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The twisted forms of a semisimple group over an \mathbb{F}_q -curve

par RONY A. BITAN, RALF KÖHL et CLAUDIA SCHOEMANN

RÉSUMÉ. Soit C une courbe projective, lisse et connexe définie sur un corps fini \mathbb{F}_q . Étant donné un $C - S$ -schéma en groupes semisimples où S est un ensemble fini de points fermés de C , nous décrivons l'ensemble de (\mathcal{O}_S -classes de) formes tordues de \underline{G} en termes d'invariants géométriques de son groupe fondamental $F(\underline{G})$.

ABSTRACT. Let C be a smooth, projective and geometrically connected curve defined over a finite field \mathbb{F}_q . Given a semisimple $C - S$ -group scheme \underline{G} where S is a finite set of closed points of C , we describe the set of (\mathcal{O}_S -classes of) twisted forms of \underline{G} in terms of geometric invariants of its fundamental group $F(\underline{G})$.

1. Introduction

Let C be a projective, smooth and geometrically connected curve defined over a finite field \mathbb{F}_q . Let Ω be the set of all closed points on C . For any $\mathfrak{p} \in \Omega$ let $v_{\mathfrak{p}}$ be the induced discrete valuation on the (global) function field $K = \mathbb{F}_q(C)$, $\widehat{\mathcal{O}}_{\mathfrak{p}}$ the ring of integers in the completion $\widehat{K}_{\mathfrak{p}}$ of K with respect to $v_{\mathfrak{p}}$, and $k_{\mathfrak{p}}$ the residue field. Any finite subset $S \subset \Omega$ gives rise to a *Dedekind scheme*, namely, a Noetherian integral scheme of dimension 1 whose local rings are regular; If S is *nonempty* it will be the spectrum of the Dedekind domain

$$\mathcal{O}_S := \{x \in K : v_{\mathfrak{p}}(x) \geq 0 \quad \forall \mathfrak{p} \notin S\}.$$

Otherwise, if $S = \emptyset$, the corresponding Dedekind scheme is the curve C itself, and we denote by \mathcal{O}_S the structural sheaf of C .

Throughout this paper \underline{G} is an \mathcal{O}_S -group scheme whose generic fiber $G := \underline{G} \otimes_{\mathcal{O}_S} K$ is almost-simple, and whose fiber $\underline{G}_{\mathfrak{p}} = \underline{G} \otimes_{\mathcal{O}_S} \widehat{\mathcal{O}}_{\mathfrak{p}}$ at any $\mathfrak{p} \in \Omega - S$ is *semisimple*, namely, (connected) reductive over $k_{\mathfrak{p}}$, and the rank of its root system equals that of its lattice of weights ([12, Exp. XIX Def. 2.7, Exp. XXI Def. 1.1.1]). Let $\underline{G}^{\text{sc}}$ be the universal (central) cover (being simply-connected) of \underline{G} , and suppose that its *fundamental group* $F(\underline{G}) := \ker[\underline{G}^{\text{sc}} \twoheadrightarrow \underline{G}]$ (cf. [10, p. 40]) is of order prime to $\text{char}(K)$.

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A *twisted form* of \underline{G} is an \mathcal{O}_S -group that is isomorphic to \underline{G} over some finite étale cover of \mathcal{O}_S . We aim to describe explicitly (in terms of some invariants of $F(\underline{G})$ and the group of outer automorphisms of \underline{G}) the finite set of all twisted forms of \underline{G} , modulo \mathcal{O}_S -isomorphisms. This is done first in Section 2 for forms arising from the torsors of the adjoint group $\underline{G}^{\text{ad}}$, and then in Section 3, through the action of the outer automorphisms of \underline{G} on its Dynkin diagram, for all twisted forms. More concrete computations are provided in Sections 4, 5 and 6. The case of type A deserves a special consideration, this is done in Section 7. The Zariski topology is treated in Section 8.

Before we start may we quote B. Conrad in the abstract of [11]: “The study of such \mathbb{Z} -groups provides concrete applications of many facets of the theory of reductive groups over rings (scheme of Borel subgroups, automorphism scheme, relative non-abelian cohomology, etc.), and it highlights the role of number theory (class field theory, mass formulas, strong approximation, point-counting over finite fields, etc.) in analyzing the possibilities”.

2. Torsors

A \underline{G} -torsor P in the étale topology is a sheaf of sets on \mathcal{O}_S equipped with a (right) \underline{G} -action, which is locally trivial in the étale topology, namely, locally for the étale topology on \mathcal{O}_S , this action is isomorphic to the action of G on itself by translation. The associated \mathcal{O}_S -group scheme ${}^P\underline{G} = \underline{G}'$, being an inner form of \underline{G} , is called the *twist* of \underline{G} by P (e.g., [24, §2.2, Lem. 2.2.3, Exs. 1, 2]). We define $H_{\text{ét}}^1(\mathcal{O}_S, \underline{G})$ to be the set of isomorphism classes of \underline{G} -torsors relative to the étale topology (or the flat one; these two cohomology sets coincide when \underline{G} is smooth; cf. [2, VIII Cor. 2.3]). This set is finite ([5, Prop. 3.9]). The sets $H^1(K, G)$ (denoting the Galois cohomology) and $H_{\text{ét}}^1(\widehat{\mathcal{O}}_{\mathfrak{p}}, \underline{G}_{\mathfrak{p}})$ are defined similarly.

There exists a canonical map of pointed-sets:

$$\lambda : H_{\text{ét}}^1(\mathcal{O}_S, \underline{G}) \rightarrow H^1(K, G) \times \prod_{\mathfrak{p} \notin S} H_{\text{ét}}^1(\widehat{\mathcal{O}}_{\mathfrak{p}}, \underline{G}_{\mathfrak{p}}).$$

defined by $[X] \mapsto [(X \otimes_{\mathcal{O}_S} \text{Sp } K) \times \prod_{\mathfrak{p} \notin S} X \otimes_{\mathcal{O}_S} \text{Sp } \widehat{\mathcal{O}}_{\mathfrak{p}}]$. Let $[\xi_0] := \lambda([G])$. The *principal genus* of \underline{G} is then $\ker(\lambda) = \lambda^{-1}([\xi_0])$, namely, the classes of \underline{G} -torsors that are generically and locally trivial at all points of \mathcal{O}_S . More generally, a *genus* of \underline{G} is any fiber $\lambda^{-1}([\xi])$ where $[\xi] \in \text{Im}(\lambda)$. The *set of genera* of \underline{G} is then:

$$\text{gen}(\underline{G}) := \{\lambda^{-1}([\xi]) : [\xi] \in \text{Im}(\lambda)\},$$

hence $H_{\text{ét}}^1(\mathcal{O}_S, \underline{G})$ is a disjoint union of all genera.

The ring of S -integral adèles $\mathbb{A}_S := \prod_{\mathfrak{p} \in S} \widehat{K}_{\mathfrak{p}} \times \prod_{\mathfrak{p} \notin S} \widehat{\mathcal{O}}_{\mathfrak{p}}$ is a subring of the adèles \mathbb{A} . A \underline{G} -torsor $P = \text{Iso}(\underline{G}, \underline{G}')$ belongs to the principal genus of

\underline{G} if it is both \mathbb{A}_S - and K -trivial, hence the principal genus bijects as a pointed-set to the S -class set of \underline{G} (see [21, Thm. I.3.5]):

$$\text{Cl}_S(\underline{G}) := \underline{G}(\mathbb{A}_S) \backslash \underline{G}(\mathbb{A}) / G(K).$$

Being finite ([5, Prop. 3.9]), its cardinality, called the S -class number of G , is denoted $h_S(\underline{G})$. As \underline{G} is assumed to have connected fibers, by Lang's Theorem (recall that all residue fields are finite) all $H_{\text{ét}}^1(\widehat{\mathcal{O}}_p, \underline{G}_p)$ vanish, which indicates that any two \underline{G} -torsors share the same genus if and only if they are K -isomorphic.

The universal cover of \underline{G} forms a short exact sequence of étale \mathcal{O}_S -groups (cf. [10, p. 40]):

$$(2.1) \quad 1 \rightarrow F(\underline{G}) \rightarrow \underline{G}^{\text{sc}} \rightarrow \underline{G} \rightarrow 1.$$

This gives rise by étale cohomology to the co-boundary map of pointed sets:

$$(2.2) \quad \delta_{\underline{G}} : H_{\text{ét}}^1(\mathcal{O}_S, \underline{G}) \twoheadrightarrow H_{\text{ét}}^2(\mathcal{O}_S, F(\underline{G}))$$

which is surjective by ([13, Cor. 1]) as \mathcal{O}_S is of *Douai-type* (see [17, Def. 5.2 and Exam. 5.4(iii),(v)]). It follows from the fact that $H_{\text{ét}}^2(\mathcal{O}_S, \underline{G}^{\text{sc}})$ (resp., $H_{\text{ét}}^2(\mathcal{O}_S, \underline{G}^{\text{sc}})$) has only trivial classes and in finite number ([13, Thm. 1.1]).

A representation $\rho : \underline{G}^{\text{sc}} \rightarrow \underline{\text{GL}}_1(A)$ where A is an Azumaya \mathcal{O}_S algebra, is said to be *center-preserving* if $\rho(Z(\underline{G})^{\text{sc}}) \subseteq Z(\underline{\text{GL}}_1(A))$. The restriction of ρ to $F(\underline{G}) \subseteq Z(\underline{G}^{\text{sc}})$, composed with the natural isomorphism $Z(\underline{\text{GL}}_1(A)) \cong \mathbb{G}_m$, is a map $\Lambda_\rho : F(\underline{G}) \rightarrow \mathbb{G}_m$, thus inducing a map: $(\Lambda_\rho)_* : H_{\text{ét}}^2(\mathcal{O}_S, F(\underline{G})) \rightarrow H_{\text{ét}}^2(\mathcal{O}_S, \mathbb{G}_m) \cong \text{Br}(\mathcal{O}_S)$. Together with the preceding map $\delta_{\underline{G}}$ we get the map of pointed-sets:

$$(2.3) \quad (\Lambda_\rho)_* \circ \delta_{\underline{G}} : H_{\text{ét}}^1(\mathcal{O}_S, \underline{G}) \rightarrow \text{Br}(\mathcal{O}_S),$$

which associates any class of \underline{G} -torsors with a class of Azuamaya \mathcal{O}_S -algebras in $\text{Br}(\mathcal{O}_S)$.

When $F(\underline{G}) = \underline{\mu}_m$, the following composition is surjective:

$$(2.4) \quad w_{\underline{G}} : H_{\text{ét}}^1(\mathcal{O}_S, \underline{G}) \xrightarrow{\delta_{\underline{G}}} H_{\text{ét}}^2(\mathcal{O}_S, F(\underline{G})) \xrightarrow{i_*^{(2)}} {}_m\text{Br}(\mathcal{O}_S),$$

and coincides with $(\Lambda_\rho)_* \circ \delta_{\underline{G}}$.

The original *Tits algebras* introduced in [26], are central simple algebras defined over a field, associated to algebraic groups defined over that field. This construction was generalized to group-schemes over rings as shown in [22, Thm. 1]. We briefly recall it here over \mathcal{O}_S : Being semisimple, \underline{G} admits an inner form \underline{G}_0 which is *quasi-split* (in the sense of [12, XXIV, 3.9], namely, not only requiring a Borel subgroup to be defined over $C - S$ but some additional data involving the scheme of Dynkin diagrams, see [10, Def. 5.2.10.]).

Definition 1. Any center-preserving representation $\rho_0 : \underline{G}_0 \rightarrow \underline{\mathbf{GL}}(V)$ gives rise to a “twisted” center-preserving representation: $\rho : \underline{G} \rightarrow \underline{\mathbf{GL}}_1(A_\rho)$, where A_ρ is an Azumaya \mathcal{O}_S -algebra, called the *Tits algebra corresponding to the representation ρ* , and its class in $\mathrm{Br}(\mathcal{O}_S)$, is its *Tits class*.

Lemma 2.1. *If \underline{G} is adjoint, then for any center-preserving representation ρ of $\underline{G}_0^{\mathrm{sc}}$, and a twisted \underline{G} -form ${}^P\underline{G}$ by a \underline{G} -torsor P , one has: $((\Lambda_\rho)_* \circ \delta_{\underline{G}})([{}^P\underline{G}]) = [{}^P A_\rho] - [A_\rho] \in \mathrm{Br}(\mathcal{O}_S)$ where $[{}^P A_\rho]$ and $[A_\rho]$ are the Tits classes of ${}^P\underline{G}$ and $\underline{G}^{\mathrm{sc}}$ corresponding to ρ , respectively.*

Proof. By descent $F(\underline{G}_0) \cong F(\underline{G})$, so we may write the short exact sequences of \mathcal{O}_S -groups:

$$(2.5) \quad \begin{aligned} 1 &\rightarrow F(\underline{G}) \rightarrow \underline{G}^{\mathrm{sc}} \rightarrow \underline{G} \rightarrow 1 \\ 1 &\rightarrow F(\underline{G}) \rightarrow \underline{G}_0^{\mathrm{sc}} \rightarrow \underline{G}_0 \rightarrow 1 \end{aligned}$$

which yield the following commutative diagram of pointed sets (cf. [16, IV, Prop. 4.3.4]):

$$(2.6) \quad \begin{array}{ccc} H_{\mathrm{\acute{e}t}}^1(\mathcal{O}_S, \underline{G}_0) & \xrightarrow{=} & H_{\mathrm{\acute{e}t}}^1(\mathcal{O}_S, \underline{G}) \\ \downarrow \delta_0 & & \downarrow \delta_{\underline{G}} \\ H_{\mathrm{\acute{e}t}}^2(\mathcal{O}_S, F(\underline{G})) & \xrightarrow[r_G]{=} & H_{\mathrm{\acute{e}t}}^2(\mathcal{O}_S, F(\underline{G})) \end{array}$$

in which $r_{\underline{G}}(x) := x - \delta_0([\underline{G}])$, so that $\delta_{\underline{G}} = r_{\underline{G}} \circ \delta_0$ maps $[\underline{G}]$ to $[0]$. The image of any twisted form ${}^P\underline{G}$ where $[P] \in H_{\mathrm{\acute{e}t}}^1(\mathcal{O}_S, \underline{G})$ (see in Section 1), under the coboundary map

$$\delta : H_{\mathrm{\acute{e}t}}^1(\mathcal{O}_S, \underline{G}_0) \rightarrow H_{\mathrm{\acute{e}t}}^2(\mathcal{O}_S, Z(\underline{G}_0^{\mathrm{sc}}))$$

induced by the universal covering of \underline{G}_0 corresponding to ρ , is $[{}^P A_\rho]$, where ${}^P A_\rho$ is the Tits-algebra of ${}^P\underline{G}$ (see [22, Thm. 1]). But \underline{G}_0 is adjoint, so $Z(\underline{G}_0^{\mathrm{sc}}) = F(\underline{G}_0) \cong F(\underline{G})$, thus the images of δ and δ_0 coincide in $\mathrm{Br}(\mathcal{O}_S)$, whence:

$$((\Lambda_\rho)_*(\delta_{\underline{G}}([\underline{G}'])) = ((\Lambda_\rho)_*(\delta_0([{}^P\underline{G}]) - \delta_0([\underline{G}])) = [{}^P A_\rho] - [A_\rho]. \quad \square$$

The fundamental group $F(\underline{G})$ is a finite, of multiplicative type (cf. [12, XXII, Cor. 4.1.7]), commutative and smooth \mathcal{O}_S -group (as its order is assumed prime to $\mathrm{char}(K)$).

Lemma 2.2. *If \underline{G} is not of type A, or $S = \emptyset$, then $H_{\mathrm{\acute{e}t}}^1(\mathcal{O}_S, \underline{G})$ is isomorphic to $H_{\mathrm{\acute{e}t}}^2(\mathcal{O}_S, F(\underline{G}))$.*

Proof. Applying étale cohomology to sequence (2.1) yields the exact sequence:

$$H_{\mathrm{\acute{e}t}}^1(\mathcal{O}_S, \underline{G}^{\mathrm{sc}}) \rightarrow H_{\mathrm{\acute{e}t}}^1(\mathcal{O}_S, \underline{G}) \xrightarrow{\delta_{\underline{G}}} H_{\mathrm{\acute{e}t}}^2(\mathcal{O}_S, F(\underline{G}))$$

in which $\delta_{\underline{G}}$ is surjective (see (2.2)). If \underline{G} is not of absolute type A, it is locally isotropic everywhere ([6, 4.3 and 4.4]), in particular at S . This is of course redundant when $S = \emptyset$. Thus $H_{\text{ét}}^1(\mathcal{O}_S, \underline{G}^{\text{sc}})$ vanishes ([4, Lem. 2.3]). Changing the base-point in $H_{\text{ét}}^1(\mathcal{O}_S, \underline{G})$ to any \underline{G} -torsor P , it is bijective to $H_{\text{ét}}^1(\mathcal{O}_S, {}^P\underline{G})$ where ${}^P\underline{G}$ is an inner form of \underline{G} (see Section 1), thus an \mathcal{O}_S -group of the same type. Similarly all fibers of $\delta_{\underline{G}}$ vanish. This amounts to $\delta_{\underline{G}}$ being injective thus an isomorphism. \square

The following two invariants of $F(\underline{G})$ were defined in [4, Def. 1]:

Definition 2. Let R be a finite étale extension of \mathcal{O}_S . We define:

$$i(F(\underline{G})) := \begin{cases} {}_m\text{Br}(R) & F(\underline{G}) = \text{Res}_{R/\mathcal{O}_S}(\underline{\mu}_m) \\ \ker({}_m\text{Br}(R) \xrightarrow{N^{(2)}} {}_m\text{Br}(\mathcal{O}_S)) & F(\underline{G}) = \text{Res}_{R/\mathcal{O}_S}^{(1)}(\underline{\mu}_m) \end{cases}$$

where for a group $*$, ${}_m*$ stands for its m -torsion part, and $N^{(2)}$ is induced by the norm map N_{R/\mathcal{O}_S} .

For $F(\underline{G}) = \prod_{k=1}^r F(\underline{G})_k$ where each $F(\underline{G})_k$ is one of the above, $i(F(\underline{G})) := \prod_{k=1}^r i(F(\underline{G})_k)$.

We also define for such R :

$$j(F(\underline{G})) := \begin{cases} \text{Pic}(R)/m & F(\underline{G}) = \text{Res}_{R/\mathcal{O}_S}(\underline{\mu}_m) \\ \ker(\text{Pic}(R)/m \xrightarrow{N^{(1)}/m} \text{Pic}(\mathcal{O}_S)/m) & F(\underline{G}) = \text{Res}_{R/\mathcal{O}_S}^{(1)}(\underline{\mu}_m) \end{cases}$$

where $N^{(1)}$ is induced by N_{R/\mathcal{O}_S} , and again

$$j\left(\prod_{k=1}^r F(\underline{G})_k\right) := \prod_{k=1}^r j(F(\underline{G})_k).$$

Definition 3. We call $F(\underline{G})$ *admissible* if it is a finite direct product of factors of the form:

- (1) $\text{Res}_{R/\mathcal{O}_S}(\underline{\mu}_m)$,
- (2) $\text{Res}_{R/\mathcal{O}_S}^{(1)}(\underline{\mu}_m)$, $[R : \mathcal{O}_S]$ is prime to m ,

where R is any finite étale extension of \mathcal{O}_S .

Lemma 2.3. *If $F(\underline{G})$ is admissible then there exists a short exact sequence of abelian groups:*

$$(2.7) \quad 1 \rightarrow j(F(\underline{G})) \rightarrow H_{\text{ét}}^2(\mathcal{O}_S, F(\underline{G})) \xrightarrow{\bar{i}_*} i(F(\underline{G})) \rightarrow 1.$$

This sequence splits thus reads: $H_{\text{ét}}^2(\mathcal{O}_S, F(\underline{G})) \cong j(F(\underline{G})) \times i(F(\underline{G}))$.

Proof. This sequence was shown in [4, Cor. 2.9] for the case S is nonempty. The proof based on applying étale cohomology to the related Kummer exact sequence is similar for $S = \emptyset$. The splitting when $F(\underline{G})$ is quasi-split was proved in [15, Thm. 1.1]. When $F(\underline{G}) = \text{Res}_{R/\mathcal{O}_S}^{(1)}(\underline{\mu}_m)$, $[R : \mathcal{O}_S]$ prime to

m , consider the exact diagram obtained by étale cohomology applied to the Kummer exact sequences related to $\underline{\mu}_m$ over \mathcal{O}_S and R :

$$(2.8) \quad \begin{array}{ccccccc} 1 & \longrightarrow & \mathrm{Pic}(R)/m & \longrightarrow & H_{\text{ét}}^2(R, \underline{\mu}_m) & \xrightarrow{i_*} & \mathrm{Br}(R)[m] \\ & & \downarrow N^{(1)}/m & & \downarrow N^{(2)} & & \downarrow N^{(2)}[m] \\ 1 & \longrightarrow & \mathrm{Pic}(\mathcal{O}_S)/m & \longrightarrow & H_{\text{ét}}^2(\mathcal{O}_S, \underline{\mu}_m) & \longrightarrow & \mathrm{Br}(\mathcal{O}_S)[m] \end{array} \longrightarrow 1.$$

The splitting of the two rows then implies the one in the assertion. \square

As a result we have two bijections as pointed-sets: the first is $\mathrm{gen}(\underline{G}) \cong i(F(\underline{G}))$; the affine case shown in [4, Cor. 3.2] holds as aforementioned for $S = \emptyset$ as well, in which case $\mathrm{Br}(C)$ is trivial ([8, Thm. 4.5.1.(v)]) thus \underline{G} admits a single genus. The second bijection is $\mathrm{Cl}_S(\underline{G}) \cong i(F(\underline{G}))$ unless \underline{G} is anisotropic at S , for which it does not have to be injective [4, Prop. 4.1]; hence when S is empty this bijection is guaranteed. Combining Lemma 2.2 with Lemma 2.3 these form (unless \underline{G} is anisotropic at S) an isomorphism of finite abelian groups:

$$(2.9) \quad H_{\text{ét}}^1(\mathcal{O}_S, \underline{G}) \cong j(F(\underline{G})) \times i(F(\underline{G})).$$

3. Twisted-forms

Before continuing with the classification of \underline{G} -forms, we would like to recall the following general construction due to Giraud and prove one related Lemma. Let R be a unital commutative ring. A central exact sequence of étale R -group schemes:

$$(3.1) \quad 1 \rightarrow A \xrightarrow{i} B \xrightarrow{\pi} C \rightarrow 1$$

induces by étale cohomology a long exact sequence of pointed-sets ([16, III, Lem. 3.3.1]):

$$(3.2) \quad 1 \rightarrow A(R) \rightarrow B(R) \rightarrow C(R) \rightarrow H_{\text{ét}}^1(R, A) \xrightarrow{i_*} H_{\text{ét}}^1(R, B) \rightarrow H_{\text{ét}}^1(R, C)$$

in which $C(R)$ acts "diagonally" on the elements of $H_{\text{ét}}^1(R, A)$ in the following way: For $c \in C(R)$, a preimage X of c under $B \rightarrow C$ is a A -bitorsor, i.e., $X = bA = Ab$ for some $b \in B(R')$, where R' is a finite étale extension of R ([16, III, 3.3.3.2]). Then given $[P] \in H_{\text{ét}}^1(R, A)$:

$$(3.3) \quad c * P = P \wedge^A X = (P \times X)/(pa, a^{-1}x).$$

The exactness of (3.2) implies that $B(R) \xrightarrow{\pi} C(R)$ is surjective if and only if $\ker(i_*) = 1$. This holds true starting with any twisted form PB of B , $[P] \in H_{\text{ét}}^1(R, A)$.

Lemma 3.1. *The following are equivalent:*

- (1) *the push-forward map $H_{\text{ét}}^1(R, A) \xrightarrow{i_*} H_{\text{ét}}^1(R, B)$ is injective,*
- (2) *the quotient map ${}^P B(R) \xrightarrow{\pi} C(R)$ is surjective for any $[P] \in H_{\text{ét}}^1(R, A)$,*
- (3) *the $C(R)$ -action on $H_{\text{ét}}^1(R, A)$ is trivial.*

Proof. Consider the exact and commutative diagram (cf. [16, III, Lem. 3.3.4])

$$\begin{array}{ccccccc} B(R) & \xrightarrow{\pi} & C(R) & \longrightarrow & H_{\text{ét}}^1(R, A) & \xrightarrow{i_*} & H_{\text{ét}}^1(R, B) \\ & & & & \cong \downarrow \theta_P & & \cong \downarrow r \\ {}^P B(R) & \xrightarrow{\pi} & C(R) & \longrightarrow & H_{\text{ét}}^1(R, {}^P A) & \xrightarrow{i'_*} & H_{\text{ét}}^1(R, {}^P B), \end{array}$$

where the map i'_* is obtained by applying étale cohomology to the sequence (3.1) while replacing B by the twisted group scheme ${}^P B$, and θ_P is the induced twisting bijection.

(1) \Leftrightarrow (2). The map i_* is injective if and only if $\ker(i'_*)$ is trivial for any A -torsor P . By exactness of the rows, this is condition (2).

(1) \Leftrightarrow (3). By [16, Prop. III.3.3.3(iv)], i_* induces an injection of $H_{\text{ét}}^1(R, A)/C(R)$ into $H_{\text{ét}}^1(R, B)$. Thus $i_* : H_{\text{ét}}^1(R, A) \rightarrow H_{\text{ét}}^1(R, B)$ is injective if and only if $C(R)$ acts on $H_{\text{ét}}^1(R, A)$ trivially. \square

Following B. Conrad in [10], we denote the group of outer automorphisms of \underline{G} by Θ .

Proposition 3.2 ([10, Prop. 1.5.1]). *Assume Φ spans $X_{\mathbb{Q}}$ and that $(X_{\mathbb{Q}}, \Phi)$ is reduced. The inclusion $\Theta \subseteq \mathbf{Aut}(\mathbf{Dyn}(\underline{G}))$ is an equality, if the root datum is adjoint or simply-connected, or if $(X_{\mathbb{Q}}, \Phi)$ is irreducible and $(\mathbb{Z}\Phi^{\vee})^*/\mathbb{Z}\Phi$ is cyclic.*

Remark 3.3. The only case of irreducible Φ in which the non-cyclicity in Proposition 3.2 occurs, is of type D_{2n} ($n \geq 2$), in which $(\mathbb{Z}\Phi^{\vee})^*/\mathbb{Z}\Phi \cong (\mathbb{Z}/2)^2$ (cf. [10, Ex. 1.5.2]).

Remark 3.4. Since \underline{G} is reductive, $\mathbf{Aut}(\underline{G})$ is representable as an \mathcal{O}_S -group and admits the short exact sequence of smooth \mathcal{O}_S -groups (see [12, XXIV, 3.10],[11, §3]):

$$(3.4) \quad 1 \rightarrow \underline{G}^{\text{ad}} \rightarrow \mathbf{Aut}(\underline{G}) \rightarrow \Theta \rightarrow 1.$$

Applying étale cohomology we get the exact sequence of pointed-sets:

$$(3.5) \quad \begin{aligned} \mathbf{Aut}(\underline{G})(\mathcal{O}_S) &\rightarrow \Theta(\mathcal{O}_S) \rightarrow H_{\text{ét}}^1(\mathcal{O}_S, \underline{G}^{\text{ad}}) \\ &\xrightarrow{i_*} H_{\text{ét}}^1(\mathcal{O}_S, \mathbf{Aut}(\underline{G})) \rightarrow H_{\text{ét}}^1(\mathcal{O}_S, \Theta) \end{aligned}$$

in which by Lemma 3.1 the $\Theta(\mathcal{O}_S)$ -action is trivial on $H_{\text{ét}}^1(\mathcal{O}_S, \underline{G}^{\text{ad}})$ if and only if i_* is injective, being equivalent to the surjectivity of

$$({}^P\mathbf{Aut}(\underline{G}))(\mathcal{O}_S) = \mathbf{Aut}({}^P\underline{G})(\mathcal{O}_S) \rightarrow \Theta(\mathcal{O}_S)$$

for all $[P] \in H_{\text{ét}}^1(\mathcal{O}_S, \Theta)$ (this action is trivial inside each genus).

It is a classical fact that $H_{\text{ét}}^1(\mathcal{O}_S, \mathbf{Aut}(\underline{G}))$ is in bijection with twisted forms of \underline{G} up to isomorphism (for a general statement of this correspondence, see [7, §2.2.4]). Therefore this pointed-set shall be denoted from now and on by $\mathbf{Twist}(\underline{G})$. This bijection is done by associating any twisted form \underline{H} of \underline{G} with the $\mathbf{Aut}(\underline{G})$ -torsor $\mathbf{Iso}(\underline{G}, \underline{H})$. If \underline{H} is an inner-form of \underline{G} , then $[\underline{H}]$ belongs to $\text{Im}(i_*)$ in (3.5).

Sequence (3.4) splits, provided that \underline{G} is quasi-split (as in Section 2). Recall that \underline{G} admits an inner form \underline{G}_0 which is quasi-split. Then $\mathbf{Aut}(\underline{G}_0) \cong \underline{G}_0^{\text{ad}} \rtimes \Theta$ (the outer automorphisms group of the two groups are canonically isomorphic). This implies by [14, Lem. 2.6.3] the decomposition

$$(3.6) \quad \begin{aligned} \mathbf{Twist}(\underline{G}_0) &= H_{\text{ét}}^1(\mathcal{O}_S, \mathbf{Aut}(\underline{G}_0)) \\ &= \coprod_{[P] \in H_{\text{ét}}^1(\mathcal{O}_S, \Theta)} H_{\text{ét}}^1(\mathcal{O}_S, {}^P(\underline{G}_0^{\text{ad}}))/\Theta(\mathcal{O}_S) \end{aligned}$$

where the quotients are taken modulo the action (3.3) of $\Theta(\mathcal{O}_S)$ on the ${}^P(\underline{G}^{\text{ad}})$ -torsors. But $\mathbf{Twist}(\underline{G}_0) = \mathbf{Twist}(\underline{G})$ and as \underline{G}_0 is inner:

$$H_{\text{ét}}^1(\mathcal{O}_S, {}^P(\underline{G}_0^{\text{ad}})) = H_{\text{ét}}^1(\mathcal{O}_S, {}^P(\underline{G}^{\text{ad}})),$$

hence (3.6) can be rewritten as:

$$(3.7) \quad \mathbf{Twist}(\underline{G}) = \coprod_{[P] \in H_{\text{ét}}^1(\mathcal{O}_S, \Theta)} H_{\text{ét}}^1(\mathcal{O}_S, {}^P(\underline{G}^{\text{ad}}))/\Theta(\mathcal{O}_S).$$

The pointed-set $H_{\text{ét}}^1(\mathcal{O}_S, \Theta)$ classifies étale extensions of \mathcal{O}_S whose automorphism group embeds into Θ . As all $H_{\text{ét}}^1(\mathcal{O}_S, {}^P(\underline{G}^{\text{ad}}))$ are finite, $\mathbf{Twist}(\underline{G})$ is finite. Together with Lemma 2.2 we get:

Proposition 3.5. *If \underline{G} is not of type A then:*

$$\mathbf{Twist}(\underline{G}) \cong \coprod_{[P] \in H_{\text{ét}}^1(\mathcal{O}_S, \Theta)} H_{\text{ét}}^2(\mathcal{O}_S, F({}^P(\underline{G}^{\text{ad}})))/\Theta(\mathcal{O}_S),$$

the $\Theta(\mathcal{O}_S)$ -action on each component is carried by Lemma 2.2 from the one on $H_{\text{ét}}^1(\mathcal{O}_S, {}^P(\underline{G}^{\text{ad}}))$, cf. (3.3).

Corollary 3.6. *When $S = \emptyset$, i.e., over C , any outer form of \underline{G} has a unique genus on which $\Theta(\mathcal{O}_S)$ acts trivially, hence one has (including for type A):*

$$\mathbf{Twist}(\underline{G}) \cong \coprod_{[P] \in H_{\text{ét}}^1(\mathcal{O}_S, \Theta)} j(F({}^P(\underline{G}^{\text{ad}}))).$$

Since C is smooth, \mathcal{O}_S is a Dedekind ring and any finite étale covering of it is the normalization of \mathcal{O}_S (or of C when $S = \emptyset$) in some finite separable extension of K , which is unramified outside S . So we may look on the fundamental groups over the according extension of fields; The following is the list of all types of absolutely almost-simple K -groups (e.g., [23, p. 333]):

Type of G	$F(G^{\text{ad}})$	$\text{Aut}(\underline{G})$
${}^1\text{A}_{n-1>0}$	μ_n	$\underline{\mathbb{Z}/2}$
${}^2\text{A}_{n-1>0}$	$R_{L/K}^{(1)}(\mu_n)$	$\underline{\mathbb{Z}/2}$
$\text{B}_n, \text{C}_n, \text{E}_7$	μ_2	0
${}^1\text{D}_n$	$\mu_4, n = 2k + 1$ $\mu_2 \times \mu_2, n = 2k$	$\underline{\mathbb{Z}/2}$
${}^2\text{D}_n$	$R_{L/K}^{(1)}(\mu_4), n = 2k + 1$ $R_{L/K}(\mu_2), n = 2k$	$\underline{\mathbb{Z}/2}$
${}^{3,6}\text{D}_4$	$R_{L/K}^{(1)}(\mu_2)$	\underline{S}_3
${}^1\text{E}_6$	μ_3	$\underline{\mathbb{Z}/2}$
${}^2\text{E}_6$	$R_{L/K}^{(1)}(\mu_3)$	$\underline{\mathbb{Z}/2}$
$\text{E}_8, \text{F}_4, \text{G}_2$	1	0

4. Split fundamental group

In the following we show Proposition 3.5. We start with the simple case in which $\Theta = 0$:

Corollary 4.1. *If \underline{G} is of the type $\text{B}_{n>1}, \text{C}_{n>1}, \text{E}_7, \text{E}_8, \text{F}_4, \text{G}_2$, for which $F(\underline{G}^{\text{ad}}) \cong \underline{\mu}_m$ there exists an isomorphism of finite abelian groups $\text{Twist}(\underline{G}) \cong \text{Pic}(\mathcal{O}_S)/m \times {}_m\text{Br}(\mathcal{O}_S)$.*

Proof. As \underline{G} is not of type A this derives from Proposition 3.5, together with the fact that $\Theta(\mathcal{O}_S) = 0$ whence there is a single component $H_{\text{ét}}^1(\mathcal{O}_S, \underline{G}^{\text{ad}})$ on which the action of $\Theta(\mathcal{O}_S)$ is trivial, and the description of the isomorphic group $H_{\text{ét}}^2(\mathcal{O}_S, F(\underline{G}^{\text{ad}}))$ is as in the split case in Lemma 2.3. \square

Example 4.2. Given a regular quadratic \mathcal{O}_S -form Q of rank $2n + 1$, its *special orthogonal group* $\underline{G} = \underline{\text{SO}}_Q$ is smooth and connected of type B_n ([9, Thm. 1.7]). Since $F(\underline{G}) = \underline{\mu}_2$ we assume $\text{char}(K)$ is odd. According to Corollary 4.1 we then get

$$\text{Twist}(\underline{G}) \cong \text{Pic}(\mathcal{O}_S)/2 \times {}_2\text{Br}(\mathcal{O}_S).$$

In case $|S| = 1$ and Q is split by an hyperbolic plane, an algorithm producing explicitly the inner forms of Q is provided in [3, Algorithm 1].

5. Quasi-split fundamental group

Unless \underline{G} is of absolute type D_4 , Θ is either trivial or equals $\{\text{id}, \tau : A \mapsto (A^{-1})^t\}$. In the latter case, τ acts on the $\underline{G}^{\text{ad}}$ -torsors via $X = \underline{G}^{\text{ad}}b$, where b is an outer automorphism of \underline{G} , defined over some finite étale extension of \mathcal{O}_S (see (3.3)). In particular:

$$\tau * \underline{G}^{\text{ad}} = (\underline{G}^{\text{ad}} \times X) / (ga, a^{-1}x),$$

which is the opposite group $(\underline{G}^{\text{ad}})^{\text{op}}$, as the action is via $a^{-1}x = x(a^t)$, (a is viewed as an element of $\underline{G}^{\text{ad}}$, not as an inner automorphism). Now if τ is defined over \mathcal{O}_S , then $\mathbf{Aut}(\underline{G})(\mathcal{O}_S) \rightarrow \Theta(\mathcal{O}_S)$ is surjective and $(\underline{G}^{\text{ad}})^{\text{op}}$ is \mathcal{O}_S -isomorphic to $\underline{G}^{\text{ad}}$, hence as τ is the only non-trivial element in $\Theta(\mathcal{O}_S)$, the map $H_{\text{ét}}^1(\mathcal{O}_S, {}^P\mathbf{Aut}(\underline{G})) \rightarrow H_{\text{ét}}^1(\mathcal{O}_S, \Theta)$ is surjective for all $[P] \in H_{\text{ét}}^1(\mathcal{O}_S, \Theta)$. This implies by Remark 3.4 that $\Theta(\mathcal{O}_S)$ acts trivially on $H_{\text{ét}}^1(\mathcal{O}_S, \underline{G}^{\text{ad}})$. Otherwise, $\underline{G}^{\text{ad}}$ and $(\underline{G}^{\text{ad}})^{\text{op}}$ represent two distinct classes in $H_{\text{ét}}^1(\mathcal{O}_S, \underline{G}^{\text{ad}})$, being identified by $\Theta(\mathcal{O}_S)$.

For any extension R of \mathcal{O}_S and L of K , we denote $\underline{G}_R := \underline{G} \otimes_{\mathcal{O}_S} R$ and $G_L := G \otimes_K L$, respectively. Let $[A_{\underline{G}}]$ be the Tits class of the universal covering $\underline{G}^{\text{sc}}$ of \underline{G} (see Definition 1). This class does not depend on the choice of the representation ρ of $\underline{G}^{\text{sc}}$, thus its notation is omitted. Recall that when $F(\underline{G})$ splits $w_{\underline{G}^{\text{ad}}}$ defined in (2.4) coincides with $\Lambda_* \circ \delta_{\underline{G}}$. Similarly, when $F(\underline{G}) = \text{Res}_{R/\mathcal{O}_S}(\underline{\mu}_m)$ (*quasi-split*) where R/\mathcal{O}_S is finite étale, $\Lambda_* \circ \delta_{\underline{G}_R}$ and $w_{\underline{G}_R^{\text{ad}}}$ defined over R , coincide.

Proposition 5.1. *Suppose $\Theta \cong \mathbb{Z}/2$ and that $F(\underline{G}^{\text{ad}}) = \text{Res}_{R/\mathcal{O}_S}(\underline{\mu}_m)$, R is finite étale over \mathcal{O}_S . Then TFAE:*

- (1) \underline{G}_R admits an outer automorphism,
- (2) $[A_{\underline{G}_R}]$ is 2-torsion in ${}_m\text{Br}(R)$,
- (3) $\Theta(R)$ acts trivially on $H_{\text{ét}}^1(R, \underline{G}_R^{\text{ad}})$.

If, furthermore, \underline{G} is not of type A, or $S = \emptyset$, then these facts are also equivalent to:

- (4) \underline{G} admits an outer automorphism,
- (5) $[A_{\underline{G}}]$ is 2-torsion in ${}_m\text{Br}(R)$,
- (6) $\Theta(\mathcal{O}_S)$ acts trivially on $H_{\text{ét}}^1(\mathcal{O}_S, \underline{G}^{\text{ad}})$.

Proof. By Lemma 2.1 the map $\Lambda_* \circ \delta_{\underline{G}_R^{\text{ad}}} : H_{\text{ét}}^1(R, \underline{G}_R^{\text{ad}}) \rightarrow \text{Br}(R)$ maps $[\underline{H}^{\text{ad}}]$ to $[A_{\underline{H}}] - [A_{\underline{G}_R}]$ where $[A_{\underline{H}}]$ is the Tits class of $\underline{H}^{\text{sc}}$ for a $\underline{G}_R^{\text{ad}}$ -torsor $\underline{H}^{\text{ad}}$. Consider this combined with the long exact sequence obtained by applying

étale cohomology to the sequence (3.4) tensored with R :

$$(5.1) \quad \begin{array}{ccccccc} & & \text{Cl}_R(\underline{G}_R^{\text{ad}}) & & & & \\ & & \downarrow & & & & \\ \text{Aut}(\underline{G}_R)(R) & \longrightarrow & \Theta(R) & \longrightarrow & H_{\text{ét}}^1(R, \underline{G}_R^{\text{ad}}) & \xrightarrow{i_*} \mathbf{Twist}(\underline{G}_R) & \\ & & & & \downarrow w_{\underline{G}_R^{\text{ad}}} = \Lambda_* \circ \delta_{\underline{G}_R^{\text{ad}}} & & \\ & & & & m\text{Br}(R) & & \end{array}$$

where $\text{Cl}_R(\underline{G}_R^{\text{ad}})$ is the principal genus of $\underline{G}_R^{\text{ad}}$ (see [4, Prop. 3.1]) noting that $F(\underline{G}_R^{\text{ad}}) = \underline{\mu}_m$. Being an inner form of $\underline{G}_R^{\text{ad}}$, $(\underline{G}_R^{\text{ad}})^{\text{op}}$ is obtained by a representative in $H_{\text{ét}}^1(R, \underline{G}_R^{\text{ad}})$. Its $w_{\underline{G}_R^{\text{ad}}}$ -image: $[A_{\underline{G}_R}^{\text{op}}] - [A_{\underline{G}_R}]$ is trivial if and only if $A_{\underline{G}_R}$ is of order ≤ 2 in $m\text{Br}(R)$, which is equivalent to $\text{Aut}(\underline{G}_R)(R)$ surjecting on $\Theta(R)$, and $\Theta(R)$ acting trivially on $H_{\text{ét}}^1(R, \underline{G}_R^{\text{ad}})$ (see at the beginning of Section 5).

If, furthermore, \underline{G} is not of type A or $S = \emptyset$, then by Lemma 2.2, together with the Shapiro Lemma we get the isomorphisms of abelian groups:

$$(5.2) \quad H_{\text{ét}}^1(\mathcal{O}_S, \underline{G}^{\text{ad}}) \cong H_{\text{ét}}^2(\mathcal{O}_S, F(\underline{G}^{\text{ad}})) \cong H_{\text{ét}}^2(R, \underline{\mu}_m) \cong H_{\text{ét}}^1(R, \underline{G}_R^{\text{ad}}).$$

So if $\Theta(R)$ acts trivially on $H_{\text{ét}}^1(R, \underline{G}_R^{\text{ad}})$, then so does $\Theta(\mathcal{O}_S)$ on $H_{\text{ét}}^1(\mathcal{O}_S, \underline{G}^{\text{ad}})$. On the other hand if it does not, this implies that $\text{Aut}(\underline{G}_R)(R) \rightarrow \Theta(R) \cong \mathbb{Z}/2$ is not surjective, thus neither is $\text{Aut}(\underline{G})(\mathcal{O}_S) \rightarrow \Theta(\mathcal{O}_S)$, which is equivalent to $\Theta(\mathcal{O}_S)$ acting non-trivially on $H_{\text{ét}}^1(\mathcal{O}_S, \underline{G}^{\text{ad}})$ by Remark 3.4. Moreover, since $i(F(\underline{G}_R^{\text{ad}})) = i(F(\underline{G}^{\text{ad}})) = m\text{Br}(R)$ (Definition 2), the identification (5.2) shows that $\text{Cl}_R(\underline{G}_R^{\text{ad}})$ bijects to $\text{Cl}_S(\underline{G}^{\text{ad}})$, whence $[A_{\underline{G}_R}]$ is 2-torsion in $m\text{Br}(R)$ if and only if $[A_{\underline{G}}]$ is. \square

If we wish to interpret a \underline{G} -torsor as a twisted form of some basic form, we shall need to describe \underline{G} first as the automorphism group of such an \mathcal{O}_S -form.

Example 5.2. Let A be a division \mathcal{O}_S -algebra of degree $n > 2$. Then $\underline{G} = \underline{\text{SL}}(A)$ of type $A_{n-1>1}$ is smooth and connected ([10, Lem. 3.3.1]). It admits a non-trivial outer automorphism τ . If the transpose anti-automorphism $A \cong A^{\text{op}}$ is defined over \mathcal{O}_S (extending τ by inverting again), then $\tau \in \text{Aut}(\underline{G})(\mathcal{O}_S)$. Otherwise, as $(\underline{G}^{\text{ad}})^{\text{op}}$ is not \mathcal{O}_S -isomorphic by some conjugation to $\underline{G}^{\text{ad}} = \underline{\text{PGL}}(A)$, it represents a non-trivial class in $H_{\text{ét}}^1(\mathcal{O}_S, \underline{G}^{\text{ad}} = \mathbf{Inn}(\underline{G}))$, whilst its image in $\mathbf{Twist}(\underline{G})$ is trivial by the inverse isomorphism $x \mapsto x^{-1}$ defined over \mathcal{O}_S (say, by the Cramer rule). So finally $\Theta(\mathcal{O}_S)$ acts trivially on $H_{\text{ét}}^1(\mathcal{O}_S, \underline{\text{PGL}}(A))$ if and only if $\text{ord}(A) \leq 2$ in $\text{Br}(\mathcal{O}_S)$, as Proposition 5.1 predicts.

5.1. Type \mathbf{D}_{2k} . Let A be an Azumaya \mathcal{O}_S -algebra ($\text{char}(K) \neq 2$) of degree $2n$ and let (f, σ) be a *quadratic pair* on A , namely, σ is an involution on A and $f : \text{Sym}(A, \sigma) = \{x \in A : \sigma(x) = x\} \rightarrow \mathcal{O}_S$ is a linear map. The scalar $\mu(a) := \sigma(a) \cdot a$ is called the *multiplier* of a . For $a \in A^\times$ we denote by $\mathbf{Int}(a)$ the induced inner automorphism. If σ is orthogonal, the associated *similitude group* is:

$$\underline{\mathbf{GO}}(A, f, \sigma) := \{a \in A^\times : \mu(a) \in \mathcal{O}_S^\times, f \circ \mathbf{Int}(a) = f\},$$

and the map $a \mapsto \mathbf{Int}(a)$ is an isomorphism of the *projective similitude group* $\underline{\mathbf{PGO}}(A, f, \sigma) := \underline{\mathbf{GO}}(A, f, \sigma)/\mathcal{O}_S^\times$ with the group of rational points $\mathbf{Aut}(A, f, \sigma)$. Such a similitude is said to be *proper* if the induced automorphism of the Clifford algebra $C(A, f, \sigma)$ is the identity on the center; otherwise it is said to be *improper*. The subgroup $\underline{G} = \underline{\mathbf{PGO}}^+(A, f, \sigma)$ of these proper similitudes is connected and adjoint, called the *projective special similitude group*. If the discriminant of σ is a square in \mathcal{O}_S^\times , then \underline{G} is of type ${}^1\mathrm{D}_n$. Otherwise of type ${}^2\mathrm{D}_n$.

When $n=2k$, in order that Θ captures the full structure of $\mathbf{Aut}(\text{Dyn}(\underline{G}))$, we would have to restrict ourselves to the two edges of simply-connected and adjoint groups (see Remark 3.3).

Corollary 5.3. *Let \underline{G} be of type ${}^2\mathrm{D}_{2k}$, $k \neq 2$, simply-connected or adjoint. For any $[P] \in H_{\text{ét}}^1(\mathcal{O}_S, \underline{\mathbb{Z}/2})$ let R_P be the corresponding quadratic étale extension of \mathcal{O}_S . Then:*

$$\mathbf{Twist}(\underline{G}) \cong \coprod_{[P] \in H_{\text{ét}}^1(\mathcal{O}_S, \underline{\mathbb{Z}/2})} \text{Pic}(R_P)/2 \times {}_2\text{Br}(R_P).$$

Proof. Any form ${}^P(\underline{G}^{\text{ad}})$ has Tits class $[A_{P\underline{G}}]$ of order ≤ 2 in ${}_2\text{Br}(R_P)$. Hence as $\Theta \cong \underline{\mathbb{Z}/2}$ and \underline{G} is not of type A, by Proposition 5.1 $\Theta(\mathcal{O}_S)$ acts trivially on $H_{\text{ét}}^1(\mathcal{O}_S, {}^P(\underline{G}^{\text{ad}}))$ for all P in $\Theta(\mathcal{O}_S)$. All fundamental groups are admissible, so the Corollary statement is Proposition 3.5 together with the description of each $H_{\text{ét}}^2(\mathcal{O}_S, F({}^P(\underline{G}^{\text{ad}})))$ as in Lemma 2.3. \square

6. Non quasi-split fundamental group

When $F(\underline{G}^{\text{ad}})$ is not quasi-split, we cannot apply the Shapiro Lemma as in (5.2) to gain control on the action of $\Theta(\mathcal{O}_S)$ on $H_{\text{ét}}^1(\mathcal{O}_S, F({}^P(\underline{G}^{\text{ad}})))$. Still under some conditions this action is provided to be trivial.

Remark 6.1. As opposed to ${}_m\text{Br}(K)$ which is infinite for any integer $m > 1$, ${}_m\text{Br}(\mathcal{O}_S)$ is finite. To be more precise, if $S \neq \emptyset$, $\text{Sp}(\mathcal{O}_S)$ is obtained by removing $|S|$ points from the projective curve C , hence $|{}_m\text{Br}(\mathcal{O}_S)| = m^{|S|-1}$ (see the proof of [4, Cor. 3.2]). When $S = \emptyset$ we have $\text{Br}(C) = 1$. In particular, if \underline{G} is not of absolute type A and $F(\underline{G}^{\text{ad}})$ splits over an extension R such that the number of places in $\text{Frac}(R)$ which lie above

places in S is 1, or when $S = \emptyset$, then $\underline{G}^{\text{ad}}$ can possess only one genus and consequently the $\Theta(\mathcal{O}_S)$ -action on $H_{\text{ét}}^1(\mathcal{O}_S, \underline{G}^{\text{ad}})$ is trivial.

E. Artin in [1] calls a Galois extension L of K *imaginary* if no prime of K is decomposed into distinct primes in L . We shall similarly call a finite étale extension of \mathcal{O}_S *imaginary* if no prime of \mathcal{O}_S is decomposed into distinct primes in it.

Lemma 6.2. *If R is imaginary over \mathcal{O}_S and m is prime to $[R : \mathcal{O}_S]$, then ${}_m\text{Br}(R) = {}_m\text{Br}(\mathcal{O}_S)$.*

Proof. If $S = \emptyset$ and R/C is imaginary then $\text{Br}(R) = \text{Br}(\mathcal{O}_S) = 1$. Otherwise, the composition of the induced norm N_{R/\mathcal{O}_S} with the diagonal morphism coming from the Weil restriction

$$\underline{\mathbb{G}}_{m,\mathcal{O}_S} \rightarrow \text{Res}_{R/\mathcal{O}_S}(\underline{\mathbb{G}}_{m,R}) \xrightarrow{N_{R/\mathcal{O}_S}} \underline{\mathbb{G}}_{m,\mathcal{O}_S}$$

is the multiplication by $n := [R : \mathcal{O}_S]$. It induces together with the Shapiro Lemma the maps:

$$H_{\text{ét}}^2(\mathcal{O}_S, \underline{\mathbb{G}}_{m,\mathcal{O}_S}) \rightarrow H_{\text{ét}}^2(R, \underline{\mathbb{G}}_{m,R}) \xrightarrow{N^{(2)}} H_{\text{ét}}^2(\mathcal{O}_S, \underline{\mathbb{G}}_{m,\mathcal{O}_S})$$

whose composition is the multiplication by n on $H_{\text{ét}}^2(\mathcal{O}_S, \underline{\mathbb{G}}_{m,\mathcal{O}_S})$. Identifying $H_{\text{ét}}^2(*, \underline{\mathbb{G}}_m)$ with $\text{Br}(*)$ and restricting to the m -torsion subgroups gives the composition

$${}_m\text{Br}(\mathcal{O}_S) \rightarrow {}_m\text{Br}(R) \xrightarrow{N^{(2)}} {}_m\text{Br}(\mathcal{O}_S)$$

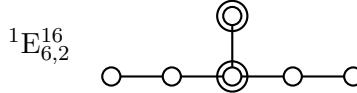
being still multiplication by n , thus an automorphism when n is prime to m . This means that ${}_m\text{Br}(\mathcal{O}_S)$ is a subgroup of ${}_m\text{Br}(R)$. As R is imaginary over \mathcal{O}_S , it is obtained by removing $|S|$ points from the projective curve defining its fraction field, so $|{}_m\text{Br}(R)| = |{}_m\text{Br}(\mathcal{O}_S)| = m^{|S|-1}$ by Remark 6.1, and the assertion follows. \square

Corollary 6.3. *If $F(\underline{G}) = \text{Res}_{R/\mathcal{O}_S}^{(1)}(\underline{\mu}_m)$ is admissible and R/\mathcal{O}_S is imaginary, then $i(F(\underline{G})) = \ker({}_m\text{Br}(R) \rightarrow {}_m\text{Br}(\mathcal{O}_S))$ (see Definition 2) is trivial, hence \underline{G} admits a single genus (cf. [4, Cor. 3.2]).*

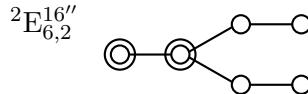
6.1. Type E₆. A *hermitian* Jordan triple over \mathcal{O}_S is a triple (A, \mathfrak{X}, U) consisting of a quadratic étale \mathcal{O}_S -algebra A with conjugation σ , a free of finite rank \mathcal{O}_S -module \mathfrak{X} , and a quadratic map $U : \mathfrak{X} \rightarrow \text{Hom}_A(\mathfrak{X}^\sigma, \mathfrak{X}) : x \mapsto U_x$, where \mathfrak{X}^σ is \mathfrak{X} with scalar multiplication twisted by σ , such that (\mathfrak{X}, U) is an (ordinary) Jordan triple as in [20]. In particular if \mathfrak{X} is an *Albert* \mathcal{O}_S -algebra, then it is called an *hermitian Albert triple*. In that case the associated trace form $T : A \times A \rightarrow \mathcal{O}_S$ is symmetric non-degenerate and it follows that the structure group of \mathfrak{X} agrees with its group of norm similarities. Viewed as an \mathcal{O}_S -group, it is reductive with center of rank 1 and its semisimple part, which we shortly denote $G(A, \mathfrak{X})$, is simply connected

of type E_6 . It is of relative type 1E_6 if $A \cong \mathcal{O}_S \times \mathcal{O}_S$ and of type 2E_6 otherwise.

Groups of type 1E_6 are classified by four relative types, among them only ${}^1E_{6,2}^{16}$ has a non-commutative Tits algebra, thus being the only type in which $\Theta(\mathcal{O}_S) \cong \mathbb{Z}/2$ may act non-trivially on $H_{\text{ét}}^1(\mathcal{O}_S, \underline{G}^{\text{ad}})$. More precisely, the Tits-algebra in that case is a division algebra D of degree 3 (cf. [25, p. 58]) and the $\Theta(\mathcal{O}_S)$ -action is trivial if and only if $\text{ord}([D]) \leq 2$ in $\text{Br}(\mathcal{O}_S)$. But $\text{ord}([D])$ is odd, thus this action is trivial if and only if D is a matrix \mathcal{O}_S -algebra.



In the case of type 2E_6 , one has six relative types (cf. [25, p. 59]), among which only ${}^2E_{6,2}^{16''}$ has a non-commutative Tits algebra (cf. [26, p. 211]). Its Tits algebra is a division algebra of degree 3 over R , and its Brauer class has trivial corestriction in $\text{Br}(\mathcal{O}_S)$. By Albert and Riehm, this is equivalent to D possessing an R/\mathcal{O}_S -involution.



From now on \sim denotes the equivalence relation on the Brauer group which identifies the class of an Azumaya algebra with the class of its opposite.

Corollary 6.4. *Let \underline{G} be of (absolute) type E_6 . For any $[P] \in H_{\text{ét}}^1(\mathcal{O}_S, \underline{\mathbb{Z}/2})$ let R_P be the corresponding quadratic étale extension of \mathcal{O}_S . Then*

$$\text{Twist}(\underline{G}) \cong \text{Pic}(\mathcal{O}_S)/3 \times {}_3\text{Br}(\mathcal{O}_S)/\sim$$

$$\coprod_{1 \neq [P]} \ker(\text{Pic}(R_P)/3 \rightarrow \text{Pic}(\mathcal{O}_S)/3) \times (\ker({}_3\text{Br}(R_P) \rightarrow {}_3\text{Br}(\mathcal{O}_S)))/\sim,$$

where $[P]$ runs over $H_{\text{ét}}^1(\mathcal{O}_S, \underline{\mathbb{Z}/2})$. The relation \sim is trivial in the first component unless $\underline{G}^{\text{ad}}$ is of type ${}^1E_{6,2}^{16}$ and is trivial in the other components unless ${}^P(\underline{G}^{\text{ad}})$ is of type ${}^2E_{6,2}^{16''}$.

Proof. The group $\Theta(\mathcal{O}_S)$ acts trivially on members of the same genus, so it is sufficient to check its action on the set of genera for each type. Since $F({}^P(\underline{G}^{\text{ad}}))$ is admissible for any $[P] \in H_{\text{ét}}^1(\mathcal{O}_S, \Theta)$, by [4, Cor. 3.2] the set of genera of each ${}^P(\underline{G}^{\text{ad}})$ bijects as a pointed-set to $i(F({}^P(\underline{G}^{\text{ad}})))$, so the assertion is Proposition 3.5 together with Lemma 2.3. The last claims are retrieved from the above discussion on the trivial action of $\Theta(\mathcal{O}_S)$ when ${}^P(\underline{G}^{\text{ad}})$ is not of type ${}^1E_{6,2}^{16}$ or ${}^2E_{6,2}^{16''}$. \square

Example 6.5. Let C be the elliptic curve $Y^2Z = X^3 + XZ^2 + Z^3$ defined over \mathbb{F}_3 . Then:

$$C(\mathbb{F}_3) = \{(1 : 0 : 1), (0 : 1 : 2), (0 : 1 : 1), (0 : 1 : 0)\}.$$

Removing the \mathbb{F}_3 -point $\infty = (0 : 1 : 0)$ the obtained smooth affine curve C^{af} is $y^2 = x^3 + x + 1$. Letting $\mathcal{O}_{\{\infty\}} = \mathbb{F}_3[C^{\text{af}}]$ we have $\text{Pic}(\mathcal{O}_{\{\infty\}}) \cong C(\mathbb{F}_3)$ (e.g., [3, Ex. 4.8]). Among the affine supports of points in $C(\mathbb{F}_3) - \{\infty\}$:

$$\{(1, 0), (0, 1/2) = (0, 2), (0, 1)\},$$

only $(1, 0)$ has a trivial y -coordinate thus being of order 2 (according to the group law there), to which corresponds the fractional ideal $P = (x - 1, y)$ of order 2 in $\text{Pic}(\mathcal{O}_{\{\infty\}})$. As $\text{Pic}(\mathcal{O}_{\{\infty\}})/3 = 1$ and $\text{Br}(\mathcal{O}_{\{\infty\}}) = 1$, a form of type 1E_6 has no non-isomorphic *inner* form, while a form of type 2E_6 may have more; for example $R = \mathcal{O}_{\{\infty\}} \oplus P$ being geometric and étale cannot be imaginary over $\mathcal{O}_{\{\infty\}}$, which means it is obtained by removing two points from a projective curve, thus

$$\begin{aligned} \ker({}_3\text{Br}(R) \rightarrow {}_3\text{Br}(\mathcal{O}_{\{\infty\}})) / \sim &= {}_3\text{Br}(R) / \sim \\ &= \{[R], [A], [A^{\text{op}}]\} / \sim = \{[R], [A]\}, \end{aligned}$$

hence an $\mathcal{O}_{\{\infty\}}$ -group of type E_6 splitting over R admits a non-isomorphic inner form.

6.2. Type D_{2k+1} . Recall from Section 5.1 that an adjoint \mathcal{O}_S -group \underline{G} of absolute type D_n can be realized as $\underline{\text{PGO}}^+(A, \sigma)$ where A is Azumaya of degree $2n$ and σ is an orthogonal involution on A . Suppose n is odd. If \underline{G} is of relative type 1D_n then $F(\underline{G}) = \underline{\mu}_4$ is admissible, thus not being of absolute type A, $\text{Cl}_S(\underline{G})$ bijects to $j(\underline{\mu}_4) = \text{Pic}(\mathcal{O}_S)/4$ and $\text{gen}(\underline{G})$ bijects to $i(\underline{\mu}_4) = {}_4\text{Br}(\mathcal{O}_S)$. Otherwise, when \underline{G} is of type 2D_n , then $F(\underline{G}) = \text{Res}_{R/\mathcal{O}_S}^{(1)}(\underline{\mu}_4)$ where R/\mathcal{O}_S is quadratic. Again not being of absolute type A, $\text{Cl}_S(\underline{G}) \cong j(F(\underline{G})) = \ker(\text{Pic}(R)/4 \rightarrow \text{Pic}(\mathcal{O}_S)/4)$, but here, as $F(\underline{G})$ is not admissible, by [4, Cor. 3.2] $\text{gen}(\underline{G})$ only injects in $i(F(\underline{G})) = \ker({}_4\text{Br}(R) \rightarrow {}_4\text{Br}(\mathcal{O}_S))$. If R/\mathcal{O}_S is imaginary, then by Lemma 6.2 $i(F(\underline{G})) = 1$. Altogether by Proposition 3.5 we get:

Corollary 6.6. *Let \underline{G} be of (absolute) type D_{2k+1} . For any $[P] \in H_{\text{ét}}^1(\mathcal{O}_S, \mathbb{Z}/2)$ let R_P be the corresponding quadratic étale extension of \mathcal{O}_S . Then there exists an exact sequence of pointed-sets*

$$\begin{aligned} \text{Twist}(\underline{G}) &\hookrightarrow \text{Pic}(\mathcal{O}_S)/4 \times \coprod_{1 \neq [P]} \ker(\text{Pic}(R_P)/4 \rightarrow \text{Pic}(\mathcal{O}_S)/4) \\ &\quad \times \left({}_4\text{Br}(\mathcal{O}_S)/\sim \times \coprod_{1 \neq [P]} (\ker({}_4\text{Br}(R_P) \rightarrow {}_4\text{Br}(\mathcal{O}_S))) / \sim \right), \end{aligned}$$

where $[P]$ runs over $H^1_{\text{ét}}(\mathcal{O}_S, \underline{\mathbb{Z}/2})$ and $[A] \sim [A^{\text{op}}]$. This map surjects onto the first component. Whenever R_P/\mathcal{O}_S is imaginary $\ker({}_4\text{Br}(R_P) \rightarrow {}_4\text{Br}(\mathcal{O}_S)) = 1$ and this map is a bijection.

Example 6.7. Let $\mathcal{O}_{\{\infty\}} = \mathbb{F}_q[x]$ (q is odd) obtained by removing $\infty = (1/x)$ from the projective line over \mathbb{F}_q . Suppose $q \in 4\mathbb{N} - 1$ so $-1 \notin \mathbb{F}_q^2$, and let $\underline{G} = \underline{\text{SO}}_{10}$ be defined over $\mathcal{O}_{\{\infty\}}$. The discriminant of an orthogonal form Q_B induced by an $n \times n$ matrix B is $\text{disc}(Q_B) = (-1)^{\frac{n(n-1)}{2}} \det(B)$. As $\text{disc}(Q_{1_{10}}) = -1$ is not a square in $\mathcal{O}_{\{\infty\}}$, \underline{G} is considered of type 2D_5 . It admits a maximal torus \underline{T} containing five 2×2 rotations blocks $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$: $a^2 + b^2 = 1$ on the diagonal. Over $R = \mathcal{O}_{\{\infty\}}[i]$ such block is diagonalizable; it becomes $\text{diag}(t, t^{-1})$. The obtained diagonal torus $\underline{T}'_s = P \underline{T}_s P^{-1}$ where $\underline{T}_s = \underline{T} \otimes R$ and P is some invertible 10×10 matrix over R , is split and 5-dimensional, so may be identified with the 5×5 diagonal torus, whose positive roots are:

$$\alpha_1 = \varepsilon_1 - \varepsilon_2, \quad \alpha_2 = \varepsilon_2 - \varepsilon_3, \quad \alpha_3 = \varepsilon_3 - \varepsilon_4, \quad \alpha_4 = \varepsilon_4 - \varepsilon_5, \quad \alpha_5 = \varepsilon_4 + \varepsilon_5.$$

Let g be the matrix differing from the 10×10 unit only at the last 2×2 block, being $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then $\det(g) = -1$ thus $\text{disc}(Q_g) = 1$ where Q_g is the induced quadratic form. This means that $\underline{G}' = \underline{\text{SO}}(Q_g)$ of type 1D_5 is the unique outer form of \underline{G} (up to \mathcal{O}_S -isomorphism). Then $\Theta = \text{Aut}(\text{Dyn}(G))$ acts on $\text{Lie}(g \underline{T}'_s g^{-1})$ by mapping the last block $\begin{pmatrix} 0 & \ln(t) \\ -\ln(t) & 0 \end{pmatrix}$ to $\begin{pmatrix} 0 & -\ln(t) \\ \ln(t) & 0 \end{pmatrix}$ and so swapping the above two roots α_4 and α_5 . Since $\mathcal{O}_{\{\infty\}}$ and R are PIDs, their Picard groups are trivial. As only one point was removed in both domains also $\text{Br}(\mathcal{O}_{\{\infty\}}) = \text{Br}(R) = 1$. We remain with only the two above forms, i.e., $\text{Twist}(\underline{G}) = \{[\underline{G}], [\underline{G}']\}$.

The same holds for $\mathcal{O}_S = \mathbb{F}_q[x, x^{-1}]$: again it is a UFD thus $\underline{G} = \underline{\text{SO}}_{10}$ defined over it still possesses only one non-isomorphic outer form. As \mathcal{O}_S is obtained by removing two points from the projective \mathbb{F}_q -line, this time ${}_4\text{Br}(\mathcal{O}_S)$ is not trivial, but still equals ${}_4\text{Br}(\mathcal{O}_S)$, so: $\ker({}_4\text{Br}(R) \rightarrow {}_4\text{Br}(\mathcal{O}_S)) = 1$.

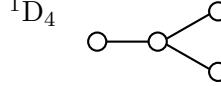
6.3. Type D₄. This case deserves a special regard as Θ is the symmetric group S_3 when \underline{G} is adjoint or simply-connected (cf. Proposition 3.2). Suppose C is an Octonion \mathcal{O}_S -algebra with norm N . For any similitude t of N (see Section 5.1) there exist similitudes t_2 and t_3 such that

$$t_1(xy) = t_2(x) \cdot t_3(y) \quad \forall x, y \in C.$$

Then the mappings:

$$(6.1) \quad \alpha : [t_1] \mapsto [t_2], \quad \beta : [t_1] \mapsto [\widehat{t}_3]$$

where $\widehat{t}(x) := \mu(t)^{-1} \cdot t(x)$, satisfy $\alpha^2 = \beta^3 = \text{id}$ and generate $\Theta = \text{Out}(\mathbf{PGO}^+(N)) \cong \underline{S}_3$.



Having three conjugacy classes, there are three classes of outer forms of G (cf. [10, p. 253]), which we denote as usual by ${}^1\text{D}_4$, ${}^2\text{D}_4$ and ${}^{3,6}\text{D}_4$. The groups in the following table are the generic fibers of these outer forms, L/K is the splitting extension of $F(G^{\text{ad}})$ (note that in the case ${}^6\text{D}_4$ L/K is not Galois):

Type of G	$F(G^{\text{ad}})$	$[L : K]$
${}^1\text{D}_4$	$\mu_2 \times \mu_2$	1
${}^2\text{D}_4$	$R_{L/K}(\mu_2)$	2
${}^{3,6}\text{D}_4$	$R_{L/K}^{(1)}(\mu_2)$	3

Starting with \underline{G} of type ${}^1\text{D}_4$, one sees that $F({}^P(\underline{G}^{\text{ad}}))$ (splitting over some corresponding extension R/\mathcal{O}_S) is admissible for any $[P] \in H_{\text{ét}}^1(\mathcal{O}_S, \Theta)$, thus according to Lemma 2.3

$$\forall [P] \in H_{\text{ét}}^1(\mathcal{O}_S, \Theta) : H_{\text{ét}}^2(\mathcal{O}_S, F({}^P(\underline{G}^{\text{ad}}))) \cong j(F({}^P(\underline{G}^{\text{ad}}))) \times i(F({}^P(\underline{G}^{\text{ad}}))).$$

The action of $\Theta(\mathcal{O}_S)$ is trivial on the first factor, classifying torsors of the same genus, so we concentrate on its action on $i(F({}^P(\underline{G}^{\text{ad}})))$. Since $\Theta \not\cong \mathbb{Z}/2$ we cannot use Proposition 5.1, but we may still imitate its arguments:

The group $\Theta(\mathcal{O}_S)$ acts non-trivially on $H_{\text{ét}}^1(\mathcal{O}_S, {}^P(\underline{G}^{\text{ad}}))$ for some $[P] \in H_{\text{ét}}^1(\mathcal{O}_S, \Theta)$ if it identifies two non isomorphic torsors of ${}^P(\underline{G}^{\text{ad}})$. The Tits algebras of their universal coverings lie in $({}_2\text{Br}(\mathcal{O}_S))^2$ if ${}^P(\underline{G}^{\text{ad}})$ is of type ${}^1\text{D}_4$, i.e., if P belongs to the trivial class in $H_{\text{ét}}^1(\mathcal{O}_S, \Theta)$, in ${}_2\text{Br}(R)$ for R quadratic étale over \mathcal{O}_S if ${}^P(\underline{G}^{\text{ad}})$ is of type ${}^2\text{D}_4$, i.e., if $[P] \in {}_2H_{\text{ét}}^1(\mathcal{O}_S, \Theta)$, and in $\ker({}_2\text{Br}(R) \rightarrow {}_2\text{Br}(\mathcal{O}_S))$ for a cubic étale extension R of \mathcal{O}_S if ${}^P(\underline{G}^{\text{ad}})$ is one of the types ${}^{3,6}\text{D}_4$, i.e., if $[P] \in {}_3H_{\text{ét}}^1(\mathcal{O}_S, \Theta)$. Therefore these Tits algebras must be 2-torsion, which means that the two torsors are \mathcal{O}_S -isomorphic in the first case and R -isomorphic in the latter three. If $F({}^P(\underline{G}^{\text{ad}}))$ is quasi-split this means (by the Shapiro Lemma) that $\Theta(\mathcal{O}_S)$ acts trivially on $H_{\text{ét}}^1(\mathcal{O}_S, {}^P(\underline{G}^{\text{ad}}))$. If $F({}^P(\underline{G}^{\text{ad}}))$ is not quasi-split, according to Corollary 6.3 if R is imaginary over \mathcal{O}_S then $i(F({}^P(\underline{G}^{\text{ad}}))) = 1$.

If a quadratic form Q has a trivial discriminant on a vector space V , the Tits algebras of the group are $\text{End}(V)$ and the two components $\mathbf{C}_+(Q)$, $\mathbf{C}_-(Q)$ of the even Clifford algebra of Q , and the triality automorphism cyclically permutes those three. More generally, if the group is represented as $\mathbf{PGO}^+(A, \sigma)$ for some orthogonal involution of trivial discriminant on a central simple algebra A of degree 8, triality automorphisms permute A

and the two components of the Clifford algebra $\mathbf{C}(A, \sigma)$; Altogether we finally get:

Corollary 6.8. *Let \underline{G} be of (absolute) type D_4 being simply-connected or adjoint. For any $[P] \in H_{\text{ét}}^1(\mathcal{O}_S, \Theta)$ let R_P be the corresponding étale extension of \mathcal{O}_S . Then:*

$$\begin{aligned} \mathbf{Twist}(\underline{G}) &\cong (\mathrm{Pic}(\mathcal{O}_S)/2 \times {}_2\mathrm{Br}(\mathcal{O}_S))^2 \\ &\quad \coprod_{1 \neq [P] \in {}_2H_{\text{ét}}^1(\mathcal{O}_S, \Theta)} \mathrm{Pic}(R_P)/2 \times {}_2\mathrm{Br}(R_P) \\ &\quad \coprod_{1 \neq [P] \in {}_3H_{\text{ét}}^1(\mathcal{O}_S, \Theta)} \ker(\mathrm{Pic}(R_P)/2 \rightarrow \mathrm{Pic}(\mathcal{O}_S)/2) \\ &\quad \quad \quad \times (\ker({}_2\mathrm{Br}(R_P) \rightarrow {}_2\mathrm{Br}(\mathcal{O}_S))) / \Theta(\mathcal{O}_S). \end{aligned}$$

If R_P is imaginary over \mathcal{O}_S , then $\ker({}_2\mathrm{Br}(R_P) \rightarrow {}_2\mathrm{Br}(\mathcal{O}_S)) = 1$.

7. The anisotropic case

Now suppose that \underline{G} does admit a twisted form such that the generic fiber of its universal covering is anisotropic at S . As previously mentioned, such group must be of absolute type A and $S \neq \emptyset$. Over a local field k , an outer form of a group of type 1A which is anisotropic, must be the special unitary group arising by some hermitian form h in r variables over a quadratic extension of k or over a quaternion k -algebra ([27, §4.4]).

A *unitary* \mathcal{O}_S -group is $\underline{\mathbf{U}}(B, \sigma) := \mathbf{Iso}(B, \sigma)$ where B is a non-split quaternion Azumaya defined over an étale quadratic extension R of \mathcal{O}_S and σ is a unitary involution on B , i.e., whose restriction to the center R is not the identity. The *special unitary group* is the kernel of the reduced norm:

$$\underline{\mathbf{SU}}(B, \sigma) := \ker(\mathrm{Nrd} : \underline{\mathbf{U}}(B, \tau) \twoheadrightarrow \underline{\mathbf{GL}}_1(R)).$$

These are of relative type ${}^2C_{2m}$ ($m \geq 2$) ([27, §4.4]) and isomorphic over R to type ${}^1A_{2m-1}$.

So in order to determine exactly when $H_{\text{ét}}^1(\mathcal{O}_S, \underline{G}^{\text{sc}})$ does not vanish, we may restrict ourselves to \mathcal{O}_S -groups whose universal covering is either $\underline{\mathbf{SL}}_1(A)$ or $\underline{\mathbf{SU}}(B, \sigma)$. In the first case, the reduced norm applied to the units of A forms the short exact sequence of smooth \mathcal{O}_S -groups:

$$(7.1) \quad 1 \rightarrow \underline{\mathbf{SL}}_1(A) \rightarrow \underline{\mathbf{GL}}_1(A) \xrightarrow{\mathrm{Nrd}} \underline{\mathbb{G}}_m \rightarrow 1.$$

Then étale cohomology gives rise to the long exact sequence:

$$\begin{aligned} (7.2) \quad 1 &\rightarrow \mathcal{O}_S^\times / \mathrm{Nrd}(A^\times) \rightarrow H_{\text{ét}}^1(\mathcal{O}_S, \underline{\mathbf{SL}}_1(A)) \xrightarrow{i_*} H_{\text{ét}}^1(\mathcal{O}_S, \underline{\mathbf{GL}}_1(A)) \\ &\xrightarrow{\mathrm{Nrd}_*} H_{\text{ét}}^1(\mathcal{O}_S, \underline{\mathbb{G}}_m) \cong \mathrm{Pic}(\mathcal{O}_S) \end{aligned}$$

in which Nrd_* is surjective since $\underline{\mathbf{SL}}_1(A)$ is simply-connected and \mathcal{O}_S is of Douai-type (see above).

Definition 4. We say that the *local-global Hasse principle* holds for \underline{G} if $h_S(\underline{G}) = |\text{Cl}_S(\underline{G})| = 1$.

Thus the Hasse principle says that an \mathcal{O}_S -group is \mathcal{O}_S -isomorphic to G if and only if it is K -isomorphic to it. This is automatic for simply-connected groups which are not of type A or when $S = \emptyset$ for which by Lemma 2.2 $H_{\text{ét}}^1(\mathcal{O}_S, \underline{G}) \cong H_{\text{ét}}^2(\mathcal{O}_S, F(\underline{G}))$ is trivial.

Corollary 7.1. Let $\underline{G} = \underline{\mathbf{SL}}_1(A)$ where A is a quaternion \mathcal{O}_S -algebra.

- (1) If $\text{Nrd} : A^\times \rightarrow \mathcal{O}_S^\times$ is not surjective, then the Hasse principle does not hold for \underline{G} .
- (2) If the generic fiber G is isotropic at S , then $\mathbf{Twist}(\underline{G})$ is in bijection as a pointed-set with the abelian group $\text{Pic}(\mathcal{O}_S)/2 \times {}_2\text{Br}(\mathcal{O}_S)$.

Proof. (1). The generic fiber $\underline{\mathbf{SL}}_1(A)$ is simply-connected thus due to Harder $H^1(K, \underline{\mathbf{SL}}_1(A)) = 1$, which indicates that $\underline{\mathbf{SL}}_1(A)$ admits a single genus (cf. Section 2), i.e., $H_{\text{ét}}^1(\mathcal{O}_S, \underline{\mathbf{SL}}_1(A))$ is equal to $\text{Cl}_S(\underline{\mathbf{SL}}_1(A))$. By the exactness of sequence (7.2), $H_{\text{ét}}^1(\mathcal{O}_S, \underline{\mathbf{SL}}_1(A))$ cannot vanish if $\text{Nrd}(A^\times) \neq \mathcal{O}_S^\times$.

(2). Being of type A_1 , $\underline{G} = \underline{\mathbf{SL}}_1(A)$ does not admit a non-trivial outer form, which implies that $\mathbf{Twist}(\underline{G}) = H_{\text{ét}}^1(\mathcal{O}_S, \underline{G}^{\text{ad}})$. The short exact sequence of the universal covering of $\underline{G}^{\text{ad}} = \underline{\mathbf{PGL}}_1(A)$ with fundamental group $\underline{\mu}_2$, induces the long exact sequence (cf. (2.2)):

$$H_{\text{ét}}^1(\mathcal{O}_S, \underline{\mathbf{SL}}_1(A)) \rightarrow H_{\text{ét}}^1(\mathcal{O}_S, \underline{\mathbf{PGL}}_1(A)) \xrightarrow{\delta_{\underline{G}^{\text{ad}}}} H_{\text{ét}}^2(\mathcal{O}_S, \underline{\mu}_2)$$

in which since $H_{\text{ét}}^1(\mathcal{O}_S, \underline{\mathbf{SL}}_1(A))$ is trivial (due to strong approximation when G is isotropic at S), the rightmost term is isomorphic by Lemma 2.3 to $\text{Pic}(\mathcal{O}_S)/2 \times {}_2\text{Br}(\mathcal{O}_S)$. \square

Example 7.2. Let C be the projective line defined over \mathbb{F}_3 and $S = \{t, t^{-1}\}$. Then $K = \mathbb{F}_3(t)$ and $\mathcal{O}_S = \mathbb{F}_3[t, t^{-1}]$. For the quaternion \mathcal{O}_S -algebra $A = (i^2 = -1, j^2 = -t)_{\mathcal{O}_S}$ we get:

$$\forall x, y, z, w \in \mathcal{O}_S : \text{Nrd}(x + yi + zj + wk) = x^2 + y^2 + t(z^2 + w^2)$$

which shows that $\text{Nrd}(A^\times) = \mathcal{O}_S^\times = \mathbb{F}_3^\times \cdot t^n, n \in \mathbb{Z}$. As \mathcal{O}_S is a UFD, the Hasse principle holds for $\underline{G} = \underline{\mathbf{SL}}_1(A)$, though its generic fiber $G \cong \text{Spin}_q$ for $q(x, y, z) = x^2 + y^2 + tz^2$ is anisotropic at S (cf. [19, Lemma 6]). We have two distinct classes in $\mathbf{Twist}(\underline{G})$, namely, $[\underline{G}]$ and $[\underline{G}^{\text{op}}]$. For $A = (-1, -1)_{\mathcal{O}_S}$, however, we get:

$$\text{Nrd}(x + yi + zj + wk) = x^2 + y^2 + z^2 + w^2$$

which does not surject on \mathcal{O}_S^\times as $t \notin \text{Nrd}(A^\times)$, so the Hasse principle does not hold for $\underline{\mathbf{SL}}_1(A)$.

Similarly, applying étale cohomology to the exact sequence of smooth \mathcal{O}_S -groups:

$$1 \rightarrow \underline{\mathbf{SU}}(B, \sigma) \rightarrow \underline{\mathbf{U}}(B, \sigma) \xrightarrow{\text{Nrd}} \underline{\mathbf{GL}}_1(R) \rightarrow 1$$

induces the exactness of:

$$\begin{aligned} 1 \rightarrow R^\times / \text{Nrd}(\underline{\mathbf{U}}(B, \sigma)(\mathcal{O}_S)) &\rightarrow H_{\text{ét}}^1(\mathcal{O}_S, \underline{\mathbf{SU}}(B, \sigma)) \\ &\rightarrow H_{\text{ét}}^1(\mathcal{O}_S, \underline{\mathbf{U}}(B, \sigma)) \xrightarrow{\text{Nrd}*} H_{\text{ét}}^1(\mathcal{O}_S, \mathbf{Aut}(R)). \end{aligned}$$

Let $A = D(B, \sigma)$ be the discriminant algebra. If R splits, namely, $R \cong \mathcal{O}_S \times \mathcal{O}_S$, then $B \cong A \times A^{\text{op}}$ and σ is the exchange involution. Then $\underline{\mathbf{U}}(B, \sigma) \cong \underline{\mathbf{GL}}_1(A)$ and $\underline{\mathbf{SU}}(B, \sigma) \cong \underline{\mathbf{SL}}_1(A)$, so we are back in the previous situation.

Corollary 7.3. *If $\underline{\mathbf{U}}(B, \sigma)(\mathcal{O}_S) \xrightarrow{\text{Nrd}} R^\times$ is not surjective then the Hasse-principle does not hold for $\underline{\mathbf{SU}}(B, \sigma)$.*

8. In the Zariski topology

A \underline{G} -torsor P is *Zariski*, if the twisted form ${}^P\underline{G}$ is generically and locally everywhere away of S isomorphic to \underline{G} , i.e., if it belongs to the principal genus of \underline{G} (see Section 2). Let \underline{G}_0 be a quasi-split semisimple \mathcal{O}_S -group with an almost-simple generic fiber. The continuous morphism between the categories of open subsets of \mathcal{O}_S : $(\mathcal{O}_S)_{\text{ét}} \rightarrow (\mathcal{O}_S)_{\text{Zar}}$ results, given a variety X defined over \mathcal{O}_S , in the opposite inclusion of cohomology sets $H_{\text{Zar}}^r(\mathcal{O}_S, X) \subseteq H_{\text{ét}}^r(\mathcal{O}_S, X)$ for all $r > 0$. The restriction of the decomposition (3.6)

$$(8.1) \quad \mathbf{Twist}(\underline{G}_0) \cong H_{\text{ét}}^1(\mathcal{O}_S, \mathbf{Aut}(\underline{G}_0)) \cong \coprod_{[P]} H_{\text{ét}}^1(\mathcal{O}_S, {}^P(\underline{G}_0^{\text{ad}})) / \Theta(\mathcal{O}_S)$$

to Zariski torsors gives (compare with [18, p. 181]):

$$(8.2) \quad \mathbf{Twist}_{\text{Zar}}(\underline{G}_0) \cong H_{\text{Zar}}^1(\mathcal{O}_S, \mathbf{Aut}(\underline{G}_0)) \cong H_{\text{Zar}}^1(\mathcal{O}_S, \underline{G}_0^{\text{ad}}) / \Theta(\mathcal{O}_S).$$

But as aforementioned, $H_{\text{Zar}}^1(\mathcal{O}_S, \underline{G}_0^{\text{ad}})$ is equal to the principal genus of $\underline{G}_0^{\text{ad}}$ on which the action of $\Theta(\mathcal{O}_S)$ is trivial, hence (8.2) refines to:

$$(8.3) \quad \mathbf{Twist}_{\text{Zar}}(\underline{G}_0) \cong H_{\text{Zar}}^1(\mathcal{O}_S, \underline{G}_0^{\text{ad}}).$$

Moreover, restricting the bijection $H_{\text{ét}}^1(\mathcal{O}_S, \underline{G}_0^{\text{ad}}) \cong H_{\text{ét}}^2(\mathcal{O}_S, F(\underline{G}_0^{\text{ad}}))$ (Lemma 2.2) to the Zariski topology, $H_{\text{Zar}}^1(\mathcal{O}_S, \underline{G}_0^{\text{ad}})$ can be replaced with $H_{\text{Zar}}^2(\mathcal{O}_S, F(\underline{G}_0^{\text{ad}}))$. All twisted forms of \underline{G}_0 in the Zariski topology being K -isomorphic are isotropic, so this time this includes groups of type A. Suppose $F(\underline{G}_0^{\text{ad}})$ is admissible, splitting over an étale extension R of \mathcal{O}_S . Then $H_{\text{Zar}}^1(\mathcal{O}_S, \mathbb{G}_m) \cong \text{Pic}(\mathcal{O}_S)$ while as R is locally factorial $H_{\text{Zar}}^2(R, \mathbb{G}_m)$ is trivial ([8, Rem. 3.5.1]) thus $i(F(\underline{G}_0))$ as well (see Definition 2). Hence

similarly as was done for $H^2_{\text{ét}}(\mathcal{O}_S, F(\underline{G}_0^{\text{ad}}))$ in Lemma 2.3, we get that $H^2_{\text{Zar}}(\mathcal{O}_S, F(\underline{G}_0^{\text{ad}})) \cong j(F(\underline{G}_0))$.

Corollary 8.1. *Let \underline{G}_0 be a semisimple \mathcal{O}_S -group with an almost-simple generic fiber and an admissible fundamental group. Then: $\mathbf{Twist}_{\text{Zar}}(\underline{G}_0) \cong j(F(\underline{G}_0))$.*

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