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Counterexamples to the Woods Conjecture in dimensions $d \geq 24$

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Résumé. Soit $\mathbf{N}_d$ le maximum des rayons de recouvrement des réseaux d-dimensionnels unimodulaires possédants d vecteurs minimaux indépendants. En 1972, A. C. Woods a conjecturé que $\mathbf{N}_d \leq \frac{\sqrt{d}}{2}$. En 2005, C. T. McMullen a démontré que la conjecture de Woods implique la célèbre conjecture de Minkowski. La conjecture de Woods est prouvée pour $d \leq 9$. En 2016, Regev, Shapira et Weiss ont trouvé des contre-exemples à la conjecture de Woods pour $d \geq 30$. Dans cet article, nous donnons des contre-exemples à la conjecture de Woods pour $d \geq 24$. La question reste donc ouverte pour les dimensions $10 \leq d \leq 23$.

Abstract. Let $\mathbf{N}_d$ be the greatest value of covering radius over all well-rounded unimodular d dimensional lattices. In 1972 A. C. Woods conjectured that $\mathbf{N}_d \leq \frac{\sqrt{d}}{2}$. C. T. McMullen proved that the Woods conjecture implies the celebrated Minkowski's conjecture in 2005. The Woods conjecture has been proved for $d \leq 9$. In 2016 Regev, Shapira and Weiss gave counterexamples for the Woods conjecture for $d \geq 30$. In this paper we give counterexamples to the Woods conjecture in dimensions $d \geq 24$. Then the unknown dimensions of the Woods conjecture are 14 dimensions $10 \leq d \leq 23$.

1. Introduction

A d dimensional lattice $\mathbf{L} \subseteq \mathbf{R}^d$ is the set of all integer coefficient linear combinations of d linearly independent vectors $\mathbf{b}_1, \dots, \mathbf{b}_d$.

$$\mathbf{L} := \{\mathbf{x} = x_1\mathbf{b}_1 + \cdots + x_d\mathbf{b}_d : x_1 \in \mathbf{Z}, \dots, x_d \in \mathbf{Z}\}.$$

The volume of this lattice is

$$\mathrm{vol}(\mathbf{L}) = |\det(\mathbf{b})|$$

where $\mathbf{b}$ is the $d \times d$ matrix with d rows $\mathbf{b}_1, \dots, \mathbf{b}_d$. $\mathbf{L}$ is called unimodular if $\mathrm{vol}(\mathbf{L}) = 1$. The length of shortest nonzero lattice vectors is $\lambda_1(\mathbf{L}) = \min_{\mathbf{x} \in \mathbf{L} - \mathbf{0}}\{\sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}\}$. If lattice vectors with the length $\lambda_1(\mathbf{L})$ in $\mathbf{L}$ span $\mathbf{R}^d$,

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it is called well-rounded. An unimodular lattice $\mathbf{L}$ is called critical if the function $\lambda_1(\mathbf{L})$ attains a local maximum at $\mathbf{L}$. We refer [1] for the detail of lattice. It was proved by Voronoi that a critical lattice is well-rounded, we refer this result to [2, Chpt. 6].

The covering radius $r(\mathbf{L}) = \max_{\mathbf{y}\in\mathbf{R}^d,\mathbf{x}\in\mathbf{L}}\{\sqrt{\langle\mathbf{y}-\mathbf{x},\mathbf{y}-\mathbf{x}\rangle}\}$. Let

$$\mathbf{N}_d = \max_{\mathbf{L}WR,\mathrm{vol}(\mathbf{L})=1}\{r(\mathbf{L})\}$$

be the greatest value of covering radius over all well-rounded unimodular d dimensional lattices. The following conjecture was due to A. C. Woods [8] and has been proved for $d \le 9$ (see [3]).

Conjecture 1.1 (The Woods conjecture)**.** $\mathbf{N}_d \le \frac{\sqrt{d}}{2}$.

Let $N(\mathbf{y}) = |y_1 \dots y_d|$ for $\mathbf{y} = (y_1, \dots, y_d) \in \mathbf{R}^d$. For a vector $\mathbf{a} = (a_1, \dots, a_d) \in \mathbf{R}^d$, the lattice $\mathbf{a}\cdot\mathbf{Z} := \{(a_1x_1, \dots, a_dx_d) : x_1 \in \mathbf{Z}, \dots, x_d \in \mathbf{Z}\}$. The following conjecture due to H. Minkowski has been studied for many years.

Conjecture 1.2 (Minkowski's conjecture)**.** *For any unimodular lattice* $\mathbf{L}$ *in* $\mathbf{R}^d$*,*

$$\sup_{\mathbf{y}\in\mathbf{R}^d}\inf_{\mathbf{x}\in\mathbf{L}} N(\mathbf{y}-\mathbf{x}) \le 2^{-d}.$$

The equality holds if and only if there exist positive $a_1, \dots, a_d$ *such that* $a_1a_2\dots a_d = 1$ *and* $\mathbf{L} = \mathbf{a}\cdot\mathbf{Z}^n$*.*

In 2005 McMullen's breakthrough JAMS paper [4] it was proved that the Woods conjecture implies Minkowski's conjecture. Hence Minkowski's conjecture is true for $d \le 9$ (see [3]). Unfortunately in 2016, counterexamples to the Woods conjecture was given for $d \ge 30$ by Regev, Shapira and Weiss in [5] and a lower bound for the Woods invariant $\mathbf{N}_d > c\frac{d}{\sqrt{\log d}}$ was proved. In the papers [6, 7] of U. Shapira and B. Weiss stable lattices were suggested to replace well-rounded lattices in McMullen's approach.

In this paper we construct counterexamples to the Woods conjecture only using simpler lattices $\mathbf{D}_n$. Then counterexamples to the Woods conjecture in dimensions $d \ge 24$ are constructed. In Regev–Shapora–Weiss's approach the 15 dimensional laminated lattice was used. From our construction the Woods conjecture is unknown now for only 14 dimensions $d = 10, 11, \dots, 23$.

2. Counterexamples to the Woods conjecture in dimensions $d \ge 24$

As in [5] set $C(\mathbf{L}) = 4r^2(\mathbf{L})$. The Woods conjecture can be re-formulated as follows.

Conjecture 2.1 (The Woods conjecture)**.** *Let* $\mathbf{L}$ *be an unimodular lattice then* $C(\mathbf{L}) \le d$*.*

The main idea of the construction of [5] is as follows. Let $\mathbf{L} = \alpha_1 \mathbf{\Lambda}_{15} + \alpha_2 \mathbf{Z}^n$ where $\mathbf{\Lambda}_{15}$ is the 15 dimensional laminated lattice, which is critical and then well-rounded. They choose suitable α_1 and α_2 such that $\mathbf{L}$ is a well-rounded unimodular lattice. Then

$$C(\mathbf{L}) = \alpha_1^2 C(\mathbf{\Lambda}_{15}) + \alpha_2^2 C(\mathbf{Z}^n) = \alpha_1^2 C(\mathbf{\Lambda}_{15}) + \alpha_2^2 n$$

This gave counterexamples for $n \geq 15$. Then counterexamples to the Woods conjecture in dimensions $d \geq 30$ were constructed. It is a surprise that actually the root lattice $\mathbf{D}_m$ as follows gives us better counterexamples to the Woods conjecture.

In the general case set $\mathbf{L} = \alpha_1 \mathbf{L}_1 \oplus \alpha_2 \mathbf{Z}^n$, where $\mathbf{L}_1$ is a dimension m lattice with volume $\mathrm{vol}(\mathbf{L}_1)$ and the length of shortest non-zero lattice vector $\lambda(\mathbf{L}_1)$. If we want the lattice $\mathbf{L}$ unimodular, then $\alpha_1^m \alpha_2^n = \frac{1}{\mathrm{vol}(\mathbf{L}_1)}$. If $\alpha_1 \lambda_1(\mathbf{L}_1) = \alpha_2$, this lattice is well-rounded. Hence when

$$\alpha_1 = \left(\frac{1}{\mathrm{vol}(\mathbf{L}_1)\lambda(\mathbf{L}_1)^n} \right)^{\frac{1}{m+n}}$$

and

$$\alpha_2 = \left(\frac{\lambda_1(\mathbf{L}_1)^m}{\mathrm{vol}(\mathbf{L}_1)} \right)^{\frac{1}{m+n}},$$

the lattice $\mathbf{L}$ is unimodular and well-rounded.

Set $\mathbf{D}'_m = \{(x_1, \dots, x_m) : x_1 + \cdots + x_m = \mathit{even}\}$. It is well-known that this is a well-rounded lattice and

$$\lambda_1(\mathbf{D}'_m) = \sqrt{2}$$

The shortest lattice vectors in $\mathbf{D}'_m$ is of the form $(0, \dots, \pm 1, 0, \dots, 0, \pm 1, 0, \dots)$. Its covering radius is $r(\mathbf{D}'_m) = \frac{\sqrt{m}}{2}$ and its volume is 2. We refer to [1] for the detail. Set $\mathbf{D}_m = \{\frac{1}{2^{1/m}}(x_1, \dots, x_m) : x_1 + \cdots + x_m = \mathit{even}\}$. It is clear that lattice vectors $\frac{1}{2^{1/m}}(0, \dots, \pm 1, 0, \dots, 0, \pm 1, 0, \dots)$ with length $\frac{\sqrt{2}}{2^{1/m}}$ are the shortest non-zero lattice vectors in $\mathbf{D}_m$. They span $\mathbf{R}^m$ and this is a well-rounded lattice in $\mathbf{R}^m$. This is an unimodular lattice. Moreover $\lambda_1(\mathbf{D}_m) = \frac{\sqrt{2}}{2^{1/m}}$, $r(\mathbf{D}_m) = \frac{\sqrt{m}}{2^{\frac{m+1}{m}}}$ and $C(\mathbf{D}_m) = \frac{m}{2^{2/m}}$.

Let $\mathbf{L} = \alpha_1 \mathbf{D}_m + \alpha_2 \mathbf{Z}^n$,$\mathbf{L}$. Then if $\alpha_1 = 2^{-\frac{n(m-2)}{2m(m+n)}}$ and $\alpha_2 = 2^{\frac{m-2}{2(m+n)}}$, $\mathbf{L}$ is a well-rounded unimodular lattice.

From α_1 and α_2 we have

$$C(\mathbf{L}) = 2^{-\frac{n+2}{m+n}} \cdot m + 2^{\frac{m-2}{m+n}} \cdot n$$

When $m = n+4$, $2^{-\frac{n+2}{m+n}} = 2^{-\frac{1}{2}}$ and $2^{\frac{m-2}{m+n}} = 2^{\frac{1}{2}}$. Then $C(\mathbf{L}) = 2^{-\frac{1}{2}}(n+4) + 2^{\frac{1}{2}} n$ and $C(\mathbf{L}) > 2.12320343n + 2\sqrt{2}$. When $n \geq 10$,

$$C(\mathbf{L}) > 2n + 4$$

So when $d = 2n + 4 \geq 24$, $C(\mathbf{L}) > d = 2n + 4$. When $m = n + 5$ and $d = 2n + 5$ and a similar result can be proved when $n \geq 12$. For dimensions $d = 25$ and $d = 27$, a simple computation as follows leads to the conclusion.

The case of $d = 25$. We set $m = 15, n = 10$. Then $C(\mathbf{L}) = 2^{-\frac{12}{25}} \cdot 15 + 2^{\frac{13}{25}} \cdot 10 > 10.75466 + 14.339 > 25$.

The case of $d = 27$. We set $m = 15, n = 12$. Then $C(\mathbf{L}) = 2^{-\frac{14}{27}} \cdot 15 + 2^{\frac{13}{27}} \cdot 12 > 10.47132 + 16.754119 > 27$.

So we have the following result.

Theorem 2.2. *Set* $\alpha_1 = 2^{-\frac{n(m-2)}{2m(m+n)}}$ *and* $\alpha_2 = 2^{\frac{m-2}{2(m+n)}}$. *Then* $\mathbf{L} = \alpha_1 \mathbf{D}_m + \alpha_2 \mathbf{Z}^n$ *is unimodular and well-rounded. Moreover*

(1) $C(\mathbf{L}) > m + n$ *when* $m = n + 4, n \geq 10$*;*
(2) $C(\mathbf{L}) > m + n$ *when* $m = n + 5, n \geq 12$*;*
(3) $C(\mathbf{L}) > m + n$ *when* $m = 15, n = 10$*;*
(4) $C(\mathbf{L}) > m + n$ *when* $m = 15, n = 12$*.*

Then the Woods conjecture is not true for dimensions $d \geq 24$*.*

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