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# Reduction of certain crystalline representations and local constancy in the weight space

par Shalini BHATTACHARYA

Résumé. Nous étudions la réduction mod $p$ des représentations galoisiennes cristallines de dimension 2. Berger a montré que lorsque la trace de l'endomorphisme de Frobenius est fixée non nulle, la réduction, sous certaines conditions, est localement constante par rapport au poids. Ici, nous donnons une estimation du rayon de constance de la réduction autour de certains points spéciaux dans l'espace de poids en calculant une majoration pour la valuation $p$-adique du rayon. Notre borne supérieure se révèle être une fonction linéaire de la pente de la représentation cristalline considérée.

Abstract. We study the mod $p$ reduction of crystalline local Galois representations of dimension 2. Berger showed that for a fixed non-zero trace of the Frobenius, the reduction process is locally constant for varying weights under certain conditions. Here we give an estimate of the radius of this local constancy around some special points in the weight space by computing an upper bound for the exponent of $p^{-1}$ in the radius. Our upper bound turns out to be a linear function of the slope of the crystalline representation under consideration.

## 1. Introduction

Let $p \geq 5$ be an odd prime number. Let $E$ be a finite extension of $\mathbb{Q}_p$ and let $v : \overline{\mathbb{Q}}_p^* \to \mathbb{Q}$ be the normalized valuation so that $v(p) = 1$. Let $\mathfrak{m}_E$ be the maximal ideal in the ring of integers $\mathcal{O}_E$ of $E$. For any integer $k \geq 2$ and any $a_p \in \mathfrak{m}_E$, let $D_{k,a_p} = Ee_1 \oplus Ee_2$ be the filtered $\varphi$-module where the Frobenius operator $\varphi$ acts by the matrix $\left(\begin{smallmatrix} 0 & -1 \\ p^{k-1} & a_p \end{smallmatrix}\right)$ with respect to the basis $\langle e_1, e_2 \rangle$, and the filtration is given by

$$\mathrm{Fil}^i(D_{k,a_p}) = \begin{cases} Ee_1 \oplus Ee_2, & \text{if } i \leq 0 \\ Ee_1, & \text{if } 1 \leq i \leq k-1 \\ 0, & \text{if } k \leq i. \end{cases}$$

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Now let $V = V_{k,a_p}$ be the unique two-dimensional irreducible crystalline representation of $G_{\mathbb{Q}_p} := \mathrm{Gal}(\overline{\mathbb{Q}}_p|\mathbb{Q}_p)$ such that $\mathrm{D}_{\mathrm{cris}}(V^*) = D_{k,a_p}$, where $V^*$ is the dual representation of $V$. The existence of such representations follows from the theory of Colmez and Fontaine [14]. We recall that $V_{k,a_p}$ has Hodge–Tate weights $(0, k-1)$ and slope $v := v(a_p) > 0$.

The semi-simplification of the mod $p$ reduction $\overline{V}_{k,a_p}$ of any $G_{\mathbb{Q}_p}$-stable integral lattice in $V_{k,a_p}$ is independent of the choice of the lattice. Despite the variety of non-isomorphic irreducible two-dimensional crystalline representations $V$ in characteristic 0 which are indexed by the tuples $(k, a_p)$ up to twists, at the mod $p$ level one has very limited choice for the semi-simplified reductions $\overline{V}$. Behaviour of the mod $p$ reductions of the representations $V_{k,a_p}$ has been studied by several mathematicians. The explicit shape of $\overline{V}_{k,a_p}$ has been computed for small weights $k \leq 2p+1$ [10, 15], small slopes $v < 2$ [8, 9, 12, 13, 16, 18], or when the slope is very large compared to the weight $k$ [7]. For results on the irreducibility of $\overline{V}_{k,a_p}$, we refer to [1]. Now one can also effectively compute these reductions using the algorithm given in [21] for small values of $k$ and $p$.

In this article we will study how the reduction behaves with varying weight $k$, where $a_p \in m_E$ is kept constant. Let us begin by recalling a result about the local constancy of the map $k \mapsto \overline{V}_{k,a_p}$, for any fixed non-zero $a_p \in \mathfrak{m}_{\bar{\mathbb{Q}}_p}$. The following theorem is due to Laurent Berger, see Theorem B of [6], together with [4].

**Theorem 1.1** (Berger)**.** *Suppose* $a_p \neq 0$ *and* $k > 3v(a_p) + \frac{(k-1)p}{(p-1)^2} + 1$. *Then there exists* $m = m(k, a_p)$ *such that if* $k' - k \in p^{m-1}(p-1)\mathbb{Z}_{>0}$*, then* $\overline{V}_{k',a_p} \cong \overline{V}_{k,a_p}$.

In the context of the theorem above, one may ask the following questions:

- Theorem 1.1 ensures the existence of $m(k, a_p)$, but no estimates are available for this constant. What are the possible values of $m(k, a_p)$? For fixed $a_p$, is it possible to choose an $m(k, a_p)$ that works for all $k$? This phenomenon, when occurs, can be referred to as "uniform local constancy" of the reduction.
- It is clear from Theorem 1.1 that local constancy in the weight space holds around the weights $k$ bigger than $\frac{3v}{1-\frac{p}{(p-1)^2}} + 1$ (also see [4]). One wonders if this bound is necessary, or whether one can improve the lower bound on $k$?

Clearly if Theorem 1.1 holds for some $m$, then so it does for all $m' \geq m$. But we will denote by $m(k, a_p)$ the smallest possible $m \in \mathbb{N}$ satisfying this property. The uniform local constancy is generically true for small slopes, where the reductions have been already computed. Let us recall some cases with known explicit values of $m(k, a_p)$:

- $v \in (0,1)$: We have $m(k,a_p) = 1$, cf. [12], unless $k \equiv 3 \mod (p-1)$ and $v = 1/2$. For $v = 1/2$ and $k \equiv 3 \mod (p-1)$, the behaviour of the reduction is complicated, and it is clear from the main theorem of [13] that $m(k,a_p)$ depends on $k$ and $a_p$ in a more serious way.
- $v = 1$: We have $m(k,a_p) = \begin{cases} 3, & \text{if } k \equiv 3 \mod (p-1) \\ 2, & \text{if } k \not\equiv 3,4 \mod (p-1). \end{cases}$
  For $k \equiv 4 \mod (p-1)$, the reductions are more complex [9].
- $v \in (1,2)$: We have $m(k,a_p) = \begin{cases} 3, & \text{if } k \equiv 3 \mod (p-1) \\ 2, & \text{if } k \not\equiv 3 \mod (p-1), \end{cases}$
  unless $v = 3/2$ and $k \equiv 5 \mod (p-1)$. For the remaining exceptional case, i.e., when $v = 3/2$ and $k \equiv 5 \mod (p-1)$, we refer to [18].

Thus for small slopes, $m(k,a_p)$ is independent of $k$ in most cases, and it does increase with the slope $v = v(a_p)$ in general. In this article we compute $m(k,a_p)$ for some small weights $k$. We also improve the lower bound on $k$ in Berger's Theorem 1.1 a bit, though could not avoid a lower bound that is linear in the slope. More precisely, we prove the following.

**Theorem 1.2.** *For $c \in \{0,1,2,3\}$, let $b \geq 2c$ and suppose $k = b + c(p-1) + 2$, $2 \leq b \leq p-1$. In the range $c < v < p/2 + c$ of slopes, if $k > 2v+2$ and $k \not\equiv 3 \mod (p+1)$, then Berger's constant $m(k,a_p)$ exists and is bounded above by $2v+1$.*

**Remark 1.3.** (a) We give an upper bound on $m(k,a_p)$ for most weights $k$ lying in the range $(2v+2, 4p-2]$. To avoid technical complications in the proof we exclude a few cases, e.g., the weights $k \equiv 3 \mod (p-1)$ or $k \equiv 3 \mod (p+1)$ or when $2 \leq b < 2c$. However, we hope a similar bound for $m(k,a_p)$ will work without these conditions.

(b) For any given finite rational number $v > c$, one can choose a prime $p > 2v$, so our theorem applies for arbitrarily large (finite) slopes. We mention here that the condition $v > c$ can be dropped for $c = 0, 1$ by the known results for $v \leq 1$. Also note that the hypothesis of Theorem 1.2 implies $a_p \neq 0$. In fact, it follows from Proposition 4.1.4 in [7] that there is no local constancy with respect to weight at $a_p = 0$.

(c) Berger proved that the constant $m(k,a_p)$ exists if $k > \frac{3v}{1-\frac{p}{(p-1)^2}} + 1$ (Theorem 1.1). However, direct computation gives us a better lower bound $2v+2$ on $k$ for local constancy. We conclude that this lower bound is strict based on the chaotic behaviour of the reduction as one $p$-adically approaches the point $k = 2v+2$, cf. [17, 18]. Looking at the constant $m(2v+2, a_p)$ in the few known cases, we note two kinds of irregularities:

(i) One can derive from [13] and [9] respectively, together with [15] that

$$\begin{aligned} m(3,a_p) &\le \lceil v(a_p^2 - p)\rceil + 1, \text{ if } v = 1/2, \quad a_p^2 \neq p \\ m(4,a_p) &\le \lceil v(a_p^2 - p^2)\rceil, \quad\ \ \text{if } v = 1, \quad a_p^2 \neq p^2. \end{aligned}$$

In these cases the constant $m(k,a_p)$ cannot be bounded in terms of the slope $v = v(a_p)$. For example, we can make $m(3,a_p)$ arbitrarily large by choosing an $a_p \neq \pm p^{1/2}$ of slope $1/2$ that is $p$-adically close enough to the point $(\pm)p^{1/2}$. In general for $k = 2v+2$, we expect the constant $m(k,a_p)$ to be determined by $v(a_p^2 - p^{k-2})$. But the quantity $v(a_p^2 - p^{k-2})$ depends on more than just the valuation of $a_p$. So even if $m(k,a_p)$ exists, it might not always be expressible as a function of the slope $v$.

(ii) The cases $a_p^2 = p^{k-2}$ are excluded in the inequalities displayed above for the following reason. If $a_p^2 = p$, one can show $m(3,a_p) = 1$ using the results in [13, 15]. But for $a_p^2 = p^2$, we have $v(a_p^2 - \binom{r}{2}p^2) = v(r-2)+2$, provided $k' = r+2$ is $p$-adically close enough to 4 and thus $p(p-1) \mid (r-2)$. This implies by Theorem 1.1 of [9] that $\overline{V}_{k',a_p}$ is reducible. On the other hand by [15] we get $\overline{V}_{4,a_p}$ is irreducible. So there does not exist any finite $m(k,a_p)$ for $k=4$ and $a_p = \pm p$. In this case, local constancy in the weight space fails around the point $k=4$!

The irregularities listed above prove that our lower bound $2v+2$ for $k$ in Theorem 1.2 is optimal. However, based on what is known for small slopes, we expect the local constancy to hold in the weight space around the points $2 \le k < 2v+2$ as well, hoping $k = 2v+2$ to be an isolated point of exception.

Let $G_{\mathbb{Q}_{p^2}}$ denote the subgroup $\mathrm{Gal}(\overline{\mathbb{Q}}_p|\mathbb{Q}_{p^2})$ of $G_{\mathbb{Q}_p}$, where $\mathbb{Q}_{p^2}$ is the unique quadratic unramified extension of $\mathbb{Q}_p$. Let $\omega : G_{\mathbb{Q}_p} \to \mathbb{F}_p^\times \hookrightarrow \bar{\mathbb{F}}_p^\times$ and $\omega_2 : G_{\mathbb{Q}_{p^2}} \to \bar{\mathbb{F}}_{p^2}^\times \hookrightarrow \bar{\mathbb{F}}_p^\times$ be fixed fundamental characters of level one and two respectively. For $p+1 \nmid a$, let $\mathrm{ind}(\omega_2^a)$ denote the unique irreducible representation of $G_{\mathbb{Q}_p}$ with determinant $\omega^a$ such that its restriction to the inertia group is isomorphic to $\omega_2^a \oplus \omega_2^{pa}$.

Theorem 1.2 is a direct consequence of the following main result in Section 3: Let us consider any weight $k' \in k+(p-1)\mathbb{N}$ for some pair $(k,a_p)$ satisfying the hypotheses of Theorem 1.2. If $t = v(k'-k) \ge 2v(a_p)$, then we show that $\overline{V}_{k',a_p}$ is irreducible of the form $\mathrm{ind}(\omega_2^{k-1})$. Thus whenever $k'$ is close enough to $k$ in the weight space, with an explicit bound $p^{-2v(a_p)}$ on their distance, $\overline{V}_{k',a_p}$ is isomorphic to $\overline{V}_{k,a_p}$. Our proof uses the compatibility of $p$-adic and mod $p$ Local Langlands correspondences following [10, 12], and we generalise some of the techniques introduced in [8] and [9]. Our methods directly determine the reduction of $V_{k',a_p}$ without knowing or computing the shape of $\overline{V}_{k,a_p}$. Therefore by using local constancy one recovers part of a general result of [7], see Remark 3.11.

More details about the proof are given in the next section.

## 2. Basics

In this section we quickly recall some notations and explain the basic principle of our proof.

**2.1. The Hecke operator.** Let $G = \mathrm{GL}_2(\mathbb{Q}_p)$, $K = \mathrm{GL}_2(\mathbb{Z}_p)$ be the standard maximal compact subgroup of $G$ and $Z \cong \mathbb{Q}_p^\times$ be the center of the group $G$. We begin by recalling the Hecke operator $T$, which acts $G$-linearly on the compact induction $\mathrm{ind}_{KZ}^G V$, for certain representations $V$ of $KZ$.

Let $R$ be a $\mathbb{Z}_p$-algebra and let $V = \mathrm{Sym}^r R^2 \otimes D^s$ be the usual symmetric power representation of $KZ$ twisted by a power of the determinant character $D$, modelled on homogeneous polynomials of degree $r$ in two variables $X$ and $Y$ over $R$. For $g \in G$, $v \in V$, let $[g, v] \in \mathrm{ind}_{KZ}^G V$ be the function with support in the coset $KZg^{-1}$ given by

$$g' \mapsto \begin{cases} g'g \cdot v, & \text{if } g' \in KZg^{-1} \\ 0, & \text{otherwise.} \end{cases}$$

Any element of $\mathrm{ind}_{KZ}^G V$ is a $V$-valued function on $G$ that is compactly supported mod $KZ$ and thus is a finite linear combination of functions of the form $[g, v]$, for $g \in G$ and $v \in V$. The Hecke operator $T$ is defined by its action on these elementary functions via the formula

$$\begin{aligned} (2.1)\quad & T([g, v(X,Y)]) \\ & = \sum_{\lambda \in \mathbb{F}_p} \left[ g \begin{pmatrix} p & [\lambda] \\ 0 & 1 \end{pmatrix},\ v\left(X, -[\lambda]X + pY\right) \right] + \left[ g \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix},\ v(pX, Y) \right], \end{aligned}$$

where $[\lambda]$ denotes the Teichmüller representative of $\lambda \in \mathbb{F}_p$.

**2.2. The Local Langlands Correspondences.** Let $\Gamma$ denote the finite group $\mathrm{GL}_2(\mathbb{F}_p)$ which naturally acts on a two-dimensional vector space over $\overline{\mathbb{F}}_p$. For any $r \geq 0$, we have the symmetric power representations

$$V_r := \mathrm{Sym}^r \, \overline{\mathbb{F}}_p^2 \in \mathrm{Rep}_{\overline{\mathbb{F}}_p}(\Gamma)$$

of dimension $r + 1$. For $0 \leq r \leq p - 1$, $\lambda \in \overline{\mathbb{F}}_p$ and $\eta : \mathbb{Q}_p^\times \to \overline{\mathbb{F}}_p^\times$ a smooth character, we know that

$$\pi(r, \lambda, \eta) := \frac{\mathrm{ind}_{KZ}^G V_r}{T - \lambda} \otimes (\eta \circ \det)$$

are smooth admissible representations of $G$, also irreducible in most cases. Recall that here $p \in KZ$ acts on $V_r := \mathrm{Sym}^r \, \overline{\mathbb{F}}_p^2$ trivially and the rest of $KZ$

acts by the inflation of $K = \mathrm{GL}_2(\mathbb{Z}_p) \twoheadrightarrow \Gamma$. These objects $\pi(r, \lambda, \eta)$ together capture all possible irreducible representations of $G$ in characteristic $p$, as proved in [2, 3, 11].

With this notation, Breuil's semi-simple mod $p$ Local Langlands Correspondence [10, Def. 1.1] is given by the map $LL$ as follows:

- $\lambda = 0$: $\mathrm{ind}_{G_{\mathbb{Q}_{p^2}}}^{G_{\mathbb{Q}_p}} (\omega_2^{r+1}) \otimes \eta \overset{LL}{\longmapsto} \pi(r, 0, \eta)$,
- $\lambda \neq 0$: $(\mu_\lambda \omega^{r+1} \oplus \mu_{\lambda^{-1}}) \otimes \eta \overset{LL}{\longmapsto} \pi(r, \lambda, \eta)^{ss} \oplus \pi([p-3-r], \lambda^{-1}, \eta\omega^{r+1})^{ss}$,

where $\{0, 1, \ldots, p-2\} \ni [p-3-r] \equiv p-3-r \mod (p-1)$.

On the other hand, by the $p$-adic Local Langlands correspondence we have the association $V_{k,a_p} \rightsquigarrow \Pi_{k,a_p}$, where $\Pi_{k,a_p}$ is the locally algebraic representation of $G$ given by

$$\Pi_{k,a_p} = \frac{\mathrm{ind}_{KZ}^G \mathrm{Sym}^r \overline{\mathbb{Q}}_p^2}{(T - a_p)},$$

where $r = k - 2 \geq 0$ and $T$ is the Hecke operator as usual. Consider the standard lattice in $\Pi_{k,a_p}$ given by

$$\begin{aligned} \Theta_{k,a_p} &:= \mathrm{image}\left(\mathrm{ind}_{KZ}^G \mathrm{Sym}^r \overline{\mathbb{Z}}_p^2 \to \Pi_{k,a_p}\right) \\ &\simeq \frac{\mathrm{ind}_{KZ}^G \mathrm{Sym}^r \overline{\mathbb{Z}}_p^2}{(T - a_p)(\mathrm{ind}_{KZ}^G \mathrm{Sym}^r \overline{\mathbb{Q}}_p^2) \cap \mathrm{ind}_{KZ}^G \mathrm{Sym}^r \overline{\mathbb{Z}}_p^2}. \end{aligned} \tag{2.2}$$

By the compatibility of the $p$-adic and mod $p$ Local Langlands Correspondence which was conjectured in [10] and proved in [5], we know that

$$\overline{\Theta}_{k,a_p}^{\mathrm{ss}} := \left(\Theta_{k,a_p} \otimes \overline{\mathbb{F}}_p\right)^{ss} \simeq LL(\overline{V}_{k,a_p}^{ss}).$$

The correspondence $LL$ at the mod $p$ level is injective, so it is enough to compute $LL(\overline{V}_{k,a_p}^{ss})$ to determine $\overline{V}_{k,a_p}^{ss}$. Therefore, we are going to study $\overline{\Theta}_{k,a_p}^{ss}$ as an object in $\mathrm{Rep}_{\overline{\mathbb{F}}_p}(G)$. The superscript "$ss$" will often be skipped, as in this article we are only concerned about the semi-simplified reduction.

## 3. Computing the reduction

**3.1. Some results in characteristic $p$.** In this subsection we prove some general lemmas in characteristic $p$ that will be useful in computing the reduction $\overline{\Theta}_{k',a_p}$, where $k'$ is as in the last paragraph of Section 1.

By the definition of $\overline{\Theta}_{k',a_p}$, for $r = k'-2 \geq 0$, we have a natural surjection

$$P : \mathrm{ind}_{KZ}^G V_r \twoheadrightarrow \overline{\Theta}_{k',a_p}.$$

Note that on the special polynomial

$$\theta(X, Y) := X^pY - Y^pX = -X \prod_{\lambda \in \mathbb{F}_p} (Y - \lambda X) \in \mathrm{Sym}^{p+1} \overline{\mathbb{F}}_p^2 = V_{p+1}, \tag{3.1}$$

$\Gamma := \mathrm{GL}_2(\mathbb{F}_p)$ acts by the determinant character. For each $m \in \mathbb{N}$, we define

$$V_r^{(m)} := \{f \in V_r : \theta^m \text{ divides } f \text{ in } \bar{\mathbb{F}}_p[X,Y]\},$$

so that $V_r \supseteq V_r^{(1)} \supseteq V_r^{(2)} \supseteq \ldots$ is a chain of $\Gamma$-submodules of length $\lfloor \frac{r}{p+1} \rfloor + 1$. Moreover, we know that $V_r^{(m)} \cong V_{r-m(p+1)} \otimes D^m$, where $D$ denotes the determinant character.

**Lemma 3.1.** *Let $F(X,Y) = \sum\limits_{i=0}^{r} a_i X^{r-i} Y^i \in V_r$ be a polynomial such that*

$$\bar{\mathbb{F}}_p \ni a_i \neq 0 \implies i \equiv a \mod (p-1),$$

*for some fixed congruence class $a \mod (p-1)$. For $0 \leq m \leq p$, we have $F(X,Y) \in V_r^{(m)}$ if and only if the following conditions are satisfied:*

- *$i < m$, or $i > r - m \implies a_i = 0 \in \bar{\mathbb{F}}_p$,*
- *$\sum\limits_i j! \binom{i}{j} a_i = 0 \in \bar{\mathbb{F}}_p$, for $0 \leq j \leq m-1$.*

*Proof.* We consider $f(z) = \sum\limits_{i=0}^{r} a_i z^i \in \bar{\mathbb{F}}_p[z]$, so that $F(X,Y) = X^r \cdot f(\frac{Y}{X})$. It follows from (3.1) that

$$\begin{aligned}
&\theta^m \mid F(X,Y) \\
&\iff F(X,Y) = F_1(X,Y) \cdot (-X)^m \prod_{\lambda \in \mathbb{F}_p} (Y - \lambda X)^m, \quad F_1 \in V_{r-(p+1)m}, \\
&\iff X^m \mid F(X,Y) \text{ and } f(Y/X) = F_1(1, Y/X)(-1)^m \prod_{\lambda \in \mathbb{F}_p} (Y/X - \lambda)^m, \\
&\iff X^m \mid F(X,Y) \text{ and } f(z) = \prod_{\lambda \in \mathbb{F}_p} (-(z-\lambda))^m F_1(1,z), \\
&\iff X^m, Y^m \mid F(X,Y) \text{ and } (z-\lambda)^m \mid f(z), \quad \forall\, \lambda \in \mathbb{F}_p^\times.
\end{aligned}$$

The conditions $X^m, Y^m \mid F(X,Y)$ are equivalent to

$$a_i \neq 0 \implies m \leq i \leq r - m,$$

and $(z-\lambda)^m$ divides $f(z)$ if and only if $f(\lambda) = f'(\lambda) = \cdots = f^{(m-1)}(\lambda) = 0 \in \bar{\mathbb{F}}_p$. Looking at the coefficients of $f(z)$, for $\lambda \in \mathbb{F}_p^\times$, we have

$$f^{(j)}(\lambda) = \sum_i a_i \cdot i(i-1)\ldots(i-j+1)\lambda^{i-j} = \lambda^{a-j} \cdot \sum_i a_i \binom{i}{j} j!,$$

using the hypothesis on the coefficients of $F(X,Y)$. This completes our proof.

Note that as we are in the situation $j < m \leq i \leq r-m$ here, the binomial coefficients $\binom{i}{j}$ above are all a priori non-zero, though some of them might vanish mod $p$. $\square$

For integers $0 \le m \le s$ let us define polynomials $F_{s,m}$ (or $F_m$) in $V_r$ as

$$F_{s,m}(X,Y) := X^m Y^{r-m} - X^{r-s+m} Y^{s-m}, \tag{3.2}$$

where $r > s$ and $r \equiv s \mod (p-1)$, so Lemma 3.1 can be applied on $F_{s,m}$ for suitable values of $m$. The congruence class of $r \mod (p-1)$ in the range $\{2,3,\dots,p\}$ will be denoted by "$b$" throughout this article. Our "$s$" here may or may not coincide with $b$, as in general $s \in \{b, b+p-1, b+2(p-1), \dots\}$. For a fixed $s$, multiple values of $m$ will be considered in our calculations. By abuse of notation, we will drop the index $s$ in $F_{s,m}$ and denote it simply by $F_m$. With this notation we prove the following key lemma:

**Lemma 3.2.** *Let $r \equiv s \mod (p-1)$, and $t = v(r-s) \ge 1$ and $1 \le m \le p-1$.*

(a) *For $s \ge 2m$, the polynomial $F_m$ is divisible by $\theta^m$ but not by $\theta^{m+1}$.*
(b) *For $s > 2m$, the image of $F_m$ generates the subquotient $\frac{V_r^{(m)}}{V_r^{(m+1)}}$ over $\Gamma$.*

*Proof.* (a). Any polynomial divisible by $\theta^{m+1}$ is a multiple of $X^{m+1}$, so $\theta^{m+1} \nmid F_m$.

To show $\theta^m \mid F_m$, by Lemma 3.1 we need to show both $m, s-m \ge m$, and further for all $0 \le j \le m-1$,

$$j!\left(\binom{r-m}{j} - \binom{s-m}{j}\right) \equiv 0 \mod p,$$

which is ensured by the fact $t = v(r-s) \ge 1$. Note that the last condition is satisfied for $j = m$ as well.

(b). We recall the structure of $\frac{V_r^{(m)}}{V_r^{(m+1)}} \cong \frac{V_{r-m(p+1)}}{V_{r-m(p+1)}^{(1)}} \otimes D^m$ given by the short exact sequence

$$\begin{aligned}(3.3)\quad 0 \to V_{s'-p+1} \otimes D^m \to \frac{V_{r-m(p+1)}}{V_{r-m(p+1)}^{(1)}} \otimes D^m \cong \frac{V_{s'}}{V_{s'}^{(1)}} \otimes D^m \\ \to V_{2p-2-s'} \otimes D^{s'+m} \to 0,\end{aligned}$$

where $s' \equiv r - m(p+1) \mod (p-1)$ in the range $s' \in \{p, p+1, \dots 2p-2\}$. This short exact sequence is non-split as $\Gamma$-representation, except for when $s' = 2p-2$. Both the non-zero maps in the sequence above are described explicitly in Lemma 5.3 of [10] in the range $p \le s' \le 2p-2$. We want to compute the image of $F_m(X,Y) \in \frac{V_r^{(m)}}{V_r^{(m+1)}}$ in the quotient above. But to do that, first we need to know its image in $\frac{V_{s'}}{V_{s'}^{(1)}} \otimes D^m$ under the isomorphism map from $\frac{V_r^{(m)}}{V_r^{(m+1)}}$. We now claim that the polynomial

$$H_m(X,Y) := F_m(X,Y) - (-1)^m \theta(X,Y)^m (Y^{r-m(p+1)} - Y^{s-2m} X^{r-s-pm+m})$$

in $V_r$ lies in the submodule $V_r^{(m+1)}$. Assuming the claim, it is enough to show the image of $\theta^m(Y^{r-m(p+1)} - Y^{s-2m}X^{r-s-pm+m})$ generates $\frac{V_r^{(m)}}{V_r^{(m+1)}}$ over $\Gamma$. By the sequence (3.3), for $s' \neq 2p-2$, it is enough to show that this image maps to a non-zero element in the quotient above. We check (for $s > 2m$) that in fact its image is

$$Y^{s'} - Y^{s'-p+1}X^{p-1} \in \frac{V_{s'}}{V_{s'}^{(1)}} \otimes D^m,$$

which maps to $-X^{2p-2-s'} (\neq 0) \in V_{2p-2-s'} \otimes D^{s'+m}$.

For $s' = 2p-2$, the sequence (3.3) splits. So we further check that the image of the polynomial $Y^{s'} - Y^{s'-p+1}X^{p-1}$ does not lie in the one-dimensional $\Gamma$-subspace $V_0 \otimes D^m$ of $\frac{V_{s'}}{V_{s'}^{(1)}} \otimes D^m$. This can be seen by applying the matrix $w = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \in \Gamma$, as

$$\begin{aligned}
&(Y^{s'} - Y^{s'-p+1}X^{p-1}) - w \cdot (Y^{s'} - Y^{s'-p+1}X^{p-1}) \\
&\quad = Y^{s'} - Y^{s'-p+1}X^{p-1} - (-1)^m (X^{s'} - X^{s'-p+1}Y^{p-1}) \notin V_{s'}^{(1)} \otimes D^m.
\end{aligned}$$

The image of $F_m(X,Y)$ in $V_{2p-2}/V_{2p-2}^{(1)} \otimes D^m$ must generate the whole module over $\Gamma$, as it is contained in none of its direct summands.

*Proof of claim.* The lowest degree of $X$ in $H_m(X,Y)$ is $\geq m+p-1 \geq m+1$, and the lowest degree of $Y$ in $H_m(X,Y)$ is $\geq s-m \geq m+1$, as $s > 2m$ by hypothesis. Following the proof of Lemma 3.1, we consider

$$\begin{aligned}
h_m(z) := H_m(1,z) &= z^{r-m} - z^{s-m} - (-1)^m (z - z^p)^m (z^{r-m(p+1)} - z^{s-2m}) \\
&= z^{r-m} - z^{s-m} - (z^{p-1} - 1)^m (z^{r-mp} - z^{s-m}).
\end{aligned}$$

We already know $X^{m+1}, Y^{m+1}$ divide $H_m(X,Y)$, hence

$$\theta^{m+1} \mid H_m(X,Y) \iff (z-\lambda)^{m+1} \mid h_m(z), \quad \forall\, \lambda \in \mathbb{F}_p^\times.$$

Equivalently, we need $\frac{d^i h_m}{dz^i}(\lambda) = 0$ for all $0 \leq i \leq m (< p)$, and all $\lambda \in \mathbb{F}_p^\times$. For the first part $F_m(1,z) = z^{r-m} - z^{s-m}$ of $h_m(z)$, this vanishing of derivatives is already proved in part (a) above. For the other part $-(z^{p-1} - 1)^m (z^{r-mp} - z^{s-m})$ of $h_m(z)$, the derivatives up to order $m$ vanish since $1 - \lambda^{p-1} = 0 = \lambda^{r-mp} - \lambda^{s-m}$, for all $\lambda \in \mathbb{F}_p^\times$. □

Let us mention here that the polynomials $F_m(X,Y)$ vanish in the case $r = s$ and thus Lemma 3.2 is not valid for $r = s$. Since this is a key lemma to be used to prove our main result, our proofs will be applicable only for large enough weights $k' > s+2$.

Now we recall a very useful fact from Remark 4.4 in [12], that if $v(a_p) < m$ and $r = k'-2 \geq m(p+1)$, then kernel of the natural map $\mathrm{ind}_{KZ}^G V_r \twoheadrightarrow \overline{\Theta}_{k',a_p}$ contains the sub-representation $\mathrm{ind}_{KZ}^G(V_r^{(m)})$, and thus $\overline{\Theta}_{k',a_p}$ is a quotient of $\mathrm{ind}_{KZ}^G(V_r/V_r^{(m)})$. We fix an $a_p$ with positive valuation, and let $n \in \mathbb{N}$ be the smallest such that $v(a_p) < n+1$, so we have

$$P : \mathrm{ind}_{KZ}^G(V_r/V_r^{(n+1)}) \twoheadrightarrow \overline{\Theta}_{k',a_p}. \tag{3.4}$$

We consider the chain of submodules of length $n+1$

$$0 \subseteq \frac{V_r^{(n)}}{V_r^{(n+1)}} \subseteq \frac{V_r^{(n-1)}}{V_r^{(n+1)}} \subseteq \cdots \subseteq \frac{V_r}{V_r^{(n+1)}},$$

inducing the chain

$$0 \subseteq M_n \subseteq M_{n-1} \subseteq \cdots \subseteq M_0 = \mathrm{ind}_{KZ}^G\left(\frac{V_r}{V_r^{(n+1)}}\right),$$

where $M_i := \mathrm{ind}_{KZ}^G\left(\frac{V_r^{(i)}}{V_r^{(n+1)}}\right)$ for $0 < i \leq n$, with respective images

$$P(M_n) \subseteq P(M_{n-1}) \subseteq \cdots \subseteq P(M_0) := \overline{\Theta}_{k',a_p}. \tag{3.5}$$

We have this chain of submodules inside $\overline{\Theta}_{k',a_p}$, and we will try to compute it piece by piece. For example, we would like to check if some of the sub-quotients $P(M_i)/P(M_{i+1})$ of $\overline{\Theta}_{k',a_p}$ are in fact zero.

**3.2. Computations in characteristic 0.** We extend the formula for the Hecke operator $T$ when acting on $\mathrm{ind}_{KZ}^G \mathrm{Sym}^r \overline{\mathbb{Q}}_p^2$ in particular, to see how $T$ acts on its explicit elements viewed as $\mathrm{Sym}^r \overline{\mathbb{Q}}_p^2$-valued functions on the Bruhat–Tits tree for $\mathrm{GL}_2$.

For $m = 0$, set $I_0 = \{0\}$, and let $I_m = \{[\lambda_0] + [\lambda_1]p + \cdots + [\lambda_{m-1}]p^{m-1} : \lambda_i \in \mathbb{F}_p\} \subset \mathbb{Z}_p$ for $m > 0$, where the square brackets denote Teichmüller representatives. For $m \geq 1$, there is a truncation map $[\quad]_{m-1} : I_m \to I_{m-1}$ given by taking the first $m-1$ terms in the $p$-adic expansion above; for $m = 1$, $[\quad]_{m-1}$ is the 0-map. Let $\alpha = \left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right)$. For $m \geq 0$ and $\lambda \in I_m$, let

$$g_{m,\lambda}^0 = \begin{pmatrix} p^m & \lambda \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad g_{m,\lambda}^1 = \begin{pmatrix} 1 & 0 \\ p\lambda & p^{m+1} \end{pmatrix},$$

noting that $g_{0,0}^0 = \mathrm{Id}$ is the identity matrix and $g_{0,0}^1 = \alpha$ in $G$. We have

$$G = \coprod_{\substack{m \geq 0,\, \lambda \in I_m, \\ i \in \{0,1\}}} KZ(g_{m,\lambda}^i)^{-1}.$$

Thus a general element in $\mathrm{ind}_{KZ}^G V$ is a finite sum of functions of the form $[g, v]$, with $g = g_{m,\lambda}^0$ or $g_{m,\lambda}^1$, for some $\lambda \in I_m$ and $v \in V$. Let

$v = \sum_{i=0}^{r} c_i X^{r-i} Y^i \in V = \mathrm{Sym}^r\, \overline{\mathbb{Q}}_p^2 \otimes D^s$. Then expanding the formula (2.1) for Hecke operator, one writes $T = T^+ + T^-$, where

$$T^+([g^0_{n,\mu}, v]) = \sum_{\lambda \in I_1} \left[ g^0_{n+1,\mu+p^n\lambda}, \sum_{j=0}^{r} \left( p^j \sum_{i=j}^{r} c_i \binom{i}{j} (-\lambda)^{i-j} \right) X^{r-j} Y^j \right] \tag{3.6}$$

$$\begin{aligned} &T^-([g^0_{n,\mu}, v]) \\ &= \left[ g^0_{n-1,[\mu]_{n-1}}, \sum_{j=0}^{r} \left( \sum_{i=j}^{r} p^{r-i} c_i \binom{i}{j} \left( \frac{\mu - [\mu]_{n-1}}{p^{n-1}} \right)^{i-j} \right) X^{r-j} Y^j \right] \quad (n > 0) \end{aligned} \tag{3.7}$$

$$T^-([g^0_{n,\mu}, v]) = [\alpha, \sum_{j=0}^{r} p^{r-j} c_j X^{r-j} Y^j] \qquad (n = 0) \tag{3.8}$$

as in Lemma 2.3 of [10]. These explicit formulas for $T^+$ and $T^-$ will be used to compute $(T - a_p)f$ for the functions $f \in \mathrm{ind}_{KZ}^G \mathrm{Sym}^r\, \overline{\mathbb{Q}}_p^2$. We will work with large values of $r$ which are $p$-adically close to some relatively small $s = b + c(p-1)$ with $2 \le b \le p-1$. Our final result is valid for $c = 0, 1, 2, 3$ and $b \ge 2c$, as mentioned in the introduction. However it will be clear from our statements that some of the intermediate lemmas are true in a more general setting.

Suppose $r \equiv s \mod p(p-1)$ for some $2 \le s \le p^2 - p$, so $t := v(r-s) > 0$. Let us define, for $0 \le i \le s$, and $0 \le m < p-1$, the sums

$$S_{r,i,m} := \sum_{\substack{j \equiv r-m \mod (p-1) \\ s-m \le j < r-m}} \binom{j}{i} \binom{r}{j} \tag{3.9}$$

and with this notation, we have the following technical lemma.

**Lemma 3.3.** *Let $r = s + dp^t(p-1)$ with $p \nmid d$, for some $s = b + c(p-1)$, $2 \le b \le p-1$ and $0 \le c \le p-1$. For $0 \le i < s$ and $0 \le m < p-1$, one has*

$$S_{r,i,m} \equiv \binom{r}{i} \left( \sum_{\substack{j \equiv s-m \mod (p-1) \\ 0 \le j < s-m}} \left( \binom{s-i}{j-i} - \binom{r-i}{j-i} \right) \right) \mod p^t,$$

*which further implies that*

$$S_{r,i,m} \equiv \begin{cases} 0 \mod p^t, & c = 0 \\ 0 \mod p^{t+1-c}, & c > 0. \end{cases}$$

The lemma can be proved using the same techniques as in [9, Proposition 2.8]. We skip the proof here to save some space.

**Proposition 3.4.** *Let $r = s + dp^t(p-1)$, where $s = b + c(p-1)$ with $2 \leq b \leq p-1$. If $b \geq c-1$, $0 \leq m < c \leq v(a_p) < p-1$ and $t > v(a_p)+c-1$, then for all $g \in G$ and for $0 \leq i \leq m$, there exists $f^i \in \mathrm{ind}_{KZ}^G \operatorname{Sym}^r \overline{\mathbb{Q}}_p^2$ such that*

$$(T - a_p)f^i \equiv \left[ g, \sum_{\substack{0<j<s-m \\ j \equiv r-m \mod (p-1)}} \binom{r-i}{j} X^{r-j}Y^j \right] \mod \wp. \tag{3.10}$$

*If $2c-1 \leq b \leq p-1$, $v(a_p) > c$ and $t > v(a_p)+c$, we can choose $f_m$ such that moreover*

$$(T - a_p)\frac{f^m}{p} \equiv \left[ g, \sum_{\substack{0<j<s-m \\ j \equiv r-m \mod (p-1)}} \frac{\binom{r-m}{j}}{p} X^{r-j}Y^j \right] \mod \wp, \tag{3.11}$$

*where $\wp$ stands for the prime ideal above $p$ in the ring of integers of $\mathbb{Q}_p(a_p)$.*

*Proof.* Since $T$ is $G$-linear operator, it is enough to prove the statement for $g = g_{1,0}^0$. We note that existence of an $m$ with $0 \leq m < c$ forces $c$ to be at least 1. Next we define $f^i = f_2 + f_1 + f_0$ as follows:

$$\begin{aligned} f_2 &= \sum_{\lambda \in \mathbb{F}_p^\times} \left[ g_{2,p[\lambda]}^0, \left(\frac{1}{[\lambda]}\right)^{m-i} \frac{F_i(X,Y)}{p^i(p-1)} \right] - \left[ g_{2,0}^0, \binom{r-i}{r-m} \frac{F_m(X,Y)}{p^m} \right], \\ f_1 &= \frac{1}{a_p} \left[ g_{1,0}^0, \sum_{\substack{s-m \leq j < r-m \\ j \equiv r-m \mod (p-1)}} \binom{r-i}{j} X^{r-j}Y^j \right], \\ f_0 &= \begin{cases} [\mathrm{Id}, F_s(X,Y)] & \text{if } r \equiv m \mod (p-1) \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{3.12}$$

Using the formulae for $T^+$ and $T^-$ we check

$$\begin{aligned} T^+ f_2 \equiv 0 \mod \wp \text{ as } t > v(a_p) \geq m \geq i, \\ \text{and } s - 2m > s - 2c + 1 > 0, \\ -a_p f_2 \equiv 0 \mod p^{v(a_p)-m} \equiv 0 \mod \wp, \text{ as } m < c \leq v(a_p) \end{aligned}$$

$$T^- f_2 - a_p f_1 + T^+ f_0 \equiv \left[ g_{1,0}^0, \sum_{\substack{0<j<s-m \\ j \equiv r-m \mod (p-1)}} \binom{r-i}{j} X^{r-j}Y^j \right] \mod \wp,$$

$$\text{as } r - s + i > i,$$

$$\begin{aligned}
-a_p f_0 &\equiv 0 \mod p^{v(a_p)} \equiv 0 \mod \wp,\\
T^- f_0 &\equiv 0 \mod p^s \equiv 0 \mod \wp, \text{ as } s>0,\\
T^- f_1 &\equiv 0 \mod p^{m+p-1-v(a_p)} \equiv 0 \mod \wp,\\
&\qquad\qquad\qquad\qquad \text{as } v(a_p) < p-1,\\
T^+ f_1 &\equiv 0 \mod \wp,
\end{aligned}$$

as $s - m = b - m + c(p-1) \geq c - 1 - m + c(p-1) \geq p - 1 > v(a_p)$, and then we use Lemma 3.3 for $S_{r-i,i',m-i}$, $0 \leq i' \leq v(a_p)$ with the fact $t > v(a_p) + c - 1 \geq v(a_p) + \lfloor \frac{s-m-i+1}{p} \rfloor - 1$, so that $t + 1 - \lfloor \frac{s-m-i+1}{p} \rfloor > 0$.

For $i = m$, the same computation works and we obtain the congruence (3.10), but Lucas' theorem tells us that each of the binomial coefficients in the right hand side are in fact divisible by $p$, if $b \geq 2c-1$. Similar careful calculation for $(T - a_p)(f^m/p)$ gives us the congruence (3.11) as claimed. □

For the rest of the article, we will use the following abuse of notation: a polynomial $f(X,Y) \in V_r$ is said to "vanish modulo $\ker P$", if for all $g \in G$ the elements $[g, f(X,Y)]$ map to 0 under the map $P : \mathrm{ind}_{KZ}^G V_r \twoheadrightarrow \overline{\Theta}_{r+2,a_p}$.

**Proposition 3.5.** *Let $r = s + dp^t(p-1)$, with $p \nmid d$ and $s = b + c(p-1)$. Further suppose that $2c - 1 \leq b \leq p-1$ and $m < c \leq v(a_p) < p-1$. If $t > v(a_p) + c - 1$, then the monomials $X^{r-b+m-j(p-1)}Y^{b-m+j(p-1)}$ for $c - m \leq j \leq c-1$ vanish modulo* $\ker P$.

*Proof.* We note that for $m = 0$, there is no integer $j$ in the given range. So we may assume $m \geq 1$ and consider the $m \times m$ matrix

$$A = (a_{ji})_{c-m \leq j \leq c-1, 0 \leq i \leq m-1}\,, \text{ with } a_{ji} := \binom{r-i}{j(p-1)+b-m}.$$

As $1 \leq m < c$, we have $c \geq 2$ and hence $t \geq 2$ by hypothesis. Using that, we have

$$\begin{aligned}
r - i = s + dp^t(p-1) - i &= b - c - i + cp + dp^t(p-1)\\
&\in (b-c-i) + cp + p^2\mathbb{Z},\\
\text{and} \quad j(p-1) + b - m &= (b-m-j) + jp.
\end{aligned}$$

We check that the entries above $b-c-i, c, b-m-j$ and $j$ all lie in the range $[0, p-1]$:

$$\begin{gathered}
0 < c - m \leq b - c - m + 1 \leq b - c - i \leq b - c \leq c - 1 < p - 1,\\
0 \leq c \leq p-1,\\
0 \leq b - 2c + 1 < b - m - (c-1) \leq b - m - j \leq b - m - (c-m) = b - c < p - 1,\\
0 < c - m \leq j \leq c - 1 < p - 1.
\end{gathered}$$

Hence by Lucas' theorem, $a_{ji} \equiv \binom{b-c-i}{b-m-j}\binom{c}{j} \mod p$, so $a_{ji} \not\equiv 0 \mod p$ if and only if $j - i \geq c - m$. Thus $A \in M_m(\mathbb{Z}_p)$ is such that modulo $p$ it reduces to a lower triangular matrix with non-zero (mod $p$) diagonal entries. Hence $\det A \in \mathbb{Z}_p^*$ and $A^{-1} \in \mathrm{GL}_m(\mathbb{Z}_p)$. For any $0 \leq l \leq m-1$, let $\underline{d} = (d_0, d_1, \ldots, d_{m-1}) \in \mathbb{Z}_p^m$ be the vector given by $\underline{d} := A^{-1}e_l$, where $e_l$ denote the column vector with 1 as the $l$-th entry and 0 elsewhere. Then by the first part of Proposition 3.4, we get

$$(T - a_p)\sum_{i=0}^{m-1}(d_i f^i) \equiv \left[g, X^{r-b+m-(c-m+l)(p-1)}Y^{b-m+(c-m+l)(p-1)}\right] \mod \wp,$$

where $f^i$'s are as in Proposition 3.4. Here we also use that the first few terms in (3.10) vanish as $\binom{r-i}{j(p-1)+b-m} \equiv 0 \mod p$ for $0 \leq j < c-m$, for any $i \geq 0$. $\square$

**Remark 3.6.** Proposition 3.5 is applicable to any $c \leq p/2$, although for the final result of this paper we have assumed $c \leq 3$ to avoid technical complications.

**Lemma 3.7.** *For $0 \leq j, m \leq c-1$, and $r = b + c(p-1) + p^2\mathbb{Z}$ with $2c - 1 \leq b \leq p-1$,*

$$\frac{\binom{r-m}{j(p-1)+b-m}}{p} \equiv \frac{\binom{p-1}{c-1-j}\binom{c}{j}}{\binom{b-m-j}{b-m-c}} \equiv (-1)^{c-1-j}\frac{\binom{c}{j}}{\binom{b-m-j}{c-j}} \mod p.$$

*Proof.* Let $A = \sum_{i=0}^{n} a_i p^i$, $B = \sum_{i=0}^{n} b_i p^i$ and $A - B = \sum_{i=0}^{n} c_i p^i$, with $0 \leq a_i, b_i, c_i < p$. If $p^e \mid \binom{A}{B}$, then by [20], we have

$$\binom{A}{B} \equiv (-p)^e \prod_{i=0}^{n} \frac{a_i!}{b_i! c_i!} \mod p^{e+1}.$$

This result can be applied for $A = r - m$, $B = j(p-1) + b - m$ and $e = 1$ to obtain the required congruence. $\square$

The next proposition is crucial for our proof, which eliminates the possible contribution of the factors $\frac{V_r^{(m)}}{V_r^{(m+1)}}$ for $0 \leq m < c$ in $\overline{\Theta}_{r+2,a_p}$, when $t = v(r-s)$ is sufficiently large.

**Proposition 3.8.** *Let $s = b + c(p-1)$ with $0 \leq c \leq 3$, $2c - 1 \leq b \leq p-1$ and let $c < v = v(a_p) < p-1$. If $r \equiv s \mod p^t(p-1)$ with $2v \leq t < \infty$, then there is a surjection*

$$\mathrm{ind}_{KZ}^G\left(\frac{V_r^{(c)}}{V_r^{(\lfloor v \rfloor + 1)}}\right) \twoheadrightarrow \overline{\Theta}_{r+2,a_p}$$

*that is induced from $P : \mathrm{ind}_{KZ}^G V_r \twoheadrightarrow \overline{\Theta}_{r+2,a_p}$.*

*Proof.* By Remark 4.4 in [12], we have $\mathrm{ind}_{KZ}^G V_r^{(n)} \subseteq \ker P$ if $r \geq n(p+1)$ and $n > v$. Using this fact for $n = \lfloor v \rfloor + 1$, we have $\mathrm{ind}_{KZ}^G V_r^{(\lfloor v \rfloor+1)} \subseteq \ker P$ for $r \geq (\lfloor v \rfloor+1)(p+1)$. For $r < (\lfloor v \rfloor+1)(p+1)$, we note that $V_r^{(\lfloor v \rfloor+1)} = 0$. Hence in any case the surjection $P$ factors through $\mathrm{ind}_{KZ}^G \left(\frac{V_r}{V_r^{(\lfloor v \rfloor+1)}}\right)$. This already proves the proposition for $c = 0$, as $V_r^{(0)} = V_r$ by definition.

So we assume $c \geq 1$ for the rest of the proof. We will show that for $0 \leq m < c$, the image of $\mathrm{ind}_{KZ}^G \left(\frac{V_r^{(m)}}{V_r^{(m+1)}}\right)$ is 0 in $\overline{\Theta}_{k,a_p}$. We begin with the simplest case $m = 0$.

*Case 1: $m = 0$.* We are under hypothesis $t \geq 2v(a_p) > 2c \geq 2$, therefore $r \equiv s \mod p^3$. By equation (3.11) we have

$$\left[ g, \sum_{\substack{0<j<s \\ j \equiv r \mod (p-1)}} \frac{\binom{r}{j}}{p} X^{r-j} Y^j \right] \in \ker P, \tag{3.13}$$

but then

$$\sum_{\substack{0<j<s \\ j \equiv r \mod (p-1)}} \frac{\binom{r}{j}}{p} X^{r-j} Y^j \equiv \nu \cdot X^{r-b} Y^b \mod \langle V_r^{(1)}, X^r \rangle_\Gamma,$$

where

$$\nu = \sum_{\substack{0<j<s \\ j \equiv r \mod (p-1)}} \frac{\binom{r}{j}}{p} \equiv \sum_{\substack{0<j<s \\ j \equiv s \mod (p-1)}} \frac{\binom{s}{j}}{p} \equiv \frac{b-s}{b} \equiv \frac{c}{b} \not\equiv 0 \mod p,$$

by Lemma 2.5 in [8]. Using (4.2) of [19] and Lemma 5.3 of [10] we can see that the monomial $X^{r-b}Y^b$ generates (over $K$ or $\Gamma$) the quotient $V_{p-1-b} \otimes D^b$ of $V_r/V_r^{(1)}$. Being generated by the image of the highest monomial $X^r$, the other factor $V_b$ of $V_r/V_r^{(1)}$ has no contribution in $\overline{\Theta}_{r+2,a_p}$. So all of the quotient $V_r/V_r^{(1)}$ has zero contribution in $\overline{\Theta}_{r+2,a_p}$.

*Case 2: $0 < m < c$.* In this case $b \geq 2c - 1 \geq 2m + 1$, so

$$\sum_{i=0}^{m+1} (-1)^i \binom{m+1}{i} X^{r-s+m+i(p-1)} Y^{s-m-i(p-1)} \in V_r^{(m+1)},$$

and by Proposition 3.5, the middle terms vanish "modulo $\ker P$", implying

$$X^{r-s+m} Y^{s-m} \equiv (-1)^m X^{r-b+m-(c-m-1)(p-1)} Y^{b-m+(c-m-1)(p-1)} \mod V_r^{(m+1)} + \text{“} \ker P \text{”}.$$

Further we recall that $m \le c-1 < v(a_p)$, so $X^m Y^{r-m}$ vanishes "mod $\ker P$", implying $F_m(X,Y) \equiv -X^{m+r-s}Y^{s-m} \mod$ "$\ker P$", where $F_m$ is the polynomial defined in (3.2). Hence it is enough to check that

$$\left[g,\ X^{r-b+m-(c-m-1)(p-1)}Y^{b-m+(c-m-1)(p-1)} \mod V_r^{(m+1)}\right] \in \ker P \tag{3.14}$$

to show $\operatorname{ind}_{KZ}^G \left(\frac{V_r^{(m)}}{V_r^{(m+1)}}\right) \ni \left[g,\ \overline{F_m(X,Y)}\right] \xmapsto{\overline{P}} 0$, killing the factor $\frac{V_r^{(m)}}{V_r^{(m+1)}}$ by Lemma 3.2 (b). As $0 \le c \le 3$, we have two possibilities here, namely $c-m=1$ or $c-m=2$.

For $c-m=1$, by congruence (3.11) and Proposition 3.5, $X^{r-b+m}Y^{b-m}$ vanishes mod $\ker P$, since by Lemma 3.7, $\frac{\binom{r-m}{b-m}}{p} \equiv \frac{(-1)^{c-1}}{\binom{b-m}{c}} \not\equiv 0 \mod p$.

For $c-m=2$, again using (3.11) and Proposition 3.5, we get that the polynomial $\frac{\binom{r-m}{b-m}}{p}X^{r-b+m}Y^{b-m} + \frac{\binom{r-m}{b-m+p-1}}{p}X^{r-b+m-p+1}Y^{b-m+p-1}$ vanishes modulo "$\ker P$". But we use the fact $\sum_{i=0}^{m+1}(-1)^i\binom{m+1}{i}X^{r-b+m-i(p-1)} \times Y^{b-m+i(p-1)}$ lies in $V_r^{(m+1)}$ together with Proposition 3.5 to conclude

$$X^{r-b+m}Y^{b-m} \equiv (m+1)X^{r-b+m-p+1}Y^{b-m+p-1} \mod V_r^{(m+1)} + \text{“}\ker P\text{”}.$$

Therefore $[g,\ \eta \cdot Y^{b-m+p-1}X^{r-b+m-p+1}] \in \ker P + P(\operatorname{ind}_{KZ}^G V_r^{(m+1)})$, where

$$\eta := (m+1)\frac{\binom{r-m}{b-m}}{p} + \frac{\binom{r-m}{b-m+p-1}}{p}.$$

By Lemma 3.7, $\eta \equiv \frac{(-1)^c}{\binom{b-m}{c}}(b-(2m+1)) \not\equiv 0 \mod p$ as $m = c-2$, proving (3.14). $\square$

The following useful lemma is a direct consequence of the explicit mod $p$ Local Langlands Correspondence.

**Lemma 3.9.** *Let $k' = r+2$ and $r \equiv b \mod (p-1)$ for some $2 \le b \le p-1$. If the map $P$ reduces to a surjection*

$$\operatorname{ind}_{KZ}^G \frac{V_r^{(c)}}{V_r^{(c+1)}} \twoheadrightarrow \overline{\Theta} := \overline{\Theta}_{k',a_p}$$

*for some $0 \le c < p/2$ and if $b \ne 2c \pm 1$, then*

$$\overline{\Theta} \cong \begin{cases} \pi(b-2c, 0, \omega^c), & b \ge 2c \\ \pi(2c-b, 0, \omega^{b-c}), & b \le 2c. \end{cases}$$

*For $b = 2c \pm 1$, in addition to the irreducible shape above one also has a reducible possibility where $\overline{\Theta}^{ss} \cong \left(\pi(p-2,\lambda,1) \oplus \pi(p-2,\lambda^{-1},1)\right) \otimes \eta$ with $\eta = \omega^{c+1}$ or $\omega^c$.*

*Proof.* Let us first consider the case $b \geq 2c$, which would be relevant for us in this paper. Using Proposition 2.1 of [8], we get

$$0 \to \mathrm{ind}_{KZ}^{G}(V_{b-2c} \otimes D^{c}) \to \mathrm{ind}_{KZ}^{G} \frac{V_r^{(c)}}{V_r^{(c+1)}} \to \mathrm{ind}_{KZ}^{G}(V_{p-1-b+2c} \otimes D^{b-c}) \to 0$$

where $\overline{\Theta}$ is a quotient of the middle term of the short exact sequence above. We check that under the hypotheses both the indices $b-2c$ and $p-1-b+2c$ lie in the range $[0, p-1]$. Since $\overline{\Theta}$ lies in the image of the mod $p$ Langlands correspondence, looking at the explicit map $LL$ defined in §2 we conclude that $\overline{\Theta}$ is supercuspidal and isomorphic to $\pi(b-2c, 0, \omega^{c}) \cong \pi(p-1-b+2c, 0, \omega^{b-c})$, unless either $b-2c$ or $p-1-b+2c$ equals $p-2$.

We note that $b-2c$ can possibly be $p-2$ only if $c=0$, but then the factor $V_{b-2c} = V_b$ is the image of the highest monomial submodule $\langle X^r \rangle$ in $V_r/V_r^{(1)}$, which is known to die under the map $P$, so cannot contribute to $\overline{\Theta}$. On the other hand if $p-1-b+2c = p-2$, i.e., $b = 2c+1$, then reducible case in the Local Langlands correspondence may occur. In that case $\overline{\Theta}$ is a quotient of $\mathrm{ind}_{KZ}^{G}(V_{p-2} \otimes D^{c+1})$, may be $\pi(p-2, 0, \omega^{c+1})$ or it may have two JH factors, namely $\pi(p-2, \lambda, \omega^{c+1})$ and $\pi(p-2, \lambda^{-1}, \omega^{c+1})$ for some $\lambda \in \overline{\mathbb{F}}_p^{\times}$.

The remaining case $b < 2c$ can be treated similarly, with

$$0 \to \mathrm{ind}_{KZ}^{G}(V_{b-2c+p-1} \otimes D^{c}) \to \mathrm{ind}_{KZ}^{G} \frac{V_r^{(c)}}{V_r^{(c+1)}} \to \mathrm{ind}_{KZ}^{G}(V_{2c-b} \otimes D^{b-c}) \to 0$$

being the relevant short exact sequence. $\square$

**Corollary 3.10.** *Let $p \geq 5$, $s = b + c(p-1)$ with $0 \leq c \leq 3$ and $2c \leq b \leq p-1$ and let $k := s+2$. If $b \neq 2c+1$ and $v = v(a_p) \in (c, c+1)$, then*

(i) *$m(k, a_p) \leq 2v+1$, and*
(ii) *by local constancy, $\overline{V}_{k,a_p} \cong \mathrm{ind}(\omega_2^{k-1})$.*

*Proof.* By Proposition 3.8, if $r \equiv s \mod p^{2v(a_p)}(p-1)$, then under the hypothesis we have a surjection

$$\mathrm{ind}_{KZ}^{G}\left(\frac{V_r^{(c)}}{V_r^{(c+1)}}\right) \twoheadrightarrow \overline{\Theta}_{k',a_p},$$

where $k' = r+2$. By Lemma 3.9, we have $\overline{\Theta}_{k',a_p} \cong \pi(b-2c, 0, \omega^{c})$. By mod $p$ Local Langlands correspondence $\overline{V}_{k',a_p} \cong \mathrm{ind}(\omega_2^{s+1})$, as $s = b-2c+c(p+1)$.

Next we apply Theorem 1.1 to $k = s+2$. One checks that $k > 3v(a_p) + \frac{(k-1)p}{(p-1)^2} + 1$ using the conditions $p \geq 5$, $v(a_p) < c+1$ and $k = b+c(p-1)+2$. So there exists $m = m(k, a_p)$, such that $k' - k \in p^{m-1}(p-1)\mathbb{Z}_{>0}$ implies $\overline{V}_{k',a_p} \cong \overline{V}_{k,a_p}$. But by the previous paragraph we know that $\overline{V}_{k',a_p} \cong$

$\mathrm{ind}(\omega_2^{k-1})$ for such $k'$ with $m \geq 2v+1$. Hence we conclude that (the smallest) $m(k,a_p) \leq 2v(a_p)+1$, and $\overline{V}_{k,a_p} \cong \mathrm{ind}(\omega_2^{k-1})$. □

**Remark 3.11.** The main point of the proof here is to determine the shape of $\overline{V}_{k',a_p}$, and then the rest follows as a consequence. The shape of $\overline{V}_{k,a_p}$ is already known for $c = 0, 1$ by the results in [10, 15], which is consistent with part (ii) above. Also since $k = b + c(p-1) + 2$, we know that $\lfloor \frac{k-2}{p-1} \rfloor = c$ unless $b = p-1$, thus we recovered the shape of $\overline{V}_{k,a_p}$ for $v(a_p) > c$ proved in [7] for the weights $k$ satisfying our hypothesis.

If $b = p-1$, then one deduces using Berger–Li–Zhu's work that for $v > c+1 = \lfloor \frac{k-2}{p-1} \rfloor$, $\overline{V}_{k,a_p} \cong \mathrm{ind}(\omega_2^{k-1})$. But we note that by Corollary 3.10 above, even in the lower range of slopes $c < v < c+1$ the reduction $\overline{V}_{k,a_p}$ has the same shape. This indicates that the bound $\lfloor \frac{k-2}{p-1} \rfloor$ may not be optimal in some cases, which is also suggested by the numerical evidences (cf. [22, §6.1]).

In fact, as it is known in most cases that $\overline{V}_{k,a_p} \cong \mathrm{ind}(\omega_2^{k-1})$ for $v(a_p) > c$, one expects Corollary 3.10 to hold without the upper bound $c+1$ on the slope. Therefore for higher slopes we attempt to eliminate the factors $\frac{V_r^{(m)}}{V_r^{(m+1)}}$ for all $m$ in the range $c < m \leq \lfloor v(a_p) \rfloor$, and we succeed under the extra condition $v(a_p) < c + p/2$.

**Proposition 3.12.** *Fix $a_p$ with $c < v(a_p) < \frac{p}{2} + c$. Let $r \equiv s \mod (p-1)$ for some $s > 2v(a_p)$, $s = c(p-1) + b < r$ with $2c \leq b \leq p-1$ and let $c < m \leq \lfloor v(a_p) \rfloor$. If $t = v(r-s) \geq 2v(a_p)$, then for $0 \leq i < m - v(\binom{r-i}{r-m})$, $\exists\, f^i \in \mathrm{ind}_{KZ}^G \mathrm{Sym}^r \overline{\mathbb{Q}}_p^2$ such that*

$$(T - a_p)f^i \equiv \frac{p^m}{a_p}\left[g_{1,0}^0, \sum_{\substack{c<j<s-m \\ j \equiv r-m \mod (p-1)}} \frac{\binom{r-i}{j}}{\binom{r-i}{r-m}} X^{r-j}Y^j\right] + \left[g_{2,0}^0, F_m(X,Y)\right] \mod \wp. \tag{3.15}$$

*Proof.* Let $f^i = f_0 + f_1 + f_2$ be given by

$$f_2 = \sum_{\lambda \in \mathbb{F}_p^\times}\left[g_{2,p[\lambda]}^0, \left(\frac{p}{[\lambda]}\right)^{m-i} \frac{F_i(X,Y)}{(p-1)\binom{r-i}{r-m}a_p}\right] + \left[g_{2,0}^0, -\frac{F_m(X,Y)}{a_p}\right]$$

$$f_1 = \left[g_{1,0}^0, \frac{p^m}{a_p^2} \sum_{\substack{s-m \leq j < r-m \\ j \equiv r-m \mod (p-1)}} \frac{\binom{r-i}{j}}{\binom{r-i}{r-m}} X^{r-j}Y^j\right]$$

$$f_0 = \begin{cases} \left[\mathrm{Id}, \frac{p^{2m-b}\binom{r-i}{b-m}}{a_p\binom{r-i}{r-m}} F_{s-b+m}(X,Y)\right] & \text{if } 0 \le b-m \le c \\ 0 & \text{otherwise.} \end{cases}$$

The binomial coefficient $\binom{r-i}{r-m}$ can be non-unit only if $b-c < m \le v(a_p)$, which never happens for $c=0$ as in that case $b = s > 2v(a_p) > v(a_p)+c$ by hypothesis. Further for $1 \le c < m$ and $0 \le i < m$, we have

$$(3.16) \quad v\left(\binom{r-i}{r-m}\right) = v\left(\binom{r-i}{s-m}\right) = \begin{cases} 0, & \text{if } c < m \le b-c \\ 1, & \text{if } i \le b-c < m \le \lfloor v(a_p) \rfloor \\ 0, & \text{if } b-c < i < m \le \lfloor v(a_p) \rfloor, \end{cases}$$

and $a_p f_0$ always vanishes mod $\wp$ as we note

$$v\left(\binom{r-i}{r-m}\right) = \begin{cases} 0,\ 1 \ \le m-c & \text{for } b-m < c, \\ 0 \qquad < m-c & \text{for } b-m = c. \end{cases}$$

$$< m-(b-m) = v\left(p^{2m-b}\right),$$

$T^- f_0$ vanishes mod $\wp$ for $0 \le b-m \le c$, as $c < m \le \lfloor v(a_p) \rfloor$ and $s > v(a_p)$ by hypothesis. Using the facts $s > 2v(a_p) \ge m + v(a_p)$ and $t > v(a_p)$, we get $T^+ f_2$ vanishes mod $\wp$ and

$$T^+ f_0 = \begin{cases} \left[g^0_{1,0}, -\frac{p^m\binom{r-i}{b-m}}{a_p\binom{r-i}{r-m}} X^{r-b+m} Y^{b-m}\right] & \text{if } 0 \le b-m \le c \\ 0 & \text{otherwise.} \end{cases}$$

Therefore we can see that

$$T^- f_2 - a_p f_1 + T^+ f_0 \equiv \frac{p^m}{a_p} \left[ g^0_{1,0}, \sum_{\substack{c<j<s-m \\ j \equiv r-m \mod (p-1)}} \frac{\binom{r-i}{j}}{\binom{r-i}{r-m}} X^{r-j} Y^j \right] \mod \wp,$$

giving the first part of the claimed congruence (3.15). As $m - i > v(\binom{r-i}{r-m})$, we have $-a_p f_2 \equiv [g^0_{2,0}, F_m(X,Y)] \mod \wp$ as needed. We further note that $T^- f_1$ vanishes mod $\wp$ as $v(a_p) < p/2 + c$ and so $2m + p - 1 \ge p + 2c + 1 > v(a_p^2\binom{r-i}{r-m})$. Finally, to compute $T^+ f_1$, we note $s > 2v(a_p)$ and Lemma 3.3 for $S_{r-i,\cdot,m-i}$ can be used. By the hypothesis $t \ge 2v(a_p)$, we have $t + 1 - c + m > v(a_p^2\binom{r-i}{r-m})$ and $T^+ f_1$ vanishes mod $\wp$. □

We will treat the first part of the right hand side of congruence (3.15) as "error", since the part supported on $g^0_{2,0}$ is what we are really interested in. We will use (3.15) for different values of $i$ (for fixed $m > c$) and add their linear combinations to cancel out those non-integral error terms, and also ensure that the main part, supported on $g^0_{2,0}$, remains integral and non-zero mod $\wp$. The next theorem is the main result of our paper.

**Theorem 3.13.** *For $c \in \{0,1,2,3\}$, suppose $c < v = v(a_p) < c + \frac{p}{2}$. Let $s = b + c(p-1) > 2v$ for $2c \leq b \leq p-1$. If $r \equiv s \mod p^t(p-1)$ and $2v \leq t < \infty$, then*

(1) *there is a surjection* $\mathrm{ind}_{KZ}^{G}\left(\frac{V_r^{(c)}}{V_r^{(c+1)}}\right) \twoheadrightarrow \overline{\Theta}_{k',a_p}$ *for* $k' = r+2$, *and*

(2) *if* $b \neq 2c+1$, *then* $\overline{V}_{k',a_p} \cong \mathrm{ind}(\omega_2^{s+1}) \cong \overline{V}_{k,a_p}$ *and* $m(k,a_p) \leq 2v+1$, *for* $k = s+2$.

*Proof.* Since the result is known for $0 < v = v(a_p) < 1$, we assume that $v \geq 1$ and so $t \geq 2$ by the hypothesis. By Proposition 3.8, we know $\overline{\Theta}_{k',a_p}$ is a quotient of $\mathrm{ind}_{KZ}^{G}\left(\frac{V_r^{(c)}}{V_r^{(\lfloor v \rfloor+1)}}\right)$. We will show that the factors $\frac{V_r^{(m)}}{V_r^{(m+1)}}$ for $c < m \leq \lfloor v \rfloor$ map to zero under the quotient map, so that it reduces to a surjection from $\mathrm{ind}_{KZ}^{G}\left(\frac{V_r^{(c)}}{V_r^{(c+1)}}\right)$ as claimed.

*Case 1: $c < m \leq b-c$.* In this case by (3.16), $\binom{r-i}{r-m} \not\equiv 0 \mod p$, for $i = 0,1,\ldots,m-1$. Then we consider the matrix $A = (a_{ji}) \in M_{c+1}(\mathbb{Z}_p)$ given by

$$a_{ji} = \begin{cases} \dfrac{\binom{r-(m-1-i)}{j(p-1)+b-m}}{\binom{r-(m-1-i)}{r-m}}, & \text{for } 0 \leq j \leq c-1, 0 \leq i \leq c \\ 1, & j = c. \end{cases}$$

We further check that $A \in \mathrm{GL}_{c+1}(\mathbb{Z}_p)$ for $0 \leq c \leq 3$, as $\det A$ is non-zero modulo $p$. So we have the column vector $\underline{d} = (d_0, d_1, \ldots, d_c) = A^{-1} \cdot (0,0,\ldots,0,1) \in \mathbb{Z}_p^{c+1}$, ensuring

$$\sum_{i=0}^{c} d_i \frac{\binom{r-(m-1-i)}{j(p-1)+b-m}}{\binom{r-(m-1-i)}{r-m}} = 0, \text{ for } 0 \leq j \leq c-1,$$
$$d_0 + d_1 + \ldots d_c = 1, \text{ for } j = c.$$

Now if we take $f = \sum_{i=0}^{c} d_i f^{m-1-i}$, where $f^i$ are as in Proposition 3.12, then

$$(T - a_p)f \equiv 0 + \left[g_{2,0}^0, F_m(X,Y)\right] \mod \wp.$$

Hence by Lemma 3.2 (b), the possibility of non-zero contribution of the factor $\frac{V_r^{(m)}}{V_r^{(m+1)}}$ to $\overline{\Theta}_{k',a_p}$ is eliminated for the range of $m$ under consideration.

*Case 2: $b - c < m \leq \lfloor v \rfloor$.* Note that this case does not arise for $c = 0$ by the condition $s > 2v$.

For $c = 1$, by Proposition 3.12 we have

$$(T - a_p)f^i \equiv \left[g_{2,0}^0, F_m(X,Y)\right] \mod \wp,$$

killing the factor $\frac{V_r^{(m)}}{V_r^{(m+1)}}$ by Lemma 3.2 (b).

For $c = 2, 3$, we attempt to use congruence (3.15) for different $i$ to eliminate the "bad" or non-integral parts of $(T - a_p)f^i$ by using invertibility of an integral ($p$-adic) matrix, as in the previous case. However, the matrix under consideration is no more integral now, by (3.16). So we consider a modified matrix $A = (a_{ji}) \in M_c(\mathbb{Z}_p)$ given by

$$a_{ji} = \begin{cases} \dfrac{\binom{m-i}{c-i-1}}{\binom{r-i-(b-m+j(p-1)}{c-i-1}}, & \text{for } 1 \leq j \leq c-1,\ 0 \leq i \leq c-1 \\ 1, & \text{for } j = 0,\ 0 \leq i \leq c-1. \end{cases}$$

Next we use the fact $r \equiv b - c \mod p$ for $c = 2$ and 3, to see that $\det A$ is congruent to $-(m-1)^{-1}$ and $2(m-2)^{-2}(m-3)^{-1} \mod p$ respectively. Thus $A \in \mathrm{GL}_c(\mathbb{Z}_p)$ and $\underline{d} = (d_0, d_1, \dots d_{c-1}) := A^{-1} \cdot (1, 0, \dots, 0) \in \mathbb{Z}_p^c$ and hence

$$\sum_{i=0}^{c-1} d_i \frac{\binom{m-i}{c-i-1}}{\binom{r-i-(b-m+j(p-1))}{c-i-1}} = 0,\ \text{for } 1 \leq j \leq c-1,$$
$$d_0 + d_1 + \cdots + d_{c-1} = 1,\ \text{for } j = 0.$$

Next, we multiply the first $c-1$ equations above by $\dfrac{\binom{r-(c-1)}{b-m+j(p-1)}}{\binom{r-(c-1)}{r-m}}$ to obtain

$$\sum_{i=0}^{c-1} d_i \frac{\binom{r-i}{b-m+j(p-1)}}{\binom{r-i}{r-m}} = 0,\ \text{for } 1 \leq j \leq c-1. \tag{3.17}$$

If we take $f = \sum_{i=0}^{c-1} d_i f^i$, where $f^i$ are as in Proposition 3.12, then

$$(T - a_p)f \equiv 0 + \left[g_{2,0}^0, F_m(X, Y)\right] \mod \wp.$$

Now that we have eliminated all $\frac{V_r^{(m)}}{V_r^{(m+1)}}$ for $c < m \leq \lfloor v \rfloor$, there is a surjection

$$\mathrm{ind}_{KZ}^G \left( \frac{V_r^{(c)}}{V_r^{(c+1)}} \right) \twoheadrightarrow \overline{\Theta}_{k', a_p}.$$

Next we use Lemma 3.9 to get to the conclusion, as in the proof of Corollary 3.10. $\square$

Note that the condition $b \neq 2c + 1$ is equivalent to $k \not\equiv 3 \mod (p+1)$. Thus we have completed the proof of Theorem 1.2 stated in the introduction.

**Remark 3.14.** Proposition 3.12 fails for $s = 2v(a_p)$ only slightly as $T^+ f_1$ does not vanish mod $\wp$, but still is integral. Further if $s$ is odd, i.e., $v(a_p) \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$, then $m \leq \lfloor v \rfloor < v$ and as a result $X^m Y^{r-m} \in \ker P$. So modulo $\ker P$, we have $F_m(X, Y) \equiv -X^{r-s+m} Y^{s-m}$ and the equation (3.15) for

$i = 0$ reduces to $(T - a_p)f^0 \equiv \left[g^0_{2,0},\ (1 - \frac{p^s}{a_p^2})F_m(X, Y)\right] \mod \wp$. Therefore following the same argument as above, Theorem 3.13 also holds for small odd weights $k = 2v + 2 < p + 2$ under the extra condition $\frac{p^{k-2}}{a_p^2} \not\equiv 1 \mod \wp$.

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