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On the spectrum of irrationality exponents of Mahler numbers

par Dzmitry BADZIAHIN

Résumé. Nous considérons les fonctions de Mahler $f(z)$ qui véri-fient l'équation fonctionnelle $f(z) = \frac{A(z)}{B(z)} f(z^d)$, où $\frac{A(z)}{B(z)}$ est dans $\mathbb{Q}(z)$ et $d \geqslant 2$ est un entier. Nous montrons que, pour tout entier b vérifiant $|b| \geqslant 2$, ou bien $f(b)$ est rationnel, ou bien son exposant d'irrationalité est rationnel. En outre, nous déterminons la valeur exacte de l'exposant d'irrationalité de $f(b)$ lorsque l'on connaît le développement en fraction continue de la fonction de Mahler $f(z)$. Cela améliore un résultat de Bugeaud, Han, Wen et Yao [6], qui ne donne qu'une borne supérieure de cet exposant.

Abstract. We consider Mahler functions $f(z)$ which satisfy the functional equation $f(z) = \frac{A(z)}{B(z)} f(z^d)$ where $\frac{A(z)}{B(z)}$ is in $\mathbb{Q}(z)$ and $d \geqslant 2$ is an integer. We prove that, for any integer b with $|b| \geqslant 2$, either $f(b)$ is rational or its irrationality exponent is rational. We also compute the exact value of the irrationality exponent of $f(b)$ as soon as the continued fraction expansion of the Mahler function $f(z)$ is known. This improves the result of Bugeaud, Han, Wen, and Yao [6] where only an upper bound of the irrationality exponent was provided.

1. Introduction

Consider a Laurent series $f(z) \in \mathbb{Q}((z^{-1}))$. It is called a *Mahler function* if for any z inside its disc of convergence f satisfies an equation of the form

$$\sum_{i=0}^{n} P_i(z) f(z^{d^i}) = Q(z)$$

for some integers $n \geqslant 1, d \geqslant 2$ and polynomials $P_0, \ldots, P_n, Q \in \mathbb{F}[z]$ with $P_0 P_n \neq 0$. The values $f(b)$ for integers b inside the disc of convergence of f are called *Mahler numbers*. In this paper we investigate the following problem:

Problem A. *Determine the set $\mathcal{L}_M$ of irrationality exponents of irrational Mahler numbers.*

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We will call the set $\mathcal{L}_M$ *the spectrum* of irrationality exponents of Mahler numbers. Recall that the irrationality exponent $\mu(\xi)$ of an irrational real number ξ is the supremum of real numbers μ such that the inequality

$$\left|\xi - \frac{p}{q}\right| < q^{-\mu}$$

has infinitely many rational solutions p/q. This is one of the most important properties of real numbers which indicates how well ξ is approached by rationals. Note that by the classical Dirichlet approximation theorem we always have $\mu(\xi) \geqslant 2$.

Similar questions have been recently asked by several authors. In 2008, Bugeaud [4] proved that for any rational $\omega \geqslant 2$ there are infinitely many automatic numbers whose irrationality exponent is equal to ω. It is well known [3] that automatic numbers are also Mahler numbers, therefore Bugeaud's result straightforwardly implies that $\mathcal{L}_M$ contains all rational numbers not smaller than two. Later in 2009, Adamczewski and Rivoal [1] commented on that result with the following question:

Problem B. *Is it true that the irrationality exponent of an automatic number is always a rational number?*

Bugeaud, Krieger and Shallit [7] extended Problem A to the set of morphic numbers. They showed that the spectrum of irrationality exponents of morphic numbers, on top of $\mathbb{Q}$, contains all Perron numbers greater than or equal to 2. Recall that a Perron number is a positive real algebraic integer, which is greater in absolute value than all of its conjugates. With respect to this result the following problem was posed:

Problem C. *Determine the set of irrationality exponents of morphic numbers. In particular, is it true that the irrationality exponent of a morphic number is always algebraic?*

Sometimes in the literature [10], Problems B and C are referred to as conjectures.

In this paper we restrict our research to solutions $f(z)$ of the following functional equation

$$f(z) = \frac{A(z)}{B(z)} f(z^d), \qquad A, B \in \mathbb{Q}[z],\ B \neq 0, \quad d \in \mathbb{Z},\ d \geqslant 2. \tag{1.1}$$

Theorem 1.1. *Let $f(z) \in \mathbb{Q}((z^{-1}))$ be a solution of (1.1) and $b \in \mathbb{Z}$ be inside the disc of convergence of $f(z)$. Assume that $A(b^{d^m})B(b^{d^m}) \neq 0$ for all $m \in \mathbb{Z}_{\geqslant 0}$ and that $f(b)$ is irrational. Then the irrationality exponent of $f(b)$ is a rational number.*

In other words, Theorem 1.1 shows that the solutions of (1.1) do not give any extra contribution to the spectrum $\mathcal{L}_M$ on top of the Mahler numbers constructed by Bugeaud.

The key ingredient in the proof of Theorem 1.1 is that $\mu(f(b))$ can be derived from the information about the continued fraction expansion of the Laurent series $f(z)$. It has been known for some time that these two notions are related. A similar idea was applied by various authors to obtain upper bounds of the irrationality exponents of certain Mahler numbers (see for example, [1, 2, 5, 8, 9]). The most general result in that direction is due to Bugeaud, Han, Wen, and Yao [6], where they manage to compute an upper bound of the irrationality exponent of $f(b)$ from the distribution of the associated non-zero Hankel determinants. In some cases they get $\mu(f(b)) \leqslant 2$ and then, due to the Dirichlet theorem, the bound is sharp. However, to the best of our knowledge, no one was able to apply these ideas and compute $\mu(f(b))$ before, if it is strictly bigger than two. In this paper we overcome that obstacle and provide the precise value of $\mu(f(b))$ for all functions f satisfying (1.1), based on the continued fraction expansion of $f(z)$.

Recall that a Laurent series $f(z) \in \mathbb{Q}((z^{-1}))$ admits the continued fraction expansion

$$f(z) = [a_0(z), a_1(z), \dots, a_k(z), \dots],$$

where $a_i \in \mathbb{Q}[z]$. It is finite if and only if $f(z)$ is a rational function. As in the case of real numbers, we call the rational function $[a_0(z), a_1(z), \dots, a_n(z)] = \frac{p_k(z)}{q_k(z)}$ the *n-th convergent of* f. Assuming that p_k and q_k are coprime, we denote by d_k the degree of the denominator q_k.

Theorem 1.2. *Let* $f(z) \in \mathbb{Q}((z^{-1})) \setminus \mathbb{Q}(z)$ *be a Laurent series which satisfies* (1.1). *Let* $b \in \mathbb{Z}$ *with* $|b| \geqslant 2$ *be inside the disc of convergence of* f *such that* $A(b^{d^m})B(b^{d^m}) \neq 0$ *for all* $m \in \mathbb{Z}_{\geqslant 0}$. *Then*

$$\mu(f(b)) = 1 + \limsup_{k\to\infty} \frac{d_{k+1}}{d_k}. \tag{1.2}$$

Unfortunately, Theorem 1.2 does not always allow to compute $\mu(f(b))$ for given polynomials A and B in (1.1). This is because the formula (1.2) requires the knowledge of the whole continued fraction of f. Finding it is usually a difficult task. However in many cases, as soon as we know that $\mu(f(b)) > 2$, we can compute the irrationality exponent of $f(b)$ after computing only finitely many convergents of f. We demonstrate the method by computing the irrationality exponents of those Mahler numbers $f(b)$ from [2], for which we know that $\mu(f(b)) > 2$.

Theorem 1.3. *Let* $f_{\mathbf{a}} = f_{a_1,a_2}(z) \in \mathbb{Z}((z^{-1}))$ *be a solution of the equation*

$$f_{\mathbf{a}}(z) = (z^2 + a_1 z + a_2) f_{\mathbf{a}}(z^3); \quad a_1, a_2 \in \mathbb{Z}. \tag{1.3}$$

For any integer b, $|b| \geqslant 2$ *one has*

(1) *for all* $s \in \mathbb{Z}$, *if* $f_{s,s^2}(b)$ *is irrational, then* $\mu(f_{s,s^2}(b)) = 3$;

(2) *for all* $s \in \mathbb{Z}$, *if* $f_{s^3,-s^2(s^2+1)}(b)$ *is irrational, then*

$$\mu(f_{s^3,-s^2(s^2+1)}(b)) = 3;$$

(3) *if* $f_{\pm 2,1}(b)$ *is irrational, then* $\mu(f_{\pm 2,1}(b)) = \frac{12}{5}$.

The paper has the following structure. Sections 2 and 3 are devoted to the proof of Theorem 1.2. In Section 4, we develop a theory of gaps in the sequence d_k of convergents degrees. We show that Theorem 1.1 follows from Proposition 4.4, the key result in that theory. The proposition itself is proven in Section 5. Finally, we end up with the proof of Theorem 1.3 in Section 6.

2. Useful estimate on irrationality exponent

The following proposition is a modification of Lemma 4.1 from [1] which we will need in the proof. But it may be of independent interest.

Proposition 2.1. *Let* $\alpha \in \mathbb{R}$. *Assume that there exist two sequences* $(\frac{p_n}{q_n})_{n\in\mathbb{N}} \in \mathbb{Q}$ *and* $(\frac{p'_n}{q'_n})_{n\in\mathbb{N}} \in \mathbb{Q}$ *of rational approximations of* α *and three sequences* θ_n, δ_n *and* τ_n *of real numbers with* $\theta_n \geqslant 1, \delta_n > 0$, $\tau_n > 0$ *such that*

(a) $q'_n \ll q_n^{\theta_n}$;

(b) $\left|\alpha - \frac{p_n}{q_n}\right| \asymp q_n^{-1-\delta_n}$; $\quad \left|\alpha - \frac{p'_n}{q'_n}\right| \asymp (q'_n)^{-1-\tau_n}$;

(c) $(q'_n)^{\tau_n} \gg q_{n+1}^{\delta_{n+1}}$, *and* $q_n^{\delta_n} \to \infty$ *as* $n \to \infty$.

for all $n \in \mathbb{N}$. *Then we have the upper bound*

$$\mu(\alpha) \leqslant \limsup_{n\to\infty} \max\left\{1 + \frac{\theta_n}{\delta_n}, \frac{(1+\tau_n)\theta_n}{\delta_n}\right\}. \tag{2.1}$$

The immediate corollary of this proposition is that if the sequences θ_n and δ_n satisfy $\theta_n/\delta_n \to 1$ as $n \to \infty$ and $\tau_n \geqslant 1$ then the sequence of approximations p'_n/q'_n to α is nearly optimal, i.e.

$$\mu(\alpha) = \limsup_{n\to\infty}(1+\tau_n).$$

Proof. Denote by $c = c(q'_n)$ a real number such that the inequality

$$\left|\alpha - \frac{p'_n}{q'_n}\right| \leqslant c(q'_n)^{-1-\tau_n} \tag{2.2}$$

is satisfied for all $n \in \mathbb{N}$. Let p/q be a rational number whose denominator q is large enough. We choose the minimal integer n such that $2cq \leqslant (q'_n)^{\tau_n}$. Condition (c) guarantees that such n exists. Then, by the choice of n and Condition (c), we have that $q \gg q_n^{\delta_n}$. By the triangle inequality we have

$$\left|\alpha - \frac{p}{q}\right| \geqslant \left|\frac{p}{q} - \frac{p'_n}{q'_n}\right| - \left|\alpha - \frac{p'_n}{q'_n}\right|.$$

Now we have two possibilities:

(1). The case $p/q \neq p'_n/q'_n$. Then $|p/q - p'_n/q'_n| \geqslant (qq'_n)^{-1}$ and from (2.2) we get that $|\alpha - p/q| \geqslant (2qq'_n)^{-1}$. We then apply Condition (a) to get

$$\left|\alpha - \frac{p}{q}\right| \gg \frac{1}{qq_n'^{\theta_n}} \gg \frac{1}{q^{1+\frac{\theta_n}{\delta_n}}}.$$

(2). The case $p/q = p'_n/q'_n$. Then we have

$$\left|\alpha - \frac{p}{q}\right| = \left|\alpha - \frac{p'_n}{q'_n}\right| \gg (q'_n)^{-1-\tau_n} \gg q^{-(1+\tau_n)\frac{\theta_n}{\delta_n}}.$$

To conclude the proof of the proposition, consider some number μ strictly bigger than the right hand side of (2.1). Then, there exists $n_0 \in \mathbb{N}$ such that $\mu > 1 + \theta_n/\delta_n$ and $\mu > (1+\tau_n)\theta_n/\delta_n$ for all $n \geqslant n_0$. Choose $n_1 \geqslant n_0$ such that for $n \leqslant n_0$ we have $(q'_n)^{\tau_n} < (q'_{n_1})^{\tau_{n_1}}$. Then, for any p/q with $2cq > (q'_{n_1})^{\tau_{n_1}}$, we have that

$$\left|\alpha - \frac{p}{q}\right| \gg q^{-\mu}$$

and hence $\mu(\alpha) \leqslant \mu$. □

3. Proof of Theorem 1.2

For convenience, denote the leading coefficients of A and B in equation (1.1) by α and β, and denote the degrees of A and B by r_a and r_b, respectively.

Consider the sequence $(p_k(z)/q_k(z))_{k\in\mathbb{Z}_{\geqslant 0}}$ of the convergents of f. Denote the degree of q_k by d_k. Then, by the standard property of convergents, we have

$$q_k(z)f(z) - p_k(z) = \sum_{i=d_{k+1}}^{\infty} c_{k,i} z^{-i}, \tag{3.1}$$

where $c_{k,j}$ are some real coefficients, and $c_{k,d_{k+1}}$ is always nonzero.

By substituting z^d in place of z in equation (3.1) and then using the functional relation (1.1) for $f(z^d)$, we get that:

$$B(z)q_k(z^d)f(z) - A(z)p_k(z^d) = A(z)\sum_{i=d_{k+1}}^{\infty} c_{k,i} z^{-di}. \tag{3.2}$$

After repeating this procedure m times we derive the following equation:

$$q_{k,m}(z)f(z) - p_{k,m}(z) = U(z)\sum_{i=d_{k+1}}^{\infty} c_{k,i} z^{-d^m i}, \tag{3.3}$$

where

$$
\begin{gathered}
q_{k,m}(z) = \prod_{t=0}^{m-1} B(z^{d^t})q_k(z^{d^m}), \quad p_{k,m}(z) = \prod_{t=0}^{m-1} A(z^{d^t})p_k(z^{d^m}), \\
U(z) = \prod_{t=0}^{m-1} A(z^{d^t}).
\end{gathered} \tag{3.4}
$$

Lemma 3.1. *Let $b \in \mathbb{R}$ with $|b| > 1$ be inside the disc of convergence of f. Assume that for all $t \in \mathbb{Z}_{\geqslant 0}$ we have $A(b^{d^t})B(b^{d^t}) \neq 0$. Then, for m large enough, we have*

$$
\begin{gathered}
|q_{k,m}(b)| \asymp \beta^m |b|^{d^m\left(\frac{r_b}{d-1}+d_k\right)} \quad \textit{and} \\
|q_{k,m}(b)f(b) - p_{k,m}(b)| \asymp \alpha^m |b|^{d^m\left(\frac{r_a}{d-1}-d_{k+1}\right)}.
\end{gathered} \tag{3.5}
$$

Here, the constants implied by the "$\asymp$" signs may depend on A, B, and k, but do not depend on m.

Proof. Since b is inside the disc of convergence of f, it is also inside the disc of convergence of

$$
z^{d_{k+1}}(q_k(z)f(z) - p_k(z)) = \sum_{i=0}^{\infty} c_{k,i+d_{k+1}} z^{-i}.
$$

By letting z tend to infinity, the right hand side tends to $c_{k,d_{k+1}} \asymp 1$. Therefore, for m large enough, one has

$$
\left| \sum_{i=d_{k+1}}^{\infty} c_{k,i} b^{-d^m i} \right| \asymp |b|^{-d^m d_{k+1}}. \tag{3.6}
$$

Next, notice that

$$
\prod_{t=0}^{\infty} \frac{A(z^{d^t})}{\alpha z^{d^t r_a}} = \prod_{t=0}^{\infty} P_A(z^{-d^t}),
$$

where $P_A(z)$ is a polynomial with $P_A(0) = 1$. One can check that the disc of convergence of this infinite product is $\{z : |z| > 1\}$. Moreover, since $A(b^{d^t}) \neq 0$ for all $t \in \mathbb{Z}$, the product

$$
\prod_{t=0}^{m-1} \frac{A(b^{d^t})}{\alpha b^{d^t r_a}}
$$

converges to a nonzero element as $m \to \infty$. In other words,

$$
\left| \prod_{t=0}^{m-1} \frac{A(b^{d^t})}{\alpha b^{d^t r_a}} \right| \asymp 1
$$

and hence

$$\left|\prod_{t=0}^{m-1} A(b^{d^t})\right| \asymp |\alpha|^m |b|^{(1+d+\cdots+d^{m-1})r_a} \asymp |\alpha|^m |b|^{\frac{d^m}{d-1}r_a}. \tag{3.7}$$

By analogous arguments we get the same estimate for the product of $B(b^{d^t})$, with t running from 0 to $m-1$.

The last ingredient of the proof is that for m large enough, $|q_k(b^{d^m})| \asymp |b|^{d^m d_k}$. Now, (3.3), (3.4), (3.6), and (3.7) give:

$$|q_{k,m}(b)| = \left|\prod_{t=0}^{m-1} B(b^{d^t}) q_k(b^{d^m})\right| \asymp |\beta|^m |b|^{d^m\left(\frac{r_b}{d-1}+d_k\right)}$$

and

$$\begin{aligned}|q_{k,m}(b)f(b) - p_{k,m}(b)|& \\ &= \left|\prod_{t=0}^{m-1} A(b^{d^t}) \sum_{i=d_{k+1}}^{\infty} c_{k,i} b^{-d^m i}\right| \asymp |\alpha|^m |b|^{d^m\left(\frac{r_a}{d-1}-d_{k+1}\right)}. \qquad \square\end{aligned}$$

As an immediate corollary of (3.5) we have that

$$|q_{k,m}(b)| = b^{d^m\left(d_k+\frac{r_b}{d-1}+o(1)\right)} \tag{3.8}$$

and

$$|q_{k,m}(b)f(b) - p_{k,m}(b)| = b^{-d^m\left(d_{k+1}-\frac{r_a}{d-1}+o(1)\right)}. \tag{3.9}$$

Since the sequence d_k tends to infinity with k, for any $\epsilon > 0$ one can choose $k = k(\epsilon)$ such that

$$\epsilon d_k > \max\left\{\frac{r_a}{d-1}+1, \frac{r_b}{d-1}+1\right\}.$$

For that k we can choose m big enough ($m > m_0(k)$) so that the absolute values of $o(1)$ in (3.8) and (3.9) are smaller than $1/2$. Then we have

$$q_{k,m}(b)^{-\frac{d_{k+1}(1+\epsilon)}{d_k(1-\epsilon)}} < |q_{k,m}(b)f(b) - p_{k,m}(b)| < q_{k,m}(b)^{-\frac{d_{k+1}(1-\epsilon)}{d_k(1+\epsilon)}}. \tag{3.10}$$

Since, by letting $k \to \infty$, we can make ϵ as small as we wish, we immediately have $\mu(f(b)) \geqslant 1 + \limsup \frac{d_{k+1}}{d_k}$. For convenience, let us denote the ratio d_{k+1}/d_k by δ_k and define

$$\rho := \limsup \frac{d_{k+1}}{d_k}.$$

3.1. Upper bound for $\mu(f(b))$. For a given $k_0 \in \mathbb{N}$, define $K = K(k_0) \in \mathbb{N}$ as the minimal possible value such that

$$d_{K+1} > d \cdot d_{k_0+1} - r_a + d + 1. \tag{3.11}$$

Consider any k in the range $k_0 \leqslant k < K$ and consider an arbitrary $m > M = M(k_0) := \max\{m_0(k_0), m_0(k_0+1), \ldots, m_0(K)\}$, so that the equation (3.10) is satisfied for all values k between k_0 and K. Equation (3.10) yields

$$\left| f(b) - \frac{p_{k,m}(b)}{q_{k,m}(b)} \right| \asymp q_{k,m}^{-1-\left(\frac{d_{k+1}}{d_k} + \epsilon_{k,m}\right)}, \tag{3.12}$$

where $\sup_{k_0 \leqslant k < K,\, m \geqslant M} |\epsilon_{k,m}|$ tends to zero as k_0 tends to infinity.

Now we construct sequences P_n/Q_n and P'_n/Q'_n in the following way:

$$\frac{P_1}{Q_1} := \frac{p_{k_0,M}(b)}{q_{k_0,M}(b)},\ \frac{P_2}{Q_2} := \frac{p_{k_0+1,M}(b)}{q_{k_0+1,M}(b)}, \ldots,\ \frac{P_{K-k_0}}{Q_{K-k_0}} := \frac{p_{K-1,M}(b)}{q_{K-1,M}(b)};$$
$$\frac{P'_1}{Q'_1} := \frac{p_{k_0+1,M}(b)}{q_{k_0+1,M}(b)},\ \frac{P'_2}{Q'_2} := \frac{p_{k_0+2,M}(b)}{q_{k_0+2,M}(b)}, \ldots,\ \frac{P'_{K-k_0}}{Q'_{K-k_0}} := \frac{p_{K,M}(b)}{q_{K,M}(b)}.$$

Then we continue defining the sequences by increasing the index M. That is, for any $u \in \mathbb{Z}_{\geqslant 0}$ and any $v \in \{1, \ldots, K - k_0\}$ we define

$$\frac{P_{u(K-k_0)+v}}{Q_{u(K-k_0)+v}} := \frac{p_{k_0+v-1,M+u}(b)}{q_{k_0+v-1,M+u}(b)}; \quad \frac{P'_{u(K-k_0)+v}}{Q'_{u(K-k_0)+v}} := \frac{p_{k_0+v,M+u}(b)}{q_{k_0+v,M+u}(b)}.$$

One can see from (3.12) that the following sequences $(\delta_n)_{n\in\mathbb{N}}$ and $(\tau_n)_{n\in\mathbb{N}}$ satisfy Condition (b) of Proposition 2.1:

$$\delta_{u(K-k_0)+v} := \frac{d_{k_0+v}}{d_{k_0+v-1}} + \epsilon_{k_0+v-1,M+u}; \quad \tau_{u(K-k_0)+v} := \frac{d_{k_0+v+1}}{d_{k_0+v}} + \epsilon_{k_0+v,M+u}.$$

Now we define a sequence $(\theta_n)_{n\in\mathbb{N}}$ so that Condition (a) is satisfied. By (3.8) we have that for any $k \in \{k_0, \ldots, K\}$ and for any $m > M$,

$$|q_{k+1,m}(b)| = |q_{k,m}(b)|^{\frac{d_{k+1}+r_b/(d-1)+o(1)}{d_k+r_b/(d-1)+o(1)}} = |q_{k,m}(b)|^{\frac{d_{k+1}}{d_k}+\epsilon^*_{k,m}},$$

where, as for $\epsilon_{k,m}$, $\sup_{k_0 \leqslant k < K,\, m \geqslant M} |\epsilon^*_{k,m}|$ tends to 0 as k_0 tends to infinity. The last equation suggests the following formula for θ_n:

$$\theta_{u(K-k_0)+v} := \frac{d_{k_0+v}}{d_{k_0+v-1}} + \epsilon^*_{k_0+v-1,M+u}.$$

It remains to verify Condition (c). The fact that $Q_n^{\delta_n}$ tends to infinity with n is obvious. Because of Condition (b), the equation $(Q'_n)^{\tau_n} \gg Q_{n+1}^{\delta_{n+1}}$ is equivalent to:

$$|Q'_n f(b) - P'_n| \ll |Q_{n+1} f(b) - P_{n+1}|. \tag{3.13}$$

By definition, for any $u \in \mathbb{Z}_{\geqslant 0}$ and $v \in \{1, \dots K - k_0 - 1\}$, we have $Q'_{u(K-k_0)+v} = Q_{u(K-k_0)+v+1}$ and $P'_{u(K-k_0)+v} = P_{u(K-k_0)+v+1}$ and both sides of (3.13) coincide for $n = u(K-k_0)+v$. Therefore it only remains to verify (3.13) for $n = (u+1)(K-k_0)$. From the estimate (3.9) and equations $Q'_{(u+1)(K-k_0)} = q_{K,M+u}(b)$, $Q_{(u+1)(K-k_0)+1} = q_{k_0,M+u+1}(b)$ we have

$$\begin{aligned}
|q_{K,M+u}(b)f(b) - p_{K,M+u}(b)| &= b^{-d^{m+u}\left(d_{K+1}-\frac{ra}{d-1}+o(1)\right)} \\
&\overset{(3.11)}{<} b^{-d^{m+u+1}\left(d_{k_0+1}-\frac{ra}{d-1}+o(1)\right)} = |q_{k_0,M+u+1}(b)f(b) - p_{k_0,M+u+1}(b)|.
\end{aligned}$$

After all conditions of Proposition 2.1 are checked, we apply it to get

$$\mu(f(b)) \leqslant \limsup_{u\to\infty} \max_{k_0 \leqslant v < K} \left\{1 + \frac{\theta_{u(K-k_0)+v}}{\delta_{u(K-k_0)+v}}, (1+\tau_{u(K-k_0)+v})\frac{\theta_{u(K-k_0)+v}}{\delta_{u(K-k_0)+v}}\right\}.$$

Notice that by construction, θ_n/δ_n tends to 1 as k_0 tends to infinity. Also, as k_0 tends to infinity, we have that

$$\tau_{u(K-k_0)+v} \to \frac{d_{k_0+v+1}}{d_{k_0+v}}.$$

for all $u \geqslant 0$ and v between k_0 and K. This leads to the upper bound

$$\mu(f(b)) \leqslant \limsup_{n\to\infty} \left\{1 + \frac{d_{n+1}}{d_n}\right\},$$

which now coincides with the lower bound for $\mu(f(b))$. That proves Theorem 1.2.

4. Gaps in the set of values d_k

Theorem 1.2 suggests that in order to compute the irrationality exponent of a Mahler number $f(b)$, we need to consider large gaps in the sequence $(d_k)_{k\in\mathbb{N}}$ of degrees of the denominators of the convergents of $f(z)$.

Define by Φ the set of all values d_k:

$$\Phi = \Phi(f) := \{d_k \ : \ k \in \mathbb{N}\}.$$

We say that $[u, v]$ is *a gap in* Φ *of size* $r > 0$ if u and v are elements of Φ, $r = v - u$ and no elements w with $u < w < v$ are in Φ. For the gap $[u, v]$ in Φ we say that $p(z)/q(z)$ is *gap's convergent* if $p(z)/q(z)$ is a convergent of f and $\deg(q) = u$. To emphasize that this convergent is associated with the gap $[u, v]$ we will use the notation $p_u(z)/q_u(z)$. To avoid abuse of notation, in the remaining part of the paper the notion $p_u(z)/q_u(z)$ will always mean the gap's convergent, i.e. $\deg(q_u) = u$.

In further discussion we always assume that the value $b \in \mathbb{N}$ satisfies the conditions of Theorem 1.2. It implies that if all gaps in Φ are of size at most $\frac{r_a+r_b}{d-1}$ then $\mu(f(b))=2$. Indeed, we have

$$\mu(f(b)) = 1 + \limsup_{\text{gaps } [u,v] \text{ of } \Phi} \frac{v}{u} \leqslant \lim_{u\to\infty}\left(1+\frac{u+\frac{r_a+r_b}{d-1}}{u}\right) = 2.$$

Therefore in order to compute the irrationality exponent of $f(b)$ it is sufficient to consider gaps in Φ of a bigger size than $\frac{r_a+r_b}{d-1}$. We call such gaps big. We introduce a partial order on the set of big gaps. We say that $[u,v] \prec [u',v']$ if there exists $m\in\mathbb{N}$ such that

$$\frac{p_{u'}(z)}{q_{u'}(z)} = \prod_{t=0}^{m-1}\frac{A(z^{d^t})}{B(z^{d^t})}\cdot\frac{p_u(z^{d^m})}{q_u(z^{d^m})}.$$

This definition is justified by the following lemma.

Lemma 4.1. *Let $[u,v]$ be a big gap in Φ. Then the fraction*

$$\frac{A(z)p_u(z^d)}{B(z)q_u(z^d)}$$

is a convergent of f. Moreover, the gap in Φ, which corresponds to this convergent, has size bigger than $v-u$.

Proof. Denote by $C(z)$ the polynomial $\gcd(A(z)p_u(z^d), B(z)q_u(z^d))$ and let $r_c := \deg(C)$. From (3.2) we have that

$$\left\|\frac{B(z)q_u(z^d)}{C(z)}f(z) - \frac{A(z)p_u(z^d)}{C(z)}\right\| = r_a - r_c - dv. \tag{4.1}$$

Here, $\|g\|$ denotes the biggest degree of z with non-zero coefficient in Laurent series g. We have that $\frac{B(z)q_u(z^d)}{C(z)}$ and $\frac{A(z)p_u(z^d)}{C(z)}$ are coprime and moreover,

$$\deg\left(\frac{B(z)q_u(z^d)}{C(z)}\right) = r_b + du - r_c < dv + r_c - r_a. \tag{4.2}$$

The last inequality is true because for big gaps we have $v-u > \frac{r_a+r_b}{d-1}$. Hence

$$\frac{A(z)p_u(z^d)}{C(z)}\Big/\frac{B(z)q_u(z^d)}{C(z)}$$

is a convergent of f and the size of its corresponding gap is

$$(dv + r_c - r_a) - (r_b + du - r_c) = 2r_c + d(v-u) - r_a - r_b > v-u. \qquad \square$$

We say that a big gap $[u,v]$ in Φ is *primitive* if there are no other big gaps $[u',v']$ in Φ such that $[u',v'] \prec [u,v]$. A primitive gap $[u,v]$ generates the ordered sequence of big gaps

$$[u,v] = [u_0,v_0] \prec [u_1,v_1] \prec [u_2,v_2] \prec \ldots$$

such that

$$\frac{p_{u_{n+1}}(z)}{q_{u_{n+1}}(z)} = \frac{A(z)p_{u_n}(z^d)}{C(z)} \Big/ \frac{B(z)q_{u_n}(z^d)}{C(z)}. \tag{4.3}$$

Then the formula (1.2) for $\mu(f(b))$ from Theorem 1.2 can be rewritten as follows:

$$\mu(f(b)) = 1 + \sup_{[u_0,v_0] \text{ is primitive}} \left\{ \limsup_{i\to\infty} \frac{v_{n_i}}{u_{n_i}} \right\} \cup \{1\}. \tag{4.4}$$

Lemma 4.2. *The size of a primitive gap in Φ does not exceed $\frac{2d-1}{d-1}(r_a+r_b)$.*

Proof. Suppose the contrary: the size of a primitive gap $[u,v]$ in Φ is bigger than $\frac{2d-1}{d-1}(r_a+r_b)$. Let w be the biggest integer such that $dw < v - r_b$ (i.e. $w = \lfloor \frac{v-r_b-1}{d} \rfloor$).

Assume that w lies inside a big gap $[s,t]$ in Φ, that is, $s \leqslant w < t$. Then, by (4.1) and (4.2) the gap, associated with the convergent

$$\frac{A(z)p_s(z^d)}{B(z)q_s(z^d)},$$

contains $[ds+r_b, dt-r_a]$. Obviously, $ds+r_b < v$ and $dt-r_a \geqslant v-r_b-r_a > u$. Therefore this gap intersects with $[u,v]$ and hence it must coincide with $[u,v]$. We get $[s,t] \prec [u,v]$, which is a contradiction.

We then deduct that w does not lie inside a big gap. In other words, there is an element $s \in \Phi$ with $0 \leqslant w - s \leqslant \frac{r_a+r_b}{d-1}$. Consider the fraction

$$\frac{p(z)}{q(z)} = \frac{A(z)p_s(z^d)}{B(z)q_s(z^d)}.$$

Then by (3.2), we have $\|q(z)f(z) - p(z)\| \leqslant r_a - d(s+1)$, which is strictly smaller than $-u$. Indeed,

$$d(s+1) - r_a \geqslant d\left(w - \frac{r_a+r_b}{d-1} + 1\right) - r_a > v - r_a - r_b - \frac{d}{d-1}(r_a+r_b) \geqslant u.$$

Divide q by q_u with the remainder: $q(z) = a(z)q_u(z) + r(z)$ and write $p(z) = a(z)p_u(z) + c(z)$. Then we have

$$\|a(z)q_u(z)f(z) - a(z)p_u(z)\| = \deg(a) - v.$$

Obviously, the degree of q is $r_b + ds$ which is strictly smaller than v and therefore $\deg(a) - v < v - u - v = -u$.

Assume that $r \neq 0$. Since the convergents of f are the best approximants to f and $\deg(r) < \deg(q_u)$, we have

$$\|r(z)f(z) - c(z)\| \geqslant \|q_{u'}(z)f(z) - p_{u'}(z)\| = -u,$$

where $p_{u'}/q_{u'}$ is the convergent of f which precedes p_u/q_u. The last two estimates imply

$$\|q(z)f(z) - p(z)\| = \|r(z)f(z) - c(z)\| \geqslant -u,$$

which contradicts the condition $r \neq 0$. Hence we get that $r = 0$ and p/q coincides with p_u/q_u. This together with (4.1) and (4.2) implies that $u = ds + r_b - r_c$ and $v = dt + r_c - r_a$ where $r_c = \deg(\gcd(A(z)p_u(z^d), B(z)q_u(z^d)))$. Since polynomials $p_u(z^d)$ and $q_u(z^d)$ are coprime, $r_c \leqslant r_a + r_b$. Finally,

$$v - u \leqslant d(t-s) + 2r_c - r_a - r_b \leqslant \frac{2d-1}{d-1}(r_a + r_b). \qquad \square$$

Lemma 4.3. *Assume that $(d-1)u > r_a$. Then*

$$\frac{v - \frac{r_a}{d-1}}{u + \frac{r_b}{d-1}} \leqslant \limsup_{n\to\infty} \left\{\frac{v_n}{u_n}\right\} \leqslant \frac{v + \frac{r_b}{d-1}}{u - \frac{r_a}{d-1}}. \tag{4.5}$$

Proof. From equations (4.1) and (4.2) we have that

$$\begin{aligned} u_{n+1} &= du_n + r_b - r_{c,n}; \\ v_{n+1} &= dv_n - r_a + r_{c,n}, \end{aligned} \tag{4.6}$$

where

$$r_{c,n} = \deg\gcd(A(z)p_{u_n}(z^d), B(z)q_{u_n}(z^d)) \tag{4.7}$$

and it is not bigger than $r_a + r_b$. This implies

$$\frac{dv_n - r_a}{du_n + r_b} \leqslant \frac{v_{n+1}}{u_{n+1}} \leqslant \frac{dv_n + r_b}{du_n - r_a}.$$

By iterating this inequality n times we get

$$\frac{d^n v - (1 + d + \cdots + d^{n-1})r_a}{d^n u + (1 + d + \cdots + d^{n-1})r_b} \leqslant \frac{v_n}{u_n} \leqslant \frac{d^n v + (1 + d + \cdots + d^{n-1})r_b}{d^n u - (1 + d + \cdots + d^{n-1})r_a}.$$

Taking limits as $n \to \infty$ yields (4.5). $\square$

Lemmata 4.2 and 4.3 together imply that only finitely many primitive gaps may contribute to the supremum in (4.4). Indeed, consider all primitive gaps $[u, v]$ in Φ with $u > \frac{r_a}{d-1}$. By Lemma 4.2, their sizes are bounded. Therefore we can choose the primitive gap $[u_0, v_0]$ in Φ with the biggest possible size S such that u_0 is smallest possible among all primitive gaps in Φ of this size. Then, by Lemma 4.3, a primitive gap $[u, v]$ in Φ can only contribute to the limsup in (4.4) if

$$\frac{u + S + \frac{r_b}{d-1}}{u - \frac{r_a}{d-1}} > \frac{v_0 - \frac{r_a}{d-1}}{u_0 + \frac{r_b}{d-1}}.$$

Since $v_0 - u_0 > \frac{r_a + r_b}{d-1}$, the right hand side of the inequality is bigger than one and therefore it gives us an upper bound for u. Denote this bound by

l_u. We deduct that only $[u,v]$ with $u \leqslant l_u$ can contribute to the limsup in (4.4) and there are obviously finitely many of them.

Now to complete the proof of Theorem 1.1, we need to show that for any primitive gap $[u,v]$ in Φ we have that $\liminf_{n\to\infty} v_n/u_n$ is a rational number. The most mysterious term in the formulae (4.6) is $r_{c,n}$. In the next section we prove the following proposition which is a key to the proof of Theorem 1.1.

Proposition 4.4. *The sequence $(r_{c,n})_{n\in\mathbb{N}}$ is eventually periodic.*

We end this section by showing how Proposition 4.4 implies Theorem 1.1. Let the sequence $(r_{c,n})_{n\in\mathbb{N}}$ be periodic, starting from the index n_0 and with the period length P, i.e. $r_{c,n_0+i} = r_{n_0+P+i}$ for every $i \in \mathbb{Z}_{\geqslant 0}$. Denote by R the following value:

$$R := d^{P-1} r_{c,n_0} + \cdots + d r_{c,n_0+P-2} + r_{c,n_0+P-1}.$$

By applying the formulae (4.6) for $u_{n_0}, v_{n_0}, u_{n_0+1}, v_{n_0+1}, \ldots$ up to u_{n_0+P}, v_{n_0+P}, we get

$$u_{n_0+P} = d^P u_{n_0} + r_b(1 + d + \cdots + d^{P-1}) - R,$$

$$v_{n_0+P} = d^P v_{n_0} - r_a(1 + d + \cdots + d^{P-1}) + R.$$

Define

$$r_u := r_b(1 + d + \cdots + d^{P-1}) - R, \quad \text{and} \quad r_v := r_a(1 + d + \cdots + d^{P-1}) - R.$$

Then we get

$$\lim_{k\to\infty} \frac{u_{n_0+kP}}{v_{n_0+kP}} = \lim_{k\to\infty} \frac{d^{kP} u_{n_0} + (1 + d^P + d^{2P} + d^{(k-1)P}) r_u}{d^{kP} v_{n_0} - (1 + d^P + d^{2P} + d^{(k-1)P}) r_v} = \frac{u_{n_0} + \frac{r_u}{d^P-1}}{v_{n_0} - \frac{r_v}{d^P-1}},$$

which is a rational number. By analogous arguments, the limits of

$$\frac{u_{n_0+1+kP}}{v_{n_0+1+kP}}, \ldots, \frac{u_{n_0+(k+1)P-1}}{v_{n_0+(k+1)P-1}}$$

as $k \to \infty$ are all rational numbers. Therefore $\limsup_{n\to\infty} v_n/u_n$, as the maximum of the limits above, is a rational number. This finishes the proof of Theorem 1.1.

5. Proof of Proposition 4.4

We split each of the polynomials A, B, p_{u_m} and q_{u_m} ($m \in \mathbb{Z}_{\geqslant 0}$) into the product of three factors: cyclotomic, non-cyclotomic and the power of z. For example, $A(z) = A_c(z) \cdot A_n(z) \cdot A_0(z)$, where all roots of $A_c(z)$ are roots of unity, $A_0(z)$ is a power of z and none of the roots of $A_n(z)$ is either zero or a root of unity. The polynomials $B_c, B_n, B_0, p_{c,u_m}, p_{n,u_m}, p_{0,u_m}, q_{c,u_m}, q_{n,u_m}$ and q_{0,u_m} are defined in the same way. Obviously,

$$\begin{aligned}\gcd(A_c(z)p_{c,u_m}(z^d),\, B_n(z)q_{n,u_m}(z^d))\\ = \gcd(A_n(z)p_{n,u_m}(z^d), B_c(z)q_{c,u_m}(z^d)) = \text{const},\end{aligned}$$

and therefore we can split $r_{c,m}$ into the sum of three parts: $r_{c,m} = r_{c,c,m} + r_{n,c,m} + r_{0,c,m}$. The first term is the degree of the cyclotomic part of the gcd in (4.7), the second one is the degree of the non-cyclotomic part of it and the third one is generated by the powers of z presented in the gcd. We will consider each term separately.

5.1. Non-cyclotomic term.

Lemma 5.1. *Let $C, D \in \mathbb{Z}[x]$ be such that none of their roots is a root of unity. Then there exists $m_0 \in \mathbb{N}$ such that for all $m > m_0$,*

$$\gcd(C(z), D(z^{d^m})) = \text{const}.$$

Proof. Assume the contrary. Then there exists a root α of C such that $z-\alpha$ divides $D(z^{d^m})$ for infinitely many values m. Hence there exists a root β of D such that

$$\beta = \alpha^{d^{m_1}} = \alpha^{d^{m_2}}$$

for some positive integers $m_1 \neq m_2$. But the latter is only possible if α is a root of unity or zero — a contradiction. $\square$

Lemma 5.2. *The sequence $(r_{n,c,m})_{m\in\mathbb{N}}$ is eventually periodic.*

Proof. From Lemma 5.1 fix m_0 such that

$$\gcd(A_n(z), B_n(z^{d^m})q_{n,u}(z^{d^m})) = \gcd(B_n(z), A_n(z^{d^m})p_{n,u}(z^{d^m})) = \text{const}$$

for all $m \geqslant m_0$. Write the non-cyclotomic part of the convergent p_{u_m}/q_{u_m} in the following form:

$$\frac{p_{n,u_m}}{q_{n,u_m}} = \frac{\prod_{t=0}^{m-1} A^*_{t,m}(z)p^*_m(z)}{\prod_{t=0}^{m-1} B^*_{t,m}(z)q^*_m(z)},$$

where the numerator and denominator of the right hand side are coprime; $A^*_{t,m}(z) \mid A(z^{d^t})$, $B^*_{t,m}(z) \mid B(z^{d^t})$, $p^*_m(z) \mid p_n(z^{d^m})$ and $q^*_m(z) \mid q_n(z^{d^m})$; the leading coefficients of $A^*_{t,m}(z), B^*_{t,m}(z), p^*_m(z), q^*_m(z)$ coincide with those of $A(z), B(z), p_n(z)$ and $q_n(z)$ respectively. Then for $m > m_0$ the degree $r_{n,c,m}$ of

$$\gcd(A_n(z)p_{n,u_{m+1}}(z^d), B_n(z)q_{n,u_{m+1}}(z^d))$$

as well as the polynomials $A^*_{t,m+1}, B^*_{t,m+1}$, $t \in \{0,\dots,m_0\}$, depend entirely on the polynomials $A^*_{0,m}, A^*_{1,m},\dots, A^*_{m_0,m}$ and $B^*_{0,m},\dots,B^*_{m_0,m}$. But there are finitely many such combinations. Therefore one can find $m_2 > m_1 > m_0$ such that $A^*_{0,m_1} = A^*_{0,m_2}, \dots, A^*_{m_0,m_1} = A^*_{m_0,m_2}$ and $B^*_{0,m_1} = B^*_{0,m_2}, \dots,$

$B^*_{m_0,m_1} = B^*_{m_0,m_2}$. Then we get $r_{n,c,m_1} = r_{n,c,m_2}, r_{n,c,m_1+1} = r_{n,c,m_2+1}$, etc. Hence the sequence of $(r_{c,n,m})_{m\in\mathbb{N}}$ is eventually periodic. $\square$

5.2. Powers of *z*. We write

$$A_0(z) = z^{s_a}; \quad B_0(z) = z^{s_b}; \quad p_{0,u}(z) = z^{s_p} \quad \text{and} \quad q_{0,u}(z) = z^{s_q}.$$

Since A and B are coprime and p_u and q_u are coprime, we have that at least one value of s_a, s_b and at least one of s_p, s_q is zero. If we have $s_a = s_p = 0$ (or $s_b = s_q = 0$) then, by (4.3), the powers of z of all numerators (denominators) are zero and hence $(r_{0,c,m})_{m\in\mathbb{Z}_{ge0}}$ is the zero sequence.

Now without loss of generality assume that $s_a > 0, s_q > 0, s_b = s_p = 0$. Denote by $s_{p,m}$ and $s_{q,m}$ the maximal powers of z of p_{u_m} and q_{u_m} respectively. Notice that, if for some $m_0 \in \mathbb{N}$ the value s_{q,m_0} is zero then, as before, the sequence $r_{0,c,m}$ becomes zero for all $m \geqslant m_0$. On the other hand, if $s_{q,m}$ is positive for all $m \in \mathbb{Z}_{\geqslant 0}$ then the power of z of $q_{u_m}(z^d)$ is always bigger than that of $A(z)$, which follows that $r_{0,c,m}$ equals s_a for all $m \in \mathbb{Z}_{\geqslant 0}$.

In all cases we have that the sequence $(r_{0,c,m})_{m\in\mathbb{Z}_{\geqslant 0}}$ is eventually periodic.

5.3. Cyclotomic term. Note that each of the polynomials A_c, B_c, $p_{c,u}$, $q_{c,u}$ is a (possibly empty) product of cyclotomic polynomials $\Phi_n(z)$. We start by investigating the structure of polynomials $\Phi_n(z^d)$ as d changes. That requires some notation. Given $n \in \mathbb{N}$, the radical of n is the product of all prime divisors of n, i.e.:

$$\operatorname{rad}(n) := \prod_{\substack{p\in\mathbb{P}\\ p\mid n}} p.$$

For two positive integers n and m, by $r(n,m)$ we denote the biggest divisor of n which is coprime with m, and $s(n,m) := n/r(n,m)$.

Lemma 5.3. *Let n, d be two positive integers. The polynomial $\Phi_n(z^d)$ is a product of cyclotomic polynomials. More precisely,*

$$\Phi_n(z^d) = \prod_{r\mid r(d,n)} \Phi_{rns(d,n)}(z).$$

Proof. All the roots of $\Phi_n(z)$ are of the form ξ_n^i, where ξ_n is n-th primitive root of unity, $0 \leqslant i < n$ and $\gcd(i,n) = 1$. Therefore the roots ξ of $\Phi_n(z^d)$ are the solutions of the equation $\xi^d = \xi_n^i$, which can be written as

$$\xi_{nd}^i \cdot \xi_d^j = \xi_{nd}^{nj+i},$$

where $0 \leqslant j < d$. The values $nj+i$ run through the set $\mathcal{N}$ of all numbers between zero and nd, which are coprime with n. Split this set into subsets

$$\mathcal{N}_t := \{x \in \mathcal{N} \ :\ \gcd(d,x) = t\}.$$

Obviously, they are non-empty only if $t \mid d$ and $\gcd(t,n) = 1$. These two conditions are equivalent to $t \mid r(d,n)$. Denote by r the fraction $r(d,n)/t$.

Notice that for any $x \in \mathcal{N}_t$ one has $\xi_{nd}^x = \xi_{nd/t}^{x/t}$ where x/t is coprime with nd/t. Finally, write $nd/t = rns(d,n)$, so the numbers ξ_{nd}^x are the roots of the polynomial $\Phi_{rns(d,n)}(z)$ and $\Phi_{rns(d,n)}(z) \mid \Phi_n(z^d)$. □

Write the polynomial $A_c(z)$ as the product:

$$A_c(z) = \prod_{\substack{r\in\mathbb{N}\\ \gcd(d,r)=1}} A_{r,c}(z),$$

where $A_{r,c}(z)$ is the product of all $\Phi_n(z)$ such that $\Phi_n(z) \mid A_c(z)$ and $r(n,d) = r$. Other polynomials $B_{r,c}(z), p_{r,c,u_m}(z), q_{r,c,u_m}(z)$ are defined analogously. Clearly, among all values r with $\gcd(r,d) = 1$ only finitely many polynomials $A_{r,c}(z)$ have positive degree.

One of the outcomes of Lemma 5.3 is that for any n and m in $\mathbb{N}$ every cyclotomic divisor $\Phi_k(z)$ of $\Phi_n(z^{d^m})$ has $r(k,d) = r(n,d)$. Therefore we can split $r_{c,c,m}$ into the sum:

$$r_{c,c,m} = \sum_{\substack{r\in\mathbb{N}\\ \gcd(d,r)=1}} r_{r,c,c,m},$$

where

$$r_{r,c,c,m} = \gcd(A_{r,c}(z)p_{r,c,u_m}(z^d), B_{r,c}(z)q_{r,c,u_m}(z^d)).$$

Only finitely many of the sequences $(r_{r,c,c,m})_{m\in\mathbb{N}}$ are non-zero.

It remains to show that every non-zero sequence $(r_{r,c,c,m})_{m\in\mathbb{N}}$ is eventually periodic.

Case 1. Assume that among the divisors of $A_{r,c}, B_{r,c}, p_{r,c,u}, q_{r,c,u}$ there are no polynomials $\Phi_r(z)$. From Lemma 5.3 we know that all divisors $\Phi_k(z)$ of $\Phi_n(z^{d^m})$ satisfy $ns(d,n)^m \mid k$. Consider a divisor $\Phi_n(z)$ of one of the polynomials $A_{r,c}, B_{r,c}, p_{r,c,u}, q_{r,c,u}$. Since $n \neq r$, and $r(n,d) = r$, we have $s(d,n) > 1$ and therefore, as m tends to infinity, all divisors $\Phi_k(z)$ of $\Phi_n(z^{d^m})$ satisfy $k \to \infty$. Therefore there exists m_0 such that for $m > m_0$

$$\gcd(A_{r,c}(z), \Phi_n(z^{d^m})) = \gcd(B_{r,c}(z), \Phi_n(z^{d^m})) = \text{const}.$$

Then the proof of Proposition 4.4 in this case is analogous to that of Lemma 5.2.

Before considering the other cases, we need more notation and lemma. Given two polynomials $f(z), g(z) \in \mathbb{Z}[z]$ with $\deg(f) > 0$ denote by $\sigma(f,g)$ the maximal power of f which divides g, i.e.

$$\sigma(f,g) := \max\{n \in \mathbb{Z}_{\geqslant 0} \;:\; (f(z))^n \mid g(z)\}.$$

Lemma 5.4. *For any $f(z) \in \mathbb{Z}[z]$ and any $k \in \mathbb{N}$ there exists a constant $c = c(f,k)$ such that for any $m \in \mathbb{N}$, $\sigma(\Phi_k(z), f(z^{d^m})) < c$.*

Proof. We write $k = rs$ where $r = r(k, d)$ is coprime with d and the radical of s divides the one of d. Split f as a product $f = f_r \cdot g$, where all the roots of g are either not roots of unity or they are roots of unity of degree k' with $r(k', d) \neq r$. The function f_r is defined as follows:

$$f_r(z) = \prod_{\operatorname{rad}(s_i) \,|\, \operatorname{rad}(d)} (\Phi_{rs_i}(z))^{\alpha(s_i)}.$$

Then $g(z^{d^m})$ is always coprime with $\Phi_k(z)$ and $\sigma(\Phi_k(z), f(z^{d^m})) = \sigma(\Phi_k(z), f_r(z^{d^m}))$.

We finish the proof of the lemma by induction on s. For $s = 1$, Lemma 5.3 implies that

$$\sigma(\Phi_r(z), f_r(z^{d^m})) = \sigma(\Phi_r(z), (\Phi_r(z^{d^m}))^{\alpha(1)}) = \alpha(1).$$

Now, consider $S \in \mathbb{N}$ with $\operatorname{rad}(S) \mid \operatorname{rad}(d)$. Assume that the statement of the lemma is satisfied for all $s < S$ with $\operatorname{rad}(s) \mid d$, i.e. for any such s there exists a constant $c(s)$ such that $\sigma(\Phi_{rs}(z), f_r(z^{d^m})) \leqslant c(s)$. Now we prove the statement for S. Lemma 5.3 implies that

$$\begin{aligned}
&\sigma(\Phi_{rS}(z), f_r(s^{d^m})) \\
&\qquad \leqslant \sigma(\Phi_{rS}(z), (\Phi_{rS}(z^{d^m}))^{\alpha(S)}) + \sum_{s|S,\ s<S} \sigma(\Phi_{rs}(z), f_r(z^{d^{m-1}})) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad \leqslant \alpha(S) + \sum_{s|S,\ s<S} c(s).
\end{aligned}$$

Since the right hand side does not depend on m, the proof is finished. $\square$

Case 2. Assume that $\Phi_r(z)$ divides $p_{r,c,u}(z)$ and

$$\gcd(\Phi_r(z), A_{r,c}(z)) = \gcd(\Phi_r(z), B_{r,c}(z)) = \text{const}.$$

Note that the case $\Phi_r(z) \mid q_{r,c,u}(z)$ can be dealt analogously: we just swap $A_{r,c}$ with $B_{r,c}$ and $p_{r,c,u}$ with $q_{r,c,u}$.

Write $A_{r,c}(z)$ and $B_{r,c}(z)$ as

$$A_{r,c}(z) = \prod_{i=1}^{n} \Phi_{rs_i}(z), \quad B_{r,c}(z) = \prod_{i=n+1}^{n+n^*} \Phi_{rs_i}(z).$$

Let $\mathcal{S}$ be the set of all positive integers s which divide one of the values s_i, $1 \leqslant i \leqslant n + n^*$, i.e.

$$\mathcal{S} := \{s \in \mathbb{N} : \exists\, i \in \{1, \ldots, n + n^*\},\ s \mid s_i\}.$$

Recall that $p_{r,c,u_{m+1}}/q_{r,c,u_{m+1}}$ can be written in the form

$$\frac{p_{r,c,u_{m+1}}(z)}{q_{r,c,u_{m+1}}(z)} = \frac{A_{r,c}(z) p_{r,c,u_m}(z^d)}{B_{r,c}(z) q_{r,c,u_m}(z^d)}.$$

Then the value $r_{r,c,c,m+1}$ is completely determined by two tuples $\Sigma_{p,m}$ and $\Sigma_{q,m}$ which are defined as follows:

$$\Sigma_{p,m} := (\sigma_{\zeta_1,m}, \dots, \sigma_{\zeta_N,m}), \quad \text{where} N = \#\mathcal{S},\ \zeta_1, \dots, \zeta_n \in \mathcal{S},$$
$$\sigma_{\zeta,m} := \sigma(\Phi_{r\zeta}(z), p_{r,c,u_m}(z^d))$$

and $\Sigma_{q,m}$ is defined analogously with p_{r,c,u_m} replaced by q_{r,c,u_m}. By Lemma 5.3, we have that $\Sigma_{p,m+1}$ and $\Sigma_{q,m+1}$ are also determined by $\Sigma_{p,m}$ and $\Sigma_{q,m}$ respectively.

It remains to show that all terms of $\Sigma_{p,m}$ and $\Sigma_{q,m}$ are bounded by a constant independent of m. That in turn will imply that there are only finitely many different values for $(\Sigma_{p,m}, \Sigma_{q,m})$ and there exist $m_1 < m_2$ such that $\Sigma_{p,m_1} = \Sigma_{p,m_2}, \Sigma_{q,m_1} = \Sigma_{q,m_2}$, hence the sequence $(r_{r,c,c,m})_{m\in\mathbb{N}}$ is eventually periodic end the proof of Proposition 4.4 is completed for this case.

Write the part $p_{r,c,u_m}/q_{r,c,u_m}$ of the convergent p_{u_m}/q_{u_m} in the following form:
$$\frac{p_{r,c,u_m}(z)}{q_{r,c,u_m}(z)} = \frac{\prod_{t=0}^{m-1} A^*_{t,m}(z)p^*_m(z)}{\prod_{t=0}^{m-1} B^*_{t,m}(z)q^*_m(z)},$$
where the numerator and the denominator of the right hand side are coprime and $A^*_{t,m}(z) \mid A_{r,c}(z^{d^t})$, $B^*_{t,m}(z) \mid B_{r,c}(z^{d^t})$, $p^*_m(z) \mid p_{r,c,u}(z^{d^m})$ and $q^*_m(z) \mid q_{r,c,u}(z^{d^m})$.

Since none of $A_{r,c}(z)$ and $B_{r,c}(z)$ are divisible by $\Phi_r(z)$ we have that there exists $m_0 \in \mathbb{N}$ such that for all $m > m_0$, the polynomials $A_{r,c}(z^{d^m})$ and $B_{r,c}(z^{d^m})$ are coprime with both $A_{r,c}(z)$ and $B_{r,c}(z)$. Therefore for each term $\sigma_{\zeta,m}$ of $\Sigma_{p,m}$ we have
$$\sigma_{\zeta,m} = \sum_{t=0}^{m_0} \sigma(\Phi_{r\zeta}(z), A^*_{t,m}(z^d)) + \sigma(\Phi_{r\zeta}(z), p^*_m(z^d)).$$
By Lemma 5.4, the right hand side is always bounded by some constant independent of m. By analogous arguments, the same is true for all terms $\sigma_{\zeta,m}$ of $\Sigma_{q,m}$.

Case 3. Assume that $\Phi_r(z)$ divides $A_{r,c}(z)$. Then, since $A_{r,c}(z)$ and $B_{r,c}(z)$ are coprime, we have that $\Phi_r(z)$ does not divide $B_{r,c}(z)$.

Note that the case $\Phi_r(z) \mid B_{r,c}(z)$ can be handled analogously. We just swap $A_{r,c}$ with $B_{r,c}$ and $p_{r,c,u}$ with $q_{r,c,u}$. Therefore Case 3 is the last one which needs to be investigated.

Lemma 5.5. *For any $n \in \mathbb{N}$ with $r(n,d) = r$ there exists $m \in \mathbb{N}$ such that $\Phi_n(z) \mid \Phi_r(z^{d^m})$.*

Proof. We write n as a product $n = rs$ and prove the lemma by induction on s. For $s = 1$ the statement is straightforward. Consider S such that

$\mathrm{rad}(S) \mid \mathrm{rad}(d)$. Assume that the statement is true for all $s < S$ with $\mathrm{rad}(s) \mid \mathrm{rad}(d)$ and prove it for S. Write the prime factorisations of S and d in the following way:

$$S = p_1^{\beta_1} \dots p_k^{\beta_k} p_{k+1}^{\beta_{k+1}} \dots p_{k+l}^{\beta_{k+l}}; \quad d = p_1^{\alpha_1} \dots p_{k+l}^{\alpha_{k+l}},$$

where $\beta_1 < \alpha_1, \dots, \beta_k < \alpha_k$ and $\beta_{k+1} \geqslant \alpha_{k+1} > 0, \dots, \beta_{k+l} \geqslant \alpha_{k+l} > 0$. Then, by Lemma 5.3, one has that $\Phi_S(z)$ divides $\Phi_s(z^d)$ for $s = p_{k+1}^{\beta_{k+1}-\alpha_{k+1}} \dots p_{k+l}^{\beta_{k+l}-\alpha_{k+l}}$. By induction assumption, we have that there exists m such that $\Phi_s(z) \mid \Phi_r(z^{d^m})$. Therefore, $\Phi_S(z) \mid \Phi_r(z^{d^{m+1}})$. □

Similarly to Case 2, define the set $\mathcal{S}$ and the following tuple:

$$\Sigma_{q,m} := (\sigma_{\zeta_1,m}, \dots, \sigma_{\zeta_N,m}), \quad \text{where } N = \#\mathcal{S}, \ \ \zeta_1, \dots, \zeta_n \in \mathcal{S} \text{ and}$$

$$\sigma_{\zeta,m} := \sigma\left(\Phi_{r\zeta}(z), \prod_{t=0}^{m-1} B_{r,c}(z^{d^t}) \cdot q_{r,c,u}(z^{d^m})\right).$$

As in Case 2, we have that all terms in $\Sigma_{q,m}$ are bounded by a constant, which is independent of m. On the other hand, by Lemma 5.3, every polynomial $A_{r,c}(z^{d^t})$ is divisible by $\Phi_r(z)$ and therefore

$$\sigma\left(\Phi_r(z), \prod_{t=0}^{m-1} A_{r,c}(z^{d^t})\right) \geqslant m.$$

In view of Lemma 5.5, there exists m_0 big enough, so that for any $\zeta \in \mathcal{S}$ and $m \geqslant m_0$ the value

$$\sigma\left(\Phi_{r\sigma}(z), \prod_{t=0}^{m-1} A_{r,c}(z^{d^t}) \cdot p_{r,c,u}(z^{d^m})\right)$$

is bigger than every term in $\Sigma_{q,m}$. That implies that for every $m > m_0$ every polynomial $(\Phi_{r\zeta}(z))^{\sigma_{\zeta,m}}$ cancels out in the expression

$$\frac{p_{r,c,u_m}(z)}{q_{r,c,u_m}(z)} = \frac{\prod_{t=0}^{m-1} A_{r,c}(z^{d^t}) p_{r,c,u}(z^{d^m})}{\prod_{t=0}^{m-1} B_{r,c}(z^{d^t}) q_{r,c,u}(z^{d^m})}.$$

Hence for $m \geqslant m_0$ the polynomial $q_{r,c,u_m}(z^d)$ is coprime with $A_{r,c}(z)$, $B_{r,c}(z)$ divides $p_{r,c,u_m}(z^d)$ and therefore the value $r_{r,c,c,m}$ is equal to r_b. Again we have that the sequence $(r_{r,c,c,m})_{m\in\mathbb{N}}$ is eventually periodic.

To finish the proof of Proposition 4.4 we observe that the sequence $r_{c,m}$ is the sum of finitely many eventually periodic sequences and hence is eventually periodic itself. □

6. Application: $d = 3$, infinite products of quadratic polynomials

Consider the set of Mahler functions $g_{\mathbf{a}}(z) = g_{a_1,a_2}(z)$ which satisfy the equation

$$g_{\mathbf{a}}(z) = (z^2 + a_1 z + a_2) q_{\mathbf{a}}(z^3); \quad a_1, a_2 \in \mathbb{Z}.$$

Such functions and their corresponding Mahler numbers were considered in [2] and it was conjectured that, given $b \in \mathbb{Z}$ with $|b| \geqslant 2$, if $g_{\mathbf{u}}(b) \notin \mathbb{Q}$ then $\mu(g_{\mathbf{a}}(b)) = 2$ for all $\mathbf{a} \in \mathbb{Z}^2$, except the following three families:

(a) $\mathbf{a} = (s, s^2)$, $s \in \mathbb{Z}$;
(b) $\mathbf{a} = (s^3, -s^2(s^2+1))$, $s \in \mathbb{Z}$;
(c) $\mathbf{a} = (\pm 2, 1)$.

In [2, Theorem 9] the lower bounds for the irrationality exponents of $g_{\mathbf{a}}(b)$ for those families is provided. Here we demonstrate how Theorem 1.2 together with Lemmata 4.2 and 4.3 can be used to show that the lower bounds in [2] are sharp.

Family (a). Let $\mathbf{a} = (s, s^2)$. Simple calculations reveal that the first convergent of $g_{\mathbf{a}}(z)$ is $1/(z-s)$ and

$$(z-s)g_{\mathbf{a}}(z) - 1 = (s - s^3)z^{-3} + \ldots$$

Therefore for $s^3 - s \neq 0$ we have that $\Phi(g_{\mathbf{a}})$ contains a primitive gap $[1,3]$ of size 2. Note that $z^{3^m} - s$ is always coprime with the polynomial $z^2 + sz + s^2$. Indeed, each root z_0 of the latter quadratic polynomial satisfies $|z_0|^3 = |s|^3$, so $|z_0|^{3^m} = |s|^{3^m} > |s|$ as soon as $|s| \geqslant 2$. But the last condition is equivalent to $s \neq \pm 1, 0$ which in turn is equivalent to $s^3 - s \neq 0$.

We thus have that the numerator and the denominator of

$$\frac{\prod_{t=0}^{m-1}(z^{2\cdot 3^t} + sz^{3^t} + s^2)}{z^{3^m} - s}$$

are always coprime. Therefore all the values $r_{c,m}$ equal zero and equations (4.6) imply that the gaps $[u_n, v_n]$ generated by $[1,3]$ satisfy

$$\frac{v_{n+1}}{u_{n+1}} = \frac{3v_n - 2}{3u_n}$$

and therefore

$$\liminf_{n\to\infty} \frac{v_n}{u_n} = \frac{v_0 - \frac{2}{2}}{u_0 + \frac{0}{2}} = 2.$$

From Lemma 4.2 we know that the size of any primitive gap in $\Phi(g_{\mathbf{a}})$ does not exceed 5. Therefore, by Lemma 4.3, only gaps with

$$2 < \frac{v}{u-1} \leqslant \frac{u+5}{u-1}$$

may contribute to the irrationality exponent of $g_{\mathbf{a}}(b)$. The last inequality is equivalent to $u < 7$. It remains to check that $\Phi(g_{\mathbf{a}}) = \{1, 3, 7, \ldots\}$ where the gap $[3, 7]$ is not primitive and is generated by $[1, 3]$. Hence there are no other primitive gaps $[u, v]$ with $u < 7$ and Theorem 1.2 implies:

Let $\mathbf{a} = (s, s^2) \in \mathbb{Z}^2$ *with* $s^3+s \neq 0$. *If* $|b| \geqslant 2$, $b \in \mathbb{Z}$ *and* $g_{\mathbf{a}}(b) \notin \mathbb{Q}$ *then* $\mu(g_{\mathbf{a}}(b)) = 3$.

For the remaining values of s we have:

$$g_{0,0}(z) = \frac{1}{z}; \quad g_{1,1}(z) = \frac{1}{z-1}; \quad g_{-1,1}(z) = \frac{1}{z+1}.$$

The function $g_{\mathbf{a}}$ is then rational, and therefore $g_{\mathbf{a}}(b) \in \mathbb{Q}$.

Family (b). Let $\mathbf{a} = (s^3, -s^2(s^2+1))$. In this case we compute

$$\frac{p_2(z)}{q_2(z)} = \frac{z + s(s^2+1)}{z^2 + sz + s^2}$$

and

$$q_2(z)g_{\mathbf{a}}(z) - p_2(z) = -(s^6 + s^4 + s^2)z^{-5} + \ldots$$

Therefore for $s^6 + s^4 + s^2 \neq 0$ we have that $\Phi(g_{\mathbf{a}})$ contains the primitive gap $[2, 5]$ of size 3. One can easily check that $z^2 + s^3 z - s^2(s^2+1) = (z - s)(z + (s^3 + s))$. On the other hand, all roots of $z^{2\cdot 3^m} + sz^{3^m} + s^2$ for $s \neq 0$ are not real. Therefore the fraction

$$\frac{\prod_{t=0}^{m-1}(z^{2\cdot 3^t} + s^3 z^{3^t} - s^2(s^2+1))p_2(z^{3^m})}{q_2(z^{3^m})}$$

is always in its reduced form, i.e. every term of $r_{c,m}$ is zero. This yields to

$$\liminf_{n\to\infty} \frac{v_n}{u_n} = \frac{v_0 - \frac{2}{2}}{u_0 + \frac{0}{2}} = 2.$$

As in the case of Family (a), we need to check that $\Phi(g_{\mathbf{a}})$ does not contain any other primitive gap $[u, v]$ with $u < 7$ which is obvious (by (4.6), we have the big gaps $[2, 5]$ and $[6, 13]$ in $\Phi(g_{\mathbf{a}})$. There is no more space for big gaps with $u < 7$). Therefore we finally get:

Let $\mathbf{a} = (s^3, -s^2(s^2+1))$ *with* $s \in \mathbb{Z}, s^6 + s^4 + s^2 \neq 0$. *If* $|b| \geqslant 2$, $b \in \mathbb{Z}$ *and* $g_{\mathbf{a}}(b) \notin \mathbb{Q}$ *then* $\mu(g_{\mathbf{a}}(b)) = 3$.

Finally notice that the equation $s^6 + s^4 + s^2 = 0$ has only one integer solution: $s = 0$. But $g_{0,0}(z)$ has already been considered in Family (a) and is equal to $1/z$.

Family (c). Let $\mathbf{a} = (2, 1)$. The case $\mathbf{a} = (-2, 1)$ is considered analogously and is left to the reader. One can check that $1, 2, 3, 4$ and 5 belong to $\Phi(g_{\mathbf{a}})$.

Direct computation shows that $[5,8]$ is the primitive gap in $\Phi(g_{\mathbf{a}})$ and the corresponding fifth convergent of $g_{\mathbf{a}}$ is

$$p_5(z) = z^4 + z^3 + 2z^2 + 4 \quad \text{and}$$
$$q_5(z) = z^5 - z^4 + z^3 - z^2 + z - 1 = (z-1)(z^2+z+1)(z^2-z+1).$$

In other words, $q_5(z) = \Phi_1(z)\Phi_3(z)\Phi_6(z)$. Lemma 5.3 implies that all cyclotomic divisors of $q_5(z^{3^m})$ are either of the form $\Phi_{3r}(z)$ with some integer r or $\Phi_1(z)$. Hence $q_5(z^{3^m})$ is always coprime with $z^2+2z+1 = \Phi_2(z)^2$, i.e. the fraction

$$\frac{\prod_{t=0}^{m-1}(z^{2\cdot 3^t} + 2z^{3^t} + 1)p_5(z^{3^m})}{q_5(z^{3^m})}$$

is always in its reduced form and every term of $r_{c,m}$ is zero. This yields to

$$\liminf_{n\to\infty} \frac{v_n}{u_n} = \frac{v_0 - 1}{u_0} = \frac{7}{5}.$$

Now from Lemma 4.2 we know that the size of any primitive gap in $\Phi(g_{\mathbf{a}})$ does not exceed 5. Therefore, by Lemma 4.3, only gaps with

$$\frac{7}{5} < \frac{v}{u-1} \leqslant \frac{u+5}{u-1}$$

may contribute to the irrationality exponent of $g_{\mathbf{a}}(b)$. The last inequality is equivalent to $u < 16$. It remains to show that all integers from 8 to 15 belong to $\Phi(g_{\mathbf{a}})$. This for example can be done by checking that the corresponding Hankel determinants

$$H_n := \det(c_{i+j-1})_{i,j\in\{1,\dots,n\}}, \quad n = 8, \dots 15$$

are not zero, where c_i are the coefficients of the series $g_{\mathbf{a}}$:

$$g_{\mathbf{a}}(z) = \sum_{i=1}^{\infty} c_i z^{-i}.$$

See [2, Corollary 1] for justification. We used Mathematika package to compute all of the required Hankel determinants.

Finally we have:

Let $\mathbf{a} = (\pm 2, 1)$. *If* $|b| \geqslant 2$, $b \in \mathbb{Z}$ *then* $\mu(g_{\mathbf{a}}(b)) = \frac{12}{5}$.

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