Noah LEBOWITZ-LOCKARD

*The distribution of numbers with many ordered factorizations*


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The distribution of numbers with many ordered factorizations

par Noah LEBOWITZ-LOCKARD

Résumé. Soit \( g(n) \) le nombre de factorisations de \( n \) en produit ordonné de facteurs plus grands que 1. On trouve des bornes précises pour les moments positifs de \( g \). On utilise ces résultats pour estimer le nombre de \( n \leq x \) tels que \( g(n) \geq x^\alpha \) pour tous les \( \alpha \) positifs. En outre, soient \( G(n) \) et \( g_P(n) \) les nombres de factorisations de \( n \) en produit ordonné de facteurs distincts plus grands que 1 et en produit ordonné de facteurs premiers respectivement. On donne des bornes inférieures pour les moments positifs de \( G \) et \( g_P \).

Abstract. Let \( g(n) \) be the number of ordered factorizations of \( n \) into numbers larger than 1. We find precise bounds on the positive moments of \( g \). We use these results to estimate the number of \( n \leq x \) satisfying \( g(n) \geq x^\alpha \) for all positive \( \alpha \). In addition, let \( G(n) \) and \( g_P(n) \) be the number of ordered factorizations of \( n \) into distinct numbers larger than 1 and primes, respectively. We also bound the positive moments of \( G \) and \( g_P \) from below.

1. Introduction

Let \( g(n) \) be the number of ordered factorizations of \( n \) into numbers larger than 1. For example, \( g(18) = 8 \) because the ordered factorizations of 18 are

\[ 18, \ 9 \cdot 2, \ 2 \cdot 9, \ 6 \cdot 3, \ 3 \cdot 6, \ 3 \cdot 3 \cdot 2, \ 3 \cdot 2 \cdot 3, \ 2 \cdot 3 \cdot 3. \]

In 1931, Kalmár [11] found an asymptotic estimate for the sum of \( g(n) \) for \( n \leq x \), namely

\[ \sum_{n \leq x} g(n) \sim -\frac{1}{\rho \zeta'(\rho)} x^\rho, \]

where \( \zeta \) is the Riemann zeta function and \( s = \rho \approx 1.73 \) is the unique solution to \( \zeta(s) = 2 \) in \( (1, \infty) \). Kalmár found the first error term for this equation, which Ikehara [9] subsequently improved. Most recently, Hwang [8] proved that

\[ \sum_{n \leq x} g(n) = -\frac{1}{\rho \zeta'(\rho)} x^\rho + O\left(x^\rho \exp\left(-c (\log_2 x)^{(3/2)-\epsilon}\right)\right) \]
for all positive $\epsilon$ where $c = c(\epsilon)$ is a positive constant. (Throughout this paper, $\log_k$ refers to the $k$th iterate of the logarithm. In addition, all error terms apply as $x \to \infty$.)

There have also been numerous results on the maximal order of $g(n)$. Clearly, $g(n) \ll n^{\rho}$ for all $n$. In 1936, Hille [7] proved that for any $\epsilon > 0$, there exist infinitely many $n$ for which $g(n) > n^{\rho-\epsilon}$. Multiple people [2, 3, 12] refined Hille’s bound. The best known result on the maximal order of $g(n)$ comes from Deléglise, Hernane, and Nicolas [2, Théorème 3], namely that there exist positive constants $C_1$ and $C_2$ such that

$$x^\rho \exp\left(-C_1 \frac{(\log x)^{1/\rho}}{\log_2 x}\right) \leq \max_{n \leq x} g(n) \leq x^\rho \exp\left(-C_2 \frac{(\log x)^{1/\rho}}{\log_2 x}\right)$$

for sufficiently large $x$. (The authors conjecture that there exists a positive constant $C$ for which

$$\max_{n \leq x} g(n) = x^\rho \exp\left(-(C + o(1)) \frac{(\log x)^{1/\rho}}{\log_2 x}\right).$$

For such a value of $C$, we would have $C_2 \leq C \leq C_1$.)

From here on, all instances of $C_1$ and $C_2$ refer to any pair of constants satisfying

$$x^\rho \exp\left(-C_1 \frac{(\log x)^{1/\rho}}{\log_2 x}\right) \leq \max_{n \leq x} g(n) \leq x^\rho \exp\left(-C_2 \frac{(\log x)^{1/\rho}}{\log_2 x}\right)$$

for sufficiently large $x$. In particular, they have the same values in Theorems 1.2 and 1.3. In Section 8, we introduce $C_3$ and $C_4$, which also have fixed values. If a result refers to a constant $C$, the value of $C$ is specific to that result. Beginning in the next section, we introduce a series of constants $c_1, c_2, \ldots$. The only constraint on a given $c_i$ is that it be large with respect to $c_{i-1}$ and $\beta$. Note that the $c_i$’s are only relevant when $\beta \leq 1/\rho$.

Throughout this paper, $o(1)$ means that a function goes to 0 as $x \to \infty$, at a rate depending on all other parameters. The rate at which this occurs is dependent upon $\beta$ unless otherwise stated. The constant multiple implied by the $\ll$ symbol also depends on $\beta$.

It is easy to bound the negative moments of $g$. If $\beta \geq 0$, then

$$\sum_{n \leq x} g(n)^{-\beta} = x^{1 + o(1)}.$$

The sum is at most $x$ because $g(n)^{-\beta} \leq 1$ for all $n$ and $\gg x / \log x$ because $g(p)^{-\beta} = 1$ for all prime $p$. In fact, Just and the author [10] recently proved that

$$\sum_{n \leq x} g(n)^{-\beta}, \quad \sum_{n \leq x} \tilde{g}(n)^{-\beta}, \quad \sum_{n \leq x} G(n)^{-\beta}$$
are all
\[ \frac{x}{\log x} \exp \left( (1 + o(1))(1 + \beta)(\log 2)^{\beta/(1+\beta)}(\log_2 x)^{1/(1+\beta)} \right), \]

where \( \tilde{g}(n) \) (resp. \( G(n) \)) is the number of ordered factorizations of \( n \) into coprime (resp. distinct) parts larger than 1. In addition, we bounded the positive moments of \( \tilde{g} \) [10, Theorem 1.8]. If \( \beta \in (0, 1) \), then
\[
\sum_{n \leq x} \tilde{g}(n)^\beta = x \exp \left( (1 + o(1)) \frac{1 - \beta}{(\log 2)^{\beta/(1-\beta)}}(\log_2 x)^{1/(1-\beta)} \right).
\]

(For the corresponding sum with \( \beta \geq 1 \), see [10, Theorems 1.2, 1.7].) Using a similar proof, we obtain a lower bound for the corresponding sum of \( g(n)^\beta \) which is larger than the bound we obtain from the sum of \( \tilde{g}(n)^\beta \). We also bound this quantity from above.

**Theorem 1.1.** If \( \beta \in (0, 1/\rho) \), then
\[
\sum_{n \leq x} g(n)^\beta \geq x \exp((C_g + o(1))(\log_2 x)^{1/(1-\beta)}),
\]

with
\[
C_g = \frac{1 - \beta}{(\log 2)^{\beta/(1-\beta)}} \exp \left( \frac{\beta}{(\log 2)(1 - \beta)} \sum_p \frac{1}{ep^{1/\beta} - 1} \right).
\]

In addition,
\[
\sum_{n \leq x} g(n)^\beta = x \exp((\log x)^{o(1)}).
\]

For the larger moments of \( g \), we obtain notably larger bounds. In particular, there is a significant increase at \( \beta = 1/\rho \). For all \( \beta < 1/\rho \), the exponent of \( \log x \) in the exponent is 0. However, at \( \beta = 1/\rho \), the exponent increases to \( 1/\rho \).

**Theorem 1.2.** If \( \beta \in [1/\rho, 1) \), then
\[
x^{\rho \beta} \exp \left( C_2(1 - \beta) \frac{\log x}{\log_2 x} \right) \leq \sum_{n \leq x} g(n)^\beta \\
\leq x^{\rho \beta} \exp \left( (1 + o(1))2\left( \frac{2}{\log 2} \right)^{1/\rho}(\log x)^{1/\rho}\log_2 x \right)
\]

for sufficiently large \( x \).

**Theorem 1.3.** If \( \beta > 1 \), then
\[
x^{\rho \beta} \exp \left( -C_1 \beta \frac{\log x}{\log_2 x} \right) \ll \sum_{n \leq x} g(n)^\beta \ll x^{\rho \beta} \exp \left( -C_2(\beta - 1) \frac{\log x}{\log_2 x} \right).
\]
Asymptotics for the moments and maximal order of the unordered factorization function are already known [1, 10, 16].

We also show that the $\beta = 1/\rho$ case of Theorem 1.2 implies the following result about the distribution of large values of $g(n)$.

**Theorem 1.4.** Fix $\epsilon > 0$. As $x \to \infty$, we have

$$\#\{n \leq x : g(n) \geq x^\alpha\} = x^{1-(\alpha/\rho)+o(1)}$$

uniformly for all $\alpha \in [0, \rho - \epsilon]$.

Let $G(n)$ and $gP(n)$ be the number of ordered factorizations of $n$ into distinct parts greater than 1 and prime parts, respectively. As with $g(n)$, asymptotic formulas for the sum and negative moments for these functions are already known [5, 10, 14]. We find lower bounds for the positive moments of these functions using techniques similar to the ones we used for $g(n)$.

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2. Preliminary results

Let $c_1$ be a large constant. For a given number $n$, let $A$ and $B$ be the $(c_1(\log x)^\beta)$-smooth and $(c_1(\log x)^\beta)$-rough parts of $n$, respectively. In other words, $n = AB$, where every prime factor of $A$ is at most $c_1(\log x)^\beta$ and every prime factor of $B$ is greater than $c_1(\log x)^\beta$. We may write

$$\sum_{n \leq x} g(n)^\beta = \sum_{A \leq x} \sum_{B \leq x/A} g(AB)^\beta.$$

Let $\Omega(n)$ be the number of (not necessarily distinct) prime factors of $n$. For a given $M$, let $\Omega > M(n)$ be the number of prime factors of $n$ which are $> M$. Before proving our main theorems, we must write a few results.

**Lemma 2.1 ([12, Lemma 2.5]).** For any two integers $n_1$ and $n_2$, we have

$$g(n_1 n_2) \leq g(n_1) \cdot (2\Omega(n_1 n_2))^{\Omega(n_2)}.$$

Because $A \leq x$, we have $\Omega(A) \leq (\log A)/(\log 2)$. Because $B \leq x$ is $(c_1(\log x)^\beta)$-rough, we have

$$\Omega(B) \leq \frac{\log B}{\log(c_1(\log x)^\beta)} \leq \frac{1}{\beta \log_2 x} \log x.$$

**Corollary 2.2.** For all $n \leq x$, we have

$$g(n) \leq g(A) \cdot \left(\frac{2}{\log_2 x}\right)^{\Omega(B)}.$$
In the proof of [15, Lemma 8], Pollack proves the following result, but does not explicitly state it.

**Lemma 2.3.** For all \( y \leq x \), we have
\[
\sum_{n \leq T} y^{\Omega > 2y(n)} \leq T \exp(2y \log x),
\]
uniformly for \( T \in [1, x] \).

We close this section with a theorem about the distribution of smooth numbers. Let \( \Psi(x, y) \) be the number of \( y \)-smooth numbers up to \( x \).

**Theorem 2.4** ([18, Theorem III.5.2]). Fix \( x \geq y \geq 2 \). We have
\[
\Psi(x, y) = \exp \left( (1 + O \left( \frac{1}{\log x} + \frac{1}{\log y} \right)) Z \right),
\]
with
\[
Z = \frac{\log x}{\log y} \log \left( 1 + \frac{y}{\log x} \right) + \frac{y}{\log y} \log \left( 1 + \frac{\log x}{y} \right).
\]

3. Large values of \( \beta \)

We establish precise bounds on the \((1/\rho)\)-th moment of \( g(n) \), which we then use to obtain bounds on the \( \beta \)-th moment of \( g \) for all \( \beta > 1/\rho \).

**Theorem 3.1.** We have
\[
\sum_{n \leq x} g(n)^{1/\rho} \leq x \exp \left( (1 + o(1))2 \left( \frac{2}{\log 2} \right)^{1/\rho} \left( \log x \right)^{1/\rho} \log 2 x \right).
\]

**Proof.** We rewrite \( g(n) \) as \( g(AB) \) and apply Corollary 2.2:
\[
\sum_{n \leq x} g(n)^{1/\rho} = \sum_{A \leq x} \sum_{B \leq x/A} g(AB)^{1/\rho}
\]
\[
\leq \sum_{A \leq x} g(A)^{1/\rho} \sum_{B \leq x/A} \left( \frac{2}{\log 2} \log x \right)^{(1/\rho)\Omega(B)}.
\]
By definition, \( \Omega(B) = \Omega_{>c_1(\log x)^{1/\rho}}(n) \). Lemma 2.3 gives us
\[
\sum_{B \leq x/A} \left( \frac{2}{\log 2} \log x \right)^{(1/\rho)\Omega_{>c_1(\log x)^{1/\rho}}(n)} \leq \frac{x}{A} \exp \left( 2 \left( \frac{2}{\log 2} \right)^{1/\rho} \left( \log x \right)^{1/\rho} \log 2 x \right),
\]
which implies that
\[
\sum_{n \leq x} g(n)^{1/\rho} \leq x \exp \left( 2 \left( \frac{2}{\log 2} \right)^{1/\rho} \log x \right) \left( \log x \right) \log_2 x \sum_{A \leq x} \frac{g(A)^{1/\rho}}{A}. 
\]

Because \( g(A) \ll A^\rho \), we have \( g(A)^{1/\rho}/A \ll 1 \). Hence,
\[
\sum_{n \leq x} g(n)^{1/\rho} \leq x \exp \left( 2 \left( \frac{2}{\log 2} \right)^{1/\rho} \log x \right) \Psi(x, c_1(\log x)^{1/\rho}). 
\]

By Theorem 2.4,
\[
\Psi(x, c_1(\log x)^{1/\rho}) = \exp(O((\log x)^{1/\rho})),
\]
which implies that
\[
\sum_{n \leq x} g(n)^{1/\rho} \leq x \exp \left( (1 + o(1))2 \left( \frac{2}{\log 2} \right)^{1/\rho} \log x \right). \quad \square
\]

While the following corollary applies to all \( \beta > 1/\rho \), it is only useful when \( \beta \leq 1 \) as well. Theorem 3.4 supersedes this result when \( \beta > 1 \).

**Corollary 3.2.** If \( \beta \geq 1/\rho \), then
\[
\sum_{n \leq x} g(n)^\beta \leq x^\rho \exp \left( (1 + o(1))2 \left( \frac{2}{\log 2} \right)^{1/\rho} \log x \right). 
\]

**Proof.** We have
\[
\sum_{n \leq x} g(n)^\beta \leq \left( \max_{n \leq x} g(n) \right)^{\beta - (1/\rho)} \sum_{n \leq x} g(n)^{1/\rho} \leq \left( x^\rho \exp \left( -C_2 \left( \frac{\log x}{\log_2 x} \right)^{1/\rho} \right) \right)^{\beta - (1/\rho)} \cdot x \exp \left( (1 + o(1))2 \left( \frac{2}{\log 2} \right)^{1/\rho} \log x \right) \log_2 x \right) \]
\[
= x^\rho \exp \left( (1 + o(1))2 \left( \frac{2}{\log 2} \right)^{1/\rho} \log x \right) \log_2 x \right). \quad \square
\]

We close this section with a few short proofs of our remaining bounds.
Theorem 3.3. For \(x\) sufficiently large, we have

\[
\sum_{n \leq x} g(n)^\beta \geq x^{\rho \beta} \exp \left( C_2 (1 - \beta) \frac{(\log x)^{1/\rho}}{\log_2 x} \right)
\]

for all \(\beta < 1\).

Proof. We have

\[
\sum_{n \leq x} g(n) \leq \left( \max_{n \leq x} g(n) \right)^{1-\beta} \sum_{n \leq x} g(n)^\beta.
\]

Therefore,

\[
\sum_{n \leq x} g(n)^\beta \geq \left( \max_{n \leq x} g(n) \right)^{(1-\beta)} \sum_{n \leq x} g(n)
\geq x^{\rho \beta} \exp \left( C_2 (1 - \beta) \frac{(\log x)^{1/\rho}}{\log_2 x} \right).
\]

Thus, though this theorem applies to all \(\beta \leq 1\), it is only useful when \(\beta \geq 1/\rho\), as we already know that the sum is at least \(\lfloor x \rfloor\).

Theorem 3.4. For \(x\) sufficiently large, we have

\[
x^{\rho \beta} \exp \left( -C_1 \beta \frac{(\log x)^{1/\rho}}{\log_2 x} \right) \leq \sum_{n \leq x} g(n)^\beta \leq x^{\rho \beta} \exp \left( -C_2 (\beta - 1) \frac{(\log x)^{1/\rho}}{\log_2 x} \right)
\]

for all \(\beta > 1\).

Proof. For the lower bound, we have

\[
\sum_{n \leq x} g(n)^\beta \geq \left( \max_{n \leq x} g(n) \right)^\beta \geq x^{\rho \beta} \exp \left( -C_1 \beta \frac{(\log x)^{1/\rho}}{\log_2 x} \right).
\]

In addition,

\[
\sum_{n \leq x} g(n)^\beta \leq \left( \max_{n \leq x} g(n) \right)^{\beta-1} \sum_{n \leq x} g(n) \leq x^{\rho \beta} \exp \left( -C_2 (\beta - 1) \frac{(\log x)^{1/\rho}}{\log_2 x} \right)
\]

gives us the upper bound.

From Theorem 3.1, we obtain Theorem 1.4.

Theorem 3.5. Fix \(\epsilon > 0\). As \(x \to \infty\),

\[
\# \{ n \leq x : g(n) \geq x^\alpha \} = x^{1-(\alpha/\rho) + o(1)}
\]

uniformly for all \(\alpha \in [0, \rho - \epsilon]\).
Proof. For a given $\alpha$, define

$$S_\alpha = \{n \leq x : g(n) \geq x^\alpha\}.$$ 

By definition,

$$\sum_{n \in S_\alpha} g(n)^{1/\rho} \geq \sum_{n \in S_\alpha} x^{\alpha/\rho} = x^{\alpha/\rho} \cdot \#S_\alpha.$$ 

From Theorem 3.1 we obtain

$$\sum_{n \in S_\alpha} g(n)^{1/\rho} \leq \sum_{n \leq x} g(n)^{1/\rho} = x \exp \left( (1 + o(1)) 2 \frac{1}{\rho} \left( \log x \right)^{1/\rho} \log_2 x \right).$$ 

Putting these inequalities together gives us

$$\#S_\alpha \leq x^{1-(\alpha/\rho)} \exp \left( (1 + o(1)) 2 \frac{1}{\rho} \left( \log x \right)^{1/\rho} \log_2 x \right) = x^{1-(\alpha/\rho)+o(1)}.$$ 

Fix $\delta > 0$. There exists some $m \leq x^{(1+\delta)\alpha/\rho}$ with the property that

$$g(m) > x^{(1+\delta)\alpha/\rho}/(1+\delta) = x^\alpha.$$ 

Therefore,

$$\#S_\alpha \geq \#\{n \leq x : m|n\} \sim x/m \geq x^{1-(1+\delta)(\alpha/\rho)}.$$ 

Taking the limit as $\delta \to 0$ shows that

$$\#S_\alpha \geq x^{1-(\alpha/\rho)+o(1)},$$

completing our proof. \qed

4. Small values of $\beta$

Using the $(1/\rho)$-th moment of $g(n)$ and the results from Section 2, we obtain the following upper bound for the small positive moments of $g(n)$. (For every result in the next two sections, we let $\beta \in (0, 1/\rho)$.)

Theorem 4.1. For all $\beta$, we have

$$\sum_{n \leq x} g(n)^\beta \leq x \exp((\log x)^{\beta^2+o(1)}).$$

In the next section, we prove the following theorem.

Theorem 4.2. If there exists a constant $C > 1$ such that

$$\sum_{n \leq x} g(n)^\beta \leq x \exp((\log x)^{\beta^C+o(1)}).$$
uniformly for all $\beta$, then

$$\sum_{n \leq x} g(n)^\beta \leq x \exp((\log x)^{\beta C+1+o(1)})$$

uniformly for all $\beta$ as well.

Applying this result arbitrarily many times allows us to obtain the upper bound in Theorem 1.1, which we rewrite here.

**Theorem 4.3.** We have

$$\sum_{n \leq x} g(n)^\beta = x \exp((\log x)^{o(1)}).$$

Before doing any of this, we write a few lemmas. We first show that we may assume that $g(n)$ is small. Afterwards, we prove that we may assume that $A$ and $\Omega(B)$ are small as well.

**Lemma 4.4.** For all $\beta$, we have

$$\sum_{n \leq x} g(n)^\beta \leq x \exp((\log x)^{o(1)})$$

for some positive constant $c_2$.

**Proof.** Fix a large number $M$. We consider

$$\sum_{k > M} \sum_{e^k \leq g(n) < e^{k+1}} g(n)^\beta.$$

We then show that for any $k$, the inner sum is sufficiently small. Note that the number of $k$ for which $g(n) < e^{k+1}$ for some $n \leq x$ is on the order of $\log x$. We have

$$\sum_{e^k \leq g(n) < e^{k+1}} g(n)^\beta \ll e^{\beta k} \#\{n \leq x : g(n) \geq e^k\}.$$

From the proof of Theorem 3.5, we see that

$$\#\{n \leq x : g(n) \geq e^k\} \leq x e^{-k/\rho} \exp \left((1 + o(1))2 \left(\frac{2}{\log 2}\right)^{1/\rho} (\log x)^{1/\rho} \log_2 x \right),$$
which implies that
\[
\sum_{x^k \leq g(n) < x^{k+1}} g(n)^\beta 
\leq x \exp \left( -k \left( \frac{1}{\rho} - \beta \right) + (1 + o(1))2 \left( \frac{2}{\log 2} \right)^{1/\rho} (\log x)^{1/\rho \log_2 x} \right).
\]
If
\[
k > \left( \frac{2\rho}{1 - \rho \beta} \left( \frac{2}{\log 2} \right)^{1/\rho} + \epsilon \right) (\log x)^{1/\rho \log_2 x}
\]
for some \( \epsilon > 0 \), then the upper bound is \( O(x) \).

From here on, we assume that
\[
g(n) \leq \exp(c_2(\log x)^{1/\rho \log_2 x}).
\]
From [12, Lemma 2.6], we have
\[
g(n) \gg 2^{\Omega(n)},
\]
allowing us to assume that
\[
\Omega(n) \leq (\log x)^{(1/\rho) + o(1)}.
\]

**Lemma 4.5.** We have
\[
\sum_{n \leq x} g(n)^\beta \leq x \exp((\log x)^{(\beta/\rho) + o(1)}) \sum_{A \leq x} \frac{g(A)^\beta}{A}.
\]

**Proof.** Recall that
\[
\sum_{n \leq x} g(n)^\beta = \sum_{A \leq x} \sum_{B \leq x/A} g(AB)^\beta.
\]
By Lemma 2.1,
\[
g(AB) \leq g(A) \cdot (2\Omega(n))^{\Omega(B)} \leq g(A) \cdot ((\log x)^{(1/\rho) + o(1)} \Omega > c_1(\log x)^{\beta}(n).
\]
Therefore,
\[
\sum_{n \leq x} g(n)^\beta 
\leq \sum_{A \leq x} g(A)^\beta \sum_{B \leq x/A} ((\log x)^{(\beta/\rho) + o(1)} \Omega > c_1(\log x)^{\beta}(n).
\]
By Lemma 2.3, the final sum in this expression is at most
\[
\frac{x}{A} \exp((\log x)^{(\beta/\rho) + o(1)}).
\]
We now have
\[ \sum_{n \leq x} g(n)^\beta \leq x \exp((\log x)^{(\beta/\rho)+o(1)}) \sum_{A \leq x} \frac{g(A)^\beta}{A}. \]
\[ \sum_{A > \exp(c_3(\log x)^\beta)} \frac{g(A)^\beta}{A}. \]
\[ \sum_{n \leq x} g(n)^\beta = o(x). \]

**Lemma 4.6.** For a sufficiently large constant \( c_3 \),
\[ \sum_{n \leq x} g(n)^\beta = o(x). \]

**Proof.** Fix a large number \( M \). By the previous result, we have
\[ \sum_{n \leq x} g(n)^\beta \leq x \exp((\log x)^{(\beta/\rho)+o(1)}) \sum_{M < A \leq x} \frac{g(A)^\beta}{A}. \]

Note that \( g(A)^\beta \ll A^{\rho\beta} \), which implies that
\[ \sum_{M < A \leq x} \frac{g(A)^\beta}{A} \ll M^{-(1-\rho\beta)} \Psi(x, c_1(\log x)^\beta). \]
By Theorem 2.4,
\[ \Psi(x, c_1(\log x)^\beta) = \exp\left( (1+o(1)) \frac{c_1(1-\beta)}{\beta}(\log x)^\beta \right). \]
If
\[ M > \exp\left( \left( \frac{c_1(1-\beta)}{\beta(1-\rho\beta)} + \epsilon \right)(\log x)^\beta \right) \]
for some \( \epsilon > 0 \), then
\[ M^{-(1-\rho\beta)} \Psi(x, c_1(\log x)^\beta) \leq \exp(-\epsilon(\log x)^\beta), \]
which implies that
\[ \sum_{n \leq x} g(n)^\beta = o(x). \]

**Lemma 4.7.** For all \( \epsilon > 0 \), we have
\[ \sum_{n \leq x} g(n)^\beta = o(x). \]

**Proof.** Once again, let \( M \) be a large number. By the previous theorem, we may assume that \( A \leq \exp(c_3(\log x)^\beta) \). We have
\[ \sum_{n \leq x} g(n)^\beta \leq \sum_{A \leq \exp(c_3(\log x)^\beta)} \frac{g(A)^\beta}{A} \sum_{B \leq x/A} \Omega(B)^{\beta\Omega(B)}. \]
By assumption, 
\[ \Omega(n) \leq (\log x)^{(1/\rho)+o(1)}. \]
By definition, \( \Omega(B) = \Omega_{>c_1(\log x)^\beta}(B) \). In addition, multiplying each term by 
\[ \exp(\beta \Omega_{>c_1(\log x)^\beta}(B) - \beta M) \]
increases the sum. Hence, 
\[
\sum_{B \leq x/A, \Omega(B) > M} (2\Omega(n))^{\beta \Omega(B)} \leq \sum_{B \leq x/A} ((\log x)^{(1/\rho)+o(1)})^{\beta \Omega_{>c_1(\log x)^\beta}(n)} \cdot \exp(\beta \Omega_{>c_1(\log x)^\beta}(B) - \beta M)
\]
\[ = \exp(-\beta M) \sum_{B \leq x/A} ((\log x)^{(1/\rho)+o(1)})^{\beta \Omega_{>c_1(\log x)^\beta}(B)} \leq \frac{x}{A} \exp((\log x)^{(\beta/\rho)+o(1)} - \beta M). \]

If \( M > (\log x)^{\beta+\epsilon} \), then this sum is at most 
\[
\frac{x \exp(-(\log x)^{\beta+\epsilon+o(1)})}{A}.
\]
Plugging this back into our original formula gives us
\[
\sum_{n \leq x \atop A \leq \exp(c_2(\log x)^\beta)} g(n)^\beta \leq x \exp(-(\log x)^{\beta+\epsilon+o(1)}) \sum_{A \leq \exp(c_3(\log x)^\beta)} \frac{g(A)^\beta}{A}.
\]
Note that the rightmost sum is \( O(\exp(c_3(\log x)^\beta)) \) because \( g(A)^\beta/A = O(1) \). We now have 
\[
\sum_{n \leq x \atop A \leq \exp(c_3(\log x)^\beta)} g(n)^\beta \leq x \exp(-(\log x)^{\beta+\epsilon+o(1)}). \tag*{\square}
\]

To summarize, we may now assume that \( \log A, \Omega(B) \leq (\log x)^{\beta+o(1)} \).
From these assumptions, we may improve Lemma 4.5.

**Theorem 4.8.** We have 
\[
\sum_{n \leq x} g(n)^\beta \leq x \exp((\log x)^{\beta^2+o(1)}) \sum_{A \leq \exp((\log x)^{\beta^2+o(1)})} \frac{g(A)^\beta}{A}.
\]

**Proof.** Once again, we have 
\[ g(AB) \leq g(A) \cdot (2\Omega(n))^{\Omega_{>c_1(\log x)^\beta}(n)}. \]
In this case, we have a more precise bound for \( \Omega(n) \). Note that 
\[ \Omega(A) \ll \log A \leq (\log x)^{\beta+o(1)}, \quad \Omega(B) \leq (\log x)^{\beta+o(1)}. \]
Therefore,
\[ g(AB) \leq g(A) \cdot ((\log x)^{\beta + o(1)})^{\Omega_{\nu > c_1 (\log x)^{\beta}} (n)}. \]

We now have
\[ \sum_{n \leq x} g(AB)^{\beta} \leq \sum_{A \leq \exp (c_2 (\log x)^{\beta})} g(A)^{\beta} \sum_{B \leq x / A} ((\log x)^{\beta + o(1)})^{\beta \Omega_{\nu > c_1 (\log x)^{\beta}} (n)}. \]

By Lemma 2.3, the rightmost sum is at most
\[ \frac{x}{A} \exp ((\log x)^{\beta^2 + o(1)}). \]

Using these results, we obtain a new upper bound on the sum of \( g(n)^{\beta} \).

**Theorem 4.9.** We have
\[ \sum_{n \leq x} g(n)^{\beta} \leq x \exp ((\log x)^{\beta^2 + o(1)}). \]

**Proof.** Because of the previous result, it is sufficient to show that
\[ \sum_{A \leq c_3 \exp (\log x)^{\beta}} g(A)^{\beta} \frac{A}{A} = \exp ((\log x)^{o(1)}). \]

We break the sum into “e-adic” intervals, based on the sizes of \( A \) and \( g(A) \):
\[ \sum_{A \leq \exp (c_3 (\log x)^{\beta})} g(A)^{\beta} \frac{A}{A} = \sum_{k \leq c_2 (\log x)^{\beta}} \sum_{\log_2 x} \sum_{m \leq \rho c_2 (\log x)^{\beta} \log_2 x} \sum_{e^k \leq A < e^{k+1}} \sum_{e^m \leq g(A) < e^{m+1}} g(A)^{\beta} \frac{A}{A}. \]

Because the number of possible \( k \) and \( m \) is sufficiently small, we only need to show that
\[ \sum_{e^k \leq A < e^{k+1}} \sum_{e^m \leq g(A) < e^{m+1}} g(A)^{\beta} \frac{A}{A} = \exp ((\log x)^{o(1)}) \]
for all possible \( k \) and \( m \).

For any \( k, m \), we have
\[ \sum_{e^k \leq A < e^{k+1}} \sum_{e^m \leq g(A) < e^{m+1}} g(A)^{\beta} \frac{A}{A} \leq \sum_{e^k \leq A < e^{k+1}} \sum_{e^m \leq g(A) < e^{m+1}} e^{m \beta - k} \leq e^{m \beta - k} \# \{ A < e^{k+1} : g(A) \geq e^m \}. \]

By Theorem 3.5,
\[ \# \{ A < e^{k+1} : g(A) \geq e^m \} \leq e^{k - (m/\rho) + o(k)}, \]
which implies that
\[ \sum_{e^k \leq A < e^{k+1}} \frac{g(A)^\beta}{A} \leq e^{m\beta - (m/\rho) + o(k)}. \]

Fix a constant \( \epsilon \). If \( k < m/\epsilon \), then the exponent is negative for \( x \) sufficiently large because \( \beta < 1/\rho \).

Suppose \( m \leq \epsilon k \). We have
\[ \sum_{e^k \leq A < e^{k+1}} \frac{g(A)^\beta}{A} \leq \sum_{e^k \leq A < e^{k+1}} A^{-(1-\epsilon + o(1))} \]
\[ \leq e^{-(1-\epsilon + o(1))k} \Psi(e^{k+1}, c_1(\log x)^\beta). \]

By assumption, \( k \leq c_3(\log x)^\beta \). If \( k = o((\log x)^\beta) \), then
\[ \Psi(e^{k+1}, c_1(\log x)^\beta) = \exp \left( (1 + o(1)) \left( k - \frac{1}{\beta} \frac{k \log k}{\log_2 x} \right) \right) \]
by Theorem 2.4. We now have
\[ \sum_{e^k \leq A < e^{k+1}} \frac{g(A)^\beta}{A} \leq \exp \left( -(1 + o(1)) \left( \frac{1}{\beta} \frac{k \log k}{\log_2 x} - \epsilon k \right) + o(k) \right). \]

If \( k > (\log x)^C \) for some constant \( C > \beta \epsilon \), then this quantity is \( o(1) \). Otherwise, the sum is still at most \( \exp((\log x)^{\beta \epsilon + o(1)}) \leq \exp((\log x)^{\epsilon/\rho + o(1)}) \).
Letting \( \epsilon \) go to 0 gives us our desired result. \( \square \)

5. Improving our bound

In the previous section, we obtained an upper bound for the sum of \( g(n)^\beta \) for all \( \beta < 1/\rho \). Using this bound, we can obtain a substantially better result using the following theorem.

**Theorem 5.1.** Let \( C > 1 \) be a constant. Suppose
\[ \sum_{n \leq x} g(n)^\beta \leq x \exp((\log x)^{3C + o(1)}) \]
for all \( \beta \in (0, 1/\rho) \). Then,
\[ \sum_{n \leq x} g(n)^\beta \leq x \exp((\log x)^{\beta C + 1 + o(1)}) \]
for all such \( \beta \) as well.
Proof. Let \( k \) be a large number. Let \( S \) be the set of all \( n \leq x \) satisfying \( e^k \leq g(n) < e^{k+1} \). We consider the sum of \( g(n)^\beta \) over all \( n \in S \). Note that
\[
\sum_{n \in S} g(n)^\beta \leq e^\beta e^{3k} \#S \leq e^{1/\rho} e^{\beta k} \#S.
\]

We bound the righthand side of the inequality above. Let \( \beta_0 \in (\beta, 1/\rho) \). We have
\[
\sum_{n \in S} g(n)^\beta \geq e^{\beta_0 k} \#S.
\]

By assumption,
\[
\sum_{n \leq x} g(n)^\beta_0 \leq x \exp((\log x)^{\beta_0^C + o(1)}).
\]
In particular, for any \( \epsilon \), there exists a number \( N \) such that if \( x > N \), then
\[
\sum_{n \leq x} g(n)^\beta_0 \leq x \exp((\log x)^{\beta_0^C + \epsilon}),
\]
which implies that
\[
\#S \leq x e^{-\beta_0 k} \exp((\log x)^{\beta_0^C + \epsilon})
\]
for all sufficiently large \( x \). (Note that \( N \) is independent of \( k \).) Plugging this into our \( g(n)^\beta \) sum gives us
\[
\sum_{n \in S} g(n)^\beta \leq x e^{-(\beta_0 - \beta)k} \exp((\log x)^{\beta_0^C + \epsilon}).
\]

If \( k > (\log x)^{\beta_0^C + \epsilon} \), then this quantity is \( o(x) \). Because the number of such \( k \) is sufficiently small, we may assume that \( k \leq (\log x)^{\beta_0^C + \epsilon} \). We may therefore assume that \( g(n) \leq \exp((\log x)^{\beta_0^C + \epsilon + o(1)}) \) and \( \Omega(n) \leq (\log x)^{\beta_0^C + \epsilon + o(1)} \). If \( x \) is sufficiently large, we have \( \Omega(n) \leq (\log x)^{\beta_0^C + 2\epsilon} \).

Using this bound on \( \Omega(n) \), we may bound the sum of \( g(n)^\beta \). We have
\[
\sum_{n \leq x} g(n)^\beta \leq \sum_{A \leq x} g(A)^\beta \sum_{B \leq x/A} (2\Omega(n))^{\beta\Omega(B)}.
\]

Suppose \( \beta_0^C + \epsilon < \beta \). Plugging in our bound on \( \Omega(n) \) allows us to bound the rightmost sum:
\[
\sum_{B \leq x/A} (2\Omega(n))^{\beta\Omega(B)} \leq \sum_{B \leq x/A} ((\log x)^{\beta_0^C + 2\epsilon})^{\beta\Omega_{>1}(\log x)^{\beta}(n)}
\]
\[
\leq \frac{x}{A} \exp((\log x)^{\beta_0^C + 2\beta\epsilon + o(1)}).
\]
For $x$ sufficiently large, we may bound the sum by
\[ \frac{x}{A} \exp((\log x)^{\beta_0^C + 3\beta \epsilon}). \]

We now have
\[ \sum_{\Omega(n) \leq (\log x)^{\beta_0^C + 2\epsilon}} g(n)^\beta \leq x \exp((\log x)^{\beta_0^C + 3\beta \epsilon}) \sum_{\Omega(A) \leq (\log x)^{\beta_0^C + 2\epsilon}} \frac{g(A)^\beta}{A}. \]

Our bound on $\Omega(A)$ gives us $A \leq \exp((\log x)^{\beta_0^C + \epsilon + o(1)}).$ In the proof of Theorem 4.9, we showed that the sum of $g(A)^\beta / A$ over all such $A$ is $\exp((\log x)^{o(1)}).$ For $x$ sufficiently large, the sum is at most $\exp((\log x)^{\beta \epsilon}).$

Putting everything together gives us
\[ \sum_{n \leq x} g(n)^\beta \leq x \exp((\log x)^{\beta_0^C + 4\beta \epsilon}). \]

Our desired result comes from the fact that we may assume that $\beta_0 - \beta$ and $\epsilon$ are both arbitrarily small. \qed

Applying this result arbitrarily many times proves that
\[ \sum_{n \leq x} g(n)^\beta = x \exp((\log x)^{o(1)}). \]

6. A lower bound for the small moments

Let $\tilde{g}(n)$ be the number of factorizations of $n$ into coprime parts greater than 1. Just and the author [10] recently proved that
\[ \sum_{n \leq x} \tilde{g}(n)^\beta = \frac{1 - \beta}{(\log 2)^{\beta/(1-\beta)}(\log_2 x)^{1/(1-\beta)}} \exp\left((\log L_0 + \log L_0 / L_0 + O\left(\frac{1}{L_0}\right)) k\right), \]
for all $\beta \in (0,1).$ Because $g(n) \geq \tilde{g}(n)$ for all $n$, this quantity is a lower bound for the sum of $g(n)^\beta.$ We provide a slightly larger lower bound for this sum. Before doing so, we write a theorem that will prove useful [17, Theorem 3.1] (see [17, Theorem 4.1] for a corresponding upper bound and [6, Corollary 2] for a more precise version of this result on a smaller interval). Let $\pi(x,k)$ be the number of $n \leq x$ with exactly $k$ distinct prime factors.

**Theorem 6.1.** For
\[ \log_2 x (\log x)^2 \leq k \leq \frac{\log x}{3 \log_2 x}, \]
we have
\[ \pi(x,k) \geq \frac{x}{k! \log x} \exp\left((\log L_0 + \frac{\log L_0}{L_0} + O\left(\frac{1}{L_0}\right)) k\right), \]
with
\[ L_0 = \log_2 x - \log k - \log_2 k. \]
We also impose a lower bound on the smallest prime factors of our values of \( n \). For a given number \( R \), we let \( \pi(x, k, R) \) be the number of \( R \)-rough \( n \leq x \) with exactly \( k \) distinct prime factors.

**Corollary 6.2.** Let \( x \) and \( k \) satisfy the conditions of the previous theorem and let \( R \) be a fixed positive number. As \( x \to \infty \), we have

\[
\pi(x, k, R) \geq \frac{x}{k! \log x} \exp\left( \left( \log L_0 + \frac{\log L_0}{L_0} + O \left( \frac{1}{L_0} \right) \right) k \right).
\]

**Proof.** Let \( n \leq x \) be an \( R \)-rough number with exactly \( k \) distinct prime factors. In the proof of the previous theorem, Pomerance already assumes that every prime factor of \( n \) is greater than or equal to \( k^2 \), which we may assume is greater than \( R \). In addition, this proof is entirely self-contained except for a reference to [17, Proposition 2.1]. However, it is straightforward to modify the proof of this result to assume that \( n \) is \( R \)-rough. \( \square \)

Using the results of [2], we bound \( g(n)^\beta \) on a suitable set of \( n \leq x \) and multiply this bound by the size of the set.

**Definition 6.3** ([2, Définition 2.1] (see also [4, Theorem 1])). For a tuple \( (a_1, \ldots, a_r) \), let \( c = c(a_1, \ldots, a_r) \) be the unique solution to the equation

\[
\prod_{i=1}^{r} \left( 1 + \frac{a_i}{c} \right) = 2.
\]

**Definition 6.4** ([2, Définition 3.1]). With \( c \) defined above, we have

\[
F := F(a_1, \ldots, a_r) = \sum_{i=1}^{r} a_i \log \left( 1 + \frac{c}{a_i} \right).
\]

**Lemma 6.5** ([2, Théorème 2]). Let \( n = p_1^{a_1} \cdots p_r^{a_r} \). We have

\[
g(n) \gg \exp\left( F - r \right) \sqrt{a_1 \cdots a_r}.
\]

**Theorem 6.6.** If \( \beta \in (0, 1/\rho) \), then

\[
\sum_{n \leq x} g(n)^\beta \geq x \exp((C_g + o(1))(\log_2 x)^{1/(1-\beta)}),
\]

with

\[
C_g = \frac{1 - \beta}{(\log 2)^{\beta/(1-\beta)}} \exp\left( \frac{\beta}{(\log 2)(1-\beta)} \sum_p \frac{1}{e^{\log^k \beta}} - 1 \right).
\]

**Proof.** Let \( k \) be a number on the order of \((\log_2 x)^C\) for some \( C > 1 \) and let \( (\alpha_1, \ldots, \alpha_r) \) be a tuple of positive real numbers which is independent of \( x \). Let \( S \) be the set of numbers \( n \leq x \) of the form \( p_1^{a_1-k} \cdots p_r^{a_r-k} m \), where \( p_i \) is the \( i \)th prime and \( m \) is a \( p_r \)-rough number with exactly \( k \) distinct prime
factors. We bound \( \#S \) from below, in addition to providing a lower bound for \( g(n) \) for all \( n \in S \).

Because the \( p_i \) and \( \alpha_i \) are fixed, the number of elements of \( S \) is equal to the number of possible values of \( m \). By assumption,

\[
m \leq \frac{x}{p_1^{\alpha_1 k} \cdots p_r^{\alpha_r k}}.
\]

Therefore,

\[
\#S = \pi \left( \frac{x}{p_1^{\alpha_1 k} \cdots p_r^{\alpha_r k}, k, p_r} \right)
\geq x \exp \left( k \log_3 x - k \log k + \left( 1 - \left( \sum_{i=1}^{r} (\log p_i) \alpha_i \right) + o(1) \right) k \right).
\]

At this point, we bound \( g(n) \). Because we only need a lower bound, we assume that \( m \) is squarefree. In our case, we have

\[
\left( \frac{1 + \frac{1}{c}}{c} \right)^k \prod_{i=1}^{r} \left( 1 + \frac{\alpha_i k}{c} \right) = 2.
\]

Though we cannot determine \( c \) exactly, we can still obtain a suitable lower bound. Because

\[
\left( \frac{1 + \frac{1}{c}}{c} \right)^k \leq 2,
\]

we have

\[
c \geq \frac{1}{2^{1/k} - 1} \sim \frac{k}{\log 2},
\]

giving us

\[
F = k \log (1 + c) + \sum_{i=1}^{r} \alpha_i k \log \left( 1 + \frac{c}{\alpha_i k} \right)
\geq k \log k + \left( \sum_{i=1}^{r} \alpha_i \log \left( 1 + \frac{1}{(\log 2) \alpha_i} \right) - \log 2 + o(1) \right) k.
\]

Note that \( (\alpha_1 k) \cdots (\alpha_r k) = \exp(o(k)) \). Hence,

\[
g(n) \geq \exp \left( k \log k + \left( \sum_{i=1}^{r} \alpha_i \log \left( 1 + \frac{1}{(\log 2) \alpha_i} \right) - 1 - \log 2 + o(1) \right) k \right)
\]

for all \( n \in S \).
We combine our estimates in order to bound the sum:
\[
\sum_{n \leq x} g(n)^\beta \geq \sum_{n \in S} g(n)^\beta \\
\geq \left( \min_{n \in S} g(n) \right)^\beta \#S \\
\geq x \exp(k \log_3 x - (1 - \beta)k \log k + (M + o(1))k),
\]
with
\[
M = 1 - (1 + \log_2 2)\beta + \sum_r \alpha_i \left( \beta \log \left( 1 + \frac{1}{(\log 2)\alpha_i} \right) - \log p_i \right).
\]
At this point, we select the \( \alpha_i \)'s in order to maximize \( M \). For all \( i \), we have
\[
\frac{\partial M}{\partial \alpha_i} = \beta \log \left( 1 + \frac{1}{(\log 2)\alpha_i} \right) - \log p_i - \frac{\beta}{1 + (\log 2)\alpha_i} = 0.
\]
Because we cannot write \( \alpha_i \) in terms of \( p_i \) nicely, we instead solve a similar equation and plug our result into our formula for \( M \). While this result is not optimal, it still provides a lower bound. As \( i \to \infty \), \( \alpha_i \to 0 \). Setting \( \alpha_i \) to 0 in the final term gives us
\[
\beta \log \left( 1 + \frac{1}{(\log 2)\alpha_i} \right) - \log p_i - \beta = 0,
\]
which implies that
\[
\alpha_i = \frac{1}{(\log 2)(e^{p_i^{1/\beta}} - 1)}.
\]
(Technically, \( \alpha_i k \) must be an integer, but rounding \( \alpha_i k \) down does not change the final result.) Hence,
\[
M = 1 - (1 + \log_2 2)\beta + \frac{\beta}{\log 2} \sum_{i=1}^r \frac{1}{e^{p_i^{1/\beta}} - 1}.
\]
Letting \( i \) go to \( \infty \) gives us
\[
M = 1 - (1 + \log_2 2)\beta + \frac{\beta}{\log 2} \sum_p \frac{1}{e^{p^{1/\beta}} - 1}.
\]
In order to finish the proof, we choose \( k \) to maximize the sum of \( g(n)^\beta \).
Recall that
\[
\sum_{n \leq x} g(n)^\beta \geq x \exp(k \log_3 x - (1 - \beta)k \log k + (M + o(1))k).
\]
If \( k > (\log_2 x)^{(1/(1-\beta)) + \epsilon} \) for some \( \epsilon > 0 \), then our bound is \( o(x) \). If \( k < (\log_2 x)^{(1/(1-\beta)) - \epsilon} \), then the bound is \( x \exp((\log_2 x)^{(1/(1-\beta)) - \epsilon + o(1)}) \). Let \( k = \frac{(1/(1-\beta))}{\log_2 x} \).
$R(\log_2 x)^{1/(1-\beta)}$ for some $R = (\log_2 x)^{o(1)}$. We have
\[
\sum_{n \leq x} g(n)^\beta \geq x \exp(R(M - (1 - \beta) \log R + o(1))(\log_2 x)^{1/(1-\beta)}).
\]
The optimal value of $R$ is the solution to the equation
\[
\frac{d}{dR}(R(M - (1 - \beta) \log R)) = M - (1 - \beta) \log R - (1 - \beta) = 0,
\]
namely
\[
R = \exp\left(\frac{M}{1 - \beta - 1}\right),
\]
which implies that
\[
\sum_{n \leq x} g(n)^\beta \geq x \exp((1 + o(1))(1 - \beta)R(\log_2 x)^{1/(1-\beta)}).
\]
We have
\[
R = \frac{1}{(\log 2)^{\beta/(1-\beta)}} \exp\left(\frac{\beta}{(\log 2)(1 - \beta)} \sum_p \frac{1}{\exp^{1/\beta} - 1}\right),
\]
completing the proof.

The lower bound for the sum of $\tilde{g}(n)^\beta$ is the result one obtains by letting $(\alpha_1, \ldots, \alpha_r)$ be the empty tuple.

7. Factorizations into distinct parts

Let $G(n)$ be the number of ordered factorizations of $n$ into distinct parts greater than 1. Warlimont \cite{19} showed that
\[
\sum_{n \leq x} G(n) = x \cdot L(x)^{O(1)},
\]
where
\[
L(x) = \exp\left(\frac{\log x \log_3 x}{\log_2 x}\right).
\]
The author and Pollack \cite{14} recently improved this result, showing that
\[
\sum_{n \leq x} G(n) = x \cdot L(x)^{1+o(1)}.
\]
In addition, we proved that for any $\epsilon > 0$, there exist infinitely many $n$ for which
\[
G(n) > n \cdot L(n)^{1-\epsilon}.
\]
A slight modification of the proof shows that
\[
\max_{n \leq x} G(n) = x \cdot L(x)^{1+o(1)}.
\]
From these bounds, we can obtain a formula for the $\beta$-th moments of $G$ for all $\beta > 1$. We have

$$\left( \max_{n \leq x} G(n) \right)^\beta \leq \sum_{n \leq x} G(n)^\beta \leq \left( \sum_{n \leq x} G(n) \right)^\beta,$$

which implies that

$$\sum_{n \leq x} G(n)^\beta = x^\beta \cdot L(x)^{\beta + o(1)}.$$

Just and the author [10] also showed that the negative moments of $G$ have the same formula as the negative moments of $g$, up to a negligible error. If $\beta > 0$, then

$$\sum_{n \leq x} G(n)^{-\beta} = \frac{x}{\log x} \exp((1 + o(1))(1 + \beta)(\log 2)^{\beta/(1 + \beta)}(\log_2 x)^{1/(1 + \beta)}).$$

All that remains is to estimate the small positive moments of $G$. We do not provide an upper bound, but we can prove a lower bound using an argument similar to the proof of Theorem 6.6. Because we do not have an asymptotic formula for $G(n)$, we use a combinatorial argument.

Once again, we let $S$ be the set of $n \leq x$ of the form $p_1^{\alpha_1 k} \cdots p_r^{\alpha_r k} m$, where $p_i$ is the $i$th prime, $m$ is a $p_r$-rough number with exactly $k$ distinct prime factors, and $k$ is on the order of $(\log_2 x)^{1/(1 - \beta)}$. In the previous section, we established that

$$\#S \geq x \exp\left(k \log_3 x - k \log k + \left(1 - \sum_{i=1}^r (\log p_i) \alpha_i + o(1)\right) k\right).$$

We now bound $G(n)$ for all $n \in S$. First, we write $m$ as a product of exactly $k$ coprime numbers greater than 1, which we can do in $k!$ ways. Then, for each $i$, we write $p_i^{\alpha_i k}$ as a product of exactly $k$ numbers (not necessarily greater than 1). For each $i$, we can do this in

$$\binom{(1 + \alpha_i)k - 1}{k}$$

ways. We then combine our factorizations into one $k$-term product. The terms are distinct because they have distinct $p_r$-rough parts. Hence,

$$G(n) \geq k! \prod_{i=1}^r \binom{(1 + \alpha_i)k - 1}{k} = \exp(k \log k + \left(\sum_{i=1}^r (1 + \alpha_i) \log(1 + \alpha_i) - \alpha_i \log \alpha_i\right) - 1 + o(1)) k).$$
Repeating the argument from the previous section gives us
\[
\sum_{n \leq x} G(n)^\beta \geq x \exp \left( (1 + o(1))(1 - \beta) \left( \prod_p \left( 1 + \frac{1}{p^{1/\beta} - 1} \right)^{\beta/(1-\beta)} \right) (\log x)^{1/(1-\beta)} \right)
\]
for all \( \beta \in (0, 1) \).

For all \( n \), \( G(n) \geq \tilde{g}(n) \). Our result is an improvement over the sum of \( \tilde{g}(n)^\beta \) when
\[
\prod_p \left( 1 + \frac{1}{p^{1/\beta} - 1} \right) > \frac{1}{\log 2},
\]
which occurs when \( \beta > 0.438 \).

8. Factorizations into prime parts

Let \( g_p(n) \) be the number of factorizations of \( n \) into prime parts. Hernane and Nicolas [5] note that a result from [13] implies that
\[
\sum_{n \leq x} g_p(n) \sim -\frac{1}{\lambda \zeta_p(\lambda)} x^\lambda,
\]
where \( \zeta_p \) is the Riemann zeta function restricted to prime terms and \( \lambda \approx 1.40 \) is the unique solution in \( (1, \infty) \) to \( \zeta_p(\lambda) = 2 \). They also showed that there exist positive constants \( C_3 \) and \( C_4 \) such that
\[
x^\lambda \exp \left( -C_3 \frac{(\log x)^\lambda}{\log_2 x} \right) \leq \max_{n \leq x} g_p(n) \leq x^\lambda \exp \left( -C_4 \frac{(\log x)^\lambda}{\log_2 x} \right)
\]
for all sufficiently large \( x \). An argument similar to the proof of Theorem 3.4 shows that if \( \beta \geq 1 \), then
\[
x^\lambda \beta \exp \left( -C_3 \beta \frac{(\log x)^{1/\lambda}}{\log_2 x} \right) \leq \sum_{n \leq x} g_p(n)^\beta \leq x^\lambda \beta \exp \left( -C_4 (\beta - 1) \frac{(\log x)^{1/\lambda}}{\log_2 x} \right)
\]
for all sufficiently large \( x \) as well.

Recall that Lemma 2.1 states that for any \( n_1, n_2 \in \mathbb{Z}_+ \), we have
\[
g(n_1 n_2) \leq g(n_1) \cdot (2\Omega(n_1 n_2))^{\Omega(n_2)}.
\]
It is straightforward to modify Klazar and Luca’s proof of this result to apply to \( g_p \).

**Lemma 8.1.** For any two integers \( n_1 \) and \( n_2 \), we have
\[
g_p(n_1 n_2) \leq g_p(n_1) \cdot (2\Omega(n_1 n_2))^{\Omega(n_2)}.
\]
From this result, we obtain variants of Corollary 3.2 and Theorem 3.3. If $\beta \in [1/\lambda, 1)$, then
\[
x^{\lambda \beta} \exp \left( C_4 (1-\beta) \frac{(\log x)^{1/\lambda}}{\log_2 x} \right) \leq \sum_{n \leq x} g_P(n)^{\beta}
\leq x^{\lambda \beta} \exp \left( (1+o(1)) \left( \frac{2}{\log 2} \right)^{1/\lambda} (\log x)^{1/\lambda \log_2 x} \right)
\]
for all sufficiently large $x$. Applying the lemma and repeating the techniques of Sections 4 and 5 shows that if $\beta \in (0, 1/\lambda)$, then
\[
\sum_{n \leq x} g_P(n)^{\beta} = x \exp((\log x)^o(1)).
\]

Finally, we note that for any tuple $(a_1, \ldots, a_r)$, we have
\[
g_P(p_1^{a_1} \cdots p_r^{a_r}) = \left( \frac{a_1 + \cdots + a_r}{a_1, \ldots, a_r} \right).
\]

Using this result, we obtain a lower bound for the small moments of $g_P$. If $\beta < 1/\lambda$, then
\[
\sum_{n \leq x} g_P(n)^{\beta} \geq x \exp \left( (1+o(1))(1-\beta) \left( 1 - \sum_p \frac{1}{p^{1/\beta}} \right)^{-\beta/(1-\beta)} (\log_2 x)^{1/(1-\beta)} \right).
\]
Noah Lebowitz-Lockard

8330 Millman St.
Philadelphia, PA, 19118, United States
E-mail: nlebovi@gmail.com
URL: https://noahlockard.wordpress.com/