \mathrm{U}_\pi f(\pi z) = 0.$$

This means that we have plenty of modular forms of level $\Gamma_0(\pi)$ and infinite slope, which is never the case for classical modular forms! In [28, Corollaire 5.10], we show that if f is an overconvergent modular form of weight k and slope smaller than $k-1$, then it is classical of level $\Gamma_0(\pi)$. A natural question is then the following:

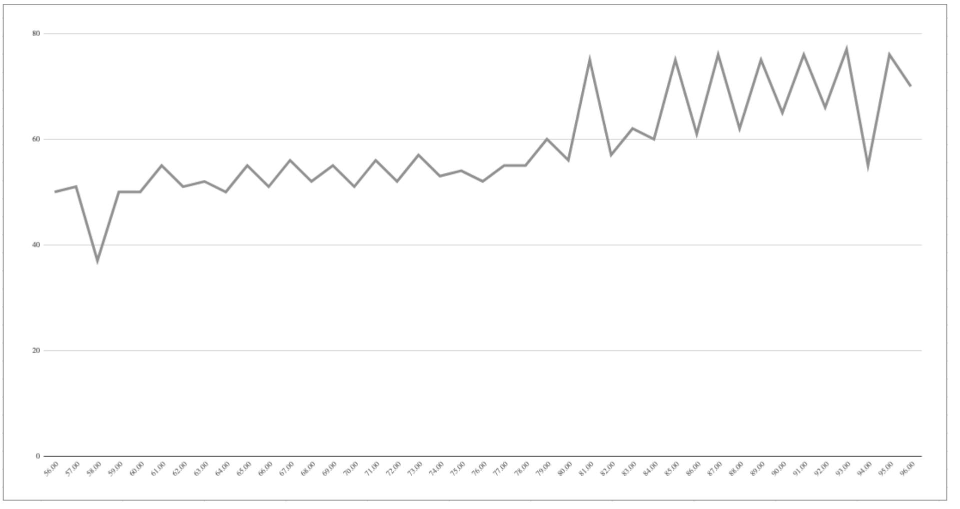

FIGURE 3.2. $q = 125$

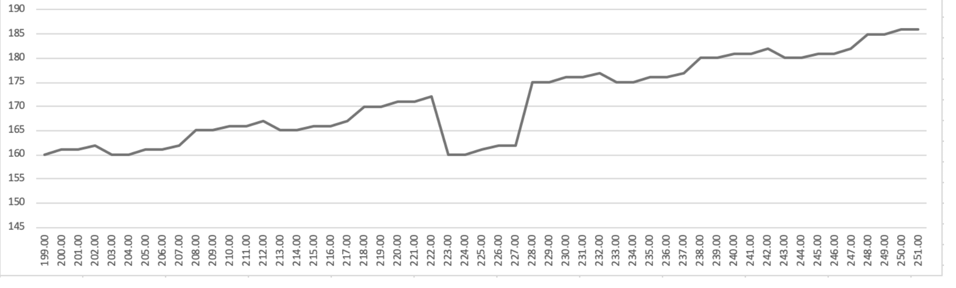

FIGURE 3.3. $q = 7$

Question 3.2. Given an overconvergent Drinfeld modular form of weight k and infinite slope, is there a criterion to decide whether f is classical?

Moreover, at least in the ordinary case, we know from [28, Théorème 3.14] that if the weight is large enough, the form f is not only classical, but it comes via π-stabilisation from a form of prime-to-π level. Hence, for a Zariski dense set of points $\{f_k\}$ in an ordinary family, we can find a corresponding classical Drinfeld modular form $\widetilde{f_k}$ of infinite slope, whose prime-to-π Hecke eigenvalues vary in a Iwasawa algebra. This leads naturally to the following:

Question 3.3. Do continuous families of infinite slope Drinfeld modular forms exist?

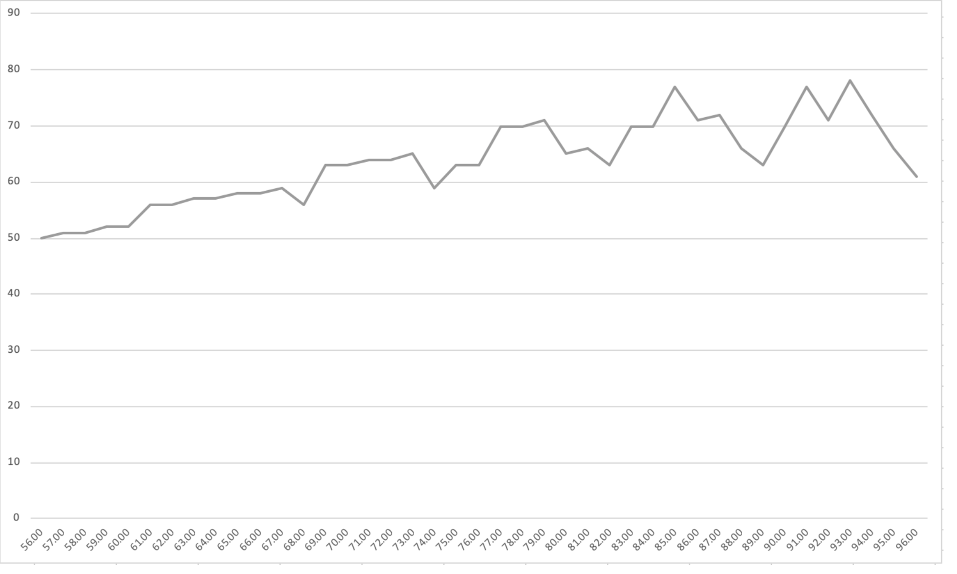

FIGURE 3.4. $q = 37$

3.6. Horizontal control theorem. An alternative approach to classical Hida theory does not vary the weight of the modular forms but varies instead the level at p of the modular curve, and then shows that the ordinary parts of the H^1 of these curves glue to a finitely generated Λ-adic module. This approach seems much harder to understand for function fields: for example, there are only $q^d - 1$ finite order characters of $(A_{\mathfrak{p}}/\mathfrak{p}^r)^\times$, independently of r.

As far as we know, the best known result towards a horizontal control theorem à la Hida as alluded to above is Marigonda's unpublished 2008 PhD thesis for Drinfeld modular curves [27, Theorem 11].

Theorem 3.4. *Let J_n be the Jacobian of $\mathcal{X}_1(\pi^n)$ and $G_r = (1+\pi A_{\mathfrak{p}})/(1+\pi^r A_{\mathfrak{p}})$. Then the ordinary part of the p-adic Tate module $T_p(J_n)$ is free over $\mathbb{Z}[G_r]$ of rank bounded by the rank of $T_p(J_n)$.*

As the rank of the latter is known to grow with the index of $\Gamma_1(\pi^r)$, this is not telling us much on the possibility of a horizontal Hida family. Moreover, the alternative approach to Hida theory due to Emerton [12] does not seem to apply here, as it relies on the fact that $\mathbb{Z}_p^\times$ is topologically generated by one element, while we are very much in a non-noetherian situation. We expect the following question has a negative answer in general:

Question 3.5. Is the inverse limit $\varprojlim_n T_p(J_n)^{\mathrm{ord}}$ a finite free Λ-module?

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