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Cornelius GREITHER et Cristian POPESCU

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Picard 1-motives and Tate sequences for function fields

par Cornelius GREITHER et Cristian POPESCU

Résumé. En utilisant nos travaux antérieurs sur la structure des réalisations ℓ-adiques des 1-motifs de Picard en tant que modules galoisiens, nous construisons des représentants explicites pour la classe de Tate ℓ-adifiée. C'est-à-dire qu'on trouve des suites de Tate explicites, comme définies dans [8], pour une extension galoisienne générale de corps globaux en caractéristique $p > 0$. En combinaison avec la Conjecture Principale Équivariante démontrée dans [4], ceci nous amène à une preuve assez directe de la Conjecture Équivariante des nombres de Tamagawa pour les motifs d'Artin à coefficients abéliens en caractéristique positive.

Abstract. We use our previous work [4] on the Galois module structure of ℓ–adic realizations of Picard 1–motives to construct explicit representatives in the ℓ–adified Tate class (i.e. explicit ℓ–adic Tate sequences, as defined in [8]) for general Galois extensions of characteristic $p > 0$ global fields. If combined with the Equivariant Main Conjecture proved in [4], these results lead to a very direct proof of the Equivariant Tamagawa Number Conjecture for characteristic $p > 0$ Artin motives with abelian coefficients.

1. Introduction

In earlier work (see in particular [4]) we studied a certain Picard 1-motive attached to a Galois cover of smooth projective curves over finite fields, which lends itself to the formulation of an Equivariant Main Conjecture in the Iwasawa Theory of function fields. This 1-motive was introduced by Deligne in the 1970s [3] and played an important role in the Deligne–Tate proof of the Brumer–Stark conjecture for function fields [9]. In [4] our further investigations of this object led to proofs of refinements of the Brumer–Stark and Coates–Sinnott conjectures for function fields. The most prominent features of the 1-motive in question are the following: everything is relatively explicit and computationally convenient, and the ℓ-adic Tate module attached to the 1-motive in question is projective over the ℓ–adic

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group ring $\mathbb{Z}_\ell[G]$ of the Galois group of the cover, as proved in [4]. In the theory of Chinburg's conjectures and their modern counterparts (the many and various instances of the Equivariant Tamagawa Number Conjecture) projective modules carrying arithmetic information are extremely important, but they are rarely made explicit, which makes calculations involving these extremely difficult. This holds in particular when one works in derived categories and has to deal with Grothendieck determinants of Tate complexes (middle parts of Tate sequences, proved to exist by Tate in [8] but are extremely elusive, in general.)

In this paper we will show that one advantage of the above mentioned 1-motive, its explicitness, also can be used profitably for constructing and understanding Tate sequences. A little more precisely: we start from the Picard 1-motive and its ℓ-adic realization (ℓ being a fixed prime) and construct an explicit four term exact sequence in the category of $\mathbb{Z}_\ell[G]$–modules. Then we show that this sequence is, up to a minor and totally explicit modification, Yoneda-equivalent to "the" Tate sequence. Of course the latter itself is only determined up to Yoneda equivalence, so it would be more accurate to say that we find a nice and explicit *representative* in the Yoneda class of Tate sequences. This class is intimately linked to local and global fundamental classes in class field theory (see [8] and [9]) and also to special values of Galois-equivariant global L–functions, as reflected by the far-reaching Equivariant Tamagawa Number Conjecture of Burns and Flach.

In the present note we are concerned with the function field (global field of positive characteristic) case. The number field case was treated in [6]. Our reason for presenting the function field case separately, even though the ideas and tools involved in [6] have some overlap with the present case, is twofold: The results in the function field case are neater and more complete, and the proofs are more straightforward and natural. (See the Remark at the end of Section 1.) We hope that this separate treatment will be useful for readers.

We begin by reviewing some notation introduced in [4]. Let us fix a finite field $\mathbb{F}_q$ of characteristic $p > 0$; in particular $p = 2$ is allowed. By $X \to Y$ we will denote a Galois cover of smooth projective curves over $\mathbb{F}_q$ with a finite Galois group G. We consider finite, disjoint G–equivariant sets S and T of closed points on X, subject to the following standard conditions: T is nonempty, and S contains all points that ramify in X/Y. Let $\overline{\mathbb{F}}$ denote the algebraic closure of $\mathbb{F}_q$. The base-changed curves $\overline{\mathbb{F}} \times_{\mathbb{F}_q} X$ and $\overline{\mathbb{F}} \times_{\mathbb{F}_q} Y$ will be written $\mathcal{X}$ and $\mathcal{Y}$; the set of points on $\mathcal{X}$ above points of S (resp. T) will be written $\mathcal{S}$ (resp. $\mathcal{T}$). The Galois group $\mathrm{Gal}(\overline{\mathbb{F}}/\mathbb{F}_q)$ is denoted by Γ. Of course there is a bijection between closed points (simply called points from now on) on X and Γ-orbits of (closed) points on $\mathcal{X}$.

Let $\mathrm{cl}_T(X)$ (resp. $\mathrm{cl}_T^0(X)$) denote the ray class group (resp. degree 0 ray class group) mod T of X, that is, the quotient of the group of divisors (resp. degree 0 divisors) on X whose support does not meet T modulo the subgroup of principal divisors $\mathrm{div}(f)$ where f is congruent to 1 "modulo T" (meaning modulo every point in T.) This means of course that f is defined at all points of T and evaluates to 1 there. Then $\mathrm{cl}_T^0(X)$ is just $J_{X,T}(\mathbb{F}_q)$, where $J_{X,T}$ is the generalized Jacobian associated to X and T.

Let L_S be the lattice attached to S. More precisely, L_S is the kernel of the map $\deg_{\mathbb{F}_q} : \mathbb{Z}S \to \mathbb{Z}$ that sends $v \in S$ to $\deg_{\mathbb{F}_q}(v) \in \mathbb{Z}$. Here $\mathbb{Z}S$ is the free $\mathbb{Z}$-module with basis S and $\deg_{\mathbb{F}_q}(v)$ is the degree relative to $\mathbb{F}_q$.

Definition 1.1. *The set S is called* large with respect to X and T *(or simply "large") if the divisor class map $d : \mathbb{Z}S \to \mathrm{cl}_T(X)$ is surjective. (Note that this is equivalent to the surjectivity of the degree* 0 *divisor class map $d_0 : L_S \to \mathrm{cl}_T^0(X)$ and the existence of a divisor of degree* 1 *in $\mathbb{Z}S$.)*

We fix the covering $X \to Y$ once for all, and we write U_S for the group of S-units in the function field $\mathbb{F}_q(X)$; similarly $U_{S,T}$ is the submodule of all S-units in that field which are congruent to 1 modulo T. There is another important lattice in the context of Tate sequences. Usually it is denoted X_S, but since X is already used as the name of a curve, we have to resort to Ξ_S. This is defined as the kernel of the map $\mathrm{aug} : \mathbb{Z}S \to \mathbb{Z}$ sending every $v \in S$ to $\mathrm{aug}(v) = 1$. Please note the difference with the definition of L_S!

We write M for the ℓ-adic Tate module of the Picard 1-motive $\mathcal{M}_{\mathcal{S},\mathcal{T}}$. For the construction and the properties of this 1-motive and of the module M, we refer to [4]; let us just recall that M is finitely generated projective over $\mathbb{Z}_\ell[G]$, as proved in [4].

Let us now describe our main result. In the first section (Proposition 2.1) we will construct, for large S, an four-term exact sequence of $\mathbb{Z}_\ell[G]$–modules

$$0 \longrightarrow \mathbb{Z}_\ell \otimes_{\mathbb{Z}} U_{S,T} \longrightarrow M \longrightarrow M \longrightarrow \mathbb{Z}_\ell \otimes_{\mathbb{Z}} \Xi_S \longrightarrow 0.$$

This sequence defines an element of

$$\mathrm{Ext}^2_{\mathbb{Z}_\ell[G]}(\mathbb{Z}_\ell \otimes_{\mathbb{Z}} \Xi_S, \mathbb{Z}_\ell \otimes_{\mathbb{Z}} U_{S,T}) \cong \mathbb{Z}_\ell \otimes_{\mathbb{Z}} \mathrm{Ext}^2_{\mathbb{Z}[G]}(\Xi_S, U_{S,T}),$$

which will be called $\rho = \rho_{X/Y,S,T}$. On the other hand, the theory of Tate classes [8] proves the existence of certain (non-explicit) 4-term exact sequences of $\mathbb{Z}[G]$–modules

$$0 \longrightarrow U_S \longrightarrow A \longrightarrow B \longrightarrow \Xi_S \longrightarrow 0$$

with A and B of finite projective dimension over $\mathbb{Z}[G]$ and whose class in the extension group $\mathrm{Ext}^2_{\mathbb{Z}[G]}(\Xi_S, U_S)$ is an explicit element denoted here $\tau = \tau_{X/Y,S}$ and called the Tate class. We will show below that there is a canonical isomorphism

$$\iota : \mathrm{Ext}^2_{\mathbb{Z}[G]}(\Xi_S, U_{S,T}) \longrightarrow \mathrm{Ext}^2_{\mathbb{Z}[G]}(\Xi_S, U_S).$$

Thus it makes sense to compare $(\iota \otimes 1_{\mathbb{Z}_\ell})(\rho)$ and the ℓ-adification $\mathbb{Z}_\ell \otimes \tau$ of the Tate class τ, viewed in $\mathrm{Ext}^2_{\mathbb{Z}_\ell[G]}(\mathbb{Z}_\ell \otimes \Xi_S, \mathbb{Z}_\ell \otimes U_S)$. Here and later on, we abuse notation, simply writing ι instead of $1_{\mathbb{Z}_\ell} \otimes \iota$. Our main result (see Theorem 3.3) then reads as follows:

Theorem 1.2. *Assume that S is large (as defined above); if $\ell = p$ (the characteristic of $\mathbb{F}_q$) then assume moreover that the order of G is not divisible by p. Then we have agreement up to a sign:*

$$\iota(\rho) = \pm(\mathbb{Z}_\ell \otimes_{\mathbb{Z}} \tau) \in \mathrm{Ext}^2_{\mathbb{Z}_\ell[G]}(\mathbb{Z}_\ell \otimes_{\mathbb{Z}} \Xi_S, \mathbb{Z}_\ell \otimes_{\mathbb{Z}} U_{S,T}).$$

We are grateful to an anonymous referee for pointing out that we can actually arrange to have a plus sign in the above formula. The crucial point is Step 6 in the proof of this main result. There we use a proposition of Burns, which has a minus sign in it, but Burns explains why "his" sign differs from the sign in a somewhat earlier paper [2] of Burns and Flach. So if we follow that latter paper, we get a plus sign.

To conclude this introduction, let us mention two links with work of other authors. In [7], Nickel compares two versions of the Equivariant Main Conjecture in Iwasawa Theory for number fields (the Ritter–Weiss version, and the version of the present authors as in [5]) in a very enlightening way. However this does not involve an explicit translation from one algebraic setup into the other. Nickel assumes both setups to be in place, plus the validity of the main theorem of [5], and then shows: The validity of the EMC for one setup is equivalent to its validity for the other. On p. 16 of [7], Nickel does mention his "guess" that the two complexes involved should be equivalent, which is more or less the statement of our main result in [6], but he gives no proof, saying that is not needed in his treatment. In this paper we confirm Nickel's "guess" in the function field setting. We would also like to mention that some proofs in this paper have some relation to deductions in recent work of Witte [10] (see also [11]), but the two approaches were developed independently of each other, and actually at roughly the same time. (The present paper has existed as a preprint for several years now.) For more detail on this last point let us refer to [6]; we should also say here that Witte treats number fields and function fields simultaneously.

2. A four term exact sequence derived from the Picard 1-motive

Write F for the arithmetic Frobenius over $\mathbb{F}_q$ and view it as a canonical topological generator of $\Gamma := G(\overline{\mathbb{F}}/\mathbb{F}_q)$. To deal with the ℓ-adic realization M of the 1-motive $\mathcal{M}_{\mathcal{S},\mathcal{T}}$, we need a little more notation. The lattice L_S was already defined in the introduction. It has a counterpart $L_{\mathcal{S}}$ at level $\overline{\mathbb{F}}$; this is simply the kernel of the map $\deg_{\overline{\mathbb{F}}} : \mathbb{Z}\mathcal{S} \to \mathbb{Z}$ sending every $w \in \mathcal{S}$ to 1. Then it is not difficult to see that we have canonical identifications

$$L_S = (L_{\mathcal{S}})^\Gamma, \qquad (\mathbb{Z}\mathcal{S})_\Gamma = \mathbb{Z}S.$$

Note that $(L_{\mathcal{S}})_\Gamma \neq L_S$, in general. In analogy with d_0, we have a divisor class map

$$\delta_0 : L_{\mathcal{S}} \longrightarrow \mathrm{cl}^0_{\mathcal{T}}(\mathcal{X}) = J_{X,T}(\overline{\mathbb{F}}).$$

Note that d_0 is the restriction of δ_0 to L_S (viewed now as a submodule of $L_{\mathcal{S}}$.)

The following proposition produces a very explicit four term exact sequence which is already reminiscent of a 2-extension defining the Tate class. The main goal of this note is to make this connection precise.

Proposition 2.1. *Assume that $1_{\mathbb{Z}_\ell} \otimes d_0$ is surjective.*

(a) *Then we have an exact sequence of $\mathbb{Z}_\ell[G]$–modules*

$$0 \longrightarrow \mathbb{Z}_\ell \otimes_{\mathbb{Z}} U_{S,T} \longrightarrow M \longrightarrow M \longrightarrow \mathbb{Z}_\ell \otimes_{\mathbb{Z}} (L_{\mathcal{S}})_\Gamma \longrightarrow 0,$$

where the map $M \to M$ in the middle is $(1-F)$, and the other maps are canonical.

(b) *Assume further that $1_{\mathbb{Z}_\ell} \otimes d$ is surjective. Then the canonical surjection $\mathbb{Z}\mathcal{S} \to \mathbb{Z}S$ induces an isomorphism $\mathbb{Z}_\ell \otimes_{\mathbb{Z}} (L_{\mathcal{S}})_\Gamma \simeq \mathbb{Z}_\ell \otimes_{\mathbb{Z}} \Xi_S$, and therefore we get an exact sequence of $\mathbb{Z}_\ell[G]$–modules*

$$0 \longrightarrow \mathbb{Z}_\ell \otimes_{\mathbb{Z}} U_{S,T} \longrightarrow M \longrightarrow M \longrightarrow \mathbb{Z}_\ell \otimes_{\mathbb{Z}} \Xi_S \longrightarrow 0,$$

whose middle map is $(1-F)$ as before.

Proof. (a). Let for the moment $M_{\mathcal{T}}$ denote the Tate module $T_\ell(\mathcal{M}_{\emptyset,\mathcal{T}})$, associated to $(\mathcal{X}/\mathcal{Y}, \mathcal{S}=\emptyset, \mathcal{T})$ (so $\mathcal{S}$ has changed to the empty set). Note that since $\mathcal{M}_{\emptyset,\mathcal{T}} = J_{X,T}(\overline{\mathbb{F}}) = \mathrm{cl}^0_{\mathcal{T}}(\mathcal{X})$ (see [4]), we have a canonical identification

$$(\mathbb{Q}/\mathbb{Z}) \otimes_{\mathbb{Z}} M_{\mathcal{T}} = \mathbb{Z}_\ell \otimes J_{X,T}(\overline{\mathbb{F}}) = \mathbb{Z}_\ell \otimes \mathrm{cl}^0_{\mathcal{T}}(\mathcal{X}).$$

The first thing to note is that one has an exact sequence of $\mathbb{Z}_\ell[G]$–modules

$$0 \longrightarrow T_\ell(\tau_{\mathcal{T}}) \longrightarrow M_{\mathcal{T}} \longrightarrow T_\ell(J_X) \longrightarrow 0,$$

where $\tau_{\mathcal{T}}$ is an $\overline{\mathbb{F}}_q$–torus depending only on $\mathcal{T}$ (see [4] for details.) As a consequence of the Weil conjectures, the eigenvalues of F acting on $T_\ell(J_X)$ are algebraic integers whose absolute values are equal to $q^{1/2}$. The eigenvalues of F acting on $\tau_{\mathcal{T}}$ are all equal to q times a root of unity (which is 1 if the torus is split). Consequently, F does not admit 1 as an eigenvalue when acting on $M_{\mathcal{T}}$ and therefore we have an isomorphism

$$(1-F) : \mathbb{Q} \otimes_{\mathbb{Z}} M_{\mathcal{T}} \cong \mathbb{Q} \otimes_{\mathbb{Z}} M_{\mathcal{T}}. \tag{2.1}$$

Note that the arguments above are valid for $\ell = p$ as well: in that case $T_p(\tau_{\mathcal{T}})$ is trivial and $T_p(J_X)$ is a $\mathbb{Z}_p$–submodule of the first crystalline homology group of $\mathcal{X}$, while F acts on the latter with the same eigenvalues as it does on any $T_\ell(J_X)$, for $\ell \neq p$. (See [4] for more details.)

Deligne constructs our module of interest M as a pull back of the canonical map

$$\mathbb{Q} \otimes_{\mathbb{Z}} M_{\mathcal{T}} \longrightarrow (\mathbb{Q}/\mathbb{Z}) \otimes_{\mathbb{Z}} M_{\mathcal{T}} = \mathbb{Z}_\ell \otimes J_{X,T}(\overline{\mathbb{F}}),$$

along the divisor map $1_{\mathbb{Z}_\ell}\otimes\delta_0 : \mathbb{Z}_\ell\otimes L_{\mathcal{S}} \to \mathbb{Z}_\ell\otimes J_{X,T}(\overline{\mathbb{F}}) = (\mathbb{Q}/\mathbb{Z})\otimes_{\mathbb{Z}} M_{\mathcal{T}}$. In other words, we have a commutative diagram of $\mathbb{Z}_\ell[G]$–modules which reads as follows (see [4].)

$$\begin{tikzcd} 0 \arrow[r] & M_{\mathcal{T}} \arrow[r] \arrow[d, "="] & M \arrow[r, "\varphi"] \arrow[d] & \mathbb{Z}_\ell\otimes L_{\mathcal{S}} \arrow[r] \arrow[d, "1_{\mathbb{Z}_\ell}\otimes\delta_0"] & 0 \\ 0 \arrow[r] & M_{\mathcal{T}} \arrow[r] & \mathbb{Q}\otimes_{\mathbb{Z}} M_{\mathcal{T}} \arrow[r, "\pi"] & (\mathbb{Q}/\mathbb{Z})\otimes_{\mathbb{Z}} M_{\mathcal{T}} \arrow[r] & 0. \end{tikzcd} \tag{2.2}$$

For simplicity, we will denote $\overline{y} := \pi(y)$, for all $y\in\mathbb{Q}\otimes_{\mathbb{Z}} M_{\mathcal{T}}$ in what follows. Also, we will use d_0 and δ_0 instead of $1_{\mathbb{Z}_\ell}\otimes d_0$ and $1_{\mathbb{Z}_\ell}\otimes\delta_0$, respectively.

The map φ in the diagram above induces a surjection $\varphi' : M_\Gamma \twoheadrightarrow (\mathbb{Z}_\ell\otimes L_{\mathcal{S}})_\Gamma$. We claim that φ' is in fact an isomorphism. Indeed, let

$$(x,\lambda)\in M\subset(\mathbb{Q}\otimes_{\mathbb{Z}} M_{\mathcal{T}})\times(\mathbb{Z}_\ell\otimes_{\mathbb{Z}} L_{\mathcal{S}})$$

be an element whose class in M_Γ is in the kernel of φ'; we want to show that $(x,\lambda)\in(1-F)M$. So we have $\overline{x}=\delta_0(\lambda)$, and there is an element $\mu\in(\mathbb{Z}_\ell\otimes_{\mathbb{Z}} L_{\mathcal{S}})$ with $\lambda=(1-F)\mu$. Now (2.1) allows us to write $x=(1-F)y$ for some $y\in\mathbb{Q}\otimes_{\mathbb{Z}} M_{\mathcal{T}}$. Then

$$(x,\lambda)=(1-F)(y,\mu).$$

Since $(1-F)(\overline{y}-\delta_0(\mu))=0$, we have $(\overline{y}-\delta_0(\mu))\in\mathbb{Z}_\ell\otimes\mathrm{cl}^0_T(X)$. The surjectivity of $1_{\mathbb{Z}_\ell}\otimes d_0$ gives an element $\nu\in\mathbb{Z}_\ell\otimes L_S$ with $d_0(\nu)=(\overline{y}-\delta_0(\mu))$. Therefore

$$(y,\mu+\nu)\in M,\qquad (x,\lambda)=(1-F)(y,\mu+\nu)\in(1-F)M,$$

as desired. This proves the injectivity of φ' and the exactness of the right half of the four term sequence in part (a).

We now look at the left half of the sequence in (a); so we have to look at the kernel of $(1-F)$ on M, i.e. at M^Γ. Since (2.1) gives $M_{\mathcal{T}}^\Gamma=(\mathbb{Q}\otimes_{\mathbb{Z}} M_{\mathcal{T}})^\Gamma=0$, the commutative diagram above leads to isomorphisms

$$\begin{aligned} M^\Gamma &\cong \ker((\mathbb{Z}_\ell\otimes_{\mathbb{Z}} L_{\mathcal{S}})^\Gamma \xrightarrow{1_{\mathbb{Z}_\ell}\otimes\delta_0} (\mathbb{Q}/\mathbb{Z}\otimes_{\mathbb{Z}} M_{\mathcal{T}})^\Gamma) \\ &\cong \ker(\mathbb{Z}_\ell\otimes L_S \xrightarrow{1_{\mathbb{Z}_\ell}\otimes d_0} \mathbb{Z}_\ell\otimes\mathrm{cl}^0_T(X)) \end{aligned}$$

Let us consider the canonical divisor map

$$\mathrm{div} : U_{S,T}\longrightarrow L_S,\qquad \mathrm{div}(x) := \sum_{w\in S}\mathrm{ord}_w(x)\cdot w,$$

where ord_w is the normalized valuation associated to w on the field $\mathbb{F}_q(X)$. First let us observe that since $T\neq\emptyset$, the map div is injective. However, we have

$$\ker(d_0)\cong\mathrm{im}(\mathrm{div}),$$

from the definitions of $U_{S,T}$ and $\mathrm{cl}^0_T(X)$. Therefore div induces an isomorphism

$$(\mathbb{Z}_\ell \otimes U_{S,T}) \cong \ker(1_{\mathbb{Z}_\ell} \otimes d_0),$$

which concludes the proof of (a).

Remark 2.2. There is a more conceptual but less explicit proof of the "right half" of the exactness statement (a). One uses the exact sequence

$$0 \longrightarrow M_{\mathcal{T}} \longrightarrow M \longrightarrow \mathbb{Z}_\ell \otimes L_{\mathcal{S}} \longrightarrow 0$$

and the ensuing long exact sequence

$$\cdots \longrightarrow \mathbb{Z}_\ell \otimes L_S = (\mathbb{Z}_\ell \otimes L_{\mathcal{S}})^\Gamma \longrightarrow (M_{\mathcal{T}})_\Gamma \longrightarrow M_\Gamma \longrightarrow (\mathbb{Z}_\ell \otimes L_{\mathcal{S}})_\Gamma \longrightarrow 0.$$

The fact that 1 is not an eigenvalue of F acting on $M_{\mathcal{T}}$ leads to an explicit (slightly tricky) isomorphism $\iota : (M_{\mathcal{T}})_\Gamma \cong (\mathbb{Z}_\ell \otimes J_T(\overline{\mathbb{F}}))^\Gamma = \mathbb{Z}_\ell \otimes \mathrm{cl}^0_T(X)$. (See [4] for details.) In the last exact sequence, if we identify $(M_{\mathcal{T}})_\Gamma$ with $\mathbb{Z}_\ell \otimes \mathrm{cl}^0_T(X)$ via ι^{-1}, the map $L^\Gamma_{\mathcal{S}} \to (M_{\mathcal{T}})_\Gamma$ is identified with $1_{\mathbb{Z}_\ell} \otimes d_0$ (this has to be checked). Then the surjectivity of $1_{\mathbb{Z}_\ell} \otimes d_0$ shows that $M_\Gamma \cong (L_{\mathcal{S}})_\Gamma$. We leave the details to the interested reader.

(b). By definition there is a short exact sequence

$$0 \longrightarrow \mathbb{Z}_\ell \otimes_{\mathbb{Z}} L_{\mathcal{S}} \longrightarrow \mathbb{Z}_\ell \mathcal{S} \longrightarrow \mathbb{Z}_\ell \longrightarrow 0.$$

The ensuing long exact sequence of Γ-invariants and covariants reads as follows:

$$0 \longrightarrow K \longrightarrow (\mathbb{Z}_\ell \otimes_{\mathbb{Z}} L_{\mathcal{S}})_\Gamma \longrightarrow (\mathbb{Z}_\ell \mathcal{S})_\Gamma \cong \mathbb{Z}_\ell S \xrightarrow{\text{aug}} \mathbb{Z}_\ell \longrightarrow 0,$$

where $K = \mathrm{coker}((\mathbb{Z}_\ell \mathcal{S})^\Gamma \to \mathbb{Z}_\ell) = \mathrm{coker}(\mathbb{Z}_\ell S \to \mathbb{Z}_\ell)$. If $1_{\mathbb{Z}_\ell} \otimes d$ is surjective, there is a point in S whose degree is prime to ℓ, and therefore $K = 0$. This proves that $(\mathbb{Z}_\ell \otimes_{\mathbb{Z}} L_{\mathcal{S}})_\Gamma$ identifies with $\ker(\mathbb{Z}_\ell S \to \mathbb{Z}_\ell)$. note that the map $(\mathbb{Z}_\ell \mathcal{S})_\Gamma = \mathbb{Z}_\ell S \to \mathbb{Z}_\ell$ in the last displayed exact sequence sends every point of S to 1, i.e. is the actual augmentation map, not the degree map. Therefore its kernel $\ker(\mathbb{Z}_\ell S \to \mathbb{Z}_\ell)$ is exactly the lattice Ξ_S, by definition. Now (b) follows from (a). $\square$

Remark 2.3. There is an analogue to the proposition above in the number field case, but it is more difficult to prove (see [6] for details), since one main ingredient is missing. Indeed, it is not true in general that γ (a chosen topological generator of the Galois group $\Gamma \cong \mathbb{Z}_p$ of the cyclotomic $\mathbb{Z}_p$–extension of a number field) avoids the eigenvalue 1 on the relevant modules. This γ is a (non-canonical) analogue of the Frobenius F in the number field setting.

3. The Tate sequence and its connection to the 1-motive

We now consider a Tate sequence attached to the data $(X/Y, S)$, satisfying the properties in the introduction. In contrast to the sequence in Proposition 2.1, the set T is not involved, and we have a sequence of $\mathbb{Z}[G]$-modules, not of $\mathbb{Z}_\ell[G]$-modules. Assuming that S is large in the absolute sense (not relative to any set T, meaning that the divisor class map $\mathbb{Z}S \to \mathrm{cl}(X)$ is surjective), Tate proves in [8] that there is an exact sequence of $\mathbb{Z}[G]$–modules, now called a Tate sequence,

$$0 \longrightarrow U_S \longrightarrow A \longrightarrow B \longrightarrow \Xi_S \longrightarrow 0 \tag{3.1}$$

with the following properties: the modules A and B are finitely generated over $\mathbb{Z}[G]$ and G-cohomologically trivial (i.e. of finite projective dimension over $\mathbb{Z}[G]$); moreover the class $\tau = \tau_{X/Y,S}$ of this sequence in $\mathrm{Ext}^2_{\mathbb{Z}[G]}(\Xi_S, U_S)$ is Tate's canonical class, which is constructed through "an interplay of local and global fundamental classes". We will not review the details of its construction and refer to the literature, particularly to Tate's original paper [8], or Chapter II.5 in Tate's book [9]) (which however is written for the number field case). In the sequel, all modules and Ext groups will be over $\mathbb{Z}[G]$ or $\mathbb{Z}_\ell[G]$; the context will make it quite clear which of the two rings is meant.

Remark 3.1. As Tate shows, the module B may be taken to be $\mathbb{Z}[G]$-projective, but A will contain torsion coming from U_S, so we cannot expect it to be projective.

Now let us assume that $(X/Y, S, T)$ are as in the introduction and that S is large relative to T. Since we have an obvious surjective morphism $\mathrm{cl}_T(X) \to \mathrm{cl}(X)$, then S is large in the absolute sense as well. Therefore we have a Tate exact sequence (3.1) and a Tate class $\tau := \tau_{X/Y,S}$ and, part (b) of Proposition 2.1 gives an exact sequence of the form

$$0 \longrightarrow \mathbb{Z}_\ell \otimes_{\mathbb{Z}} U_{S,T} \longrightarrow M \longrightarrow M \longrightarrow \mathbb{Z}_\ell \otimes_{\mathbb{Z}} \Xi_S \longrightarrow 0,$$

for all primes ℓ. Let $\rho = \rho_{X/Y,S,T,\ell}$ denote the class of this 2-extension in the group $\mathrm{Ext}^2_{\mathbb{Z}_\ell[G]}(\mathbb{Z}_\ell \otimes_{\mathbb{Z}} \Xi_S, \mathbb{Z}_\ell \otimes_{\mathbb{Z}} U_{S,T})$. Before we can compare τ and ρ, we need to clear up a small technical point.

Lemma 3.2. *If S is large relative to T, then the canonical map*

$$\iota : \mathrm{Ext}^2_{\mathbb{Z}[G]}(\Xi_S, U_{S,T}) \longrightarrow \mathrm{Ext}^2_{\mathbb{Z}[G]}(\Xi_S, U_S)$$

induced by the inclusion $U_{S,T} \to U_S$ is an isomorphism.

Proof. In general, we have a canonical exact sequence of $\mathbb{Z}[G]$–modules

$$0 \longrightarrow U_{S,T} \longrightarrow U_S \longrightarrow \bigoplus_{w\in T} \kappa(w)^\times \longrightarrow \mathrm{cl}^0_T(X)/\operatorname{im}(d_0), \tag{3.2}$$

where $\kappa(w)$ is the residue field corresponding to w. Now, if d_0 is surjective, this induces an isomorphism of $\mathbb{Z}[G]$–modules

$$U_S/U_{S,T} \cong \bigoplus_{w\in T} \kappa(w)^\times.$$

A standard application of Shapiro's lemma combined with Hilbert's theorem 90 shows that the module on the right is G–cohomologically trivial, therefore of projective dimension 1 over $\mathbb{Z}[G]$. This implies that

$$\mathrm{Ext}^i_{\mathbb{Z}[G]}(N, (U_S/U_{S,T})) = 0, \qquad \text{for all } i \geq 1,$$

and all $\mathbb{Z}[G]$–modules N with no $\mathbb{Z}$–torsion. This vanishing for $i = 1, 2$ and $N := \Xi_S$ plus the long Ext exact sequence associated to (3.2) is just what one needs to show that the canonical map in the lemma is an isomorphism. □

It is also easy to see that Ext commutes with the functor $\mathbb{Z}_\ell \otimes_{\mathbb{Z}} -$, as long as we stick to finitely generated modules. Let us repeat this more formally: If N and N' are finitely generated $\mathbb{Z}[G]$–modules, then the canonical map

$$\mathrm{Ext}^i_{\mathbb{Z}[G]}(N, N') \longrightarrow \mathrm{Ext}^i_{\mathbb{Z}_\ell[G]}(\mathbb{Z}_\ell \otimes_{\mathbb{Z}} N, \mathbb{Z}_\ell \otimes_{\mathbb{Z}} N')$$

(which will be written as $1\otimes_{\mathbb{Z}} *$ or sometimes suppressed entirely in context) induces an isomorphism

$$\mathbb{Z}_\ell \otimes_{\mathbb{Z}} \mathrm{Ext}^i_{\mathbb{Z}[G]}(N, N') \cong \mathrm{Ext}^i_{\mathbb{Z}_\ell[G]}(\mathbb{Z}_\ell \otimes_{\mathbb{Z}} N, \mathbb{Z}_\ell \otimes_{\mathbb{Z}} N').$$

Hence the comparison in the following theorem makes sense.

Theorem 3.3. *Assume that S is large relative to T. Then we have*

$$(1_{\mathbb{Z}_\ell} \otimes \iota)(\rho) = \pm 1 \otimes_{\mathbb{Z}} \tau \in \mathrm{Ext}^2_{\mathbb{Z}_\ell[G]}(\mathbb{Z}_\ell \otimes_{\mathbb{Z}} \Xi_S, \mathbb{Z}_\ell \otimes_{\mathbb{Z}} U_S),$$

for all primes ℓ, assuming that $\ell \nmid |G|$ if $\ell = p$.

The proof of this result will occupy the entire next section.

Remark 3.4. As will become clear below, one essential ingredient in the proof is the link between Weil-étale cohomology (with $\mathbb{Z}(1)$–coefficients) and ℓ–adic étale cohomology with coefficients in $\mathbb{Z}_\ell(1)$, for $\ell \neq p$. This is the main reason why we are imposing the condition $p \nmid |G|$, when $\ell = p$. A method we are envisaging for dealing with the general case when $\ell = p$ will involve links between crystalline cohomology and Weil-étale cohomology; we hope to be able to come back to this at a later occasion.

4. Proof of Theorem 3.3

The argument will proceed in several steps. As a “zeroth step” let us get the case $\ell = p$ out of the way; this case eludes the arguments given below. So assume $\ell = p$; we made the very strict hypothesis that in this case, p does not divide $|G|$. This will allow to dispatch of this case quickly. Indeed, under this hypothesis the module $\mathbb{Z}_p \otimes \Xi_S$ will be projective over $\mathbb{Z}_p[G]$ since it is G–cohomologically trivial and has no $\mathbb{Z}_p$–torsion. But then we have $\mathrm{Ext}^2_{\mathbb{Z}_p[G]}(\mathbb{Z}_p \otimes \Xi_S, \mathbb{Z}_p \otimes U_S) = 0$, and therefore the Theorem above is trivially true in this case.

So let us assume $\ell \neq p$ from now on. This hypothesis will be heavily used in the arguments which follow, which are based on previous work by Deligne [3] and Burns [1]. In the first three steps we will work over $\overline{\mathbb{F}}$. Recall that calligraphic letters stand for objects over $\overline{\mathbb{F}}$: $\mathcal{X} = \overline{\mathbb{F}} \times_{\mathbb{F}_q} X$, $\mathcal{T}$, $\mathcal{S}$ and so on. Let us denote by j and i the open, respectively closed immersion

$$j : \mathcal{X} \setminus (\mathcal{S} \cup \mathcal{T}) \longrightarrow \mathcal{X} \setminus \mathcal{S}, \qquad i : \mathcal{T} \longrightarrow \mathcal{X} \setminus \mathcal{S}.$$

Step 1: We claim that there is a canonical isomorphism $M \cong \mathrm{H}^1_{et}(\mathcal{X} \setminus \mathcal{S}, j_!\mathbb{Z}_\ell(1))$. This first step essentially comes from 10.3.6 in [3]. As in loc. cit., let us fix a positive integer n coprime to p. Eventually, n will equal an arbitrary power of ℓ.

Let $(\mathcal{X} \setminus \mathcal{S})_{\mathcal{T}}$ and $\mathcal{X}_{\mathcal{T}}$ be the singular curves obtained from $\mathcal{X} \setminus \mathcal{S}$ and $\mathcal{X}$ by contracting $\mathcal{T}$ to a single point, call this point t. (In Deligne's notation in loc. cit. $X := (\mathcal{X} \setminus \mathcal{S})_{\mathcal{T}}$ and $\overline{X} := \mathcal{X}_{\mathcal{T}}$.) Note that the Picard group of the singular curve $\mathcal{X}_{\mathcal{T}}$ is precisely the generalized Jacobian $J_T(\overline{\mathbb{F}}) = \mathrm{cl}^0_{\mathcal{T}}(\mathcal{X})$. This is a well-known consequence of a Meyer–Vietoris sequence. Hence, we have an equality

$$T_{\mathbb{Z}/n\mathbb{Z}}(\mathrm{H}^1_m((\mathcal{X} \setminus \mathcal{S})_{\mathcal{T}})(1)) = \mathcal{M}_{\mathcal{S},\mathcal{T}}[n]$$

between Deligne's 1-motivic term $T_{\mathbb{Z}/n\mathbb{Z}}(\mathrm{H}^1_m((\mathcal{X} \setminus \mathcal{S})_{\mathcal{T}})(1))$ and the n–torsion $\mathcal{M}_{\mathcal{S},\mathcal{T}}[n]$ of our 1–motive $\mathcal{M}_{\mathcal{S},\mathcal{T}}$. (See [4] for a very concrete definition of $\mathcal{M}_{\mathcal{S},\mathcal{T}}[n]$.)

Now in 10.3.6 of [3] Deligne establishes a canonical isomorphism

$$T_{\mathbb{Z}/n\mathbb{Z}}(\mathrm{H}^1_m((\mathcal{X} \setminus \mathcal{S})_{\mathcal{T}})(1)) \cong \mathrm{H}^1_{et}((\mathcal{X} \setminus \mathcal{S})_{\mathcal{T}}, \mathbb{Z}/n(1)).$$

Let us consider the open immersion of singular curves $j_t : (\mathcal{X} \setminus \mathcal{S})_{\mathcal{T}} \setminus \{t\} \to (\mathcal{X} \setminus \mathcal{S})_{\mathcal{T}}$. By a direct argument (similar to Step 2 below but easier) we observe that

$$\mathrm{H}^1_{et}((\mathcal{X} \setminus \mathcal{S})_{\mathcal{T}}, \mathbb{Z}/n(1)) \cong \mathrm{H}^1_{et}((\mathcal{X} \setminus \mathcal{S})_{\mathcal{T}}, (j_t)_!\mathbb{Z}/n(1)).$$

Let us consider the natural map $\pi : (\mathcal{X} \setminus \mathcal{S}) \to (\mathcal{X} \setminus \mathcal{S})_{\mathcal{T}}$. Then we have an equality of sheaves $\pi_*(j_t)_!\mathbb{Z}/n(1) = j_!\mathbb{Z}/n(1)$ and we get

$$\mathrm{H}^1_{et}((\mathcal{X} \setminus \mathcal{S})_{\mathcal{T}}, (j_t)_!\mathbb{Z}/n(1)) \cong \mathrm{H}^1_{et}((\mathcal{X} \setminus \mathcal{S}), j_!\mathbb{Z}/n(1)).$$

If we combine the isomorphisms above, we obtain

$$\mathcal{M}_{\mathcal{S},\mathcal{T}}[n] \cong \mathrm{H}^1_{et}((\mathcal{X} \setminus \mathcal{S}), j_!\mathbb{Z}/n(1)),$$

for all n coprime to p. Now we let $n := \ell^\nu$ and pass to a projective limit as $\nu \to \infty$ to obtain the desired canonical isomorphism

$$M \cong \mathrm{H}^1_{et}(\mathcal{X} \setminus \mathcal{S}, j_!\mathbb{Z}_\ell(1)).$$

Step 2: We claim that the étale sheaf $j_!\mathbb{Z}_\ell(1)$ on $\mathcal{X} \setminus \mathcal{S}$ has nonzero cohomology only in degree one. To show this, one first looks at the étale sheaf $\mathbb{Z}_\ell(1)$ on $\mathcal{X} \setminus \mathcal{S}$. We have

$$\mathrm{H}^0(\mathcal{X} \setminus \mathcal{S}, \mathbb{Z}_\ell(1)) = \mathbb{Z}_\ell(1), \qquad \mathrm{H}^2(\mathcal{X} \setminus \mathcal{S}, \mathbb{Z}_\ell(1)) = 0. \tag{4.1}$$

The first equality above is obvious, while the second requires a proof. For that, one considers the exact sequence of étale sheaves on $\mathcal{X} \setminus \mathcal{S}$

$$0 \longrightarrow \mathbb{Z}/\ell^n\mathbb{Z}(1) \longrightarrow \mathbb{G}_m \xrightarrow{\ell^n} \mathbb{G}_m \longrightarrow 0.$$

Since $\mathrm{H}^1(\mathcal{X} \setminus \mathcal{S}, \mathbb{G}_m) = \mathrm{Pic}(\mathcal{X} \setminus \mathcal{S})$, which is a quotient of $\mathrm{Pic}^0 \mathcal{X}$ and therefore divisible, we obtain an isomorphism

$$\mathrm{H}^2(\mathcal{X} \setminus \mathcal{S}, \mathbb{Z}/\ell^n\mathbb{Z}(1)) \cong \mathrm{H}^2(\mathcal{X} \setminus \mathcal{S}, \mathbb{G}_m)[\ell^n].$$

However, $\mathrm{H}^2(\mathcal{X} \setminus \mathcal{S}, \mathbb{G}_m) = \mathrm{Br}(\mathcal{X} \setminus \mathcal{S})$, which injects in the Brauer group $\mathrm{Br}(\overline{\mathbb{F}}(\mathcal{X}))$ of the function field of $\mathcal{X}$, which is trivial by Tsen's theorem. This gives equalities $\mathrm{H}^2(\mathcal{X} \setminus \mathcal{S}, \mathbb{Z}/\ell^n\mathbb{Z}(1)) = 0$, for all n. By taking a projective limit when $n \to \infty$ this concludes the proof of (4.1).

Now we consider the long exact sequence in étale cohomology corresponding to the following short exact sequence of sheaves on $\mathcal{X} \setminus \mathcal{S}$.

$$0 \longrightarrow j_!\mathbb{Z}_\ell(1) \longrightarrow \mathbb{Z}_\ell(1) \longrightarrow i_*\mathbb{Z}_\ell(1) \longrightarrow 0. \tag{4.2}$$

In light of (4.1) the corresponding long exact sequence in cohomology reads as follows.

$$0 \longrightarrow \mathrm{H}^0(\mathcal{X} \setminus \mathcal{S}, j_!\mathbb{Z}_\ell(1)) \longrightarrow \mathbb{Z}_\ell(1) \xrightarrow{d} \bigoplus_{v \in \mathcal{T}} \mathbb{Z}_\ell(1) \longrightarrow \mathrm{H}^1(\mathcal{X} \setminus \mathcal{S}, j_!\mathbb{Z}_\ell(1))$$
$$\longrightarrow \mathrm{H}^1(\mathcal{X} \setminus \mathcal{S}, \mathbb{Z}_\ell(1)) \longrightarrow \mathrm{H}^1(\mathcal{X} \setminus \mathcal{S}, i_*\mathbb{Z}_\ell(1)) \longrightarrow \mathrm{H}^2(\mathcal{X} \setminus \mathcal{S}, j_!\mathbb{Z}_\ell(1)) \longrightarrow 0.$$

Since the map d is just a diagonal embedding and therefore injective, we have

$$\mathrm{H}^0(\mathcal{X} \setminus \mathcal{S}, j_!\mathbb{Z}_\ell(1)) = 0.$$

On the other hand, since the functor i_* is exact, we have

$$\mathrm{H}^1(\mathcal{X} \setminus \mathcal{S}, i_*\mathbb{Z}_\ell(1)) \cong \mathrm{H}^1(\mathcal{T}, \mathbb{Z}_\ell(1)) = 0,$$

where the vanishing of the second cohomology group is a consequence of the fact that $\mathcal{T}$ is just a finite union of copies of $\mathrm{Spec}(\overline{\mathbb{F}})$ (spectrum of an algebraically closed field). Hence we also have $\mathrm{H}^2(\mathcal{X}\setminus\mathcal{S}, j_!\mathbb{Z}_\ell(1)) = 0$, which concludes Step 2.

Remark 4.1. From general principles (see for example [1]) it is known that $R\Gamma(\mathcal{X}\setminus\mathcal{S}, j_!\mathbb{Z}_\ell(1))$ is represented by a perfect complex of $\mathbb{Z}_\ell[G]$-modules. This general principle is very nicely illustrated here: the complex has only one nonzero module M in degree 1, and that module is indeed $\mathbb{Z}_\ell[G]$-projective.

Step 3: Hence $R\Gamma(\mathcal{X}\setminus\mathcal{S}, j_!\mathbb{Z}_\ell(1))$ is represented by the complex which has M in degree one and zero in all other degrees. In fact Step 1 tells us that the first cohomology of any complex (say $C^\bullet$) which represents this $R\Gamma$, is M, and all other cohomology groups vanish. From this it is rather easy to construct a quasi-isomorphism between the complexes $C^\bullet$ and $[\cdots \to 0 \to M \to 0 \to 0 \cdots]$ with M in degree 1.

Step 4: Descent to the $\mathbb{F}_q$-level: a representative for $R\Gamma(X\setminus S, j_!\mathbb{Z}_\ell(1))$. Now we work at the $\mathbb{F}_q$–level, and by abuse of notation we use j and i again for the open and closed immersion at the X–level, respectively:

$$j : X\setminus(S\cup T) \longrightarrow X\setminus S, \qquad i : T \longrightarrow X\setminus S.$$

If we combine [1] with our Step 1 above we get that $R\Gamma(X\setminus S, j_!\mathbb{Z}_\ell(1))$ is represented by the complex

$$\cdots \longrightarrow 0 \longrightarrow \overset{1}{M} \overset{1-F}{\longrightarrow} \overset{2}{M} \longrightarrow 0 \longrightarrow \cdots, \tag{4.3}$$

where M shows up in degrees 1 and 2, and F is the Frobenius morphism as usual.

In more detail: Let $\mathcal{F}$ be an etale sheaf on $X\setminus S$ and $C^\bullet$ a complex representing $R\Gamma(\mathcal{X}\setminus\mathcal{S},\mathcal{F})$, which has an appropriate action of G and Frobenius, in the sense of loc. cit. p. 370 bottom. Then $R\Gamma(X\setminus S,\mathcal{F})$ is quasi-isomorphic to the shifted mapping cone

$$\mathrm{Cone}(C^\bullet \overset{1-F}{\longrightarrow} C^\bullet)[-1],$$

as shown in loc. cit. p. 371 (one also has to use Lemma 4 of loc. cit.). We now take $C^\bullet = [M]$ concentrated in degree 1 and use the result above combined with our Step 3. The shifted mapping cone above is exactly the complex (4.3).

Step 5: Removing $j_!$: a representative for $R\Gamma(X\setminus S,\mathbb{Z}_\ell(1))$. We look at the standard exact sequence of sheaves in Step 2, now at level $\mathbb{F}_q$, that is, over $X\setminus S$; its first (respectively second) term is the sheaf $\mathbb{Z}_\ell(1)$ with (respectively without) $j_!$, and the third term has cohomology in dimension one only.

We have determined a simple complex that represents $R\Gamma(X \setminus S, j_!\mathbb{Z}_\ell(1))$, and we want to deduce a complex almost as simple that represents $R\Gamma(X \setminus S, \mathbb{Z}_\ell(1))$. Let ι denote the inclusion of sheaves $j_!\mathbb{Z}_\ell(1) \to \mathbb{Z}_\ell(1)$. (Note that this map, somewhat abusively, also stands in Lemma 3.2 for the map on Ext^2 level induced by the inclusion $U_{S,T} \to U_S$.) One checks that $\mathrm{H}^1(X \setminus S, \iota)$ is injective with cokernel $U_S/U_{S,T} \cong \mathrm{H}^1_{et}(X, i^*\mathbb{Z}_\ell(1))$, and that $\mathrm{H}^2(X \setminus S, \iota)$ is an isomorphism.

There is a map f of complexes from some complex $C^\bullet$ which represents $R\Gamma(X \setminus S, j_!\mathbb{Z}_\ell(1))$ (as in Step 3) to some complex $D^\bullet$ which represents $R\Gamma(X \setminus S, \mathbb{Z}_\ell(1))$, such that f induces $\mathrm{H}^\bullet(X \setminus S, \iota)$ on cohomology. In particular it gives the inclusion $U_{S,T} \to U_S$ on H^1, and an isomorphism on H^2. Let $C'^\bullet$ be the complex given by pushing out:

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathbb{Z}_\ell U_{S,T} & \longrightarrow & M & \longrightarrow & M \\
\downarrow & & \downarrow & & \downarrow & & \| \\
0 & \longrightarrow & \mathbb{Z}_\ell U_S & \longrightarrow & (C')^1 & \longrightarrow & (C')^2
\end{array}$$

Then f extends to a map of complexes f' from $C'^\bullet$ to $D^\bullet$, just by the universal property of the pushout. One verifies that f' is now an isomorphism on H^1, and nothing has changed on H^2, so f' is a quasi-isomorphism. On the other hand, the pushed-out complex $C'^\bullet$ is quasi-isomorphic to a complex arising from $[M \to M]$ by the same pushout procedure, call the resulting complex $[M' \to M]$.

In a slightly different language: if we view representing complexes of $R\Gamma(X \setminus S, \mathcal{F})$ as Yoneda classes of 2-extensions of H^1 by H^0, with $\mathcal{F}$ being either $j_!\mathbb{Z}_\ell(1)$ or $\mathbb{Z}_\ell(1)$, then the pushout with ι sends a representative of $R\Gamma(X \setminus S, j_!\mathbb{Z}_\ell(1))$ to a representative of $R\Gamma(X \setminus S, \mathbb{Z}_\ell(1))$. In other words, the complex $[M' \to M]$ represents $R\Gamma(X \setminus S, \mathbb{Z}_\ell(1))$ and can be seen as a 2-extension of $(\mathbb{Z}_\ell L_{\mathcal{S}})_\Gamma$ by $\mathbb{Z}_\ell U_S$.

Step 6: Putting things together. We invoke [1] again, to obtain that (maybe up to a sign) $R\Gamma(X \setminus S, \mathbb{Z}_\ell(1))$ is the ℓ-adified Tate class. In more detail: We use Lemma 3 of loc. cit. in order to identify ℓ-adified Weil cohomology with etale cohomology with $\mathbb{Z}_\ell(1)$-coefficients, and then Proposition 4.1 of loc. cit. Two things should be noted:

(i) When quoting the aforementioned lemma we again use $\ell \neq p$.

(ii) Things are simpler in our case than in the situation of Proposition 4.1 in [1], since Burns is allowing $A_{K,S}$ (his notation for the class group of $X \setminus S$) to be nonzero and c.t., whereas our assumption "d_0 surjective" actually makes it zero at least in the ℓ-part, and we may forget about the set S' and the maps ι_S, $\iota_{S,S'}$ in that proposition.

So the ℓ-adified Tate class is represented by the complex $[M' \to M]$ constructed in the previous step. Since on the Ext level this complex is gotten by applying ι (in the sense of Lemma 3.2) to $[M \to M]$, we are done with the proof of Theorem 3.3.

5. An example

Finally, we would like to exhibit an example which is as simple as possible without being trivial. Even this very simple setup presents interesting complexities.

Let $\mathbb{F}_q = \mathbb{F}_{11}$, Y the projective line (with variable x) and $X \to Y$ the degree 2 cover given by $y^2 = x^3 - x$. Of course X is an elliptic curve. It has 12 points over $\mathbb{F}_{11}$, so $|\mathrm{cl}^0(X)| = 12$. We take $\ell = 2$ and $S \subset X$ to be the set of ramified points. It consists of ∞ and the three points $P_r := (r, 0)$ with $r = -1, 0, 1$. For T we may take any pair of Galois-conjugate $\mathbb{F}_{11}$-rational points of X outside S; but since we will need to do explicit calculations, we make a choice: $T = \{(4, \pm 4)\} = \{t_+, t_-\}$.

Lemma 5.1. *$d_0 : L_S \to \mathrm{cl}^0_T(X)$ induces an isomorphism*

$$\delta : L_S/2L_S \cong \mathbb{Z}_2 \otimes \mathrm{cl}^0_T(X).$$

Proof. At the finite level we have the usual exact sequence

$$1 \longrightarrow \mathbb{Z}_2 \otimes \frac{\mathbb{F}_{11}^\times \times \mathbb{F}_{11}^\times}{\mathrm{diag}(\mathbb{F}_{11}^\times)} \xrightarrow{\rho} \mathbb{Z}_2 \otimes \mathrm{cl}^0_T(X) \longrightarrow \mathbb{Z}_2 \otimes \mathrm{cl}^0(X) \longrightarrow 1,$$

where $\rho((\alpha, \beta)) = \mathrm{div}(f)$, with $f \in \mathbb{F}_{11}(X)$ such that $f(t_+) = \alpha$ and $f(t_-) = \beta$, for all $(\alpha, \beta) \in (\mathbb{F}_{11}^\times \times \mathbb{F}_{11}^\times)$. On the one hand, $\mathbb{Z}_2 \otimes \mathrm{cl}^0(X)$ has order 4, therefore we have

$$\mathbb{Z}_2 \otimes \mathrm{cl}^0(X) = \mathrm{cl}^0(X)[2] = \langle [P_0 - \infty], [P_1 - \infty] \rangle.$$

On the other hand, the domain of ρ has order 2 and is generated by $(4, -4)$. Also,

$$\mathrm{div}(y) = P_{-1} + P_0 + P_1 - 3\infty$$

and $(y(t_+), y(t_-)) = (4, -4)$. Consequently, the exact sequence above (kernel and cokernel are elementary 2-groups) shows that $\mathbb{Z}_2 \otimes \mathrm{cl}^0_T(X)$ is an elementary 2-group of order 8 and

$$\mathbb{Z}_2 \otimes \mathrm{cl}^0_T(X) = \mathrm{cl}^0_T(X)[2] = \langle [P_1 - \infty], [P_0 - \infty], [P_{-1} + P_0 + P_1 - 3\infty] \rangle.$$

This implies that the map $\delta : L_S/2L_S \to \mathbb{Z}_2 \otimes \mathrm{cl}^0_T(X)$ exists; it is a surjective morphisms of two groups of order 8, and therefore a group isomorphism. $\square$

Corollary 5.2. *The divisor map induces an isomorphism*

$$1 \otimes \mathrm{div} : \mathbb{Z}_2 \otimes U_{S,T} \cong 2(\mathbb{Z}_2 \otimes L_S).$$

Proof. This is a direct consequence of the above Lemma and the exact sequence

$$0 \longrightarrow \mathbb{Z}_2 \otimes U_{S,T} \xrightarrow{\mathrm{div}} \mathbb{Z}_2 \otimes L_S \xrightarrow{1\otimes d_0} \mathbb{Z}_2 \otimes \mathrm{cl}_T^0 \longrightarrow 0.$$

(The surjectivity of $1 \otimes d_0$ is a consequence of the Lemma.) □

Let us note that since in this case all points in S are $\mathbb{F}_{11}$–rational, we have an equality $\Xi_S = L_S$.

Definition 5.3. *We let* $\psi := 1/2(1\otimes \mathrm{div}) : \mathbb{Z}_2 \otimes U_{S,T} \cong \mathbb{Z}_2 \otimes L_S = \mathbb{Z}_2 \otimes \Xi_S$.

Let us write $G = \mathrm{Gal}(X/Y) = \{\mathrm{id}, \sigma\}$. We recall $M_{\mathcal{T}}$ is the 2-adic Tate module $T_2(\mathrm{cl}_{\mathcal{T}}^0(\mathcal{X}))$. So, in this case we have

$$M_{\mathcal{T}}/2 \cong \mathrm{cl}_{\mathcal{T}}^0(\mathcal{X})[2] = \mathrm{cl}_T^0(X)[2].$$

The isomorphism above is canonical (and general), while the equality (in general just an inclusion from right to left) is a consequence of the fact that in this case, since T consists of two $\mathbb{F}_{11}$–rational points (therefore $|\mathcal{T}| = 2$) and $\mathrm{genus}(X) = 1$, we have $|\mathrm{cl}_{\mathcal{T}}^0(\mathcal{X})[2]| = 8 = |\mathrm{cl}_T^0(X)[2]|$.

Since S and T consist of $\mathbb{F}_q$–rational points and $\mathrm{genus}(X) = 1$, we have

$$\mathrm{rank}_{\mathbb{Z}_2}(M_{\mathcal{T}}) = \mathrm{rank}_{\mathbb{Z}_2}(\mathbb{Z}_2 \otimes L_{\mathcal{S}}) = 3,$$

and σ acts on $M_{\mathcal{T}}$ as multiplication by (-1) and on $\mathbb{Z}_2 \otimes L_{\mathcal{S}}$ as multiplication by $(+1)$. Moreover, since $\mathbb{Z}_2[G]$ is a local ring (as G is a 2–group) and M is $\mathbb{Z}_2[G]$–projective (as proved in [4]), M has to be $\mathbb{Z}_2[G]$–free of rank 3. (It does have rank 6 over $\mathbb{Z}_2$, according to the top exact sequence of (2.2).) Consequently, we have the following isomorphisms of $\mathbb{Z}_2[G]$–modules:

$$M \cong \mathbb{Z}_2[G]^3, \qquad M_{\mathcal{T}} \cong I_G^3, \qquad \mathbb{Z}_2 \otimes L_{\mathcal{S}} \simeq \mathbb{Z}_2^3,$$

where $I_G = \mathbb{Z}_2 \cdot (\sigma - 1)$ is the augmentation ideal in $\mathbb{Z}_2[G]$ and $\mathbb{Z}_2$ is viewed with the trivial G–action. Moreover, we have $\mathbb{Z}_2[G]$–module isomorphisms

$$\mathbb{Z}_2 \otimes U_{S,T} \cong \mathbb{Z}_2 L_S = \mathbb{Z}_2 \otimes \Xi_S \simeq \mathbb{Z}_2^3,$$

because $\mathbb{Z}_2 \otimes L_S = \mathbb{Z}_2 \otimes L_{\mathcal{S}}$ (again, the $\mathbb{F}_q$–rationality of points in S plays a role.)

In this particular situation one has group isomorphisms

$$\begin{aligned}
\mathrm{Ext}^2_{\mathbb{Z}_2[G]}(\mathbb{Z}_2 \otimes L_S, \mathbb{Z}_2 \otimes U_{S,T}) &\cong \widehat{\mathrm{H}}^2(G, \mathrm{Hom}_{\mathbb{Z}_2}(\mathbb{Z}_2 \otimes \Xi_S, \mathbb{Z}_2 \otimes U_{S,T}) \\
&\cong \widehat{\mathrm{H}}^0(G, \mathrm{Hom}_{\mathbb{Z}_2}(\mathbb{Z}_2 \otimes \Xi_S, \mathbb{Z}_2 \otimes U_{S,T}) \\
&\cong \mathrm{Hom}_{\mathbb{Z}_2}(\mathbb{Z}_2 \otimes \Xi_S, \mathbb{Z}_2 \otimes U_{S,T})/2 \\
&\cong M_3(\mathbb{Z}/2\mathbb{Z}).
\end{aligned}$$

(As usual, the notation $\widehat{\mathrm{H}}^i$ stands for the i-th Tate cohomology.) The first isomorphism above is general (see Chapter II, Section 5 in [9]); the second isomorphism is due to the cyclicity of G which makes its Tate cohomology periodic of period 2; the third and fourth isomorphism are consequences of the $\mathbb{Z}_2[G]$–module structure of $\mathbb{Z}_2 \otimes \Xi_S$ and $\mathbb{Z}_2 \otimes U_{S,T}$ (free $\mathbb{Z}_2$–modules of rank 3 with trivial G–action, see above.)

It is a bit tedious to write down explicitly the morphism in

$$\mathrm{Hom}_{\mathbb{Z}_2}(\mathbb{Z}_2 \otimes \Xi_S, \mathbb{Z}_2 \otimes U_{S,T})/2$$

which corresponds to the extension class ρ we constructed above via the above sequence of isomorphisms. However, we did this calculation and we concluded that the morphism in question is the composition $(\psi^{-1}\delta^{-1}\frac{1-F}{2}\delta)$ of isomorphisms

$$\Xi_S/2\Xi_S \overset{\delta}{\simeq} M_{\mathcal{T}}/2M_{\mathcal{T}} \overset{\frac{1-F}{2}}{\simeq} M_{\mathcal{T}}/2M_{\mathcal{T}} \overset{\delta^{-1}}{\simeq} \Xi_S/2\Xi_S \overset{\psi^{-1}}{\simeq} U_{S,T}/2U_{S,T}.$$

The only isomorphism which requires an explanation is $\frac{1-F}{2}$. Due to the fact that

$$\ker(\mathbb{Z}_2 \otimes \mathrm{cl}^0_{\mathcal{T}}(\mathcal{X}) \overset{1-F}{\longrightarrow} \mathbb{Z}_2 \otimes \mathrm{cl}^0_{\mathcal{T}}(\mathcal{X})) = \mathbb{Z}_2 \otimes \mathrm{cl}^0_T(X) = \mathrm{cl}^0_{\mathcal{T}}(\mathcal{X})[2]$$

we can immediately conclude that we have an exact sequence

$$0 \longrightarrow M_{\mathcal{T}} \overset{1-F}{\longrightarrow} M_{\mathcal{T}} \longrightarrow M_{\mathcal{T}}/2M_{\mathcal{T}} \longrightarrow 0.$$

Therefore $(1-F)$ viewed as an endomorphism of $M_{\mathcal{T}}$ has no kernel, and image equal to $2M_{\mathcal{T}}$. It is therefore equal to "2 times an isomorphism". That isomorphism reduced mod2 is what we call $\frac{1-F}{2}$ above.

Now we will describe the matrix in $M_3(\mathbb{Z}/2\mathbb{Z})$ of the action of $\frac{1-F}{2}$ on a carefully chosen basis of $M_{\mathcal{T}}/2$, leaving the (much easier) calculation of the matrix for ψ to the interested reader. For that purpose we computed F modulo 4 on the free $\mathbb{Z}_2$–module $M_{\mathcal{T}}$ of rank 3. In other words, we calculated F on the 4–torsion $\mathrm{cl}^0_{\mathcal{T}}(\mathcal{X})[4]$. The whole calculation is a little cumbersome since (as far as we know) PARI does not have a function that allows to calculate in generalized Jacobians. As a $\mathbb{Z}/4$-basis for $\mathrm{cl}^0_{\mathcal{T}}(\mathcal{X})[4]$ we took divisor classes $\{d_0, d_1, \mathrm{div}(f)\}$, where

$$2d_i = [P_i] - [\infty], \quad i = 0, 1, \text{ and } f \in \overline{\mathbb{F}}_{11}(\mathcal{X}), \quad f(t_+) = 2, \quad f(t_-) = 2 \cdot \zeta_4,$$

with ζ_4 a primitive root of unity of order 4 in $\overline{\mathbb{F}}_{11}$. Recall that $T = \mathcal{T} = \{t_+, t_-\}$ and that $y(t_+) = 4 = f(t_+)^2$ and $y(t_-) = -4 = f(t_-)^2$. Therefore

$$2\,\mathrm{div}(f) = \mathrm{div}(y) = P_0 + P_1 + P_{-1} - 3\infty.$$

We found the following matrix for F, the columns giving the coefficients of F applied to the above basis elements:

$$\begin{pmatrix} 3 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix} = I_3 + 2A, \text{ with } A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Hence the isomorphism $(1-F)/2$ on the 2-torsion $M_{\mathcal{T}}/2$ is given by the matrix A, in the basis given by the classes

$$\{P_0 - \infty, P_1 - \infty, \mathrm{div}(y)\}.$$

Perhaps the basis $\{P_0 - \infty, P_1 - \infty, P_{-1} - \infty\}$ for $\mathrm{cl}_T(X)[2] = M_{\mathcal{T}}/2$ (in that order!) is more natural. In that basis, $(1-F)/2$ is given by the matrix

$$A' = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

This completes our determination of the class of the 2-extension ρ. Our main result says that ρ agrees (up to a sign, see Lemma 3.2) with the Tate class. Even given this very explicit knowledge, it would appear very hard to show this directly even in this particular case.

Remark 5.4. Since there are compatible automorphisms of Y and X which exchange 1 and -1 (P_1 and P_{-1} respectively) and fix the points 0 and P_0, one might expect a symmetry: exchanging simultaneously the 2nd against the 3rd row (column respectively) should leave A' unchanged, which is obviously not the case. But there is no contradiction. Indeed, this tentative symmetry argument is unsound, because it fails to take T into account; and in fact the presence of T in the way we chose it spoils the symmetry.

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Cornelius Greither
Institut für Theoretische Informatik und Mathematik
Universität der Bundeswehr, München
85577 Neubiberg, Germany
E-mail: cornelius.greither@unibw.de

Cristian Popescu
Department of Mathematics, University of California
San Diego, La Jolla, CA 92093-0112, USA
E-mail: cpopescu@math.ucsd.edu