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On binary cubic and quartic forms

par Stanley Yao XIAO

Résumé. Dans cet article, nous décrivons le groupe d'automorphismes rationnels d'une forme binaire cubique ou quartique à coefficients entiers et à discriminant non nul en termes de certains covariants quadratiques des formes cubiques et quartiques. Cela nous permet d'étendre les travaux de Hooley et de donner des formules asymptotiques précises pour le nombre d'entiers appartenant à un intervalle et représentables par une forme cubique ou quartique donnée. En outre, nous déterminons le corps de définition des droites contenues dans certaines surfaces cubiques et quartiques associées à des formes cubiques et quartiques binaires.

Abstract. In this paper we determine the group of rational automorphisms of binary cubic and quartic forms with integer coefficients and non-zero discriminant in terms of certain quadratic covariants of cubic and quartic forms. This allows one to give precise asymptotic formulae for the number of integers in an interval representable by a binary cubic or quartic form and extends work of Hooley. Further, we give the field of definition of lines contained in certain cubic and quartic surfaces related to binary cubic and quartic forms.

1. Introduction

Let F be a binary form with integer coefficients, non-zero discriminant, and degree $d \geq 3$. For each positive number Z, put $R_F(Z)$ for the number of integers of absolute value at most Z which is representable by the binary form F. In [7], [8], and [9] C. Hooley gave explicitly the asymptotic formula for the quantity $R_F(Z)$ when F is an irreducible binary cubic form or a biquadratic quartic form. Various authors have dealt with the case when F is a diagonal form; see [15] for a summary of these results.

In [15], Stewart and Xiao proved the existence of an asymptotic formula for $R_F(Z)$ for all F with integer coefficients, non-zero discriminant, and degree $d \geq 3$. More precisely, they proved that for each such F, there exists a positive rational number W_F which depends only on F for which the asymptotic formula

$$R_F(Z) \sim W_F A_F Z^{\frac{2}{d}} \tag{1.1}$$

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holds with a power-saving error term. Here A_F is the area of the region

$$\{(x,y) \in \mathbb{R}^2 : |F(x,y)| \le 1\}.$$

The power-saving error term is obtained from the p-adic determinant method developed by Heath–Brown in [6], and its subsequent refinement by Salberger [13].

Stewart and Xiao showed that W_F is an explicit function of the *rational automorphism group* of F. To define this group, consider the substitution action of $\mathrm{GL}_2(\mathbb{C})$ on binary forms given as follows: for $T = \left(\begin{smallmatrix} t_1 & t_2 \\ t_3 & t_4 \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{C})$, put

$$F_T(x,y) = F(t_1x + t_2y, t_3x + t_4y). \tag{1.2}$$

Then the rational automorphism group $\operatorname{Aut} F$ of a binary form F is defined to be:

$$\operatorname{Aut} F = \{T \in \mathrm{GL}_2(\mathbb{Q}) : F_T(x,y) = F(x,y)\}. \tag{1.3}$$

We shall also denote by $\operatorname{Aut}_{\mathbb{F}} F$ to be the maximal subgroup of $\mathrm{GL}_2(\mathbb{F})$ which fixes F via the action (1.2), for any subring $\mathbb{F}$ of the complex numbers $\mathbb{C}$.

In [15], it was not shown how to obtain $\operatorname{Aut} F$, and therefore W_F, explicitly from the coefficients of F, except for the case of diagonal forms of the shape $Ax^d + By^d$. In general, this can be quite difficult. In this paper, we show how to compute $\operatorname{Aut}_{\mathbb{F}} F$ of F for any subring $\mathbb{F} \subset \mathbb{C}$ when F is a binary cubic or quartic form with complex coefficients. As one will see, our arguments apply equally well to many other rings, for example fields of characteristic 0. Since our primary interest is in $\operatorname{Aut} F$ we have decided to keep the exposition succinct at the cost of the extra generality. As an immediate application we show how to determine W_F when $d = 3, 4$, and thereby completing the work of Hooley in [7], [8], and [9] for degrees 3 and 4. We thus obtain the following theorem:

Theorem 1.1. *Let F be a binary form with integer coefficients, non-zero discriminant, and degree $d \in \{3,4\}$. Then for each $d = 3,4$ there exists a positive number $\beta_d < 2/d$ which depends only on d and a positive rational number W_F such that for all $\varepsilon > 0$, the asymptotic formula*

$$R_F(Z) = W_F A_F Z^{\frac{2}{d}} + O_{F,\varepsilon}\left(Z^{\beta_d + \varepsilon}\right) \tag{1.4}$$

holds. Moreover, the quantity W_F can be explicitly determined in terms of the coefficients of F.

Theorem 1.1 will follow from Theorem 1.2 in [15] and Theorems 3.1 and 4.1, which give explicit methods to determine $\operatorname{Aut} F$ from the coefficients of F when $\deg F = 3, 4$ respectively, provided that we can compute the area A_F.

It remains to give an explicit expression for the quantity A_F in terms of the coefficients of F. In the cubic case this was done by Hooley himself in [7] and [8], where he showed that A_F is a constant times a power of the discriminant of F, and that this constant is expressible in terms of values of the gamma function. Bean gave an explicit formula for A_F in terms of hypergeometric functions in [1]. We further note that for all $d \geq 3$ (not necessarily $d \in \{3, 4\}$), A_F was determined by Stewart and Xiao in [15] in the case of diagonal forms of the shape $Ax^d + By^d$, where they gave A_F in terms of the discriminant $\Delta(F)$ of F and the gamma function.

Our explicit characterization of automorphism groups of binary cubic and quartic forms even over $\mathbb{C}$ allows us to study lines on algebraic surfaces of the shape

$$X_F : F(x_1, x_2) - F(x_3, x_4) = 0. \tag{1.5}$$

For cubic surfaces, it is a celebrated theorem of Cayley and Salmon that cubic surfaces contain exactly 27 lines over an algebraically closed field. However, for a cubic surface defined over $\mathbb{Q}$, these lines are typically not defined over $\mathbb{Q}$. There exists a unique smallest finite extension $K/\mathbb{Q}$ such that all 27 lines are defined. In particular, for the generic cubic surface defined over $\mathbb{Q}$, this field is Galois over $\mathbb{Q}$ and its Galois group is isomorphic to $W(E_6)$, the Weyl group for the E_6 root system. Ekedahl [5] found an explicit example of a cubic surface which realizes this bound. We shall prove that when F is a cubic form the field of definition of the lines on the surface X_F given by (1.5) is very small.

For quartic surfaces, it is not known in general how many lines they contain. The generic quartic surface contains no lines; see [3]. Recall that the $\mathrm{PGL}_2(\mathbb{C})$-automorphism group of a binary form F is the maximal subgroup of $\mathrm{PGL}_2(\mathbb{C})$ which permutes the projective roots of F via action by Möbius transformation. It is a consequence of Theorem 3.1 in [3] that the surface X_F given in (1.5) contains exactly $d(d + \upsilon_F)$ many lines, where υ_F is the number of elements in the $\mathrm{PGL}_2(\mathbb{C})$-automorphism group of F and d is the degree of F. For the quartic case this was already known to Segre; see [14].

Put

$$F(x, y) = a_4x^4 + a_3x^3y + a_2x^2y^2 + a_1xy^3 + a_0y^4.$$

The ring of polynomial invariants of binary quartic forms with respect to the action (1.2) is generated by two elements, usually denoted by $I(F)$ and $J(F)$, given by

$$I(F) = 12a_4a_0 - 3a_3a_1 + a_2^2 \tag{1.6}$$

and

$$J(F) = 72a_4a_2a_0 + 9a_3a_2a_1 - 27a_4a_1^2 - 27a_0a_3^2 - 2a_2^3. \tag{1.7}$$

It is known from the work of Klein [11] and later Segre [14] that the $\mathrm{PGL}_2(\mathbb{C})$-automorphism group of a binary quartic form F with complex coefficients and non-zero discriminant is isomorphic to the Klein group $\mathcal{C}_2 \times \mathcal{C}_2$ unless the invariants $I(F), J(F)$ vanishes. Specifically, the $\mathrm{PGL}_2(\mathbb{C})$-automorphism group of a quartic form F is isomorphic to the dihedral group $\mathcal{D}_4$ if $J(F) = 0$ and isomorphic to the alternating group $\mathcal{A}_4$ if $I(F) = 0$; see Proposition 2.1. We do not know how to explicitly determine the field of definitions of the lines on X_F corresponding to the extra automorphisms when $I, J = 0$, but in the generic case when $I(F) \cdot J(F) \neq 0$, we can determine the field of definition of all lines on X_F. We obtain the following theorem:

Theorem 1.2. *Let F be a binary cubic or quartic form with non-zero discriminant and integer coefficients. Let X_F be the algebraic surface defined by* (1.5)*. Then*

(1) *for $\deg F = 3$, X_F contains exactly 27 distinct lines over $\overline{\mathbb{Q}}$, and these lines are defined over a field of degree at most 12 over $\mathbb{Q}$.*
(2) *for $\deg F = 4$, X_F contains exactly 32 distinct lines over $\overline{\mathbb{Q}}$ if both $I(F)$ and $J(F)$ are non-zero, 48 lines when $J(F) = 0$, and 64 lines when $I(F) = 0$. Further, when $I(F) \cdot J(F) \neq 0$, these lines are defined over a field of degree at most 48 over $\mathbb{Q}$.*

We shall denote by $\mathcal{C}_n$ the cyclic group of order n, $\mathcal{D}_n$ for the dihedral group of order $2n$, $\mathcal{A}_n$ for the alternating group on n letters, and $\mathcal{S}_n$ for the symmetric group on n letters throughout this paper. Moreover, for a binary form F, we shall denote its discriminant by $\Delta(F)$.

Remark 1.3. Throughout the paper we work over subfields of the complex numbers, but as will be apparent most of the arguments can be carried out over fields of characteristic zero and for all sufficiently large characteristic.

2. Automorphism groups of binary cubic and quartic forms over large fields

For a binary form F of degree d with complex coefficients, let $\mathcal{B}_F$ denote the set of roots of F in $\mathbb{P}^1(\mathbb{C})$. An element $T = \left(\begin{smallmatrix} t_1 & t_2 \\ t_3 & t_4 \end{smallmatrix}\right) \in \mathrm{PGL}_2(\mathbb{C})$ acts on a point $\theta \in \mathbb{P}^1(\mathbb{C})$ via the Möbius action

$$T\theta = \frac{t_1\theta + t_2}{t_3\theta + t_4}. \tag{2.1}$$

For a finite set $\mathcal{S} \subset \mathbb{P}^1(\mathbb{C})$, put

$$T\mathcal{S} = \{T\theta : \theta \in \mathcal{S}\}.$$

Define the $\mathrm{PGL}_2(\mathbb{C})$-*automorphism group of* F to be

$$\mathrm{Aut}^*_{\mathbb{C}} F = \{T \in \mathrm{PGL}_2(\mathbb{C}) : T\mathcal{B}_F = \mathcal{B}_F\}. \tag{2.2}$$

It is easily seen that

$$\mathrm{Aut}^*_{\mathbb{C}} F = \{T \in \mathrm{PGL}_2(\mathbb{C}) : F_T = \lambda F \text{ for some } \lambda \in \mathbb{C}^\times\}. \tag{2.3}$$

It is well-known that $\mathrm{Aut}^*_{\mathbb{C}}(F)$ can be embedded into the symmetric group $\mathcal{S}_d$ via the action (2.1) of PGL_2 on the roots of $F(x,1)$, viewed as elements in $\mathbb{P}^1(\mathbb{C})$. Moreover, the natural homomorphism

$$\mathrm{Aut}_{\mathbb{C}}(F) \to \mathrm{Aut}^*_{\mathbb{C}}(F)$$

has kernel given by $\{\mu_d I_{2\times 2} : \mu_d \text{ is a } d\text{-th root of unity}\}$.

We shall prove the following Proposition:

Proposition 2.1. *Let F be a binary form with non-zero discriminant, complex coefficients, and $\deg F \in \{3,4\}$. Then*

$$\mathrm{Aut}^*_{\mathbb{C}}(F) \cong \begin{cases} \mathcal{S}_3, & \text{if } d=3 \\ \mathcal{C}_2 \times \mathcal{C}_2, \mathcal{D}_4, \mathcal{A}_4, & \text{if } d=4. \end{cases}$$

Moreover $\mathrm{Aut}^*_{\mathbb{C}}(F) \cong \mathcal{D}_4$ when $d=4$ if and only if $J(F)=0$ and $\mathrm{Aut}^*_{\mathbb{C}}(F) \cong \mathcal{A}_4$ if and only if $I(F)=0$.

We have the following lemma for binary cubic forms with non-zero discriminant:

Lemma 2.2. *Let F be a binary cubic form with complex coefficients and non-zero discriminant. Then F is $\mathrm{GL}_2(\mathbb{C})$ equivalent to $xy(x+y)$.*

Proof. This follows from the fact that the action of $\mathrm{PGL}_2(\mathbb{C})$ on $\mathbb{P}^1(\mathbb{C})$ is 3-transitive. ☐

A similar lemma, due to Cayley (see [12] for an account), holds in the quartic case:

Lemma 2.3. *Let F be a binary quartic form with complex coefficients and non-zero discriminant. Then there exists a complex number A such that F is $\mathrm{GL}_2(\mathbb{C})$ equivalent to $x^4+Ax^2y^2+y^4$. Moreover, every binary quartic form F with complex coefficients and non-zero discriminant with $I(F)=0$ is $\mathrm{GL}_2(\mathbb{C})$-equivalent to*

$$x(x^3+y^3)$$

and equivalent to

$$x^4+y^4$$

if $J(F)=0$.

We shall next require the following lemma, which follows from simple group theory:

Lemma 2.4. *Let F be a binary form with complex coefficients. Then for $T \in \mathrm{GL}_2(\mathbb{C})$, we have*

$$\mathrm{Aut}_{\mathbb{C}} F_T = T^{-1}(\mathrm{Aut}_{\mathbb{C}} F)T$$

and likewise

$$\mathrm{Aut}^*_{\mathbb{C}} F_T = T^{-1}(\mathrm{Aut}^*_{\mathbb{C}} F)T.$$

This allows us to prove the following:

Lemma 2.5. *Let F be a binary quartic form with complex coefficients and non-zero discriminant. Then $\mathrm{Aut}^*_{\mathbb{C}} F$ contains an element of order 3 if and only if $I(F) = 0$, and contains an element of order 4 if and only if $J(F) = 0$.*

Proof. Suppose that $\mathrm{Aut}^*_{\mathbb{C}} F$ contains an element T of order 3. By Lemma 2.4, we may assume that

$$T = \begin{pmatrix} 1 & 0 \\ 0 & \mu_3 \end{pmatrix},$$

where μ_3 is a primitive third root of unity. It then follows that F is $\mathrm{GL}_2(\mathbb{C})$-equivalent to a form of the shape

$$\mathcal{F}(x, y) = Ax(x^3 + y^3)$$

for some complex number A, and from (1.6) one immediately sees that $I(F) = 0$. A similar argument shows that if $\mathrm{Aut}^*_{\mathbb{C}} F$ contains an element of order 4, then F is equivalent to $x^4 + y^4$ and thus $J(F) = 0$.

For the converse, if $\Delta(F) \neq 0$ then Lemma 2.3 implies that $\mathrm{Aut}^*_{\mathbb{C}} F$ contains elements of order 3 and 4 when $I(F) = 0$ and $J(F) = 0$, respectively. □

Now we may give a proof of Proposition 2.1.

Proof of Proposition 2.1. Let F be a binary cubic form with complex coefficients and non-zero discriminant. Then, by Lemma 2.2, it follows that F is $\mathrm{GL}_2(\mathbb{C})$-equivalent to $F_0 = xy(x + y)$. A quick calculation reveals that $\mathrm{Aut}^*_{\mathbb{C}} F_0$ is generated by the pair of matrices

$$\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

and it is routine to check that $\mathrm{Aut}^*_{\mathbb{C}} F_0$ is isomorphic to $\mathcal{S}_3$. Thus Lemma 2.4 shows that $\mathrm{Aut}^*_{\mathbb{C}} F \cong \mathcal{S}_3$, as desired.

Now let F be a binary quartic form with non-zero discriminant. By Lemma 2.3 it follows that F is $\mathrm{GL}_2(\mathbb{C})$-equivalent to $F_1 = x^4 + Ax^2y^2 + y^4$ for some complex number A. It is easily checked that $\mathrm{Aut}^*_{\mathbb{C}} F_1$ contains the set

$$\left\{ I_{2\times 2}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\},$$

which is a set of representatives of a group isomorphic to $\mathcal{C}_2 \times \mathcal{C}_2$ in $\mathrm{PGL}_2(\mathbb{C})$.

Note that $\mathrm{Aut}^*_{\mathbb{C}}(F)$ is a subgroup of the symmetric group $\mathcal{S}_4$. By Lemma 2.5, $\mathrm{Aut}^*_{\mathbb{C}}(F)$ contains an element of order 3 when $I(F) = 0$. The only subgroups of $\mathcal{S}_4$ containing both $\mathcal{C}_2 \times \mathcal{C}_2$ and an element of order 3 are the alternating group $\mathcal{A}_4$ and $\mathcal{S}_4$ itself. By Lemma 2.5 we see that $\mathrm{Aut}^*_{\mathbb{C}} F$ cannot equal $\mathcal{S}_4$, since otherwise $\mathrm{Aut}^*_{\mathbb{C}} F$ contains an element of order 4, which implies that $J(F) = 0$; and since $27\Delta(F) = 4I(F)^3 - J(F)^2$, this contradicts the assumption that F has non-zero discriminant. Similarly, the only subgroups of $\mathcal{S}_4$ which contain $\mathcal{C}_2 \times \mathcal{C}_2$ and an element of order 4 are $\mathcal{D}_4$ and $\mathcal{S}_4$ itself, and the latter contains an element of order 3; hence cannot be isomorphic to $\mathrm{Aut}^*_{\mathbb{C}} F$ for F with non-zero discriminant by Lemma 2.5. This completes the proof of Proposition 2.1. $\square$

3. Binary cubic forms

Suppose

$$F(x, y) = b_3x^3 + b_2x^2y + b_1xy^2 + b_0y^3$$

is a binary cubic form with integer coefficients and non-zero discriminant. We shall assume, after applying a $\mathrm{GL}_2(\mathbb{Z})$-action if necessary, that $b_3 \neq 0$. It is known that there is a single *rational* quadratic covariant of F, given by the Hessian $q_F(x, y) = Ax^2 + Bxy + Cy^2$, where A, B, C are as below:

$$A = b_2^2 - 3b_3b_1, B = b_2b_1 - 9b_3b_0, C = b_1^2 - 3b_2b_0. \tag{3.1}$$

Put $D = B^2 - 4AC$. It is known that $D = -3\Delta(F)$.

In his thesis, G. Julia identified three additional *irrational*, or algebraic, quadratic covariants which depend on the roots $\theta_1, \theta_2, \theta_3$ of $F(x, 1)$ in [10]. We shall write the Julia covariant with respect to a root θ of $F(x, 1)$ as follows:

$$J_\theta(x, y) = h_2x^2 + h_1xy + h_0y^2, \tag{3.2}$$

where

$$\begin{aligned} h_2 &= 9b_3^2\theta^2 + 6b_3b_2\theta + 6b_3b_1 - b_2^2, \\ h_1 &= 6b_3b_2\theta^2 + 6(b_2^2 - b_3b_1)\theta + 2b_2b_1, \\ h_0 &= 3b_3b_1\theta^2 + 3(b_2b_1 - 3b_3b_0)\theta + 2b_1^2 - 3b_2b_0. \end{aligned}$$

Cremona showed that h_2, h_1, h_0 are algebraic integers in [4], in the discussion immediately following equation (11). Thus, whenever θ is rational, J_θ has rational integral coefficients.

For a binary quadratic form $f(x, y) = ax^2 + bxy + cy^2$ with complex coefficients, define

$$\mathcal{M}_f = \begin{pmatrix} b & 2c \\ -2a & -b \end{pmatrix} \tag{3.3}$$

and

$$\mathcal{N}_f = \frac{1}{2\Delta(f)} \begin{pmatrix} b\sqrt{-3\Delta(f)} - \Delta(f) & 2c\sqrt{-3\Delta(f)} \\ -2a\sqrt{-3\Delta(f)} & -b\sqrt{-3\Delta(f)} - \Delta(f) \end{pmatrix}. \tag{3.4}$$

Here the square root of a complex number is the principal square root with non-negative real part and positive imaginary part if the real part vanishes.

Define

$$\mathcal{T}_\theta = \frac{-1}{6\Delta(F)} \mathcal{M}_{J_\theta} \mathcal{M}_{q_F}. \tag{3.5}$$

When θ is rational, the matrix $6\Delta(F)\mathcal{T}_\theta$ has integer entries since $\Delta(q_F) = -3\Delta(F)$ and $\Delta(J_\theta) = 12\Delta(F)$.

We have the following theorem:

Theorem 3.1. *Let F be a binary cubic form with integer coefficients and non-zero discriminant. Then:*

1. $\operatorname{Aut} F = \{I_{2\times 2}\}$ *if and only if F is irreducible and $\Delta(F)$ is not a square.*
2. $\operatorname{Aut} F$ *is generated by* $\mathcal{N}_{q_F} \in \mathrm{GL}_2(\mathbb{Q})$ *and is isomorphic to* $\mathcal{C}_3$ *if and only if F is irreducible and $\Delta(F)$ is a square.*
3. $\operatorname{Aut} F$ *is generated by $\mathcal{T}_\theta$ for the unique rational root θ of $F(x, 1)$ and is isomorphic to $\mathcal{C}_2$ if and only if F has exactly one rational linear factor over $\mathbb{Q}$, corresponding to the root θ.*
4. $\operatorname{Aut} F = \{I_{2\times 2}, \mathcal{N}_{q_F}, \mathcal{N}_{q_F}^2, \mathcal{T}_{\theta_1}, \mathcal{T}_{\theta_2}, \mathcal{T}_{\theta_3}\} \cong \mathcal{D}_3$ *if and only if F splits completely over $\mathbb{Q}$.*

We shall prove the following result, from which Theorem 3.1 will follow:

Proposition 3.2. *Let F be a binary cubic form with complex coefficients and non-zero discriminant. Suppose that the x^3-coefficient of F is non-zero and let $\theta_1, \theta_2, \theta_3$ be the three distinct roots of $F(x, 1)$. Then a set of representatives of* $\operatorname{Aut}^*_{\mathbb{C}}(F)$ *in* $\mathrm{GL}_2(\mathbb{C})$ *is given by*

$$\left\{ I_{2\times 2}, \mathcal{T}_{\theta_1}, \mathcal{T}_{\theta_2}, \mathcal{T}_{\theta_3}, \mathcal{N}_{q_F}, \mathcal{N}_{q_F}^2 \right\}.$$

Proof. By Lemma 2.2 and the observation that q_F and $\mathcal{J}_{\theta_i}, i = 1, 2, 3$ are covariants of F, it suffices to prove Proposition 3.2 for any binary cubic form with non-zero discriminant and non-zero leading coefficient. We choose

$$F(x, y) = 2x^3 + 3x^2 y + xy^2.$$

The roots of $F(x, 1)$ are then $\theta_1 = 0, \theta_2 = -1, \theta_3 = -1/2$. Computing the Julia covariants we then see that they are given by

$$J_1(F) = 3x^2 + 6xy + 2y^2, J_2(F) = 3x^2 - y^2, J_3(F) = -6x^2 - 6xy - y^2,$$

whence

$$\mathcal{M}_{\mathcal{J}_1} = \begin{pmatrix} 6 & 4 \\ -6 & -6 \end{pmatrix}, \quad \mathcal{M}_{\mathcal{J}_2} = \begin{pmatrix} 0 & -2 \\ -6 & 0 \end{pmatrix}, \quad \mathcal{M}_{\mathcal{J}_3} = \begin{pmatrix} -6 & -2 \\ 12 & 6 \end{pmatrix}.$$

The Hessian covariant of F is given by

$$q_F(x,y) = 3x^2 + 3xy + y^2$$

and

$$\mathcal{M}_{q_F} = \begin{pmatrix} 3 & 2 \\ -6 & -3 \end{pmatrix}.$$

It thus follows that

$$\mathcal{T}_1 = \begin{pmatrix} 1 & 0 \\ -3 & -1 \end{pmatrix}, \quad \mathcal{T}_2 = \begin{pmatrix} -2 & -1 \\ 3 & 2 \end{pmatrix}, \quad \mathcal{T}_3 = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}.$$

A quick calculation then shows that $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3$ fix F by substitution. Similarly, one can check that

$$\mathcal{N}_{q_F} = \begin{pmatrix} -2 & -1 \\ 3 & 1 \end{pmatrix}$$

and its square both fix F via substitution. This completes the proof. $\square$

We remark that the requirement for the x^3-coefficient of F be non-zero is merely in place because of how the Julia covariants are defined. Indeed the statement holds for all binary cubic forms with non-zero discriminant, because the Julia covariants are covariants.

We may now prove Theorem 3.1.

Proof of Theorem 3.1. Put $\operatorname{Aut}^* F$ for the subset of $\operatorname{Aut}^*_{\mathbb{C}} F$ defined over the rationals. We note that the natural map

$$\operatorname{Aut}^* F \to \operatorname{Aut} F$$

is an isomorphism, since $\mathbb{Q}$ does not contain any non-trivial cube roots of unity. Thus, the elements of $\operatorname{Aut} F$ must come from the set

$$\{I_{2\times 2}, \mathcal{T}_{\theta_1}, \mathcal{T}_{\theta_2}, \mathcal{T}_{\theta_3}, \mathcal{N}_{q_F}, \mathcal{N}_{q_F}^2\}.$$

Observe that $\mathcal{M}_{q_F} \in \mathrm{GL}_2(\mathbb{Q})$ for all binary cubic forms with integer coefficients. From here it is plain that $\mathcal{T}_\theta$ can be in $\mathrm{GL}_2(\mathbb{Q})$ only if $\mathcal{M}_{\mathcal{J}_\theta}$ has rational coefficients, and from (3.2) we see that this can only occur when θ is at most a quadratic irrational. Therefore if F is irreducible, then $\mathcal{T}_\theta$ does not lie in $\operatorname{Aut} F$.

We show that in fact J_θ has integral coefficients if and only if $\theta \in \mathbb{Q}$. Put $\theta = u + v\sqrt{k}$, with $u, v \in \mathbb{Q}, k \in \mathbb{Z}$. Since $h_2, h_1, h_0 \in \mathbb{Z}$, it follows that

$$9b_3^2(2uv\sqrt{k}) + 6b_3b_2(v\sqrt{k}) = 0,$$
$$6b_3b_2(2uv\sqrt{k}) + 6(b_2^2 - b_3b_1)(v\sqrt{k}) = 0,$$
$$3b_3b_1(2uv\sqrt{k}) + 3(b_2b_1 - 3b_3b_0)(v\sqrt{k}) = 0.$$

If $v = 0$, then we are done. Otherwise, we see that u must satisfy

$$3b_3^2u = -b_3b_2, 2b_3b_2u = b_3b_1 - b_2^2, 2b_3b_1u = 3b_3b_0 - b_1b_2.$$

If $b_3 = 0$, then the first and second equations imply that $b_2 = 0$, so $\Delta(F) = 0$. Therefore $b_3 \neq 0$ and we get that $u = \frac{-b_2}{3b_3}$. This then gives $b_2^2 = 3b_3b_1$ and $b_1b_2 = 9b_3b_0$. From (3.1) we see again that q_F must be singular, which implies that F is singular. Therefore whenever F is non-singular and $J_\theta \in \mathbb{Z}[x,y]$, we must have $\theta \in \mathbb{Q}$.

By examining the explicit formula in (3.4), it follows that $\mathcal{N}_{q_F} \in \mathrm{GL}_2(\mathbb{Q})$ only when $-3\Delta(q_F)$ is a square, which is equivalent to $\Delta(F)$ being a square. Thus, when $\Delta(F)$ is not a square and F is irreducible, $\operatorname{Aut} F$ contains just the identity matrix.

When F is reducible, say θ is a rational root of $F(x,1)$, we see that $\mathcal{T}_\theta$ does indeed lie in $\mathrm{GL}_2(\mathbb{Q})$. An elementary calculation shows that if $F(x,1)$ has a unique rational root then $\Delta(F)$ is not a square, and thus $\mathcal{N}_{q_F} \notin \mathrm{GL}_2(\mathbb{Q})$. Therefore, $\mathcal{T}_\theta$ is the only non-trivial element of $\operatorname{Aut} F$. Finally, if $F(x,1)$ has three rational roots, it is obvious from the definition of the discriminant that $\Delta(F)$ is a square and hence $\mathcal{T}_{\theta_i}, i = 1,2,3$ and $\mathcal{N}_{q_F}, \mathcal{N}_{q_F}^2$ are all rational. □

4. Binary quartic forms

Suppose

$$F(x,y) = a_4x^4 + a_3x^3y + a_2x^2y^2 + a_1xy^3 + a_0y^4$$

is a binary quartic form with integer coefficients and non-zero discriminant. For a binary quadratic form $f(x,y) = ax^2 + bxy + cy^2$ with real coefficients and non-zero discriminant, put

$$U_f = \frac{1}{\sqrt{|\Delta(f)|}}\mathcal{M}_f. \tag{4.1}$$

We say that a binary quadratic form f with complex coefficients is *rationally good* if it is proportional over $\mathbb{C}$ to a quadratic form g with integer coefficients and $|\Delta(g)|$ is the square of an integer. Otherwise, we say that f is *rationally bad*.

Binary quartic forms have a degree 6 covariant given by

$$\begin{aligned} F_6(x,y) = (a_3^3 + 8a_4^2a_1 - 4a_4a_3a_2)x^6 \\ + 2(16a_4^2a_0 + 2a_4a_3a_1 - 4a_4a_2^2 + a_3^2a_2)x^5y \\ + 5(8a_4a_3a_0 + a_3^2a_1 - 4a_4a_2a_1)x^4y^2 \\ + 20(a_3^2a_0 - a_4a_1^2)x^3y^3 \\ - 5(8a_4a_1a_0 + a_3a_1^2 - 4a_3a_2a_0)x^2y^4 \\ - 2(16a_4a_0^2 + 2a_3a_1a_0 - 4a_2^2a_0 + a_2a_1^2)xy^5 \\ - (a_1^3 + 8a_3a_0^2 - 4a_2a_1a_0)y^6. \end{aligned} \tag{4.2}$$

We call a quadratic form divisor f of F_6 *significant* if the quartic form $G = F_6/f$ satisfies $J(G) = 0$.

It turns out that the covariant F_6 and its significant factors controls the behaviour of $\operatorname{Aut} F$. We have the following theorem:

Theorem 4.1. *Let F be a binary quartic form with integer coefficients and non-zero discriminant.*

(1) $\operatorname{Aut} F = \{\pm I_{2\times 2}\}$ *if and only if F_6 does not have any real rationally good significant quadratic factors.*
(2) $\operatorname{Aut} F$ *is generated by $U_f \in \mathrm{GL}_2(\mathbb{Q})$ and $-I_{2\times 2}$ if and only if F_6 has a unique real rationally good significant factor f. In this case* $\operatorname{Aut} F$ *is isomorphic to $\mathcal{C}_2 \times \mathcal{C}_2$ or $\mathcal{C}_4$.*
(3) $\operatorname{Aut} F = \{\pm I_{2\times 2}, \pm U_{f_1}, \pm U_{f_2}, \pm U_{f_3}\} \cong \mathcal{D}_4$ *if and only if F_6 can be written as $F_6 = f_1 f_2 f_3$ where f_i is a real rationally good significant factor of F_6 for $i = 1, 2, 3$.*

We remark that the sextic covariant F_6 of a binary quartic form F is always a *Klein form*; see Lemma 4.2. This fact does not appear to be well-known. Given the significance of Klein forms in problems involving the super-elliptic equation (see [2]), this phenomenon may be of independent interest.

4.1. Binary sextic Klein forms and significant quadratic factors. There is a simple characterization of the elements in $\operatorname{Aut} F$ in terms of *significant* quadratic factors of the sextic covariant F_6 given in (4.2).

A degree six binary form

$$G(x,y) = g_6x^6 + g_5x^5y + g_4x^4y^2 + g_3x^3y^3 + g_2x^2y^4 + g_1xy^5 + g_0y^6$$

is said to be a *Klein form* if its coefficients satisfy the following quadratic equations (see [2]):

$$\begin{aligned} 10g_6g_2 - 5g_5g_3 + 2g_4^2 &= 0 \\ 25g_6g_1 - 5g_5g_2 + g_3g_4 &= 0 \\ 50g_6g_0 - 2g_2g_4 + g_3^2 &= 0. \end{aligned} \tag{4.3}$$

Moreover it is known that all binary sextic Klein forms with complex coefficients and non-zero discriminant are $\mathrm{GL}_2(\mathbb{C})$-equivalent to each other, a fact already known to Klein [11].

We have the following fact, which appears to be new:

Lemma 4.2. *Let F be a binary quartic form with complex coefficients and non-zero discriminant. Then its sextic covariant F_6, given in* (4.2), *is a Klein form with non-zero discriminant.*

Proof. Since all binary quartic forms with non-zero discriminant are equivalent to a form of the shape $x^4 + Ax^2y^2 + y^4$ for some complex number A, it suffices to verify that the sextic covariant of $F(x,y) = x^4 + Ax^2y^2 + y^4$ is a Klein form. A quick calculation shows that F_6 is proportional over $\mathbb{C}$ to

$$G(x,y) = xy(x^4 - y^4),$$

which is independent of A. We then see that $\Delta(G) \neq 0$ and that the coefficients of G satisfy the quadratic equations in (4.3). $\square$

By Lemma 4.2, the deduction of Theorem 4.1 from Proposition 4.6 will follow from the following lemmas.

Lemma 4.3. *Let G be a sextic Klein form with non-zero discriminant. Then G can be written as $G = G_1G_2G_3$, where each G_i is a significant quadratic factor of G, in only one way up to permutation of the factors and up to homothety over $\mathbb{C}$.*

Proof. Since all binary sextic Klein forms with non-zero discriminant lie in a single $\mathrm{GL}_2(\mathbb{C})$-orbit, it suffices to prove Lemma 4.3 for just the Klein form $G(x,y) = xy(x^4 - y^4)$. We see that factoring G as $G = G_1G_2G_3$, with

$$G_1 = xy, G_2 = x^2 - y^2, G_3 = x^2 + y^2$$

that each G_i is a significant factor of G; that is, the quartic form $\mathcal{G}_1 = G/G_1 = x^4 - y^4$ satisfies $J(\mathcal{G}_1) = 0$, and similarly for $\mathcal{G}_i = G/G_i$ for $i = 2, 3$.

Now pick another quadratic factor of G, say $V(x,y) = x(x+y)$. Then $\mathcal{V} = G/V = y(x-y)(x^2+y^2)$ has

$$J(\mathcal{V}) = 72(0)(-1)(-1) + 9(1)(-1)(1) - 27(0)(1)^2 - 27(-1)(1)^2 - 2(-1)^3 = 20.$$

A similar calculation shows that for any other quadratic factor V distinct from G_1, G_2, G_3, that $J(G/V) \neq 0$, whence V is not a significant factor of G. $\square$

4.2. $\mathrm{Aut}^*_{\mathbb{C}}$ *F* for binary quartic forms. In this section we aim to show that $\mathrm{Aut}^*_{\mathbb{C}}(F)$ is determined explicitly by certain quadratic covariants of a binary quartic form F, called the *Cremona covariants*, which are significant divisors of F_6. Let

$$F(x,y) = a_4x^4 + a_3x^3y + a_2x^2y^2 + a_1xy^3 + a_0y^4$$

be a binary quartic form. We shall assume, by applying a $\mathrm{GL}_2(\mathbb{Z})$-action if necessary, that $a_4 \neq 0$. Unlike the cubic case, there are no rational quadratic covariants for binary quartic forms. However, there are three *irrational* quadratic covariants discovered by Cremona [4]. These covariants can be given explicitly in terms of the roots of $F(x,1)$. Define $\chi(F)$ to be the number of real roots of $F(x,1)$. We will then label the roots θ_i, $i = 1,2,3,4$

of $F(x,1)$ as in [1]:

$$\begin{cases} \theta_1 > \theta_2 > \theta_3 > \theta_4, & \text{if } \chi(F) = 4, \\ \theta_1 > \theta_2, \theta_3 = \overline{\theta_4}, \Im(\theta_3) > 0, & \text{if } \chi(F) = 2, \\ \theta_1 = \overline{\theta_2}, \theta_3 = \overline{\theta_4}, \Im(\theta_1) > 0, \Im(\theta_3) < 0, & \text{if } \chi(F) = 0. \end{cases} \tag{4.4}$$

Here $\Im(z)$ refers to the imaginary part of the complex number z. Put

$$\begin{aligned} A_1 &= a_4(\theta_1+\theta_2-\theta_3-\theta_4), \quad B_1 = 2a_4(\theta_3\theta_4-\theta_1\theta_2), \\ &C_1 = a_4(\theta_1\theta_2(\theta_3+\theta_4)-\theta_3\theta_4(\theta_1+\theta_2)), \\ A_2 &= a_4(\theta_1+\theta_3-\theta_2-\theta_4), \quad B_2 = 2a_4(\theta_2\theta_4-\theta_1\theta_3), \\ &C_2 = a_4(\theta_1\theta_3(\theta_2+\theta_4)-\theta_2\theta_4(\theta_1+\theta_3)), \\ A_3 &= a_4(\theta_1+\theta_4-\theta_2-\theta_3), \quad B_3 = 2a_4(\theta_2\theta_3-\theta_1\theta_4), \\ &C_3 = a_4(\theta_1\theta_4(\theta_2+\theta_3)-\theta_2\theta_3(\theta_1+\theta_4)) \end{aligned} \tag{4.5}$$

and define the i-th *Cremona covariant* to be

$$\mathfrak{C}_i(x,y) = A_i x^2 + B_i xy + C_i y^2, i = 1,2,3. \tag{4.6}$$

Put

$$D_i = \Delta(\mathfrak{C}_i) \text{ for } i = 1,2,3. \tag{4.7}$$

One checks that the D_i's satisfy

$$\begin{aligned} D_1 &= 4a_4^2(\theta_1-\theta_3)(\theta_1-\theta_4)(\theta_2-\theta_3)(\theta_2-\theta_4), \\ D_2 &= 4a_4^2(\theta_1-\theta_2)(\theta_1-\theta_4)(\theta_3-\theta_2)(\theta_3-\theta_4), \\ D_3 &= 4a_4^2(\theta_1-\theta_2)(\theta_1-\theta_3)(\theta_4-\theta_2)(\theta_4-\theta_3). \end{aligned} \tag{4.8}$$

We note that (4.8) implies that $D_i \neq 0$ for $i = 1,2,3$ whenever $\Delta(F) \neq 0$. In [4], Cremona showed that the Cremona covariants $\mathfrak{C}_i$ satisfies

$$F_6(x,y) = \mathfrak{C}_1(x,y)\mathfrak{C}_2(x,y)\mathfrak{C}_3(x,y). \tag{4.9}$$

Put

$$\mathcal{U}_i = \frac{1}{\sqrt{D_i}}\mathcal{M}_{\mathfrak{C}_i}, i = 1,2,3.$$

We have the following proposition:

Proposition 4.4. *Let F be a binary quartic form with complex coefficients and non-zero discriminant. Suppose that the x^4-coefficient of F is non-zero and that $I(F)J(F) \neq 0$. Then a set of representatives of $\mathrm{Aut}^*_{\mathbb{C}}(F)$ in $\mathrm{GL}_2(\mathbb{C})$ is given by*

$$\{I_{2\times 2}, \mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3\}.$$

Moreover, for each $i = 1,2,3$ we have $F_{\mathcal{U}_i} = F$ with respect to the action (1.2)*.*

Proof. By Proposition 2.1 and its proof, it follows that $\operatorname{Aut}^*_{\mathbb{C}}(F) \cong \mathcal{C}_2 \times \mathcal{C}_2$. Therefore, it suffices to check that $\mathcal{M}_{\mathfrak{C}_i} \in \operatorname{Aut}^*_{\mathbb{C}}(F)$ for each $i = 1, 2, 3$. Let us consider the action of $\mathcal{M}_{\mathfrak{C}_1}$ on θ_1, via the action in (2.1). We have

$$U_1 : \theta_1 \mapsto \frac{B_1\theta_1 + 2C_1}{-2A_1\theta_1 - B_1}.$$

Expanding using (4.5), we obtain

$$\frac{B_1\theta_1 + 2C_1}{-2A_1\theta_1 - B_1} = \frac{-2\theta_2(\theta_1 - \theta_3)(\theta_1 - \theta_4)}{-2(\theta_1 - \theta_3)(\theta_1 - \theta_4)} = \theta_2.$$

Next we see that

$$\frac{B_1\theta_3 + 2C_1}{-2A_1\theta_3 - B_1} = \frac{2\theta_4(\theta_3 - \theta_1)(\theta_3 - \theta_2)}{2(\theta_3 - \theta_1)(\theta_3 - \theta_2)} = \theta_4.$$

A similar calculation shows that U_1 sends θ_2 to θ_1 and θ_4 to θ_3. This shows that $\mathcal{M}_{\mathfrak{C}_1}$ permutes the roots of F. A similar calculation shows that $\mathcal{M}_{\mathfrak{C}_2}, \mathcal{M}_{\mathfrak{C}_3}$ similarly permute the roots of $F(x, 1)$.

To confirm that $F_{\mathcal{U}_1} = F$ say, we further need to check that $\mathcal{U}_1$ fixes the leading coefficient of F. This is equivalent to checking that a_4 is equal to

$$\frac{1}{D_1^2}\Big(a_4B_1^4 + a_3B_1^3(-2A_1) + a_2B_1^2(-2A_1)^2 + a_1B_1(-2A_1)^3 + a_0(-2A_1)^4\Big). \tag{4.10}$$

Using the fact that $a_4 \neq 0$ and the Vieta relations

$$\begin{aligned} \frac{a_3}{a_4} &= -(\theta_1 + \theta_2 + \theta_3 + \theta_4), \\ \frac{a_2}{a_4} &= \theta_1\theta_2 + \theta_1\theta_3 + \theta_1\theta_4 + \theta_2\theta_3 + \theta_2\theta_4 + \theta_3\theta_4, \\ \frac{a_1}{a_4} &= -(\theta_1\theta_2\theta_3 + \theta_1\theta_2\theta_4 + \theta_1\theta_3\theta_4 + \theta_2\theta_3\theta_4), \end{aligned}$$

and

$$\frac{a_0}{a_4} = \theta_1\theta_2\theta_3\theta_4,$$

we see that (4.10) is equivalent to checking that

$$\begin{aligned} &(\theta_3\theta_4 - \theta_1\theta_2)^4 + (\theta_1 + \theta_2 + \theta_3 + \theta_4)(\theta_3\theta_4 - \theta_1\theta_2)^3(\theta_1 + \theta_2 - \theta_3 - \theta_4) \\ &+ (\theta_1\theta_2 + \theta_1\theta_3 + \theta_1\theta_4 + \theta_2\theta_3 + \theta_2\theta_4 + \theta_3\theta_4) \\ &\qquad \times (\theta_3\theta_4 - \theta_1\theta_2)^2(\theta_1 + \theta_2 - \theta_3 - \theta_4)^2 \\ &+ (\theta_1\theta_2\theta_3 + \theta_1\theta_2\theta_4 + \theta_1\theta_3\theta_4 + \theta_2\theta_3\theta_4)(\theta_3\theta_4 - \theta_1\theta_2)(\theta_1 + \theta_2 - \theta_3 - \theta_4)^3 \\ &+ \theta_1\theta_2\theta_3\theta_4(\theta_1 + \theta_2 - \theta_3 - \theta_4)^4 \end{aligned}$$

is equal to

$$(\theta_1 - \theta_3)^2(\theta_1 - \theta_4)^2(\theta_2 - \theta_3)^2(\theta_2 - \theta_4)^2.$$

This can be done using any standard computer algebra package (in particular, we used Sage). Thus $\mathcal{U}_1 \in \operatorname{Aut}_{\mathbb{C}} F$. The verification that $\mathcal{U}_2, \mathcal{U}_3 \in \operatorname{Aut}_{\mathbb{C}} F$ follows similarly. $\square$

Again we remark that the requirement for the x^4-coefficient to be non-zero is not essential, since the $\mathfrak{C}_i$'s are covariants. There is a more intrinsic way to define the Cremona covariants in terms of the *cubic resolvent* of F and the Hessian covariant; see [4].

The following lemma shows that the Cremona covariants $\mathfrak{C}_i$ are precisely the significant factors of F_6.

Lemma 4.5. *Let F be a binary quartic form with complex coefficients and non-zero discriminant. Then for each $i = 1, 2, 3$, the Cremona covariant $\mathfrak{C}_i$ of F is a significant factor of the sextic covariant F_6.*

Proof. Recall that each binary quartic form F with complex coefficients and non-zero discriminant is equivalent to $F_A = x^4 + Ax^2y^2 + y^4$ for some complex number A, and that the Cremona covariants of F_A are proportional to $xy, x^2 - y^2, x^2 + y^2$. Lemma 4.5 then follows from Lemma 4.3. $\square$

4.3. $\operatorname{Aut}_{\mathbb{R}} F$ for real binary quartic forms. Even though we are primarily interested in $\operatorname{Aut} F$, which is defined to be the set of $T \in \mathrm{GL}_2(\mathbb{Q})$ which fixes F via the action (1.2), it will be convenient to first consider the larger group $\operatorname{Aut}_{\mathbb{R}} F$. It is clear that $\operatorname{Aut} F \subset \operatorname{Aut}_{\mathbb{R}} F$. Proposition 4.4 shows that the matrices $\mathcal{U}_i, i = 1, 2, 3$ are in $\operatorname{Aut}_{\mathbb{C}} F$, it thus remains to check whether it is possible that $\mathcal{U}_i \in \mathrm{GL}_2(\mathbb{R})$, possibly up to multiplying by a 4-th root of unity. We have the following proposition:

Proposition 4.6. *Let F be a binary quartic form with real coefficients and non-zero discriminant. Then $\operatorname{Aut}_{\mathbb{R}} F$ is given by:*

$$\begin{cases} \{\pm I_{2\times 2}, \pm U_{\mathfrak{C}_1}, \pm U_{\mathfrak{C}_2}, \pm U_{\mathfrak{C}_3}\} & \text{if } \chi(F) = 4, \\ \{\pm I_{2\times 2}, \pm U_{\mathfrak{C}_1}\} & \text{if } \chi(F) = 2, \\ \{\pm I_{2\times 2}, \pm U_{\mathfrak{C}_1}, \pm \mu_4 U_{\mathfrak{C}_2}, \pm \mu_4 U_{\mathfrak{C}_3}\} & \text{if } \chi(F) = 0. \end{cases}$$

Proof. When $\chi(F) = 4$, it is obvious that each $\mathfrak{C}_i$ is real and thus $\mathcal{U}_i$ is real as long as $\Delta(\mathfrak{C}_i)$ is positive. This holds for $i = 1, 3$ but $D_2 < 0$, whence $\sqrt{D_2} = \mu_4 \sqrt{|D_2|}$. Therefore $\mu_4 \mathcal{U}_2 = U_{\mathfrak{C}_2} \in \operatorname{Aut}_{\mathbb{R}} F$, as desired.

When $\chi(F) = 2$, from (4.4) and (4.5) we see that $\mathfrak{C}_1$ is real with positive discriminant, while $\mathfrak{C}_2, \mathfrak{C}_3$ are neither real nor purely imaginary. Moreover, neither can be proportional over $\mathbb{C}$ to a real form. To see this, observe that from an examination of (4.4) we see that $\mathfrak{C}_2, \mathfrak{C}_3$ have coefficients which are conjugate in $\mathbb{C}$. Thus, $\mathfrak{C}_2$ is proportional to a real form if and only if $\mathfrak{C}_3$ is proportional to a real form; and moreover, they must be proportional to each other. This implies that $F_6 = \mathfrak{C}_1 \mathfrak{C}_2 \mathfrak{C}_3$ is a singular form, which by Lemma 2.3 and the proof of Lemma 4.2 shows that F itself must have vanishing discriminant.

When $\chi(F) = 0$, we see that $\mathfrak{C}_1$ is real with positive discriminant while $\mathfrak{C}_2, \mathfrak{C}_3$ have coefficients which are purely imaginary, and thus multiplying by $i = \mu_4$ turns them into real quadratic forms. It thus follows that in each case, $\operatorname{Aut}_{\mathbb{R}} F$ contains the sets given in the proposition.

It remains to check that $\operatorname{Aut}_{\mathbb{R}} F$ cannot be any larger in the cases when $I(F)$ or $J(F)$ vanishes. When $I(F) = 0$ this easily follows since $\operatorname{Aut}_{\mathbb{R}} F$, being a finite subgroup of $\mathrm{GL}_2(\mathbb{R})$, cannot contain a copy of $\mathcal{A}_4$. When $J(F) = 0$ we see that the preimage of an order 4 element in $\operatorname{Aut}^*_{\mathbb{C}}(F)$ is necessarily an element T of order 8 in $\mathrm{GL}_2(\mathbb{R})$. Since all elements of order 8 in $\mathrm{GL}_2(\mathbb{R})$ are conjugate, we may then assume $T \in \mathrm{SO}_2(\mathbb{R})$. But then by letting T permute the roots of a binary quartic form we see that T necessarily sends F to $-F$, hence $T \notin \operatorname{Aut}_{\mathbb{R}} F$. □

We may now give a proof of Theorem 4.1.

Proof of Theorem 4.1. By Proposition 4.6, the potential non-trivial elements of $\operatorname{Aut} F$ are given explicitly in terms of the Cremona covariants. For each $\varepsilon U_{\mathfrak{C}_i} \in \operatorname{Aut}_{\mathbb{R}} F$, where $\varepsilon \in \{1, \mu_4\}$, we have that $\varepsilon U_{\mathfrak{C}_i} \in \operatorname{Aut} F$ only if $\mathfrak{C}_i$ is proportional over $\mathbb{C}$ to an integral binary quadratic form $f_i(x, y) = f_{i,2}x^2 + f_{i,1}xy + f_{i,0}y^2$. In this case we have

$$\varepsilon U_{\mathfrak{C}_i} = \frac{1}{\sqrt{|\Delta(f_i)|}} \begin{pmatrix} f_{i,1} & 2f_{i,0} \\ -2f_{i,2} & -f_{i,1} \end{pmatrix}.$$

Then we see that $\varepsilon U_{\mathfrak{C}_i} \in \operatorname{Aut} F$ only if $|\Delta(f_i)|$ is a square; that is, $\mathfrak{C}_i$ is rationally good. We then see that these conditions are also sufficient for $\varepsilon U_{\mathfrak{C}_i} \in \mathrm{GL}_2(\mathbb{Q})$.

Therefore, when F_6 has no real quadratic rationally good significant factors, $\operatorname{Aut}_{\mathbb{R}} F = \{\pm I_{2\times 2}\}$. When F_6 has exactly one real quadratic rationally good significant factor g, it can have positive or negative discriminant, which will determine the order of U_f in $\mathrm{GL}_2(\mathbb{R})$. If $\Delta(f) < 0$ then U_f will have order 4 and $\operatorname{Aut}_{\mathbb{R}} F \cong \mathcal{C}_4$, and if $\Delta(f) > 0$ then U_f has order 2 and $\operatorname{Aut}_{\mathbb{R}} F \cong \mathcal{C}_2 \times \mathcal{C}_2$. Finally, if F_6 has three real quadratic rationally good significant factors $f_i, i = 1, 2, 3$ then f_i is proportional over $\mathbb{R}$ to $\varepsilon U_{\mathfrak{C}_i}$ for $i = 1, 2, 3$ and so $\operatorname{Aut} F$ is as given by Proposition 4.6. □

5. Cubic and quartic surfaces defined by binary forms and Theorem 1.2

In this section we apply our theorems characterizing the automorphism groups of binary cubic and quartic forms F to study lines on the surface X_F given in (1.5), and to prove Theorem 1.2.

For a binary form F, denote by $\mathcal{B}_F$ the set of projective roots of F in $\mathbb{P}^1(\mathbb{C})$. Let F_1, F_2 be two binary forms with complex coefficients and non-zero discriminant. Put $\mathcal{G}(F_1, F_2)$ for the set of elements in $\mathrm{PGL}_2(\mathbb{C})$ which map $\mathcal{B}_{F_1}$ to $\mathcal{B}_{F_2}$. When $\deg F_1 \neq \deg F_2$, it is obvious that $\mathcal{G}(F_1, F_2)$ is empty. When $\deg F_1 = \deg F_2 = 3$, the set $\mathcal{G}(F_1, F_2)$ always consists of six elements since $\mathrm{PGL}_2(\mathbb{C})$ is 3-transitive on $\mathbb{P}^1(\mathbb{C})$. When $\deg F_1 = \deg F_2 = 4$, the cardinality of $\mathcal{G}(F_1, F_2)$ can be $0, 4, 8, 12$. Put

$$\upsilon_{F_1,F_2} = \#\mathcal{G}(F_1, F_2) \tag{5.1}$$

and

$$\upsilon_F = \upsilon_{F,F}.$$

Consider the surface X_{F_1,F_2} defined by

$$F_1(x_1, x_2) - F_2(x_3, x_4) = 0.$$

We then have the following, which is Theorem 3.1 in [3]:

Proposition 5.1. *Let F_1, F_2 be two binary forms with non-zero discriminant and equal degree d. Then the number of lines on the surface X_{F_1,F_2} is equal to $d(d + \upsilon_{F_1,F_2})$.*

By Proposition 5.1, the number of lines on a surface X_F with F a binary form with non-zero discriminant is completely determined by $\mathrm{Aut}^*_{\mathbb{C}} F$. To prove Theorem 1.2, however, we shall need the following refinement of Proposition 5.1, which is contained in the proof of Theorem 3.1 in [3].

We shall denote a projective point in $\mathbb{P}^3$ by $[x_1 : x_2 : x_3 : x_4]$. Let F be a binary form of degree d with complex coefficients and non-zero discriminant. By applying a $\mathrm{GL}_2(\mathbb{C})$ transformation, we may assume that the leading coefficient of F is non-zero.

Lemma 5.2. *Let $\psi_1, \cdots, \psi_d$ denote the roots of $F(x, 1)$. Then all lines on the surface X_F are in exactly one of the following two categories:*

(1) *(Root lines) $L = \{[s\psi_j : s : t\psi_k : t] \in \mathbb{P}^3 : s, t \in \mathbb{C}\}$ for some $1 \leq j, k \leq d$, or*
(2) *(Automorphism lines) There exists an automorphism*

$$T = \begin{pmatrix} t_1 & t_2 \\ t_3 & t_4 \end{pmatrix} \in \mathrm{Aut}_{\mathbb{C}} F$$

such that

$$L = \{[u : v : t_1u + t_2v : t_3u + t_4v] \in \mathbb{P}^3 : u, v \in \mathbb{C}\}.$$

Proof. The proof of Lemma 5.2 follows from the proof of Theorem 3.1 in [3]. The second part of Lemma 5.2 was done by Heath–Brown in [6]. □

5.1. Cubic surfaces. We shall state a more precise version of part (1) of Theorem 1.2. For a given binary form F with integer coefficients and non-zero discriminant, write K for the field of smallest degree for which all lines contained in the surface X_F defined by (1.5) are defined over K. We put $\mathbb{F}$ for the splitting field of F, and we shall denote by μ_3 for a primitive third root of unity. We will prove the following for cubic surfaces X_F defined (1.5) and a binary cubic form $F(x, y)$:

Proposition 5.3. *Let F be a binary cubic form with integer coefficients and non-zero discriminant. Then $K = \mathbb{F}(\mu_3)$.*

Proof. From Proposition 5.1 we see that the root lines of X_F are defined over $\mathbb{F}$. For the automorphism lines, we see that $\mathcal{M}_{q_F}$ and $\mathcal{M}_{\mathcal{J}_\theta}$ are defined over $\mathbb{F}$. Hence, $\mathcal{T}_\theta$ is defined over $\mathbb{F}$. Note that the definition $\mathcal{N}_{q_F}$ and $\mathcal{N}_{q_F}^2$ involves the term $\sqrt{-3\Delta(F)}$, which may not lie in $\mathbb{F}$. We note however that $\sqrt{-3} \in \mathbb{Q}(\mu_3)$ and $\sqrt{\Delta(F)} \in \mathbb{F}$, since it is the product of the differences of the roots of $F(x, 1)$. Therefore, all of the automorphisms of F are defined over $\mathbb{F}(\mu_3)$ and hence all of the automorphism lines are defined over $\mathbb{F}(\mu_3)$. $\square$

Finally, it is clear that Proposition 5.3 implies the cubic case of Theorem 1.2, since $[\mathbb{F}(\mu_3) : \mathbb{Q}] \leq 12$.

5.2. Quartic surfaces. Let F be a binary quartic form with integer coefficients and non-zero discriminant. By Propositions 2.1 and 5.1, the surface X_F contains either 32, 48, or 64 lines depending on whether I, J vanish. Put σ_F for the number of lines contained in the surface X_F given in (1.5). We then have the following proposition:

Proposition 5.4. *Let F be a binary quartic form with integer coefficients and non-zero discriminant.*

(1) *If $I(F), J(F)$ are both non-zero, then $\sigma_F = 32$. Moreover, $[K : \mathbb{Q}] \leq 48$ with equality holding whenever $\operatorname{Gal} F \cong \mathcal{S}_4$ and $\Delta(F)$ is not the negative of a square integer.*
(2) *If $I(F) = 0$, then $\sigma_F = 64$.*
(3) *If $J(F) = 0$, then $\sigma_F = 48$.*

Proof. By Propositions 2.1, the size of $\operatorname{Aut}_{\mathbb{C}}^* F$ is $4, 8, 12$ respectively when $I(F), J(F)$ are both non-zero, when $J(F) = 0$ and $I(F) \neq 0$, and $I(F) = 0$ with $J(F) \neq 0$. Thus, by Proposition 5.1, we have

$$\sigma_F = \begin{cases} 4(4+4) = 32 & \text{if } I(F)J(F) \neq 0, \\ 4(4+8) = 48 & \text{if } J(F) = 0, I(F) \neq 0, \\ 4(4+12) = 64 & \text{if } I(F) = 0, J(F) \neq 0. \end{cases}$$

We now prove part (1) of the proposition. The treatment of the root lines is the same as the cubic case, and it is clear that the root lines are defined over $\mathbb{F}$. For the automorphism lines, the lines corresponding to $\mathcal{M}_{\mathfrak{C}_1}, \mathcal{M}_{\mathfrak{C}_2}, \mathcal{M}_{\mathfrak{C}_3}$ are defined over $\mathbb{F}$ by (4.5). The remaining automorphism lines are defined after adjoining μ_4 to $\mathbb{F}$, whence $K = \mathbb{F}(\mu_4)$. Moreover $\mathbb{F}$ is at most a degree 24 extension over $\mathbb{Q}$, thus $[K : \mathbb{Q}] \leq 48$. Observe that equality holds only when $[\mathbb{F} : \mathbb{Q}] = 24$, which implies that $\operatorname{Gal} F \cong \mathcal{S}_4$, and that $\mathbb{F}(\mu_4) \neq \mathbb{F}$. This condition is equivalent to $\mu_4 \notin \mathbb{F}$, and an elementary exercise in Galois theory yields that this happens if and only if $\Delta(F)$ is not the negative of a square integer. $\square$

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