JOURNAL
de Théorie des Nombres
de BORDEAUX

*anciennement Séminaire de Théorie des Nombres de Bordeaux*

Noah LEBOWITZ-LOCKARD

**The distribution of numbers with many ordered factorizations**

Tome 33, n$^{o}$ 2 (2021), p. 583-606.

<http://jtnb.centre-mersenne.org/item?id=JTNB_2021__33_2_583_0>

*Journal de Théorie des Nombres de Bordeaux* **33** (2021), 583–606

# The distribution of numbers with many ordered factorizations

par Noah LEBOWITZ-LOCKARD

Résumé. Soit $g(n)$ le nombre de factorisations de $n$ en produit ordonné de facteurs plus grands que 1. On trouve des bornes précises pour les moments positifs de $g$. On utilise ces résultats pour estimer le nombre de $n \leq x$ tels que $g(n) \geq x^{\alpha}$ pour tous les $\alpha$ positifs. En outre, soient $G(n)$ et $g_{\mathcal{P}}(n)$ les nombres de factorisations de $n$ en produit ordonné de facteurs distincts plus grands que 1 et en produit ordonné de facteurs premiers respectivement. On donne des bornes inférieures pour les moments positifs de $G$ et $g_{\mathcal{P}}$.

Abstract. Let $g(n)$ be the number of ordered factorizations of $n$ into numbers larger than 1. We find precise bounds on the positive moments of $g$. We use these results to estimate the number of $n \leq x$ satisfying $g(n) \geq x^{\alpha}$ for all positive $\alpha$. In addition, let $G(n)$ and $g_{\mathcal{P}}(n)$ be the number of ordered factorizations of $n$ into distinct numbers larger than 1 and primes, respectively. We also bound the positive moments of $G$ and $g_{\mathcal{P}}$ from below.

## 1. Introduction

Let $g(n)$ be the number of ordered factorizations of $n$ into numbers larger than 1. For example, $g(18) = 8$ because the ordered factorizations of 18 are

$$18, \quad 9 \cdot 2, \quad 2 \cdot 9, \quad 6 \cdot 3, \quad 3 \cdot 6, \quad 3 \cdot 3 \cdot 2, \quad 3 \cdot 2 \cdot 3, \quad 2 \cdot 3 \cdot 3.$$

In 1931, Kalmár [11] found an asymptotic estimate for the sum of $g(n)$ for $n \leq x$, namely

$$\sum_{n \leq x} g(n) \sim -\frac{1}{\rho\zeta'(\rho)} x^{\rho},$$

where $\zeta$ is the Riemann zeta function and $s = \rho \approx 1.73$ is the unique solution to $\zeta(s) = 2$ in $(1, \infty)$. Kalmár found the first error term for this equation, which Ikehara [9] subsequently improved. Most recently, Hwang [8] proved that

$$\sum_{n \leq x} g(n) = -\frac{1}{\rho\zeta'(\rho)} x^{\rho} + O\Big(x^{\rho} \exp(-c(\log_2 x)^{(3/2)-\epsilon})\Big)$$

Manuscrit reçu le 15 juillet 2020, révisé le 2 mai 2021, accepté le 5 juin 2021.
*Mathematics Subject Classification.* 11A25, 11A51, 11N37.
*Mots-clefs.* Ordered factorizations.

for all positive $\epsilon$ where $c = c(\epsilon)$ is a positive constant. (Throughout this paper, $\log_k$ refers to the $k$th iterate of the logarithm. In addition, all error terms apply as $x \to \infty$.)

There have also been numerous results on the maximal order of $g(n)$. Clearly, $g(n) \ll n^\rho$ for all $n$. In 1936, Hille [7] proved that for any $\epsilon > 0$, there exist infinitely many $n$ for which $g(n) > n^{\rho-\epsilon}$. Multiple people [2, 3, 12] refined Hille's bound. The best known result on the maximal order of $g(n)$ comes from Deléglise, Hernane, and Nicolas [2, Théorème 3], namely that there exist positive constants $C_1$ and $C_2$ such that

$$x^\rho \exp\left(-C_1 \frac{(\log x)^{1/\rho}}{\log_2 x}\right) \leq \max_{n \leq x} g(n) \leq x^\rho \exp\left(-C_2 \frac{(\log x)^{1/\rho}}{\log_2 x}\right)$$

for sufficiently large $x$. (The authors conjecture that there exists a positive constant $C$ for which

$$\max_{n \leq x} g(n) = x^\rho \exp\left(-(C + o(1)) \frac{(\log x)^{1/\rho}}{\log_2 x}\right).$$

For such a value of $C$, we would have $C_2 \leq C \leq C_1$.)

From here on, all instances of $C_1$ and $C_2$ refer to any pair of constants satisfying

$$x^\rho \exp\left(-C_1 \frac{(\log x)^{1/\rho}}{\log_2 x}\right) \leq \max_{n \leq x} g(n) \leq x^\rho \exp\left(-C_2 \frac{(\log x)^{1/\rho}}{\log_2 x}\right)$$

for sufficiently large $x$. In particular, they have the same values in Theorems 1.2 and 1.3. In Section 8, we introduce $C_3$ and $C_4$, which also have fixed values. If a result refers to a constant $C$, the value of $C$ is specific to that result. Beginning in the next section, we introduce a series of constants $c_1, c_2, \ldots$ The only constraint on a given $c_i$ is that it be large with respect to $c_{i-1}$ and $\beta$. Note that the $c_i$'s are only relevant when $\beta \leq 1/\rho$.

Throughout this paper, $o(1)$ means that a function goes to 0 as $x \to \infty$, at a rate depending on all other parameters. The rate at which this occurs is dependent upon $\beta$ unless otherwise stated. The constant multiple implied by the $\ll$ symbol also depends on $\beta$.

It is easy to bound the negative moments of $g$. If $\beta \geq 0$, then

$$\sum_{n \leq x} g(n)^{-\beta} = x^{1+o(1)}.$$

The sum is at most $x$ because $g(n)^{-\beta} \leq 1$ for all $n$ and $\gg x/\log x$ because $g(p)^{-\beta} = 1$ for all prime $p$. In fact, Just and the author [10] recently proved that

$$\sum_{n \leq x} g(n)^{-\beta}, \quad \sum_{n \leq x} \widetilde{g}(n)^{-\beta}, \quad \sum_{n \leq x} G(n)^{-\beta}$$

are all

$$\frac{x}{\log x}\exp\Big((1+o(1))(1+\beta)(\log 2)^{\beta/(1+\beta)}(\log_2 x)^{1/(1+\beta)}\Big),$$

where $\widetilde{g}(n)$ (resp. $G(n)$) is the number of ordered factorizations of $n$ into coprime (resp. distinct) parts larger than 1. In addition, we bounded the positive moments of $\widetilde{g}$ [10, Theorem 1.8]. If $\beta \in (0,1)$, then

$$\sum_{n\le x}\widetilde{g}(n)^\beta = x\exp\left((1+o(1))\frac{1-\beta}{(\log 2)^{\beta/(1-\beta)}}(\log_2 x)^{1/(1-\beta)}\right).$$

(For the corresponding sum with $\beta \ge 1$, see [10, Theorems 1.2, 1.7].) Using a similar proof, we obtain a lower bound for the corresponding sum of $g(n)^\beta$ which is larger than the bound we obtain from the sum of $\widetilde{g}(n)^\beta$. We also bound this quantity from above.

**Theorem 1.1.** *If $\beta \in (0, 1/\rho)$, then*

$$\sum_{n\le x} g(n)^\beta \ge x\exp((C_g+o(1))(\log_2 x)^{1/(1-\beta)}),$$

*with*

$$C_g = \frac{1-\beta}{(\log 2)^{\beta/(1-\beta)}}\exp\left(\frac{\beta}{(\log 2)(1-\beta)}\sum_p \frac{1}{ep^{1/\beta}-1}\right).$$

*In addition,*

$$\sum_{n\le x} g(n)^\beta = x\exp((\log x)^{o(1)}).$$

For the larger moments of $g$, we obtain notably larger bounds. In particular, there is a significant increase at $\beta = 1/\rho$. For all $\beta < 1/\rho$, the exponent of $\log x$ in the exponent is 0. However, at $\beta = 1/\rho$, the exponent increases to $1/\rho$.

**Theorem 1.2.** *If $\beta \in [1/\rho, 1)$, then*

$$x^{\rho\beta}\exp\left(C_2(1-\beta)\frac{(\log x)^{1/\rho}}{\log_2 x}\right) \le \sum_{n\le x} g(n)^\beta \le x^{\rho\beta}\exp\left((1+o(1))2\left(\frac{2}{\log 2}\right)^{1/\rho}(\log x)^{1/\rho}\log_2 x\right)$$

*for sufficiently large $x$.*

**Theorem 1.3.** *If $\beta > 1$, then*

$$x^{\rho\beta}\exp\left(-C_1\beta\frac{(\log x)^{1/\rho}}{\log_2 x}\right) \ll \sum_{n\le x} g(n)^\beta \ll x^{\rho\beta}\exp\left(-C_2(\beta-1)\frac{(\log x)^{1/\rho}}{\log_2 x}\right).$$

Asymptotics for the moments and maximal order of the unordered factorization function are already known [1, 10, 16].

We also show that the $\beta = 1/\rho$ case of Theorem 1.2 implies the following result about the distribution of large values of $g(n)$.

**Theorem 1.4.** *Fix $\epsilon > 0$. As $x \to \infty$, we have*

$$\#\{n \le x : g(n) \ge x^{\alpha}\} = x^{1-(\alpha/\rho)+o(1)}$$

*uniformly for all $\alpha \in [0, \rho - \epsilon]$.*

Let $G(n)$ and $g_{\mathcal{P}}(n)$ be the number of ordered factorizations of $n$ into distinct parts greater than 1 and prime parts, respectively. As with $g(n)$, asymptotic formulas for the sum and negative moments for these functions are already known [5, 10, 14]. We find lower bounds for the positive moments of these functions using techniques similar to the ones we used for $g(n)$.

**Acknowledgments.** The author wishes to thank Gérald Tenenbaum for his assistance with the smooth number computations.

## 2. Preliminary results

Let $c_1$ be a large constant. For a given number $n$, let $A$ and $B$ be the $(c_1(\log x)^{\beta})$-smooth and $(c_1(\log x)^{\beta})$-rough parts of $n$, respectively. In other words, $n = AB$, where every prime factor of $A$ is at most $c_1(\log x)^{\beta}$ and every prime factor of $B$ is greater than $c_1(\log x)^{\beta}$. We may write

$$\sum_{n \le x} g(n)^{\beta} = \sum_{\substack{A \le x \\ A\ (c_1(\log x)^{\beta})\text{-smooth}}} \sum_{\substack{B \le x/A \\ B\ (c_1(\log x)^{\beta})\text{-rough}}} g(AB)^{\beta}.$$

Let $\Omega(n)$ be the number of (not necessarily distinct) prime factors of $n$. For a given $M$, let $\Omega_{>M}(n)$ be the number of prime factors of $n$ which are $> M$. Before proving our main theorems, we must write a few results.

**Lemma 2.1** ([12, Lemma 2.5])**.** *For any two integers $n_1$ and $n_2$, we have*

$$g(n_1 n_2) \le g(n_1) \cdot (2\Omega(n_1 n_2))^{\Omega(n_2)}.$$

Because $A \le x$, we have $\Omega(A) \le (\log A)/(\log 2)$. Because $B \le x$ is $(c_1(\log x)^{\beta})$-rough, we have

$$\Omega(B) \le \frac{\log B}{\log(c_1(\log x)^{\beta})} \le \frac{1}{\beta}\frac{\log x}{\log_2 x}.$$

**Corollary 2.2.** *For all $n \le x$, we have*

$$g(n) \le g(A) \cdot \left(\frac{2}{\log 2}\log x\right)^{\Omega(B)}.$$

In the proof of [15, Lemma 8], Pollack proves the following result, but does not explicitly state it.

**Lemma 2.3.** *For all $y \le x$, we have*

$$\sum_{n\le T} y^{\Omega_{>2y}(n)} \le T\exp(2y\log_2 x),$$

*uniformly for $T \in [1,x]$.*

We close this section with a theorem about the distribution of smooth numbers. Let $\Psi(x,y)$ be the number of $y$-smooth numbers up to $x$.

**Theorem 2.4** ([18, Theorem III.5.2])**.** *Fix $x \ge y \ge 2$. We have*

$$\Psi(x,y) = \exp\left(\left(1 + O\left(\frac{1}{\log_2 x} + \frac{1}{\log y}\right)\right)Z\right),$$

*with*

$$Z = \frac{\log x}{\log y}\log\left(1+\frac{y}{\log x}\right) + \frac{y}{\log y}\log\left(1+\frac{\log x}{y}\right).$$

## 3. Large values of $\beta$

We establish precise bounds on the $(1/\rho)$-th moment of $g(n)$, which we then use to obtain bounds on the $\beta$-th moment of $g$ for all $\beta > 1/\rho$.

**Theorem 3.1.** *We have*

$$\sum_{n\le x} g(n)^{1/\rho} \le x\exp\left((1+o(1))2\left(\frac{2}{\log 2}\right)^{1/\rho}(\log x)^{1/\rho}\log_2 x\right).$$

*Proof.* We rewrite $g(n)$ as $g(AB)$ and apply Corollary 2.2:

$$\sum_{n\le x} g(n)^{1/\rho} = \sum_{\substack{A\le x\\ A\ (c_1(\log x)^{1/\rho})\text{-smooth}}}\ \sum_{\substack{B\le x/A\\ B\ (c_1(\log x)^{1/\rho})\text{-rough}}} g(AB)^{1/\rho}$$

$$\le \sum_{\substack{A\le x\\ A\ (c_1(\log x)^{1/\rho})\text{-smooth}}} g(A)^{1/\rho} \sum_{\substack{B\le x/A\\ B\ (c_1(\log x)^{1/\rho})\text{-rough}}} \left(\frac{2}{\log 2}\log x\right)^{(1/\rho)\Omega(B)}.$$

By definition, $\Omega(B) = \Omega_{>c_1(\log x)^{1/\rho}}(n)$. Lemma 2.3 gives us

$$\sum_{\substack{B\le x/A\\ B\ (c_1(\log x)^{1/\rho})\text{-rough}}} \left(\frac{2}{\log 2}\log x\right)^{(1/\rho)\Omega_{>c_1(\log x)^{1/\rho}}(n)}$$

$$\le \frac{x}{A}\exp\left(2\left(\frac{2}{\log 2}\right)^{1/\rho}(\log x)^{1/\rho}\log_2 x\right),$$

which implies that

$$\sum_{n\le x} g(n)^{1/\rho} \le x\exp\left(2\left(\frac{2}{\log 2}\right)^{1/\rho}(\log x)^{1/\rho}\log_2 x\right)\sum_{\substack{A\le x\\ A\ (c_1(\log x)^{1/\rho})\text{-smooth}}}\frac{g(A)^{1/\rho}}{A}.$$

Because $g(A) \ll A^\rho$, we have $g(A)^{1/\rho}/A \ll 1$. Hence,

$$\sum_{n\le x} g(n)^{1/\rho} \le x\exp\left(2\left(\frac{2}{\log 2}\right)^{1/\rho}(\log x)^{1/\rho}\log_2 x\right)\Psi(x, c_1(\log x)^{1/\rho}).$$

By Theorem 2.4,

$$\Psi(x, c_1(\log x)^{1/\rho}) = \exp(O((\log x)^{1/\rho})),$$

which implies that

$$\sum_{n\le x} g(n)^{1/\rho} \le x\exp\left((1+o(1))2\left(\frac{2}{\log 2}\right)^{1/\rho}(\log x)^{1/\rho}\log_2 x\right). \qquad \square$$

While the following corollary applies to all $\beta > 1/\rho$, it is only useful when $\beta \le 1$ as well. Theorem 3.4 supersedes this result when $\beta > 1$.

**Corollary 3.2.** *If $\beta \ge 1/\rho$, then*

$$\sum_{n\le x} g(n)^{\beta} \le x^{\rho\beta}\exp\left((1+o(1))2\left(\frac{2}{\log 2}\right)^{1/\rho}(\log x)^{1/\rho}\log_2 x\right).$$

*Proof.* We have

$$\begin{aligned}\sum_{n\le x} g(n)^{\beta} &\le \left(\max_{n\le x} g(n)\right)^{\beta-(1/\rho)}\sum_{n\le x} g(n)^{1/\rho}\\ &\le \left(x^\rho \exp\left(-C_2\frac{(\log x)^{1/\rho}}{\log_2 x}\right)\right)^{\beta-(1/\rho)}\\ &\qquad \cdot x\exp\left((1+o(1))2\left(\frac{2}{\log 2}\right)^{1/\rho}(\log x)^{1/\rho}\log_2 x\right)\\ &= x^{\rho\beta}\exp\left((1+o(1))2\left(\frac{2}{\log 2}\right)^{1/\rho}(\log x)^{1/\rho}\log_2 x\right). \qquad \square\end{aligned}$$

We close this section with a few short proofs of our remaining bounds.

**Theorem 3.3.** *For $x$ sufficiently large, we have*

$$\sum_{n\le x} g(n)^\beta \ge x^{\rho\beta}\exp\left(C_2(1-\beta)\frac{(\log x)^{1/\rho}}{\log_2 x}\right)$$

*for all $\beta < 1$.*

*Proof.* We have

$$\sum_{n\le x} g(n) \le \left(\max_{n\le x} g(n)\right)^{1-\beta}\sum_{n\le x} g(n)^\beta.$$

Therefore,

$$\sum_{n\le x} g(n)^\beta \ge \left(\max_{n\le x} g(n)\right)^{-(1-\beta)}\sum_{n\le x} g(n)$$
$$\ge x^{\rho\beta}\exp\left(C_2(1-\beta)\frac{(\log x)^{1/\rho}}{\log_2 x}\right). \qquad \square$$

Though this theorem applies to all $\beta \le 1$, it is only useful when $\beta \ge 1/\rho$, as we already know that the sum is at least $\lfloor x\rfloor$.

**Theorem 3.4.** *For $x$ sufficiently large, we have*

$$x^{\rho\beta}\exp\left(-C_1\beta\frac{(\log x)^{1/\rho}}{\log_2 x}\right) \le \sum_{n\le x} g(n)^\beta \le x^{\rho\beta}\exp\left(-C_2(\beta-1)\frac{(\log x)^{1/\rho}}{\log_2 x}\right)$$

*for all $\beta > 1$.*

*Proof.* For the lower bound, we have

$$\sum_{n\le x} g(n)^\beta \ge \left(\max_{n\le x} g(n)\right)^{\beta} \ge x^{\rho\beta}\exp\left(-C_1\beta\frac{(\log x)^{1/\rho}}{\log_2 x}\right).$$

In addition,

$$\sum_{n\le x} g(n)^\beta \le \left(\max_{n\le x} g(n)\right)^{\beta-1}\sum_{n\le x} g(n) \le x^{\rho\beta}\exp\left(-C_2(\beta-1)\frac{(\log x)^{1/\rho}}{\log_2 x}\right)$$

gives us the upper bound. $\square$

From Theorem 3.1, we obtain Theorem 1.4.

**Theorem 3.5.** *Fix $\epsilon > 0$. As $x \to \infty$,*

$$\#\{n \le x : g(n) \ge x^\alpha\} = x^{1-(\alpha/\rho)+o(1)}$$

*uniformly for all $\alpha \in [0, \rho-\epsilon]$.*

*Proof.* For a given $\alpha$, define

$$S_\alpha = \{n \leq x : g(n) \geq x^\alpha\}.$$

By definition,

$$\sum_{n \in S_\alpha} g(n)^{1/\rho} \geq \sum_{n \in S_\alpha} x^{\alpha/\rho} = x^{\alpha/\rho} \cdot \# S_\alpha.$$

From Theorem 3.1 we obtain

$$\sum_{n \in S_\alpha} g(n)^{1/\rho} \leq \sum_{n \leq x} g(n)^{1/\rho}$$
$$= x \exp\left((1+o(1)) 2 \left(\frac{2}{\log 2}\right)^{1/\rho} (\log x)^{1/\rho} \log_2 x\right).$$

Putting these inequalities together gives us

$$\# S_\alpha \leq x^{1-(\alpha/\rho)} \exp\left((1+o(1)) 2 \left(\frac{2}{\log 2}\right)^{1/\rho} (\log x)^{1/\rho} \log_2 x\right)$$
$$= x^{1-(\alpha/\rho)+o(1)}.$$

Fix $\delta > 0$. There exists some $m \leq x^{(1+\delta)\alpha/\rho}$ with the property that

$$g(m) > (x^{(1+\delta)\alpha/\rho})^{\rho/(1+\delta)} = x^\alpha.$$

Therefore,

$$\# S_\alpha \geq \#\{n \leq x : m | n\} \sim x/m \geq x^{1-(1+\delta)(\alpha/\rho)}.$$

Taking the limit as $\delta \to 0$ shows that

$$\# S_\alpha \geq x^{1-(\alpha/\rho)+o(1)},$$

completing our proof. □

## 4. Small values of $\beta$

Using the $(1/\rho)$-th moment of $g(n)$ and the results from Section 2, we obtain the following upper bound for the small positive moments of $g(n)$. (For every result in the next two sections, we let $\beta \in (0, 1/\rho)$.)

**Theorem 4.1.** *For all $\beta$, we have*

$$\sum_{n \leq x} g(n)^\beta \leq x \exp((\log x)^{\beta^2 + o(1)}).$$

In the next section, we prove the following theorem.

**Theorem 4.2.** *If there exists a constant $C > 1$ such that*

$$\sum_{n \leq x} g(n)^\beta \leq x \exp((\log x)^{\beta^C + o(1)})$$

*uniformly for all $\beta$, then*

$$\sum_{n\le x} g(n)^\beta \le x\exp((\log x)^{\beta^{C+1}+o(1)})$$

*uniformly for all $\beta$ as well.*

Applying this result arbitrarily many times allows us to obtain the upper bound in Theorem 1.1, which we rewrite here.

**Theorem 4.3.** *We have*

$$\sum_{n\le x} g(n)^\beta = x\exp((\log x)^{o(1)}).$$

Before doing any of this, we write a few lemmas. We first show that we may assume that $g(n)$ is small. Afterwards, we prove that we may assume that $A$ and $\Omega(B)$ are small as well.

**Lemma 4.4.** *For all $\beta$, we have*

$$\sum_{\substack{n\le x\\ g(n)>\exp(c_2(\log x)^{1/\rho}\log_2 x)}} g(n)^\beta \le x\exp((\log x)^{o(1)})$$

*for some positive constant $c_2$.*

*Proof.* Fix a large number $M$. We consider

$$\sum_{k>M}\sum_{\substack{n\le x\\ e^k\le g(n)<e^{k+1}}} g(n)^\beta.$$

We then show that for any $k$, the inner sum is sufficiently small. Note that the number of $k$ for which $g(n)<e^{k+1}$ for some $n\le x$ is on the order of $\log x$. We have

$$\sum_{\substack{n\le x\\ e^k\le g(n)<e^{k+1}}} g(n)^\beta \ll e^{\beta k}\#\{n\le x : g(n)\ge e^k\}.$$

From the proof of Theorem 3.5, we see that

$$\#\{n\le x : g(n)\ge e^k\} \\ \le xe^{-k/\rho}\exp\left((1+o(1))2\left(\frac{2}{\log 2}\right)^{1/\rho}(\log x)^{1/\rho}\log_2 x\right),$$

which implies that

$$\sum_{\substack{n\le x\\ e^k\le g(n)<e^{k+1}}} g(n)^\beta \le x\exp\left(-k\left(\frac{1}{\rho}-\beta\right)+(1+o(1))2\left(\frac{2}{\log 2}\right)^{1/\rho}(\log x)^{1/\rho}\log_2 x\right).$$

If

$$k>\left(\frac{2\rho}{1-\rho\beta}\left(\frac{2}{\log 2}\right)^{1/\rho}+\epsilon\right)(\log x)^{1/\rho}\log_2 x$$

for some $\epsilon>0$, then the upper bound is $O(x)$. $\square$

From here on, we assume that

$$g(n)\le \exp(c_2(\log x)^{1/\rho}\log_2 x).$$

From [12, Lemma 2.6], we have

$$g(n)\gg 2^{\Omega(n)},$$

allowing us to assume that

$$\Omega(n)\le(\log x)^{(1/\rho)+o(1)}.$$

**Lemma 4.5.** *We have*

$$\sum_{n\le x} g(n)^\beta \le x\exp((\log x)^{(\beta/\rho)+o(1)})\sum_{\substack{A\le x\\ A\ (c_1(\log x)^\beta)\text{-smooth}}}\frac{g(A)^\beta}{A}.$$

*Proof.* Recall that

$$\sum_{n\le x} g(n)^\beta=\sum_{\substack{A\le x\\ A\ (c_1(\log x)^\beta)\text{-smooth}}}\ \sum_{\substack{B\le x/A\\ B\ (c_1(\log x)^\beta)\text{-rough}}} g(AB)^\beta.$$

By Lemma 2.1,

$$g(AB)\le g(A)\cdot(2\Omega(n))^{\Omega(B)}\le g(A)\cdot((\log x)^{(1/\rho)+o(1)})^{\Omega_{>c_1(\log x)^\beta}(n)}.$$

Therefore,

$$\sum_{n\le x} g(n)^\beta \le \sum_{\substack{A\le x\\ A\ (c_1(\log x)^\beta)\text{-smooth}}} g(A)^\beta \sum_{\substack{B\le x/A\\ B\ (c_1(\log x)^\beta)\text{-rough}}}((\log x)^{(\beta/\rho)+o(1)})^{\Omega_{>c_1(\log x)^\beta}(n)}.$$

By Lemma 2.3, the final sum in this expression is at most

$$\frac{x}{A}\exp((\log x)^{(\beta/\rho)+o(1)}).$$

We now have

$$\sum_{n\le x} g(n)^\beta \le x\exp((\log x)^{(\beta/\rho)+o(1)}) \sum_{\substack{A\le x\\ A\ (c_1(\log x)^\beta)\text{-smooth}}} \frac{g(A)^\beta}{A}. \qquad \square$$

This result allows us to bound $A$ and $\Omega(B)$.

**Lemma 4.6.** *For a sufficiently large constant $c_3$,*

$$\sum_{\substack{n\le x\\ A>\exp(c_3(\log x)^\beta)}} g(n)^\beta = o(x).$$

*Proof.* Fix a large number $M$. By the previous result, we have

$$\sum_{\substack{n\le x\\ A>M}} g(n)^\beta \le x\exp((\log x)^{(\beta/\rho)+o(1)}) \sum_{M<A\le x} \frac{g(A)^\beta}{A}.$$

Note that $g(A)^\beta \ll A^{\rho\beta}$, which implies that

$$\sum_{M<A\le x} \frac{g(A)^\beta}{A} \ll M^{-(1-\rho\beta)}\Psi(x, c_1(\log x)^\beta).$$

By Theorem 2.4,

$$\Psi(x, c_1(\log x)^\beta) = \exp\left((1+o(1))\frac{c_1(1-\beta)}{\beta}(\log x)^\beta\right).$$

If

$$M > \exp\left(\left(\frac{c_1(1-\beta)}{\beta(1-\rho\beta)}+\epsilon\right)(\log x)^\beta\right)$$

for some $\epsilon > 0$, then

$$M^{-(1-\rho\beta)}\Psi(x, c_1(\log x)^\beta) \le \exp(-(\epsilon+o(1))(\log x)^\beta),$$

which implies that

$$\sum_{\substack{n\le x\\ A>M}} g(n)^\beta = o(x). \qquad \square$$

**Lemma 4.7.** *For all $\epsilon > 0$, we have*

$$\sum_{\substack{n\le x\\ \Omega(B)>(\log x)^{\beta+\epsilon}}} g(n)^\beta = o(x).$$

*Proof.* Once again, let $M$ be a large number. By the previous theorem, we may assume that $A \le \exp(c_3(\log x)^\beta)$. We have

$$\sum_{\substack{n\le x\\ A\le\exp(c_3(\log x)^\beta)\\ \Omega(B)>M}} g(n)^\beta \le \sum_{A\le\exp(c_3(\log x)^\beta)} g(A)^\beta \sum_{\substack{B\le x/A\\ \Omega(B)>M}} (2\Omega(n))^{\beta\Omega(B)}.$$

By assumption,

$$\Omega(n) \leq (\log x)^{(1/\rho)+o(1)}.$$

By definition, $\Omega(B) = \Omega_{>c_1(\log x)^\beta}(B)$. In addition, multiplying each term by

$$\exp(\beta\Omega_{>c_1(\log x)^\beta}(B) - \beta M)$$

increases the sum. Hence,

$$\begin{aligned}\sum_{\substack{B\leq x/A\\ \Omega(B)>M}} (2\Omega(n))^{\beta\Omega(B)} &\leq \sum_{B\leq x/A} \left((\log x)^{(1/\rho)+o(1)}\right)^{\beta\Omega_{>c_1(\log x)^\beta}(n)}\\ &\quad \cdot \exp(\beta\Omega_{>c_1(\log x)^\beta}(B) - \beta M)\\ &= \exp(-\beta M) \sum_{B\leq x/A} \left((\log x)^{(1/\rho)+o(1)}\right)^{\beta\Omega_{>c_1(\log x)^\beta}(B)}\\ &\leq \frac{x}{A}\exp((\log x)^{(\beta/\rho)+o(1)} - \beta M).\end{aligned}$$

If $M > (\log x)^{\beta+\epsilon}$, then this sum is at most

$$\frac{x\exp(-(\log x)^{\beta+\epsilon+o(1)})}{A}.$$

Plugging this back into our original formula gives us

$$\sum_{\substack{n\leq x\\ A\leq\exp(c_2(\log x)^\beta)\\ \Omega(B)>M}} g(n)^\beta \leq x\exp(-(\log x)^{\beta+\epsilon+o(1)}) \sum_{A\leq\exp(c_3(\log x)^\beta)} \frac{g(A)^\beta}{A}.$$

Note that the rightmost sum is $O(\exp(c_3(\log x)^\beta))$ because $g(A)^\beta/A = O(1)$. We now have

$$\sum_{\substack{n\leq x\\ A\leq\exp(c_3(\log x)^\beta)\\ \Omega(B)>M}} g(n)^\beta \leq x\exp(-(\log x)^{\beta+\epsilon+o(1)}). \qquad \square$$

To summarize, we may now assume that $\log A, \Omega(B) \leq (\log x)^{\beta+o(1)}$. From these assumptions, we may improve Lemma 4.5.

**Theorem 4.8.** *We have*

$$\sum_{n\leq x} g(n)^\beta \leq x\exp((\log x)^{\beta^2+o(1)}) \sum_{A\leq\exp((\log x)^{\beta+o(1)})} \frac{g(A)^\beta}{A}.$$

*Proof.* Once again, we have

$$g(AB) \leq g(A)\cdot(2\Omega(n))^{\Omega_{>c_1(\log x)^\beta}(n)}.$$

In this case, we have a more precise bound for $\Omega(n)$. Note that

$$\Omega(A) \ll \log A \leq (\log x)^{\beta+o(1)}, \qquad \Omega(B) \leq (\log x)^{\beta+o(1)}.$$

Therefore,

$$g(AB) \le g(A) \cdot \left((\log x)^{\beta+o(1)}\right)^{\Omega_{>c_1(\log x)^\beta}(n)}.$$

We now have

$$\sum_{n\le x} g(AB)^\beta \le \sum_{A\le \exp(c_2(\log x)^\beta)} g(A)^\beta \sum_{B\le x/A} \left((\log x)^{\beta+o(1)}\right)^{\beta\Omega_{>c_1(\log x)^\beta}(n)}.$$

By Lemma 2.3, the rightmost sum is at most

$$\frac{x}{A}\exp((\log x)^{\beta^2+o(1)}).$$

□

Using these results, we obtain a new upper bound on the sum of $g(n)^\beta$.

**Theorem 4.9.** *We have*

$$\sum_{n\le x} g(n)^\beta \le x\exp((\log x)^{\beta^2+o(1)}).$$

*Proof.* Because of the previous result, it is sufficient to show that

$$\sum_{A\le c_3\exp((\log x)^\beta)} \frac{g(A)^\beta}{A} = \exp((\log x)^{o(1)}).$$

We break the sum into "$e$-adic" intervals, based on the sizes of $A$ and $g(A)$:

$$\sum_{A\le \exp(c_3(\log x)^\beta)} \frac{g(A)^\beta}{A}$$

$$= \sum_{k\le c_2(\log x)^\beta \log_2 x} \quad \sum_{m\le \rho c_2(\log x)^\beta \log_2 x} \quad \sum_{\substack{e^k\le A<e^{k+1}\\ e^m\le g(A)<e^{m+1}}} \frac{g(A)^\beta}{A}.$$

Because the number of possible $k$ and $m$ is sufficiently small, we only need to show that

$$\sum_{\substack{e^k\le A<e^{k+1}\\ e^m\le g(A)<e^{m+1}}} \frac{g(A)^\beta}{A} = \exp((\log x)^{o(1)})$$

for all possible $k$ and $m$.

For any $k$, $m$, we have

$$\sum_{\substack{e^k\le A<e^{k+1}\\ e^m\le g(A)<e^{m+1}}} \frac{g(A)^\beta}{A} \ll \sum_{\substack{e^k\le A<e^{k+1}\\ e^m\le g(A)<e^{m+1}}} e^{m\beta-k}$$

$$\le e^{m\beta-k}\#\{A<e^{k+1} : g(A)\ge e^m\}.$$

By Theorem 3.5,

$$\#\{A<e^{k+1} : g(A)\ge e^m\} \le e^{k-(m/\rho)+o(k)},$$

which implies that

$$\sum_{\substack{e^k \le A < e^{k+1} \\ e^m \le g(A) < e^{m+1}}} \frac{g(A)^\beta}{A} \le e^{m\beta - (m/\rho) + o(k)}.$$

Fix a constant $\epsilon$. If $k < m/\epsilon$, then the exponent is negative for $x$ sufficiently large because $\beta < 1/\rho$.

Suppose $m \le \epsilon k$. We have

$$\begin{aligned}\sum_{\substack{e^k \le A < e^{k+1} \\ g(A) < e^{\epsilon k+1}}} \frac{g(A)^\beta}{A} &\le \sum_{e^k \le A < e^{k+1}} A^{-(1-\epsilon+o(1))} \\ &\le e^{-(1-\epsilon+o(1))k} \Psi(e^{k+1}, c_1 (\log x)^\beta).\end{aligned}$$

By assumption, $k \le c_3 (\log x)^\beta$. If $k = o((\log x)^\beta)$, then

$$\Psi(e^{k+1}, c_1(\log x)^\beta) = \exp\left((1+o(1))\left(k - \frac{1}{\beta}\frac{k \log k}{\log_2 x}\right)\right)$$

by Theorem 2.4. We now have

$$\sum_{\substack{e^k \le A < e^{k+1} \\ g(A) = e^{o(k)}}} \frac{g(A)^\beta}{A} \le \exp\left(-(1+o(1))\left(\frac{1}{\beta}\frac{k\log k}{\log_2 x} - \epsilon k\right) + o(k)\right).$$

If $k > (\log x)^C$ for some constant $C > \beta\epsilon$, then this quantity is $o(1)$. Otherwise, the sum is still at most $\exp((\log x)^{\beta\epsilon + o(1)}) \le \exp((\log x)^{(\epsilon/\rho)+o(1)})$. Letting $\epsilon$ go to 0 gives us our desired result. □

## 5. Improving our bound

In the previous section, we obtained an upper bound for the sum of $g(n)^\beta$ for all $\beta < 1/\rho$. Using this bound, we can obtain a substantially better result using the following theorem.

**Theorem 5.1.** *Let $C > 1$ be a constant. Suppose*

$$\sum_{n \le x} g(n)^\beta \le x \exp((\log x)^{\beta^C + o(1)})$$

*for all $\beta \in (0, 1/\rho)$. Then,*

$$\sum_{n \le x} g(n)^\beta \le x \exp((\log x)^{\beta^{C+1} + o(1)})$$

*for all such $\beta$ as well.*

*Proof.* Let $k$ be a large number. Let $S$ be the set of all $n \le x$ satisfying $e^k \le g(n) < e^{k+1}$. We consider the sum of $g(n)^\beta$ over all $n \in S$. Note that

$$\sum_{n \in S} g(n)^\beta \le e^\beta e^{\beta k} \# S \le e^{1/\rho} e^{\beta k} \# S.$$

We bound the righthand side of the inequality above. Let $\beta_0 \in (\beta, 1/\rho)$. We have

$$\sum_{n \in S} g(n)^{\beta_0} \ge e^{\beta_0 k} \# S.$$

By assumption,

$$\sum_{n \le x} g(n)^{\beta_0} \le x \exp((\log x)^{\beta_0^C + o(1)}).$$

In particular, for any $\epsilon$, there exists a number $N$ such that if $x > N$, then

$$\sum_{n \le x} g(n)^{\beta_0} \le x \exp((\log x)^{\beta_0^C + \epsilon}),$$

which implies that

$$\# S \le x e^{-\beta_0 k} \exp((\log x)^{\beta_0^C + \epsilon})$$

for all sufficiently large $x$. (Note that $N$ is independent of $k$.) Plugging this into our $g(n)^\beta$ sum gives us

$$\sum_{n \in S} g(n)^\beta \le x e^{-(\beta_0 - \beta) k} \exp((\log x)^{\beta_0^C + \epsilon}).$$

If $k > (\log x)^{\beta_0^C + \epsilon}$, then this quantity is $o(x)$. Because the number of such $k$ is sufficiently small, we may assume that $k \le (\log x)^{\beta_0^C + \epsilon}$. We may therefore assume that $g(n) \le \exp((\log x)^{\beta_0^C + \epsilon + o(1)})$ and $\Omega(n) \le (\log x)^{\beta_0^C + \epsilon + o(1)}$. If $x$ is sufficiently large, we have $\Omega(n) \le (\log x)^{\beta_0^C + 2\epsilon}$.

Using this bound on $\Omega(n)$, we may bound the sum of $g(n)^\beta$. We have

$$\sum_{\substack{n \le x \\ \Omega(n) \le (\log x)^{\beta_0^C + 2\epsilon}}} g(n)^\beta \le \sum_{\substack{A \le x \\ \Omega(A) \le (\log x)^{\beta_0^C + 2\epsilon}}} g(A)^\beta \sum_{\substack{B \le x/A \\ \Omega(B) \le (\log x)^{\beta_0^C + 2\epsilon}}} (2\Omega(n))^{\beta \Omega(B)}.$$

Suppose $\beta_0^C + \epsilon < \beta$. Plugging in our bound on $\Omega(n)$ allows us to bound the rightmost sum:

$$\begin{aligned} \sum_{\substack{B \le x/A \\ \Omega(B) \le (\log x)^{\beta_0^C + 2\epsilon}}} (2\Omega(n))^{\beta \Omega(B)} &\le \sum_{B \le x/A} ((\log x)^{\beta_0^C + 2\epsilon})^{\beta \Omega_{> c_1 (\log x)^\beta}(n)} \\ &\le \frac{x}{A} \exp((\log x)^{\beta \beta_0^C + 2\beta\epsilon + o(1)}). \end{aligned}$$

For $x$ sufficiently large, we may bound the sum by

$$\frac{x}{A}\exp((\log x)^{\beta\beta_0^C+3\beta\epsilon}).$$

We now have

$$\sum_{\substack{n\le x\\ \Omega(n)\le(\log x)^{\beta_0^C+2\epsilon}}} g(n)^\beta \le x\exp((\log x)^{\beta\beta_0^C+3\beta\epsilon}) \sum_{\substack{A\le x\\ \Omega(A)\le(\log x)^{\beta_0^C+2\epsilon}}} \frac{g(A)^\beta}{A}.$$

Our bound on $\Omega(A)$ gives us $A \le \exp((\log x)^{\beta_0^C+\epsilon+o(1)})$. In the proof of Theorem 4.9, we showed that the sum of $g(A)^\beta/A$ over all such $A$ is $\exp((\log x)^{o(1)})$. For $x$ sufficiently large, the sum is at most $\exp((\log x)^{\beta\epsilon})$. Putting everything together gives us

$$\sum_{n\le x} g(n)^\beta \le x\exp((\log x)^{\beta\beta_0^C+4\beta\epsilon}).$$

Our desired result comes from the fact that we may assume that $\beta_0-\beta$ and $\epsilon$ are both arbitrarily small. □

Applying this result arbitrarily many times proves that

$$\sum_{n\le x} g(n)^\beta = x\exp((\log x)^{o(1)}).$$

## 6. A lower bound for the small moments

Let $\widetilde{g}(n)$ be the number of factorizations of $n$ into coprime parts greater than 1. Just and the author [10] recently proved that

$$\sum_{n\le x}\widetilde{g}(n)^\beta = x\exp\left((1+o(1))\frac{1-\beta}{(\log 2)^{\beta/(1-\beta)}}(\log_2 x)^{1/(1-\beta)}\right)$$

for all $\beta\in(0,1)$. Because $g(n)\ge\widetilde{g}(n)$ for all $n$, this quantity is a lower bound for the sum of $g(n)^\beta$. We provide a slightly larger lower bound for this sum. Before doing so, we write a theorem that will prove useful [17, Theorem 3.1] (see [17, Theorem 4.1] for a corresponding upper bound and [6, Corollary 2] for a more precise version of this result on a smaller interval). Let $\pi(x,k)$ be the number of $n\le x$ with exactly $k$ distinct prime factors.

**Theorem 6.1.** *For*

$$\log_2 x(\log_3 x)^2 \le k \le \frac{\log x}{3\log_2 x},$$

*we have*

$$\pi(x,k)\ge\frac{x}{k!\log x}\exp\left(\left(\log L_0+\frac{\log L_0}{L_0}+O\left(\frac{1}{L_0}\right)\right)k\right),$$

*with*

$$L_0=\log_2 x-\log k-\log_2 k.$$

We also impose a lower bound on the smallest prime factors of our values of $n$. For a given number $R$, we let $\pi(x,k,R)$ be the number of $R$-rough $n \le x$ with exactly $k$ distinct prime factors.

**Corollary 6.2.** *Let $x$ and $k$ satisfy the conditions of the previous theorem and let $R$ be a fixed positive number. As $x \to \infty$, we have*

$$\pi(x,k,R) \ge \frac{x}{k!\log x}\exp\left(\left(\log L_0 + \frac{\log L_0}{L_0} + O\left(\frac{1}{L_0}\right)\right)k\right).$$

*Proof.* Let $n \le x$ be an $R$-rough number with exactly $k$ distinct prime factors. In the proof of the previous theorem, Pomerance already assumes that every prime factor of $n$ is greater than or equal to $k^2$, which we may assume is greater than $R$. In addition, this proof is entirely self-contained except for a reference to [17, Proposition 2.1]. However, it is straightforward to modify the proof of this result to assume that $n$ is $R$-rough. □

Using the results of [2], we bound $g(n)^\beta$ on a suitable set of $n \le x$ and multiply this bound by the size of the set.

**Definition 6.3** ([2, Définition 2.1] (see also [4, Theorem 1]))**.** For a tuple $(a_1,\dots,a_r)$, let $c = c(a_1,\dots,a_r)$ be the unique solution to the equation

$$\prod_{i=1}^{r}\left(1+\frac{a_i}{c}\right) = 2.$$

**Definition 6.4** ([2, Définition 3.1])**.** With $c$ defined above, we have

$$F := F(a_1,\dots,a_r) = \sum_{i=1}^{r} a_i \log\left(1+\frac{c}{a_i}\right).$$

**Lemma 6.5** ([2, Théorème 2])**.** *Let $n = p_1^{a_1}\cdots p_r^{a_r}$. We have*

$$g(n) \gg \frac{\exp(F-r)}{\sqrt{a_1\cdots a_r}}.$$

**Theorem 6.6.** *If $\beta \in (0, 1/\rho)$, then*

$$\sum_{n\le x} g(n)^\beta \ge x\exp((C_g + o(1))(\log_2 x)^{1/(1-\beta)}),$$

*with*

$$C_g = \frac{1-\beta}{(\log 2)^{\beta/(1-\beta)}}\exp\left(\frac{\beta}{(\log 2)(1-\beta)}\sum_p \frac{1}{ep^{1/\beta}-1}\right).$$

*Proof.* Let $k$ be a number on the order of $(\log_2 x)^C$ for some $C > 1$ and let $(\alpha_1,\dots,\alpha_r)$ be a tuple of positive real numbers which is independent of $x$. Let $S$ be the set of numbers $\le x$ of the form $p_1^{\alpha_1 k}\cdots p_r^{\alpha_r k} m$, where $p_i$ is the $i$th prime and $m$ is a $p_r$-rough number with exactly $k$ distinct prime factors. We bound $\#S$ from below, in addition to providing a lower bound for $g(n)$ for all $n \in S$.

Because the $p_i$ and $\alpha_i$ are fixed, the number of elements of $S$ is equal to the number of possible values of $m$. By assumption,

$$m \leq \frac{x}{p_1^{\alpha_1 k} \cdots p_r^{\alpha_r k}}.$$

Therefore,

$$\begin{aligned}\#S &= \pi\left(\frac{x}{p_1^{\alpha_1 k} \cdots p_r^{\alpha_r k}}, k, p_r\right) \\ &\geq x \exp\left(k \log_3 x - k \log k + \left(1 - \left(\sum_{i=1}^{r} (\log p_i)\alpha_i\right) + o(1)\right) k\right).\end{aligned}$$

At this point, we bound $g(n)$. Because we only need a lower bound, we assume that $m$ is squarefree. In our case, we have

$$\left(1 + \frac{1}{c}\right)^k \prod_{i=1}^{r} \left(1 + \frac{\alpha_i k}{c}\right) = 2.$$

Though we cannot determine $c$ exactly, we can still obtain a suitable lower bound. Because

$$\left(1 + \frac{1}{c}\right)^k \leq 2,$$

we have

$$c \geq \frac{1}{2^{1/k} - 1} \sim \frac{k}{\log 2},$$

giving us

$$\begin{aligned}F &= k \log(1 + c) + \sum_{i=1}^{r} \alpha_i k \log\left(1 + \frac{c}{\alpha_i k}\right) \\ &\geq k \log k + \left(\left(\sum_{i=1}^{r} \alpha_i \log\left(1 + \frac{1}{(\log 2)\alpha_i}\right)\right) - \log_2 2 + o(1)\right) k.\end{aligned}$$

Note that $(\alpha_1 k) \cdots (\alpha_r k) = \exp(o(k))$. Hence,

$$g(n) \geq \exp\left(k \log k + \left(\left(\sum_{i=1}^{r} \alpha_i \log\left(1 + \frac{1}{(\log 2)\alpha_i}\right)\right) - 1 - \log_2 2 + o(1)\right) k\right)$$

for all $n \in S$.

We combine our estimates in order to bound the sum:

$$\begin{aligned}\sum_{n\le x} g(n)^\beta &\ge \sum_{n\in S} g(n)^\beta \\ &\ge \left(\min_{n\in S} g(n)\right)^\beta \#S \\ &\ge x\exp(k\log_3 x-(1-\beta)k\log k+(M+o(1))k),\end{aligned}$$

with

$$M=1-(1+\log_2 2)\beta+\sum_{i=1}^r \alpha_i\left(\beta\log\left(1+\frac{1}{(\log 2)\alpha_i}\right)-\log p_i\right).$$

At this point, we select the $\alpha_i$'s in order to maximize $M$. For all $i$, we have

$$\frac{\partial M}{\partial\alpha_i}=\beta\log\left(1+\frac{1}{(\log 2)\alpha_i}\right)-\log p_i-\frac{\beta}{1+(\log 2)\alpha_i}=0.$$

Because we cannot write $\alpha_i$ in terms of $p_i$ nicely, we instead solve a similar equation and plug our result into our formula for $M$. While this result is not optimal, it still provides a lower bound. As $i\to\infty$, $\alpha_i\to 0$. Setting $\alpha_i$ to 0 in the final term gives us

$$\beta\log\left(1+\frac{1}{(\log 2)\alpha_i}\right)-\log p_i-\beta=0,$$

which implies that

$$\alpha_i=\frac{1}{(\log 2)(ep_i^{1/\beta}-1)}.$$

(Technically, $\alpha_i k$ must be an integer, but rounding $\alpha_i k$ down does not change the final result.) Hence,

$$M=1-(1+\log_2 2)\beta+\frac{\beta}{\log 2}\sum_{i=1}^r\frac{1}{ep_i^{1/\beta}-1}.$$

Letting $i$ go to $\infty$ gives us

$$M=1-(1+\log_2 2)\beta+\frac{\beta}{\log 2}\sum_p\frac{1}{ep^{1/\beta}-1}.$$

In order to finish the proof, we choose $k$ to maximize the sum of $g(n)^\beta$. Recall that

$$\sum_{n\le x} g(n)^\beta\ge x\exp(k\log_3 x-(1-\beta)k\log k+(M+o(1))k).$$

If $k>(\log_2 x)^{(1/(1-\beta))+\epsilon}$ for some $\epsilon>0$, then our bound is $o(x)$. If $k<(\log_2 x)^{(1/(1-\beta))-\epsilon}$, then the bound is $x\exp((\log_2 x)^{(1/(1-\beta))-\epsilon+o(1)})$. Let $k=$

$R(\log_2 x)^{1/(1-\beta)}$ for some $R = (\log_2 x)^{o(1)}$. We have

$$\sum_{n \le x} g(n)^\beta \ge x \exp(R(M - (1-\beta)\log R + o(1))(\log_2 x)^{1/(1-\beta)}).$$

The optimal value of $R$ is the solution to the equation

$$\frac{\mathrm{d}}{\mathrm{d}R}(R(M - (1-\beta)\log R)) = M - (1-\beta)\log R - (1-\beta) = 0,$$

namely

$$R = \exp\left(\frac{M}{1-\beta} - 1\right),$$

which implies that

$$\sum_{n \le x} g(n)^\beta \ge x \exp((1 + o(1))(1-\beta)R(\log_2 x)^{1/(1-\beta)}).$$

We have

$$R = \frac{1}{(\log 2)^{\beta/(1-\beta)}} \exp\left(\frac{\beta}{(\log 2)(1-\beta)} \sum_p \frac{1}{ep^{1/\beta} - 1}\right),$$

completing the proof. □

The lower bound for the sum of $\widetilde{g}(n)^\beta$ is the result one obtains by letting $(\alpha_1, \ldots, \alpha_r)$ be the empty tuple.

## 7. Factorizations into distinct parts

Let $G(n)$ be the number of ordered factorizations of $n$ into distinct parts greater than 1. Warlimont [19] showed that

$$\sum_{n \le x} G(n) = x \cdot L(x)^{O(1)},$$

where

$$L(x) = \exp\left(\frac{\log x \log_3 x}{\log_2 x}\right).$$

The author and Pollack [14] recently improved this result, showing that

$$\sum_{n \le x} G(n) = x \cdot L(x)^{1+o(1)}.$$

In addition, we proved that for any $\epsilon > 0$, there exist infinitely many $n$ for which

$$G(n) > n \cdot L(n)^{1-\epsilon}.$$

A slight modification of the proof shows that

$$\max_{n \le x} G(n) = x \cdot L(x)^{1+o(1)}.$$

From these bounds, we can obtain a formula for the $\beta$-th moments of $G$ for all $\beta > 1$. We have

$$\left(\max_{n\le x} G(n)\right)^\beta \le \sum_{n\le x} G(n)^\beta \le \left(\sum_{n\le x} G(n)\right)^\beta,$$

which implies that

$$\sum_{n\le x} G(n)^\beta = x^\beta \cdot L(x)^{\beta+o(1)}.$$

Just and the author [10] also showed that the negative moments of $G$ have the same formula as the negative moments of $g$, up to a negligible error. If $\beta > 0$, then

$$\sum_{n\le x} G(n)^{-\beta} = \frac{x}{\log x}\exp((1+o(1))(1+\beta)(\log 2)^{\beta/(1+\beta)}(\log_2 x)^{1/(1+\beta)}).$$

All that remains is to estimate the small positive moments of $G$. We do not provide an upper bound, but we can prove a lower bound using an argument similar to the proof of Theorem 6.6. Because we do not have an asymptotic formula for $G(n)$, we use a combinatorial argument.

Once again, we let $S$ be the set of $n \le x$ of the form $p_1^{\alpha_1 k}\cdots p_r^{\alpha_r k} m$, where $p_i$ is the $i$th prime, $m$ is a $p_r$-rough number with exactly $k$ distinct prime factors, and $k$ is on the order of $(\log_2 x)^{1/(1-\beta)}$. In the previous section, we established that

$$\#S \ge x\exp\left(k\log_3 x - k\log k + \left(1 - \left(\sum_{i=1}^r (\log p_i)\alpha_i\right) + o(1)\right)k\right).$$

We now bound $G(n)$ for all $n\in S$. First, we write $m$ as a product of exactly $k$ coprime numbers greater than 1, which we can do in $k!$ ways. Then, for each $i$, we write $p_i^{\alpha_i k}$ as a product of exactly $k$ numbers (not necessarily greater than 1). For each $i$, we can do this in

$$\binom{(1+\alpha_i)k-1}{k}$$

ways. We then combine our factorizations into one $k$-term product. The terms are distinct because they have distinct $p_r$-rough parts. Hence,

$$\begin{aligned} G(n) &\ge k!\prod_{i=1}^r \binom{(1+\alpha_i)k-1}{k} \\ &= \exp(k\log k \\ &\qquad + \left(\left(\sum_{i=1}^r (1+\alpha_i)\log(1+\alpha_i) - \alpha_i\log\alpha_i\right) - 1 + o(1)\right)k\right). \end{aligned}$$

Repeating the argument from the previous section gives us

$$\sum_{n\leq x} G(n)^\beta$$
$$\geq x\exp\left((1+o(1))(1-\beta)\left(\prod_p\left(1+\frac{1}{p^{1/\beta}-1}\right)^{\beta/(1-\beta)}\right)(\log_2 x)^{1/(1-\beta)}\right)$$

for all $\beta \in (0,1)$.

For all $n$, $G(n) \geq \widetilde{g}(n)$. Our result is an improvement over the sum of $\widetilde{g}(n)^\beta$ when

$$\prod_p\left(1+\frac{1}{p^{1/\beta}-1}\right) > \frac{1}{\log 2},$$

which occurs when $\beta > 0.438$.

## 8. Factorizations into prime parts

Let $g_{\mathcal{P}}(n)$ be the number of factorizations of $n$ into prime parts. Hernane and Nicolas [5] note that a result from [13] implies that

$$\sum_{n\leq x} g_{\mathcal{P}}(n) \sim -\frac{1}{\lambda\zeta_{\mathcal{P}}'(\lambda)}x^\lambda,$$

where $\zeta_{\mathcal{P}}$ is the Riemann zeta function restricted to prime terms and $\lambda \approx 1.40$ is the unique solution in $(1,\infty)$ to $\zeta_{\mathcal{P}}(\lambda) = 2$. They also showed that there exist positive constants $C_3$ and $C_4$ such that

$$x^\lambda \exp\left(-C_3\frac{(\log x)^\lambda}{\log_2 x}\right) \leq \max_{n\leq x} g_{\mathcal{P}}(n) \leq x^\lambda \exp\left(-C_4\frac{(\log x)^\lambda}{\log_2 x}\right)$$

for all sufficiently large $x$. An argument similar to the proof of Theorem 3.4 shows that if $\beta \geq 1$, then

$$x^{\lambda\beta}\exp\left(-C_3\beta\frac{(\log x)^{1/\lambda}}{\log_2 x}\right) \leq \sum_{n\leq x} g_{\mathcal{P}}(n)^\beta$$
$$\leq x^{\lambda\beta}\exp\left(-C_4(\beta-1)\frac{(\log x)^{1/\lambda}}{\log_2 x}\right)$$

for all sufficiently large $x$ as well.

Recall that Lemma 2.1 states that for any $n_1, n_2 \in \mathbb{Z}_+$, we have

$$g(n_1n_2) \leq g(n_1)\cdot(2\Omega(n_1n_2))^{\Omega(n_2)}.$$

It is straightforward to modify Klazar and Luca's proof of this result to apply to $g_{\mathcal{P}}$.

**Lemma 8.1.** *For any two integers $n_1$ and $n_2$, we have*

$$g_{\mathcal{P}}(n_1n_2) \leq g_{\mathcal{P}}(n_1)\cdot(2\Omega(n_1n_2))^{\Omega(n_2)}.$$

From this result, we obtain variants of Corollary 3.2 and Theorem 3.3. If $\beta \in [1/\lambda, 1)$, then

$$x^{\lambda\beta} \exp\left(C_4(1-\beta)\frac{(\log x)^{1/\lambda}}{\log_2 x}\right) \leq \sum_{n\leq x} g_{\mathcal{P}}(n)^\beta$$
$$\leq x^{\lambda\beta} \exp\left((1+o(1))2\left(\frac{2}{\log 2}\right)^{1/\lambda}(\log x)^{1/\lambda}\log_2 x\right)$$

for all sufficiently large $x$. Applying the lemma and repeating the techniques of Sections 4 and 5 shows that if $\beta \in (0, 1/\lambda)$, then

$$\sum_{n\leq x} g_{\mathcal{P}}(n)^\beta = x\exp((\log x)^{o(1)}).$$

Finally, we note that for any tuple $(a_1, \dots, a_r)$, we have

$$g_{\mathcal{P}}(p_1^{a_1}\cdots p_r^{a_r}) = \binom{a_1+\cdots+a_r}{a_1,\dots,a_r}.$$

Using this result, we obtain a lower bound for the small moments of $g_{\mathcal{P}}$. If $\beta < 1/\lambda$, then

$$\sum_{n\leq x} g_{\mathcal{P}}(n)^\beta$$
$$\geq x\exp\left((1+o(1))(1-\beta)\left(1-\sum_p \frac{1}{p^{1/\beta}}\right)^{-\beta/(1-\beta)}(\log_2 x)^{1/(1-\beta)}\right).$$

## References


[1] E. R. Canfield, P. Erdős & C. Pomerance, "On a problem of Oppenheim concerning 'Factorisatio Numerorum'", *J. Number Theory* **17** (1983), no. 1, p. 1-28.

[2] M. Deléglise, M.-O. Hernane & J.-L. Nicolas, "Grandes valeurs et nombres champions de la fonction arithmétique de Kalmár", *J. Number Theory* **128** (2008), no. 6, p. 1676-1716.

[3] P. Erdős, "On some asymptotic formulas in the theory of the 'Factorisatio Numeroum'", *Ann. Math.* **42** (1941), p. 989-993, corrigendum in *ibid.* **44**, p. 647-651.

[4] R. Evans, "An asymptotic formula for extended Eulerian numbers", *Duke Math. J.* **41** (1974), no. 1, p. 161-175.

[5] M.-O. Hernane & J.-L. Nicolas, "Grandes valeurs du nombre de factorisations d'un entier en produit ordonné de facteurs premiers", *Ramanujan J.* **14** (2007), no. 2, p. 277-304.

[6] A. Hildebrand & G. Tenenbaum, "On the number of prime factors of an integer", *Duke Math. J.* **56** (1988), no. 3, p. 471-501.

[7] E. Hille, "A problem in 'Factorisatio Numerorum'", *Acta Arith.* **2** (1936), p. 134-144.

[8] H.-K. Hwang, "Distribution of the number of factors in random ordered factorizations of integers", *J. Number Theory* **81** (2000), no. 1, p. 61-92.

[9] S. Ikehara, "On Kalmár's problem in 'Factorisatio Numerorum'. II", *Proc. Phys.-Math. Soc. Japan, III. Ser.* **23** (1941), p. 767-774.

[10] M. Just & N. Lebowitz-Lockard, "On factorizations into coprime parts", to appear in *Int. J. Number Theory*, 2021.

[11] L. Kalmár, "Über die mittlere Anzahl der Produktdarstellungen der Zahlen, erste Mitteilung", *Acta Litt. Sci. Szeged* **5** (1931), p. 95-107.

[12] M. Klazar & F. Luca, "On the maximal order of numbers in the 'Factorisatio Numerorum' problem", *J. Number Theory* **124** (2007), no. 2, p. 470-490.
[13] A. Knopfmacher, J. Knopfmacher & R. Warlimont, "Ordered factorizations for integers and arithmetical semigroups", in *Advances in Number Theory (Kingston, Ontario, 1991)*, Clarendon Press, 1993, p. 151-165.
[14] N. Lebowitz-Lockard & P. Pollack, "On ordered factorizations into distinct parts", *Proc. Amer. Math. Soc.* **148** (2020), no. 4, p. 1447-1453.
[15] P. Pollack, "How often is Euler's totient a perfect power?", *J. Number Theory* **197** (2019), p. 1-12.
[16] ———, "The distribution of numbers with many factorizations", to appear in *Math. Z.*, 2021.
[17] C. Pomerance, "On the distribution of round numbers", in *Number Theory Proceedings, Ootacamund, India, 1984*, Lecture Notes in Mathematics, vol. 1122, Springer, 1985, p. 173-200.
[18] G. Tenenbaum, *Introduction to Analytic and Probabilistic Number Theory*, 3rd expanded ed., Graduate Studies in Mathematics, vol. 163, American Mathematical Society, 2015.
[19] R. Warlimont, "Factorisatio numerorum with constraints", *J. Number Theory* **45** (1993), no. 2, p. 186-199.

Noah Lebowitz-Lockard
8330 Millman St.
Philadelphia, PA, 19118, United States
*E-mail*: nlebowi@gmail.com
*URL*: https://noahlockard.wordpress.com/