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Deligne–Illusie Classes as Arithmetic Kodaira–Spencer Classes

par Taylor DUPUY et David ZUREICK-BROWN

Résumé. Faltings a montré qu'il n'y a pas de « classes de Kodaira–Spencer arithmétiques » satisfaisant à un certain axiome de compatibilité. En modifiant légèrement ses définitions, nous montrons que les classes de Deligne–Illusie satisfont à ce que l'on pourrait considérer comme « condition de compatibilité de Kodaira–Spencer arithmétique ».

Abstract. Faltings showed that "arithmetic Kodaira–Spencer classes" satisfying a certain compatibility axiom cannot exist. By modifying his definitions slightly, we show that the Deligne–Illusie classes satisfy what could be considered an "arithmetic Kodaira–Spencer" compatibility condition.

1. Introduction

The abstract of the paper "Does there exist an Arithmetic Kodaira–Spencer class?" [9] is the following: "We show that an analog of the Kodaira–Spencer class for curves over number-fields cannot exist." In the present paper we show that if we modify the axioms in [9] slightly such classes can exist; motivated by work of Buium and by work of Mochizuki, we give a candidate for such a class.

Remark 1.1. The term "arithmetic Kodaira–Spencer class" is vague and the definition varies from paper to paper. In this paper we use the Deligne–Illusie class (see Section 2.5). More distinct "arithmetic Kodaira–Spencer theory" can be found in [7], [5] and [10, §1.4].

We recall the setup of [9]. For schemes S and X of finite type over a base B and a smooth map of B-schemes $\pi\colon X \to S$, we have an exact sequence

$$0 \to \pi^*(\Omega_{S/B}) \to \Omega_{X/B} \to \Omega_{X/S} \to 0 \tag{1.1}$$

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giving rise to a class $\kappa(X) \in \operatorname{Ext}^1(\Omega_{X/S}, \pi^*\Omega_{S/B}) = H^1(X, T_{X/S} \otimes \pi^*\Omega_{S/B})$ which [9] calls the Kodaira–Spencer class. The image of $\kappa(X)$ under the natural map

$$H^1(X, T_{X/S} \otimes \pi^*\Omega_{S/B}) \to \operatorname{Hom}_{\mathcal{O}_S}(T_{S/B}, R^1\pi_* T_{X/S})$$

(arising from the projection formula [11, Tag 01E6] and the 5-term exact sequence arising from spectral sequence for the composition $\Gamma \circ \pi_*$ of functors) is the Kodaira–Spencer map. Such classes are important for many diophantine reasons and we refer the reader to [9] for a discussion.

The problem observed in [9] (and elsewhere) is that if S is the spectrum of the ring of integers of a number field then there are no derivations and hence the Kodaira–Spencer map doesn't make sense.[1] Although no map can exist, it is (a priori) possible for extensions corresponding to (1.1) to exist in a canonical way (they don't as Faltings observes). For such extension classes to be canonical [9] posits that for morphisms $f\colon X \to Y$ of smooth S-schemes, "Kodaira–Spencer classes with values in ω" (where $\omega = \Omega_{S/B}$) should satisfy

$$f^*(\kappa(Y)) = df_*(\kappa(X)) \in H^1\left(X, f^*(T_{Y/S}) \otimes \omega\right). \tag{1.2}$$

Although [9] shows no such classes may exist, we show (using Buium's "wittferential algebra" [5], which formalizes the analogy between Witt vectors and power series) that there exist classes $\mathrm{DI}_{X_1/S_1}(\delta) \in H^1(X_0, F^*T_{X_0})$ which we call "Deligne–Illusie classes", and which satisfy a condition similar to (1.2). Here subscripts n denote a reduction modulo p^{n+1} and the recipient sheaf here is the Frobenius tangent sheaf, whose local sections are Frobenius semi-linear derivations. The name stems from their implicit use in [6]. We show the following.

Theorem 1.2. *For a morphism $f\colon X \to Y$ of smooth p-formal schemes over $S = \operatorname{Spf} \mathbf{Z}_p$ which is either smooth or a closed immersion we have*

$$f^* \mathrm{DI}_{Y_1/S_1}(\delta) = df_* \mathrm{DI}_{X_1/S_1}(\delta) \in H^1(X_0, F_{X_0}^* T_{X_0}). \tag{1.3}$$

In Section 2 we give the analogies with the Kodaira–Spencer map, and we prove (1.3) in Section 3. In a separate paper we study the vector bundles coming from these extensions [8].

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[1] Actually, $\Omega_{\mathcal{O}_K/\mathbf{Z}}$ exists and the annihilator is the different, which controls ramification. This means for all but finitely many primes its localization will be zero. The theory we give presently gives something for unramified primes.

Christelle Vincent, for enlightening conversations. We especially thank Piotr Achinger for pointing out an alternative approach to this proof via [1, Lemma 3.3.3]. This paper started during the first author's visit to the 2014 Spring Semester at MSRI on Model Theory.

2. Notation and analogies

2.1. Classical derivations/differentiation and π-derivations/wittferentiation. Let CRing denote the category of commutative rings. For $R \in \mathsf{CRing}$ we let CRing_R denote the category of R-algebras.

Let $A \in \mathsf{CRing}$ and $B \in \mathsf{CRing}_A$. We have a correspondence between the module of derivations $\partial_f \colon A \to B$, which we denote by $\mathrm{Der}(A, B)$, and functions $f \colon A \to B[\varepsilon]/(\varepsilon^2)$ given by

$$\begin{array}{ccc} A & \xrightarrow{f} & B[\varepsilon]/(\varepsilon^2) \\ & {\scriptstyle \partial_f} \searrow & \downarrow {\scriptstyle \mathrm{pr}_1} \\ & & B \end{array}$$

where $\mathrm{pr}_i \colon B[\varepsilon]/(\varepsilon^2) \to B$ are given by $\mathrm{pr}_0(a + \varepsilon b) = a$ and $\mathrm{pr}_1(a + \varepsilon b) = b$. The map from the collection of such f's to the collection of derivations is given by

$$f \mapsto \partial_f = \mathrm{pr}_1 \circ f.$$

If X is a scheme over a ring R, we will let $\mathrm{Der}(\mathcal{O}_X/R)$ denote the sheaf of $R-$linear derivations on X; this sheaf is isomorphic to $T_{X/R}$.

Now for the arithmetic version. The idea in what follows is to replace $B \mapsto B[\varepsilon]/(\varepsilon^2)$ with other ring schemes to get "new derivations". In the same way that derivations are in correspondence with maps to the ring of dual numbers, π-derivations are defined via maps to rings of truncated witt vectors of length two.

Let R be a finite extension of $\mathbf{Z}_p$ with uniformizer $\pi \in R$. Let q denote the cardinality of the residue field of R. For an R-algebra A we define $W_{\pi,1}(A)$ to be the set $A \times A$ with addition and multiplication rules given by

$$(a_0, a_1)(b_0, b_1) = (a_0 b_0, a_1 b_0^q + b_1 a_0^q + \pi a_1 b_1),$$
$$(a_0, a_1) + (b_0, b_1) = \left(a_0 + b_0, a_1 + b_1 - \frac{1}{\pi} \sum_{j=1}^{q-1} \binom{q}{j} a_0^{q-j} b_0^j \right);$$

these are the so-called *ramified witt vectors of length two*. When the π is understood we will just denote this ring by W_1.

Let $A \in \mathsf{CRing}_R$ and $B \in \mathsf{CRing}_A$, with structure map $g \colon A \to B$. We define a π- *derivation* to be a function $\delta \colon A \to B$ such that the map

$$f \colon A \to W_1(B),\ x \mapsto f(x) := (g(x), \delta(x))$$

is a ring homomorphism. Given a ring homomorphism $f\colon A \to W_1(B)$, the composition $\delta_f = \mathrm{pr}_1 \circ f$ is a π-derivation. From the sum and product rules for Witt vectors we may derive the sum, product and identity rules for p-derivations. We denote the collection of π-*derivations* from $\delta\colon A \to B$ by $\pi\text{-Der}(A, B)$.

Example 2.1. In the examples below we will let $\pi = p$ a rational prime.

(1) $\delta\colon \mathbf{Z}_p \to \mathbf{Z}_p$ given by $\delta(x) = (x - x^p)/p$;
(2) $\delta\colon \mathbf{Z}/p^2 \to \mathbf{Z}/p$ given by the same formula. Note now that division by p is a map $p\mathbf{Z}/p^2 \to \mathbf{Z}/p$.

Finally note that given a π-derivation δ, the map $\phi(x) = x^q + \pi\delta(x)$ is a lift of the Frobenius (a ring homomorphism whose reduction modulo π coincides with a qth power map).

2.2. Notation for reductions mod powers of primes. We start with a field K of characteristic zero, complete under a discrete valuation v, with residue field k of characteristic $p > 0$. We assume v is normalized such that $v(K^\times) = \mathbf{Z}$ and we denote by $e := v(p)$ the absolute ramification index. Let R be the valuation ring of K. Assume now that we are given a prime element $\pi \in R$ which is algebraic over $\mathbf{Q}_p$. Having fixed K and π as above we shall define a map $\delta\colon R \to R$ which will play the role of a "derivation with respect to π". Let q be the cardinality of the residue field of $\mathbf{Q}_p(\pi)$. Then, by standard local field theory, there exists a ring automorphism $\phi\colon R \to R$ that lifts the Frobenius automorphism $F\colon k \to k$, $F(x) := x^q$. We define the map $\delta\colon R \to R$ by the formula

$$\delta(x) = \frac{\phi(x) - x^q}{\pi}$$

for $x \in R$. We shall usually write $x', x'', \ldots, x^{(n)}$ in place of $\delta(x), \delta^2(x), \ldots, \delta^n(x)$.

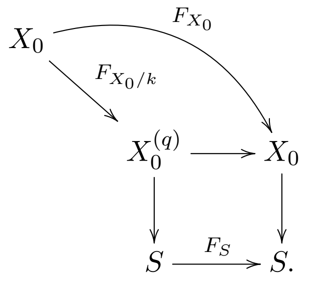
There exists a unique lift of the Frobenius $\phi = \phi_{R,\pi}$ which acts as $\phi(\zeta_n) = \zeta_n^q$ (for $(n, q) = 1$) and satisfies $\phi(\pi) = \pi$. We will let

$$R_n = R/\pi^{n+1}$$

and for X/R a scheme we let

$$X_n = X \otimes R_n = X \bmod \pi^{n+1}.$$

2.3. Absolute and relative Frobenius. For X_0/S a scheme over a base S of characteristic p we will let $F_{X_0} = F_{X_0,q}$ denote the absolute Frobenius and $F_{X_0/S} = F_{X_0/S,q}$ denote the relative Frobenius. They fit into a diagram



Here $X_0^{(q)} = X_0 \times_{S,F_S} S$ is the Frobenius twist of X_0, which is just the pullback of X_0 by the Frobenius on the base. In terms of equations, we simply raise to qth power the coefficients of the defining equations of X_0. On sections we have $F_{X_0}^{\#}(f) = f^q$ and $F_S^{\#}(a) = a^q$. When no confusion arises, we may just denote a Frobenius as F.

Let X and X' be schemes or π-formal schemes over R which lift X_0. A *lift of the Frobenius* is a morphism

$$\phi\colon X \to X'$$

such that $\phi \otimes_R R/\pi \cong F_{X_0}$.

2.4. Frobenius derivations. For X_0 a scheme over a field k of characteristic p we define the sheaf $\mathrm{FDer}(\mathcal{O}_{X_0})$ of *Frobenius semi-linear derivations* or *F-derivations* to be $\mathrm{FDer}(\mathcal{O}_{X_0}) := F_{X_0}^* T_{X_0/k}$; note that these can be either the p-Frobenius or a p^a-Frobenius depending on the context. It follows directly from the definition that local section D has the property that D acts as

$$D(xy) = x^q D(y) + D(x) y^q,$$

where x and y are local sections of $\mathcal{O}_{X_0}$.

2.5. Deligne–Illusie classes. Let X/R be a smooth scheme. As in the above setup, let $\delta\colon R \to R$ be the unique π-derivation such that the induced Frobenius fixes a chosen uniformizer π. We define the *Deligne–Illusie class* to be the Čech cohomology class

$$\mathrm{DI}_{X_1/R_1}(\delta) = [\delta_i - \delta_j \mod \pi] \in H^1(X_0, F_{X_0}^* T_{X_0/k})$$

where $\delta_i\colon \mathcal{O}_{U_{i,1}} \to \mathcal{O}_{U_{i,0}}$ are local prolongations of p-derivations on the base and $(U_i \to X)_{i \in I}$ is a cover by Zariski affine opens with lifts of the π-derivations. Such lifts exist locally due the infinitesimal lifting property. See, for example, [4, Lemma 1.3]. When the derivation on the base R is understood we will use the notation

$$\mathrm{DI}_{X_1/R_1}(\delta) = \mathrm{DI}(X_1).$$

When we want to signify that $\mathrm{DI}(X_1)$ is an obstruction to lifting the mth power Frobenius we use the notation $\mathrm{DI}^m(X_1)$.

2.6. Kodaira–Spencer classes and three properties of Kodaira–Spencer classes. Let X/K be a smooth projective variety. Let $\partial_K \in \mathrm{Der}(K)$ be a derivation on the base. Let $(U_i \to X)_{i\in I}$ be a cover by Zariski opens. The *Kodaira–Spencer class* is defined by

$$\mathrm{KS}_{X/K}(\partial_K) = [\partial_i - \partial_j] \in H^1(X, T_{X/K})$$

where $\partial_i \in \mathrm{Der}(\mathcal{O}_X(U_i))$ are prolongations of the derivation on the base: $\partial_i|_K = \partial_K$. We present three properties which will have arithmetic analogs.

2.6.1. ***Property 1: Representability of sheaf of prolongations of derivations.*** The first jet space is defined to be the representative of the sheaf of prolonged derivatives $\mathrm{Der}(\mathcal{O}_X/(K,\partial_K))$ on X:

$$\mathrm{Der}(\mathcal{O}_X/(K,\partial_K)) \cong \Gamma_X(-, J^1(X/(K,\partial_K))).$$

Here $g\colon J^1(X/(K,\partial_K)) \to X$ is the first jet space on X and the right hand side denotes the sheaf of sections of g.[2] Local sections of this space are local lifts of the derivation. One may observe that $J^1(X/(K,\partial_K))$ is a torsor under $T_{X/K}$, and is thus classified by $\mathrm{KS}_{X/K}(\partial_K) \in H^1(X, T_{X/K})$ (the difference of two derivations prolonging a derivation on the base field is zero on the base field since they agree there).

2.6.2. ***Property 2: Buium–Ehresmann Theorem.*** Let K be an algebraically closed field equipped with a derivation ∂. In what follows we let $K^\partial = \{a \in K : \partial(a) = 0\}$ denote the field of constants. The following are equivalent for X/K projective:

(1) $\mathrm{KS}_{X/K}(\partial) = 0$,
(2) X/K admits a global lift of ∂, and
(3) $X \cong X_0 \otimes_{K^\partial} K$ for some scheme X_0 defined over K^∂;

see [3, Chapter II, Section 1].

2.6.3. ***Property 3: Kodaira–Spencer Compatibility.*** In [9] it was asked if there exists an arithmetic Kodaira–Spencer class. He isolated the following key property: let K be a field with a derivation. If $f\colon X \to Y$ is a morphism over K (for example, a smooth morphism or a closed immersion) then

$$f^* \mathrm{KS}_{Y/K}(\partial) = df_* \mathrm{KS}_{X/K}(\partial) \in H^1(X, f^*T_{Y/K}),$$

where

$$df\colon T_{X/K} \to f^*T_{Y/K}$$

is the natural map and df_* is the induced map on cohomology.

[2] In the special case that $\partial_K = 0$ we have $J^1(X/(K,\partial_K)) = T_{X/K}$ and the functor of points of J^1 is just X composed with the dual number functor; i.e., $J^1(X/(K,\partial_K))(A) \to X(A[\epsilon]/\epsilon^2)$.

2.7. Three analogous properties for Deligne–Illusie classes. We now present three properties (one of which is new and stated as a theorem) which are analogs of the three properties for Kodaira–Spencer classes.

2.7.1. ***Property 1: Representability of sheaf of prolongations of p-deriva- tions.*** We now work over R a finite extension of $\mathbf{Z}_p$ with prime element $\pi \in R$. Let X be a π-formal scheme over R as in Section 2.2. We define the first π-jet space ([5, 4]) to represent the sheaf of π-derivations on X. More precisely the map $g\colon J^1(X) \to X$ represents the sheaf of π-derivations (in characteristic zero). That is, local sections of g correspond to local lifts of π-derivations. When talking about the first π-jet space of a scheme we will always mean the first π-jet space of its π-formal completion.

We can consider the above situation modulo π^2. Here, the sheaf

$$\pi\text{-Der}(\mathcal{O}_{X_1}, \mathcal{O}_{X_0})$$

of prolongations of the π-derivation $\delta_1\colon R_1 \to R_0$ is representated by sections of a map

$$g_0\colon J^1(X)_0 \to X_0.$$

Here $J^1(X)_0$ is the reduction mod π of the first arithmetic jet space. Local sections of g_0 correspond to local lifts of the Frobenius on $\mathcal{O}_{X_1} \to \mathcal{O}_{X_1}$, or equivalently π-derivations $\mathcal{O}_{X_1} \to \mathcal{O}_{X_0}$. The scheme $J^1(X)_0$ is a torsor under $F^*T_{X_0}$ whose class is classified by $\mathrm{DI}_{X_1/R_1}(\delta_1)$ (this can be seen by just subtracting two π-derivations pointwise and obtaining a derivation of the Frobenius).

Locally, the constructions looks as follows: for a ring $A = R\langle X\rangle/(G) = \varprojlim R[X]/(G, \pi^n)$, where $X = (x_1, \dots, x_m)$ and $G = (f_1, \dots, f_e)$, we have

$$\mathcal{O}(J^1(\mathrm{Spec}(A))) = R\langle X, \dot{X}\rangle/(G, \delta(G))$$

where $\delta(G)$ denotes the tuple of formal π-derivations of the elements $f_1, \dots,$ f_e which we understand as expanding using the sum and product rules to arrive at elements of $R[X, \dot{X}]$. For example

$$\delta(x^2 + rx_1) = 2x_2^q\dot{x}_2 + \delta(r)x_1^q + \dot{x}_1 r^q + \pi\dot{x}_1\delta(r)C_\pi(x_2^2, rx_1)$$

where $C_\pi(a,b) = \frac{a^q+b^q-(a+b)^q}{\pi} \in R[a,b]$ is the polynomial in the addition rule for Witt vectors. Here the universal formal π-derivation $\delta\colon R\langle X\rangle \to R\langle X, \dot{X}\rangle$ prolongs the fixed π-derivation on the base. This construction globalizes to give a π-formal scheme $g\colon J^1(X) \to X$.

2.7.2. ***Property 2: Buium–Ehresmann Theorem/Descent Philosophy.*** Recall that

$$\mathrm{DI}_{X_1/R_1}(\delta_1) = 0$$

if and only if X_1 has a lift of the Frobenius modulo π^2. In view of the analogy with Buium–Ehresmann theorem this should be viewed as a sort-of descent. In fact, Borger defines a category of Λ_p-schemes where the objects are pairs

(X, ϕ_X) consisting of schemes or p-formal schemes together with lifts of the Frobenius and whose morphisms $(X, \phi_X) \to (Y, \phi_Y)$ are morphisms $f\colon X \to Y$ which are equivariant with respect to ϕ_X and ϕ_Y. We think of this as a sort of descent to the field with one element in view of [2].

2.7.3. *Property 3: Deligne–Illusie Compatibility.* In the present paper we prove the following.

Theorem 2.2. *Let $f\colon X \to Y$ be a morphism of formally smooth π-formal schemes over R (a finite extension of $\mathbf{Z}_p$ with specified prime element π). If f is smooth or a closed immersion then*

$$df_* \operatorname{DI}_{X_1/R_1}(\delta_1) = f^* \operatorname{DI}_{Y_1/R_1}(\delta_1) \in H^1(X, f^*F^*T_{X_0/R_0}). \tag{2.1}$$

This property is new and is proved in Section 3. The proof uses affine bundle structures of $J^1(X/R)$—the first π-arithmetic jet space of Buium, the fact that smooth morphisms locally decompose as étale morphism followed by projections from an affine space, and properties of jet spaces and étale morphisms $X \to Z$ of p-formal schemes $J^1(X) \cong J^1(Z) \times_Z X$ to build "local Frobenius compatibility data".

3. Proof of compatibility

In what follows we will fix R a finite extension of $\mathbf{Z}_p$ with uniformizer π and residue field k of cardinality q. We will fix a π-derivation on the base.

Definition 3.1.

(1) A morphism of R schemes $f\colon X \to Y$ is *Deligne–Illusie compatible* provided

$$df_*(\operatorname{DI}(X_1)) = f^* \operatorname{DI}(Y_1) \in H^1(X_0, f^*FT_{Y_0}).$$

(2) Let $f\colon X \to Y$ be a morphism of π-formal schemes. By locally *local Frobenius compatibility data* for f we will mean two covers

$$(U_i \to X)_{i\in I} \text{ and } (V_i \to Y)_{i \in I}$$

with lifts of the Frobenius ϕ_{U_i} and ϕ_{V_i} (with the second cover possibly having repeat open sets) such that for each i,

$$f(U_i) \subset V_i$$

and $f|_{U_i}$ is compatible with ϕ_{U_i} and ϕ_{V_i}.

(3) If f admits local Frobenius compatibility data we will say f is *locally Frobenius compatible.*

Lemma 3.2. Let $f\colon X \to Y$ be a morphism of smooth π-formal schemes over $\operatorname{Spf}(R)$.

(1) If f is a closed immersion then f is locally Frobenius compatible.
(2) If f is étale then f is locally Frobenius compatible.

(3) If f is a projection of the form $\mathbf{A}^n_Y \to Y$ then f is locally Frobenius compatible.

In the proofs, we will repeatedly use the fact that a scheme X admits a Frobeinus lift if and only if the map $J^1(X) \to X$ admits a section, and that two lifts are compatible if and only if the induced diagram

$$\begin{array}{ccc} J^1(X) & \longrightarrow & J^1(Y) \\ \downarrow\uparrow & & \downarrow\uparrow \\ X & \longrightarrow & Y \end{array} \tag{3.1}$$

commutes.

Proof. We begin with case 1. We will work with π-formal schemes and omit the hats. Let X have dimension n and Y have dimension $n+m$. The problem is affine local, so by [5, Chapter 3, Proposition 3.13, p. 75] we may assume without loss of generality that X and Y are affine and that $J^1(X) \cong X \times \mathbf{A}^n$ and $J^1(Y) \cong Y \times \mathbf{A}^{n+m}$.Compatible lifts of the Frobenius ϕ_X and ϕ_Y are thus equivalent to compatible sections of the diagram

$$\begin{array}{ccc} X \times \mathbf{A}^n & \longrightarrow & Y \times \mathbf{A}^{n+m} \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y. \end{array} \tag{3.2}$$

On coordinate rings, the map $X \times \mathbf{A}^n \to Y \times \mathbf{A}^{n+m}$ is given by a map

$$\mathcal{O}(Y)\langle s_1, \dots, s_{n+m}\rangle \xrightarrow{\alpha} \mathcal{O}(X)\langle t_1, \dots, t_n\rangle$$

where the s_i and t_j are coordinates on each affine space, and our desired sections correspond to a commutative diagram

$$\begin{array}{ccc} \mathcal{O}(Y)\langle s_1, \dots, s_{n+m}\rangle & \xrightarrow{\alpha} & \mathcal{O}(X)\langle t_1, \dots, t_n\rangle \\ \downarrow{\scriptstyle \sigma_Y} & & \downarrow{\scriptstyle \sigma_X} \\ \mathcal{O}(Y) & \xrightarrow{\beta} & \mathcal{O}(X) \end{array} \tag{3.3}$$

where σ_Y and σ_X are the natural maps given by $\sigma_Y(s_i) = \delta_Y(s_i)$ and $\sigma_X(t_i) = \delta_X(t_i)$ where δ_X and δ_Y are the π-derivations associated to ϕ_X and ϕ_Y (c.f. [5, Chapter 3, Section 3.2]). Observe that the map α is determined by a formula of the form

$$\alpha(s_i) = \sum_J a_{i,J} t^J, \quad 1 \le i \le m$$

where $J = (j_1, \dots, j_n) \in \mathbf{N}^n$, $t^J = \prod t_i^{j_i}$ is multi-index notation, and $a_{i,J} \in \mathcal{O}(X)$ π-adically tend to zero as $|J| \to \infty$.

Suppose $\sigma_X(t_i)$ is defined by $\sigma_X(t_i) = A_i \in \mathcal{O}(X)$ for some choices of $A_i \in \mathcal{O}(X)$. We will prove that there exists a lift of the Frobenius of Y which is compatible with this one. Observe the compatibility condition $\alpha \circ \phi_Y = \phi_X \circ \alpha$ implies $\beta \circ \sigma_Y = \sigma_X \circ \alpha$, which implies that

$$\beta\sigma_Y(s_i) = \sum_J a_{i,J} A^J := \overline{B_i}.$$

Here $A = (A_1, \ldots, A_m)$. Constructing σ_Y to make the diagram (3.3) commute is now simple: for any $B_i \in \mathcal{O}(Y)$ with image $\overline{B}_i$ in $\mathcal{O}(X)$, the morphism σ_Y defined by

$$\sigma_Y(s_i) = B_i$$

works (i.e. defines a commutative diagram). Note that such B_i always exist because $\mathcal{O}(Y) \to \mathcal{O}(X)$ was assumed to be surjective.

Next we prove the second claim. Suppose f is étale. By [5, Chapter 3, Corollary 3.16, p. 77] we have

$$J^1(X) \cong X \times_Y J^1(Y) \tag{3.4}$$

as π-formal schemes. In this case, the diagram

$$\begin{array}{ccc} J^1(X) = X \times_Y J^1(Y) & \longrightarrow & J^1(Y) \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array} \tag{3.5}$$

is cartesian, and given a section of $\sigma_Y \colon J^1(Y) \to Y$ we can simply take σ_X to be (id, σ_Y).

For the third claim, let $m = \dim(Y)$. While it is not in general true that $J^1(X_1 \times X_2) \cong J^1(X_1) \times J^1(X_2)$, this isomorphism does hold if X_2 is affine space. We consider the diagram

$$\begin{array}{ccc} J^1(X) \cong J^1(Y) \times J^1(\mathbf{A}^n) & \longrightarrow & J^1(Y) \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y. \end{array}$$

Since $J^1(\mathbf{A}^n) \cong \mathbf{A}^{2n}$, any section of $Y \to J^1(X)$ extends to a section of $Y \to J^1(X)$, completing the proof. $\square$

Lemma 3.3. The following are true.

(1) If $f \colon X \to Y$ admits local Frobenius compatibility data, it is Deligne–Illusie compatible.
(2) If $f \colon X \to Z$ is Deligne–Illusie compatible and $g \colon Z \to Y$ is Deligne–Illusie compatible then their composition is.

Proof. We will work π-formally and omit hats everywhere. To begin the proof of the first claim, we fix local Frobenius compatibility data (Definition 2): i.e., we fix open covers $(U_i \to X)_{i\in I}$ and $(V_i \to Y)_{i\in I}$ such that $f(U_i) \subset V_i$ together with $\phi_i^X \colon \mathcal{O}(U_i) \to \mathcal{O}(U_i)$ and $\phi_i^Y \colon \mathcal{O}(V_i) \to \mathcal{O}(V_i)$ such that $\phi_X^i f^\# = f^\# \phi_Y^i$. Observe that this last condition is equivalent to $\delta_i^X f^\# = \delta_Y^i f^\#$ as elements of π-$\mathrm{Der}(\mathcal{O}_Y, f_*\mathcal{O}_X)(U_i)$. This implies for each $U_{ij} = U_i \cap U_j$ we have

$$D_{ij}^X f^\# = f^\# D_{ij}^Y \in \mathrm{FDer}(\mathcal{O}_Y(V_{ij}), f_*\mathcal{O}_X(U_{ij})), \tag{3.6}$$

where $D_{ij}^X := \delta_i^X - \delta_j^X$ and $D_{ij}^Y := \delta_i^Y - \delta_Y^J$. Note that the right hand side of (3.6) induces $df\,\mathrm{DI}(X)$ and the right hand side of (3.6) induces $f^*\,\mathrm{DI}(Y)$. [3] [4]

The proof of the second claim requires the identities

$$d(g \circ h) = h^*(dg_*) \circ dh_*, (g\circ h)^* = h^* g^*.$$

It then follows that

$$\begin{aligned} f^*\,\mathrm{DI}(Y) &= (g\circ h)^*\,\mathrm{DI}(Y) \\ &= h^* g^*\,\mathrm{DI}(Y) \\ &= h^*(dg_*\,\mathrm{DI}(Z)) \\ &= (h^* dg_*)(h^*\,\mathrm{DI}(Z)) \\ &= (h^* dg_*)(dh\,\mathrm{DI}(X)) \\ &= df_*\,\mathrm{DI}(X). \end{aligned}$$

The fourth equality follows from the diagram

$$\begin{array}{ccc} H^i(Z, \mathrm{FDer}(Z)) & \xrightarrow{dg_*} & H^i(Z, g^*\,\mathrm{FDer}(Y)) \\ \downarrow{\scriptstyle h^*} & & \downarrow{\scriptstyle h*} \\ H^i(X, h^*\,\mathrm{FDer}(Z)) & \xrightarrow{h^* dg_*} & H^i(X, h^* g^*\,\mathrm{FDer}(Y)). \end{array} \qquad \square$$

Theorem 3.4. *Let $f\colon X_1 \to Y_1$ be a smooth morphism of smooth R_1-schemes. Then*

$$df_*(\mathrm{DI}(X_1)) = f^*(\mathrm{DI}(Y_1)) \in H^1(X_0, f^* F T_{Y_0}).$$

Proof. We first prove the theorem locally and assume we can factor the morphism $f\colon X \to Y$ as

$$X \to \mathbf{A}_Y^n \to Y,$$

[3] Since $\mathcal{O}(f^{-1}(U_{ij})) \to \mathcal{O}(V_{ij})$ we may view this as giving a map on X and hence giving a cocycle for a sheaf on X.

[4] In general, for F a quasi-cohrent sheaf on Y, the map $f^*\colon H^i(Y, F) \to H^i(X, f^*F)$ can be performed locally by just identifying sections of F with sections of f^*F with new coefficients.

where the first map is étale and the second map is the standard projection (see e.g. [11, Tag 039P]). This can be done locally where by "locally" we mean that there exists a cover by affine open subsets $X' \subset X$ and $Y' \subset Y$ with $f(X') \subset Y'$ with this factorization.

We will now express f as a composition of Deligne–Illusie compatible morphisms. We apply Lemma 3.2 part 2 and Lemma 3.2 part 3 together with Lemma 3.3 part 1 to get the outer morphisms of the composition to be Deligne–Illusie compatible. Lemma 3.3 part 2 says the composition of compatible morphisms is compatible.

We now show compatibility globally. Consider a covering $(U_{i,0} \to X_0)_{i \in I}$ such that

$$df_*(\mathrm{DI}(X_1))|_{U_{i,0}} = f^*(\mathrm{DI}(Y_1))|_{U_{i,0}} \in \underline{H}^1(X_0, f^*FT_{Y_0})(U_{i,0}).$$

Putting these together gives an element

$$c \in H^0(X_0, \underline{H}^1(f^*FT_{Y_0})).$$

The comparison between the cohomology sheaf $\underline{H}^1(X_0, f^*FT_{Y_0})$ and the cohomology $H^1(X_0, f^*FT_{Y_0})$ comes from the low degree exact sequence of the spectral sequence comparing sheafy cohomology and cohomology (see for example [11, 01ES] for the spectral sequence). The convergent spectral sequence is given by

$$E_2^{i,j} = H^i(X_0, \underline{H}^j(f^*FT_{Y_0})) \implies H^{i+j}(X_0, f^*FT_{Y_0})$$

and the low degree exact sequence gives

$$\begin{aligned} 0 &\to H^1(X_0, \underline{H}^0(FT_{Y_0})) \to H^1(X_0, f^*FT_{Y_0}) \to H^0(X, \underline{H}^1(f^*FT_{Y_0})) \\ &\to H^1(X_0, \underline{H}^0(f^*FT_{Y_0})) \to H^2(X_0, f^*FT_{Y_0}) \end{aligned}$$

which reduces to

$$0 \to H^1(X_0, f^*FT_{Y_0}) \to H^0(X, \underline{H}^1(f^*FT_{Y_0})) \to H^2(X_0, f^*FT_{Y_0}) \to 0.$$

By local compatibility we have that $f^*\,\mathrm{DI}(Y_1)$ and $df\,\mathrm{DI}(X_1)$ in $H^1(X_0, f^*FT_{Y_0})$ map to the same element in $H^0(X_0, \underline{H}^1(f^*FT_{Y_0}))$; since the map

$$H^1(X_0, f^*FT_{Y_0}) \to H^0(X_0, \underline{H}^1(f^*FT_{Y_0}))$$

is injective, the desired equality follows. □

Proof of Theorem 1.2. By Theorem 3.4 we know that compatibility holds for a smooth morphism. By combining Lemma 3.2 with Lemma 3.3 we have compatibility for closed immersions. □

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